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Commutative subalgebras of the algebra of smooth operators

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Abstract We consider the Fréchet *-algebra $\mathcal{L}(s', s) \subseteq \mathcal{L}(\ell_2)$ of the so-called smooth operators, i.e. continuous linear operators from the dual s' of the space s of rapidly decreasing sequences to s. This algebra is a non-commutative analogue of the algebra s. We characterize closed *-subalgebras of $\mathcal{L}(s', s)$ which are at the same time isomorphic to closed *-subalgebras of s and we provide an example of a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ which cannot be embedded into s.

Keywords Topological algebras of operators · Nuclear Fréchet spaces · Smooth operators

Mathematics Subject Classification 46H35 · 46J40 · 46A11 · 46A63

1 Introduction

The algebra $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to the algebra

$$\mathcal{K}_\infty := \left\{ (x_{j,k})_{j,k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}^2} : \sup_{j,k\in\mathbb{N}} |x_{j,k}| j^q k^q < \infty \text{ for all } q \in \mathbb{N}_0 \right\}$$

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of rapidly decreasing matrices (with matrix multiplication and matrix complex conjugation). Another representation of $\mathcal{L}(s', s)$ is the algebra $\mathcal{S}(\mathbb{R}^2)$ of Schwartz functions on $\mathbb{R}^2$ with the Volterra convolution

$$(f \cdot g)(x, y) := \int_{\mathbb{R}} f(x, z)g(z, y)\mathrm{d}z$$

as multiplication and the involution

$$f^*(x, y) := \overline{f(y, x)}.$$

In these forms, the algebra $\mathcal{L}(s', s)$ usually appears and plays a significant role in K-theory of Fréchet algebras (see Bhatt and Inoue [1, Ex. 2.12], Cuntz [6, p. 144], [7, p. 64–65], Glöckner and Langkamp [11], Phillips [14, Def. 2.1]) and in C^*-dynamical systems (Elliot, Natsume and Nest [9, Ex. 2.6]). Very recently, Piszczek obtained several results concerning closed ideals, automatic continuity (for positive functionals and derivations), amenability and Jordan decomposition in $\mathcal{K}_\infty$ (see Piszczek [16–19] and his forthcoming paper "The noncommutative Schwartz space is weakly amenable"). Moreover, in the context of algebras of unbounded operators, the algebra $\mathcal{L}(s', s)$ appears in the book [20] as

$$\mathbb{B}_1(s) := \{x \in \mathcal{L}(\ell_2)\colon x\ell_2 \subseteq s, x^*\ell_2 \subseteq s \text{ and } \overline{axb} \text{ is nuclear for all } a, b \in \mathcal{L}^*(s)\},$$

where $\mathcal{L}^*(s)$ is the so-called maximal O^*-algebra on s (see also [20, Def. 2.1.6, Prop. 2.1.8, Def. 5.1.3, Cor. 5.1.18, Prop. 5.4.1 and Prop. 6.1.5]).

The algebra of smooth operators can be seen as a noncommutative analogue of the commutative algebra s. The most important features of this algebra are the following:

- it is isomorphic as a Fréchet space to the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth rapidly decreasing functions on the real line;
- it is isomorphic as a Fréchet *-algebra to many algebras of operators acting between natural spaces of distributions and functions, e.g. to the algebra of operators from the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions on the real line to the space $\mathcal{S}(\mathbb{R})$ (see also [8, Th. 1.1]);
- it is a dense *-subalgebra of the C^*-algebra $\mathcal{K}(\ell_2)$ of compact operators on ℓ_2;
- it is (properly) contained in the intersection of all Schatten classes $\mathcal{S}_p(\ell_2)$ over $p > 0$; in particular $\mathcal{L}(s', s)$ is contained in the class $\mathcal{HS}(\ell_2)$ of Hilbert-Schmidt operators, and thus it is a unitary space;
- the operator C^*-norm $||\cdot||_{\ell_2\to\ell_2}$ is the so-called dominating norm on that algebra (the dominating norm property is a key notion in the structure theory of nuclear Fréchet spaces – see [3, Prop. 3.2] and [13, Prop. 31.5]).

The main result of the present paper is a characterization of closed *-subalgebras of $\mathcal{L}(s', s)$ which are at the same time isomorphic as Fréchet *-algebras to closed *-subalgebras of the algebra s (Theorem 6.2). It turns out that these are exactly those subalgebras which satisfy the classical condition (Ω) of Vogt, i.e. which are isomorphic (as Fréchet spaces) to complemented subspaces of s. Then in Theorem 6.9 we give an example of a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ which does not satisfy this condition.

To prove this result we characterize in Sect. 4 closed *-subalgebras of Köthe sequence algebras (Proposition 4.3). In particular, we give such a description for closed *-subalgebras of s (Corollary 4.4). In Sect. 5 we describe all closed *-subalgebras of $\mathcal{L}(s', s)$ as suitable Köthe sequence algebras (see Corollary 5.4 and compare with [3, Th.4.8]).

The present paper is a continuation of [3,8] and it focuses on descriptions of closed commutative *-subalgebras of $\mathcal{L}(s', s)$ (especially those with the property (Ω)). Most of the results have been already presented in the author's PhD dissertation [2].

2 Notation and terminology

Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

By a *projection* on the complex separable Hilbert space ℓ_2 we always mean a continuous orthogonal (i.e. self-adjoint) projection.

By e_k we denote the vector in $\mathbb{C}^{\mathbb{N}}$ whose k-th coordinate equals 1 and the others equal 0.

By a *Fréchet space* we mean a complete metrizable locally convex space over $\mathbb{C}$ (we will not use locally convex spaces over $\mathbb{R}$). A *Fréchet algebra* is a Fréchet space which is an algebra with continuous multiplication. A *Fréchet* *it algebra is a Fréchet algebra with continuous involution.

For locally convex spaces E, F, we denote by $\mathcal{L}(E, F)$ the space of all continuous linear operators from E to F. To shorten notation, we write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

We use standard notation and terminology. All the notions from functional analysis are explained in [4,13] and those from topological algebras in [10,24].

3 Preliminaries

3.1 The space s and its dual

We recall that the *space of rapidly decreasing sequences* is the Fréchet space

$$s := \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_q := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{2q} \right)^{1/2} < \infty \text{ for all } q \in \mathbb{N}_0 \right\}$$

with the topology corresponding to the system $(|\cdot|_q)_{q\in\mathbb{N}_0}$ of norms. We may identify the strong dual of s (i.e. the space of all continuous linear functionals on s with the topology of uniform convergence on bounded subsets of s, see e.g. [13, Definition on p. 267]) with the *space of slowly increasing sequences*

$$s' := \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|'_q := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{-2q} \right)^{1/2} < \infty \text{ for some } q \in \mathbb{N}_0 \right\}$$

equipped with the inductive limit topology given by the system $(|\cdot|'_q)_{q\in\mathbb{N}_0}$ of norms (note that for a fixed q, $|\cdot|'_q$ is defined only on a subspace of s'). More precisely, every $\eta \in s'$ corresponds to the continuous linear functional on s:

$$\xi \mapsto \langle \xi, \eta \rangle := \sum_{j=1}^{\infty} \xi_j \overline{\eta_j}$$

(note the conjugation on the second variable). These functionals are continuous, because, by the Cauchy–Schwartz inequality, for all $q \in \mathbb{N}_0$, $\xi \in s$ and $\eta \in s'$ we have

$$|\langle \xi, \eta \rangle| \le |\xi|_q |\eta|'_q.$$

Conversely, one can show that for each continuous linear functional y on s there is $\eta \in s'$ such that $y = \langle \cdot, \eta \rangle$.

Similarly, we identify each $\xi \in s$ with the continuous linear functional on s':

$$\eta \mapsto \langle \eta, \xi \rangle := \sum_{j=1}^{\infty} \eta_j \overline{\xi_j}.$$

In particular, for each continuous linear functional y on s' there is $\xi \in s$ such that $y = \langle \cdot, \xi \rangle$.

We emphasize that the "scalar product" $\langle \cdot, \cdot \rangle$ is well-defined on $s \times s' \cup s' \times s$ and, of course, on $\ell_2 \times \ell_2$.

3.2 The property *(DN)* for the space *s*

Closed subspaces of the space s can be characterized by the so-called property (DN).

Definition 3.1 A Fréchet space $(X, (||\cdot||_q)_{q\in\mathbb{N}_0})$ has the *property* (DN) (see [13, Definition on p. 359]) if there is a continuous norm $||\cdot||$ on X such that for all $q \in \mathbb{N}_0$ there is $r \in \mathbb{N}_0$ and $C > 0$ such that

$$||x||_q^2 \le C||x|| \, ||x||_r$$

for all $x \in X$. The norm $||\cdot||$ is called a *dominating norm*.

Vogt (see [23] and [13, Ch. 31]) proved that a Fréchet space is isomorphic to a closed subspace of s if and only if it is nuclear and it has the property (DN).

The property (DN) for the space s reads as follows (see [13, Lemma 29.2(3)] and its proof).

Proposition 3.2 *For every $p \in \mathbb{N}_0$ and $\xi \in s$ we have*

$$|\xi|_p^2 \le ||\xi||_{\ell_2} |\xi|_{2p}.$$

In particular, the norm $||\cdot||_{\ell_2}$ is a dominating norm on s.

3.3 The algebra $\mathcal{L}(s', s)$

It is a simple matter to show that $\mathcal{L}(s', s)$ with the topology of uniform convergence on bounded sets in s' is a Fréchet space. It is isomorphic to $s \widehat{\otimes} s$, the completed tensor product of s (see [12, §41.7(5)] and note that, s being nuclear, there is only one tensor topology), and thus $\mathcal{L}(s', s) \cong s$ as Fréchet spaces (see e.g. [13, Lemma 31.1]). Moreover, it is easily seen that $(||\cdot||_q)_{q \in \mathbb{N}_0}$,

$$||x||_q := \sup_{|\xi|_q' \le 1} |x\xi|_q,$$

is a fundamental sequence of norms on $\mathcal{L}(s', s)$.

Let us introduce multiplication and involution on $\mathcal{L}(s', s)$. First observe that s is a dense subspace of ℓ_2. Moreover, ℓ_2 is a dense subspace of s', and, finally, the inclusion maps $j_1 : s \hookrightarrow \ell_2$, $j_2 : \ell_2 \hookrightarrow s'$ are continuous. Hence,

$$\iota : \mathcal{L}(s', s) \hookrightarrow \mathcal{L}(\ell_2), \quad \iota(x) := j_1 \circ x \circ j_2,$$

is a well-defined (continuous) embedding of $\mathcal{L}(s', s)$ into the C^*-algebra $\mathcal{L}(\ell_2)$, and thus it is natural to define a multiplication on $\mathcal{L}(s', s)$ by

$$xy := \iota^{-1}(\iota(x) \circ \iota(y)),$$

i.e.

$$xy = x \circ j \circ y,$$

where $j := j_2 \circ j_1 : s \hookrightarrow s'$. Similarly, an involution on $\mathcal{L}(s', s)$ is defined by

$$x^* := \iota^{-1}(\iota(x)^*),$$

where $\iota(x)^*$ is the hermitian adjoint of $\iota(x)$. One can show that these definitions are correct, i.e. $\iota(x) \circ \iota(y), \iota(x)^* \in \iota(\mathcal{L}(s', s))$ for all $x, y \in \mathcal{L}(s', s)$ (see also [3, p.148]).

From now on, we will identify $x \in \mathcal{L}(s', s)$ and $\iota(x) \in \mathcal{L}(\ell_2)$ (we omit ι in the notation).

A Fréchet algebra E is called *locally m-convex* if E has a fundamental system of submultiplicative seminorms. It is well-known that $\mathcal{L}(s', s)$ is locally m-convex (see e.g. [14, Lemma 2.2]), and moreover, the norms $||\cdot||_q$ are submultiplicative (see [3, Proposition 2.5]). This shows simultaneously that the multiplication introduced above is separately continuous, and thus, by [24, Theorem 1.5], it is jointly continuous. Moreover, by [10, Corollary 16.7], the involution on $\mathcal{L}(s', s)$ is continuous.

We may summarize this paragraph by saying that $\mathcal{L}(s', s)$ is a noncommutative *-subalgebra of the C^*-algebra $\mathcal{L}(\ell_2)$ which is (with its natural topology) a locally m-convex Fréchet *-algebra isomorphic as a Fréchet space to s.

4 Köthe algebras

In this section we collect and prove some folklore facts on Köthe algebras which are known for specialists but probably never published.

Definition 4.1 A matrix $A = (a_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$ of non-negative numbers such that

(i) for each $j \in \mathbb{N}$ there is $q \in \mathbb{N}_0$ such that $a_{j,q} > 0$
(ii) $a_{j,q} \le a_{j,q+1}$ for $j \in \mathbb{N}$ and $q \in \mathbb{N}_0$

is called a *Köthe matrix*.

For $1 \le p < \infty$ and a Köthe matrix A we define the *Köthe space*

$$\lambda^p(A) := \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_{p,q}^p := \sum_{j=1}^{\infty} |\xi_j|^p a_{j,q}^p < \infty \text{ for all } q \in \mathbb{N}_0 \right\}$$

and for $p = \infty$

$$\lambda^\infty(A) := \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| a_{j,q} < \infty \text{ for all } q \in \mathbb{N}_0 \right\}$$

with the locally convex topology given by the seminorms $(|\cdot|_{p,q})_{q\in\mathbb{N}_0}$ (see e.g. [13, Definition p. 326]).

Sometimes, for simplicity, we will write $\lambda^\infty(a_{j,q})$ (i.e. only the entries of the matrix) instead of $\lambda^\infty(A)$.

It is well-known (see [13, Lemma 27.1]) that the spaces $\lambda^p(A)$ are Fréchet spaces and sometimes they are Fréchet *-algebras with pointwise multiplication and conjugation (e.g. if $a_{j,q} \ge 1$ for all $j \in \mathbb{N}$ and $q \in \mathbb{N}_0$, see also [15, Proposition 3.1]); in that case they are called *Köthe algebras*.

Clearly, s is the Köthe space $\lambda^2(A)$ for $A = (j^q)_{j\in\mathbb{N},q\in\mathbb{N}_0}$ and it is a Fréchet *-algebra. Moreover, since the matrix A satisfies the so-called Grothendieck–Pietsch condition (see e.g. [13, Proposition 28.16(6)]), s is nuclear, and thus it has also other Köthe space representations (see again [13, Proposition 28.16 and Example 29.4(1)]), i.e. for all $1 \le p \le \infty$, $s = \lambda^p(A)$ as Fréchet spaces.

We use ℓ_2-norms in the definition of s to clarify our ideas, for example we have $|\xi|_0 = ||\xi||_{\ell_2}$ for $\xi \in s$ and $|\eta|_0' = ||\eta||_{\ell_2}$ for $\eta \in \ell_2$. However, in some situations the supremum norms $|\cdot|_{\infty,q}$ (as they are relatively easy to compute) or the ℓ_1-norms will be more convenient.

Proposition 4.2 *Let* $A = (a_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$, $B = (b_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$ *be Köthe matrices and for a bijection* $\sigma\colon \mathbb{N}\to\mathbb{N}$ *let* $A_\sigma := (a_{\sigma(j),q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$. *Assume that* $\lambda^\infty(A)$ *and* $\lambda^\infty(B)$ *are Fréchet* **-algebras. Then the following assertions are equivalent:*

(i) $\lambda^\infty(A) \cong \lambda^\infty(B)$ *as Fréchet* **-algebras;*
(ii) *there is a bijection* $\sigma\colon \mathbb{N}\to\mathbb{N}$ *such that* $\lambda^\infty(A_\sigma) = \lambda^\infty(B)$ *as Fréchet* **-algebras;*
(iii) *there is a bijection* $\sigma\colon \mathbb{N}\to\mathbb{N}$ *such that* $\lambda^\infty(A_\sigma) = \lambda^\infty(B)$ *as sets;*
(iv) *there is a bijection* $\sigma\colon \mathbb{N}\to\mathbb{N}$ *such that*
(α) $\forall q\in\mathbb{N}_0\ \exists r\in\mathbb{N}_0\ \exists C>0\ \forall j\in\mathbb{N}\quad a_{\sigma(j),q}\le Cb_{j,r}$,
(β) $\forall r'\in\mathbb{N}_0\ \exists q'\in\mathbb{N}_0\ \exists C'>0\ \forall j\in\mathbb{N}\quad b_{j,r'}\le C'a_{\sigma(j),q'}$.

Proof (i) $\Rightarrow$ (ii) Assume that there is an isomorphism $\Phi\colon \lambda^\infty(A)\to\lambda^\infty(B)$ of Fréchet *-algebras. Clearly, if $\xi^2=\xi$, then $\Phi(\xi)=\Phi(\xi^2)=(\Phi(\xi))^2$, and the same is true of Φ^{-1}, i.e. Φ maps the idempotents of $\lambda^\infty(A)$ onto the idempotents of $\lambda^\infty(B)$. Hence for a fixed $k\in\mathbb{N}$, there is $I\subset\mathbb{N}$ such that

$$\Phi(e_k)=e_I,$$

where e_I is a sequence which has 1 on an index set $I\subset\mathbb{N}$ and 0 otherwise. Suppose that $|I|\ge 2$ and let $j\in I$. Then $e_I=e_j+e_{I\setminus\{j\}}$, where $e_j\in\lambda^\infty(B)$ and $e_{I\setminus\{j\}}=e_I-e_j\in\lambda^\infty(B)$. Therefore, there are nonempty subsets $I_j, I_j'\subset\mathbb{N}$ such that $\Phi(e_{I_j})=e_j$ and $\Phi(e_{I_{j'}})=e_{I\setminus\{j\}}$. We have

$$e_{I_j}e_{I_j'}=\Phi^{-1}(e_j)\Phi^{-1}(e_{I\setminus\{j\}})=\Phi^{-1}(e_je_{I\setminus\{j\}})=0,$$

and thus $I_j\cap I_j'=\emptyset$. Consequently,

$$\Phi(e_k)=e_j+e_{I\setminus\{j\}}=\Phi(e_{I_j})+\Phi(e_{I_j'})=\Phi(e_{I_j\cup I_j'}),$$

whence $1=|\{k\}|=|I_j\cup I_j'|\ge 2$, a contradiction. Hence $\Phi(e_k)=e_{n_k}$ for some $n_k\in\mathbb{N}$, i.e. for the bijection $\sigma\colon\mathbb{N}\to\mathbb{N}$ defined by $n_{\sigma(k)}:=k$ we have $\Phi(e_{\sigma(k)})=e_k$. Therefore, a Fréchet *-isomorphism Φ is given by $(\xi_{\sigma(k)})_{k\in\mathbb{N}}\mapsto(\xi_k)_{k\in\mathbb{N}}$ for $(\xi_{\sigma(k)})_{k\in\mathbb{N}}\in\lambda^\infty(A)$, and thus $\lambda^\infty(A_\sigma)=\lambda^\infty(B)$ as Fréchet *-algebras.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (iv) The proof follows from the observation that the identity map $\mathrm{Id}\colon\lambda^\infty(A_\sigma)\to\lambda^\infty(B)$ is continuous (use the closed graph theorem).

(iv) $\Rightarrow$ (i) It is easy to see that $\Phi\colon\lambda^\infty(A)\to\lambda^\infty(B)$ defined by $e_{\sigma(k)}\mapsto e_k$ is an isomorphism of Fréchet *-algebras. □

In the following proposition we characterize infinite-dimensional closed *-subalgebras of nuclear Köthe algebras whose elements tend to zero (note that if a Köthe space is contained in ℓ_∞ then it is a Köthe algebra). Consequently, we obtain a characterization of closed *-subalgebras of s (Corollary 4.4).

Proposition 4.3 *For* $\mathcal{N}\subset\mathbb{N}$ *let* $e_\mathcal{N}$ *denote a sequence which has 1 on* $\mathcal{N}$ *and 0 otherwise. Let* $A=(a_{j,q})_{j\in\mathbb{N},q\in\mathbb{N}_0}$ *be a Köthe matrix such that* $\lambda^\infty(A)$ *is nuclear and* $\lambda^\infty(A)\subset c_0$. *Let* E *be an infinite-dimensional closed* **-subalgebra of* $\lambda^\infty(A)$. *Then*

(i) *there is a family* $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ *of finite nonempty pairwise disjoint sets of natural numbers such that* $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ *is a Schauder basis of* E*;*
(ii) $E \cong \lambda^\infty\left(\max_{j\in\mathcal{N}_k} a_{j,q}\right)$ *as Fréchet* **-algebras and the isomorphism is given by* $e_{\mathcal{N}_k} \mapsto e_k$ *for* $k \in \mathbb{N}$.
Conversely, if $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ *is a family of finite nonempty pairwise disjoint sets of natural numbers and* F *is the closed* **-subalgebra of* $\lambda^\infty(A)$ *generated by the set* $\{e_{\mathcal{N}_k}\}_{k\in\mathbb{N}}$*, then*
(iii) $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ *is a Schauder basis of* F*;*
(iv) $F \cong \lambda^\infty(\max_{j\in\mathcal{N}_k} a_{j,q})$ *as Fréchet* **-algebras and the isomorphism is given by* $e_{\mathcal{N}_k} \mapsto e_k$ *for* $k \in \mathbb{N}$.

Proof In order to prove (i) and (ii) set

$$\mathcal{N}_0 := \{j \in \mathbb{N}\colon \xi_j = 0 \quad \text{for all}\, \xi \in E\}$$

and define an equivalence relation $\sim$ on $\mathbb{N}\backslash\mathcal{N}_0$ by

$$i \sim j \Leftrightarrow \xi_i = \xi_j \text{ for all } \xi \in E.$$

Since E is infinite-dimensional, our relation produces infinitely many equivalence classes $\mathcal{N}_k$, say

$$\mathcal{N}_k := [\min(\mathbb{N}\backslash\mathcal{N}_0 \cup \cdots \cup \mathcal{N}_{k-1})]_{/\sim}$$

for $k \in \mathbb{N}$.

Fix $\kappa \in \mathbb{N}$ and take $\xi \in E$ such that $\xi_j \neq 0$ for $j \in \mathcal{N}_\kappa$. Denote $\eta_k := \xi_j$ if $j \in \mathcal{N}_k$. Let

$$\mathcal{M}_1 := \left\{ j \in \mathbb{N}\colon |\eta_j| = \sup_{i\in\mathbb{N}} |\eta_i| \right\}.$$

Assume we have already defined $\mathcal{M}_1, \ldots, \mathcal{M}_{l-1}$. If there is $j \in \mathbb{N}\backslash\{\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{l-1}\}$ such that $\eta_j \neq 0$ then we define

$$\mathcal{M}_l := \{j \in \mathbb{N}\colon |\eta_j| = \sup\{|\eta_i|\colon i \in \mathbb{N}\backslash\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{l-1}\}\}.$$

Otherwise, denote $\mathcal{I} := \{1, \ldots, l-1\}$. If this procedure leads to infinitely many sets $\mathcal{M}_l$ then we set $\mathcal{I} := \mathbb{N}$. It is easily seen that for each $l \in \mathcal{I}$ there is $\mathcal{I}_l \subset \mathbb{N}$ such that $\mathcal{M}_l = \bigcup_{k\in\mathcal{I}_l} \mathcal{N}_k$. By assumption $\xi \in c_0$, hence $(|\eta_k|)_{k\in\mathbb{N}} \in c_0$ as well, and thus each $\mathcal{M}_l$ is a finite nonempty set.

We first show that $e_{\mathcal{M}_l} \in E$ for $l \in \mathcal{I}$. For $l \in \mathcal{I}$ fix $m_l \in \mathcal{M}_l$. If $\mathcal{I} = \{1\}$, then $\xi_j = 0$ for $j \notin \mathcal{M}_1$, and $e_{\mathcal{M}_1} = \frac{\xi\overline{\xi}}{|\eta_{m_1}|^2} \in E$. Let us consider the case $|\mathcal{I}| > 1$. Since in nuclear Fréchet spaces every basis is absolute (and thus unconditional), we have

$$\sum_{l\in\mathcal{I}} |\eta_l|^2 e_{\mathcal{M}_l} = \sum_{j=1}^{\infty} |\xi_j|^2 e_j = \xi\overline{\xi} \in E,$$

and, consequently,

$$x_n := \sum_{l\in\mathcal{I}} \left(\frac{|\eta_l|}{|\eta_{m_1}|}\right)^{2n} e_{\mathcal{M}_l} = \left(\frac{\xi\overline{\xi}}{|\eta_{m_1}|^2}\right)^n \in E$$

for all $n \in \mathbb{N}$. Then for q and n we get

$$\begin{aligned}|x_n - e_{\mathcal{M}_1}|_{\infty,q} &= \left|\sum_{l\in\mathcal{I}}^{\infty} \left(\frac{|\eta_l|}{|\eta_{m_1}|}\right)^{2n} e_{\mathcal{M}_l} - e_{\mathcal{M}_1}\right|_{\infty,q} \\ &= \left|\sum_{l\in\mathcal{I}\setminus\{1\}} \left(\frac{|\eta_l|}{|\eta_{m_1}|}\right)^{2n} e_{\mathcal{M}_l}\right|_{\infty,q} \le \sum_{l\in\mathcal{I}\setminus\{1\}} \left(\frac{|\eta_l|}{|\eta_{m_1}|}\right)^{2n} |e_{\mathcal{M}_l}|_{\infty,q} \\ &\le \frac{1}{|\eta_{m_1}|}\left(\frac{|\eta_{m_2}|}{|\eta_{m_1}|}\right)^{2n-1} \sum_{l\in\mathcal{I}\setminus\{1\}} |\eta_l|\, |e_{\mathcal{M}_l}|_{\infty,q}.\end{aligned}$$

Since $(e_j)_{j\in\mathbb{N}}$ is an absolute basis in $\lambda^\infty(A)$, the above series is convergent. Note also that $|\eta_{m_2}| < |\eta_{m_1}|$. This shows that $x_n \to e_{\mathcal{M}_1}$ in $\lambda^\infty(A)$, and $e_{\mathcal{M}_1} \in E$. Assume that $e_{\mathcal{M}_1}, \ldots, e_{\mathcal{M}_{l-1}} \in E$. If $|\mathcal{I}| = l-1$ then we are done. Otherwise, $\eta_{m_l} \neq 0$ and

$$x_n^{(l)} := \left(\frac{\xi\overline{\xi} - \xi\overline{\xi}\sum_{j=1}^{l-1} e_{\mathcal{M}_j}}{|\eta_{m_l}|^2}\right)^n \in E$$

for $n \in \mathbb{N}$. As above we show that $x_n^{(l)} \to e_{\mathcal{M}_l}$ in $\lambda^\infty(A)$, and thus $e_{\mathcal{M}_l} \in E$. Proceeding by induction, we prove that $e_{\mathcal{M}_l} \in E$ for $l \in \mathcal{I}$.

Now, we shall prove that $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ is a Schauder basis of E. Choose $\iota \in \mathcal{I}$ such that $\kappa \in \mathcal{I}_\iota$ and for $k \in \mathcal{I}_\iota$ let n_k be an arbitrary element of $\mathcal{N}_k$. Then $\sum_{k\in\mathcal{I}_\iota} \eta_{n_k} e_{\mathcal{N}_k} = \xi e_{\mathcal{M}_\iota} \in E$. Consequently, by [3, Lemma 4.1], $e_{\mathcal{N}_\kappa} \in E$. Since κ was arbitrarily choosen, each $e_{\mathcal{N}_k}$ is in E and it is a simple matter to show that $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ is a Schauder basis of E.

Moreover, $|e_{\mathcal{N}_k}|_{\infty,q} = \max_{j\in\mathcal{N}_k} a_{j,q}$ hence, by [13, Corollary 28.13] and nuclearity, E is isomorphic as a Fréchet space to $\lambda^\infty(\max_{j\in\mathcal{N}_k} a_{j,q})$. The analysis of the proof of [13, Corollary 28.13] shows that this isomorphism is given by $e_{\mathcal{N}_k} \mapsto e_k$ for $k \in \mathbb{N}$, and thus it is also a Fréchet *-algebra isomorphism.

Now, we prove (iii) and (iv). First note that every element of F is the limit of elements of the form $\sum_{k=1}^M c_k e_{\mathcal{N}_k}$, where $M \in \mathbb{N}$ and $c_1, \ldots, c_M \in \mathbb{C}$. Therefore, if $\xi \in F$, then $\xi_i = \xi_j$ for $k \in \mathbb{N}$ and $i, j \in \mathcal{N}_k$. This shows that each $\xi \in F$ has the unique series representation $\xi = \sum_{k=1}^\infty \xi_{n_k} e_{\mathcal{N}_k}$, where $(n_k)_{k\in\mathbb{N}}$ is an arbitrarily choosen sequence such that $n_k \in \mathcal{N}_k$ for $k \in \mathbb{N}$. Since the series is absolutely convergent, $(e_{\mathcal{N}_k})_{k\in\mathbb{N}}$ is a Schauder basis of F. Statement (iv) follows by the same method as in (ii). □

Corollary 4.4 *Every infinite-dimensional closed *-subalgebra of s is isomorphic as a Fréchet *-algebra to $\lambda^\infty(n_k^q)$ for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of natural* *numbers. Conversely, if* $(n_k)_{k\in\mathbb{N}}$ *is a strictly increasing sequence of natural numbers, then* $\lambda^\infty(n_k^q)$ *is isomorphic as a Fréchet* **-algebra to some infinite-dimensional closed* **-subalgebra of* s*. Moreover, every closed* **-subalgebra of* s *is a complemented subspace of* s.

Proof We apply Proposition 4.3 to the Köthe matrix $(j^q)_{j\in\mathbb{N},q\in\mathbb{N}_0}$. Let $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ be a family of finite nonempty pairwise disjoint sets of natural numbers. We have

$$\max_{j\in\mathcal{N}_k} j^q = (\max\{j : j \in \mathcal{N}_k\})^q \tag{1}$$

for all $q \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be the bijection for which $(\max\{j : j \in \mathcal{N}_{\sigma(k)}\})_{k\in\mathbb{N}}$ is (strictly) increasing and let $n_k := \max\{j : j \in \mathcal{N}_{\sigma(k)}\}$ for $k \in \mathbb{N}$. Then, by Proposition 4.2,

$$\lambda^\infty\left(\max_{j\in\mathcal{N}_k} j^q\right) \cong \lambda^\infty(n_k^q)$$

as Fréchet *-algebras, and therefore the first two statements follow from Proposition 4.3.

Now, let E be a closed *-subalgebra of s. If E is finite dimensional then, clearly, E is complemented in s. Otherwise, by Proposition 4.3(i), E is a closed linear span of the set $\{e_{\mathcal{N}_k}\}_{k\in\mathbb{N}}$ for some family $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ of finite nonempty pairwise disjoint sets of natural numbers. Define $\pi : s \to E$ by

$$(\pi x)_j := \begin{cases} x_{n_k} & \text{for } j \in \mathcal{N}_{\sigma(k)} \\ 0 & \text{otherwise} \end{cases}$$

where $(n_k)_{k\in\mathbb{N}}$ and σ are as above. From (1) we have for every $q \in \mathbb{N}_0$

$$\begin{aligned} |\pi x|_{\infty,q} &= \sup_{j\in\mathbb{N}} |(\pi x)_j| j^q \le \sup_{k\in\mathbb{N}} |x_{n_k}| \max_{j\in\mathcal{N}_{\sigma(k)}} j^q = \sup_{k\in\mathbb{N}} |x_{n_k}| (\max\{j : j \in \mathcal{N}_k\})^q \\ &= \sup_{k\in\mathbb{N}} |x_{n_k}| n_k^q \le \sup_{j\in\mathbb{N}} |x_j| j^q = |x|_{\infty,q}, \end{aligned}$$

and thus π is well-defined and continuous. Since π is a projection, our proof is complete. □

5 Representations of closed commutative *-subalgebras of $\mathcal{L}(s', s)$ by Köthe algebras

The aim of this section is to describe all closed commutative *-subalgebras of $\mathcal{L}(s', s)$ as Köthe algebras $\lambda^\infty(A)$ for matrices A determined by orthonormal sequences whose elements belong to the space s (Theorem 5.3 and Corollaries 5.4 and 5.5). For the convenience of the reader, we quote two results from [3] (with minor modifications which do not require extra arguments).

For a subset Z of $\mathcal{L}(s', s)$ we will denote by $\operatorname{alg}(Z)$ ($\overline{\operatorname{lin}}(Z)$, resp.) the closed *-subalgebra of $\mathcal{L}(s', s)$ generated by Z (the closed linear span of Z, resp.).

By [3, Lemma 4.4], every closed commutative *-subalgebra E of $\mathcal{L}(s', s)$ admits a special Schauder basis. This basis consists of all nonzero minimal projections in E ([3, Lemma 4.4] shows that these projections are pairwise orthogonal) and we call it the *canonical Schauder basis* of E.

Proposition 5.1 [3, Proposition 4.7] *Every sequence* $\{P_k\}_{k\in\mathcal{N}} \subset \mathcal{L}(s', s)$ *of nonzero pairwise orthogonal projections is the canonical Schauder basis of the algebra* $\operatorname{alg}(\{P_k\}_{k\in\mathcal{N}})$*. In particular,* $\{P_k\}_{k\in\mathcal{N}}$ *is a basic sequence in* $\mathcal{L}(s', s)$*, i.e. it is a Schauder basis of the Fréchet space* $\overline{\operatorname{lin}}(\{P_k\}_{k\in\mathcal{N}})$.

Theorem 5.2 [3, Theorem 4.8] *Let E be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and let $\{P_k\}_{k\in\mathbb{N}}$ be the canonical Schauder basis of E. Then*

$$E = \operatorname{alg}(\{P_k\}_{k\in\mathbb{N}}) \cong \lambda^\infty(||P_k||_q)$$

as Fréchet *-algebras and the isomorphism is given by $P_k \mapsto e_k$ for $k \in \mathbb{N}$.

Please note that a projection P is in $\mathcal{L}(s', s)$ if and only if it is of the form

$$P\xi = \sum_{k\in I} \langle \xi, f_k\rangle f_k$$

for some finite set I and an orthonormal sequence $(f_k)_{k\in I} \subset s$.

We will also use the identity

$$\lambda^\infty(||\langle \cdot, f_k\rangle f_k||_q) = \lambda^\infty(|f_k|_q) \tag{2}$$

which holds for every orthonormal sequence $(f_k)_{k\in\mathbb{N}} \subset s$. (see [3, Rem. 4.11]).

Now we are ready to state and prove the main result of this section.

Theorem 5.3 *Every closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of the algebra $\lambda^\infty(|f_k|_q)$ for some orthonormal sequence $(f_k)_{k\in\mathbb{N}} \subset s$. More precisely, if E is an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and $(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ is its canonical Schauder basis for some family of finite pairwise disjoint subsets $(\mathcal{N}_k)_{k\in\mathbb{N}}$ of natural numbers and an orthonormal sequence $(f_j)_{j\in\mathbb{N}} \subset s$, then E is isomorphic as a Fréchet *-algebra to the closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ generated by $\{\sum_{j\in\mathcal{N}_k} e_j\}_{k\in\mathbb{N}}$ and the isomorphism is given by $\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j \mapsto \sum_{j\in\mathcal{N}_k} e_j$ for $k \in \mathbb{N}$.*

Conversely, if $(f_k)_{k\in\mathbb{N}} \subset s$ is an orthonormal sequence, then every closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to some closed commutative *-subalgebra of $\mathcal{L}(s', s)$.

Proof By Theorem 5.2, $E = \operatorname{alg}\left(\left\{\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j\right\}_{k\in\mathbb{N}}\right)$ for $(\mathcal{N}_k)_{k\in\mathbb{N}}$ and $(f_j)_{j\in\mathbb{N}} \subset s$ as in the statement. Let F be the closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ generated by $\{\sum_{j\in\mathcal{N}_k} e_j\}_{k\in\mathbb{N}}$. Define

$$\Phi\colon \mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathcal{N}}) \to \lambda^\infty(|f_k|_q)$$

by $\langle\cdot, f_k\rangle f_k \mapsto e_k$, where $\mathcal{N} := \bigcup_{k\in\mathbb{N}} \mathcal{N}_k$. By Proposition 5.1, $\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathcal{N}}$ is the canonical Schauder basis of $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathcal{N}})$, and thus Theorem 5.2 and (2) imply that Φ is a Fréchet *-algebra isomorphism. Hence, $(\sum_{j\in\mathcal{N}_k} e_j)_{k\in\mathbb{N}} = (\Phi(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j))_{k\in\mathbb{N}}$ is a Schauder basis of $\Phi(E)$ and $\Phi(E)$ is a closed *-subalgebra of $\lambda^\infty(|f_k|_q)$. Therefore,

$$\Phi(E) = \overline{\mathrm{lin}}\left(\left\{\sum_{j\in\mathcal{N}_k} e_j\right\}_{k\in\mathbb{N}}\right) \subset F \subset \Phi(E),$$

whence $\Phi(E) = F$. In consequence $\Phi_{|E}$ is a Fréchet *-algebra isomorphism of E and F, which completes the proof of the first statement.

If now $(f_k)_{k\in\mathbb{N}} \subset s$ is an arbitrary orthonormal sequence then, according to Proposition 5.1, Theorem 5.2 and identity (2), $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$. Consequently, every closed *-subalgebra of $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$. □

The following characterization of infinite-dimensional closed commutative *-subalgebras of $\mathcal{L}(s', s)$ is a straightforward consequence of Proposition 4.3 and Theorem 5.3.

Corollary 5.4 *Every infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to the algebra $\lambda^\infty(\max_{j\in\mathcal{N}_k}|f_j|_q)$ for some orthonormal sequence $(f_k)_{k\in\mathbb{N}} \subset s$ and some family $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ of finite nonempty pairwise disjoint sets of natural numbers. In fact, if E is an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and $(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ is its canonical Schauder basis, then*

$$E \cong \lambda^\infty\left(\max_{j\in\mathcal{N}_k}|f_j|_q\right)$$

as Fréchet *-algebras and the isomorphism is given by $\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j \mapsto e_k$ for $k \in \mathbb{N}$.

Conversely, if $(f_k)_{k\in\mathbb{N}} \subset s$ is an orthonormal sequence and $\{\mathcal{N}_k\}_{k\in\mathbb{N}}$ is a family of finite nonempty pairwise disjoint sets of natural numbers, then $\lambda^\infty(\max_{j\in\mathcal{N}_k}|f_j|_q)$ is isomorphic as a Fréchet *-algebra to some infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$.

At the end of this section we consider the case of maximal commutative subalgebras of $\mathcal{L}(s', s)$. A closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is said to be *maximal commutative* if it is not properly contained in any larger closed commutative * -subalgebra of $\mathcal{L}(s', s)$.

We say that an orthonormal system $(f_k)_{k\in\mathbb{N}}$ of ℓ_2 is *s-complete*, if every f_k belongs to s and for every $\xi \in s$ the following implication holds: if $\langle\xi, f_k\rangle = 0$ for every

$k \in \mathbb{N}$, then $\xi = 0$. A sequence $\{P_k\}_{k\in\mathbb{N}}$ of nonzero pairwise orthogonal projections belonging to $\mathcal{L}(s', s)$ is called $\mathcal{L}(s', s)$-*complete* if there is no nonzero projection P belonging to $\mathcal{L}(s', s)$ such that $P_k P = 0$ for every $k \in \mathbb{N}$.

One can easily show that an orthonormal system $(f_k)_{k\in\mathbb{N}}$ is s-complete if and only if the sequence of projections $(\langle\cdot, f_k\rangle f_k)_{k\in\mathbb{N}}$ is $\mathcal{L}(s', s)$-complete. Hence, by [3, Theorem 4.10], closed commutative *-subalgebra E of $\mathcal{L}(s', s)$ is maximal commutative if and only if there is an s-complete sequence $(f_k)_{k\in\mathbb{N}}$ such that $(\langle\cdot, f_k\rangle f_k)_{k\in\mathbb{N}}$ is the canonical Schauder basis of E. Combining this with Corollary 5.4, we obtain the first statement of the following corollary.

Corollary 5.5 *Every closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to the algebra $\lambda^\infty(|f_k|_q)$ for some s-complete orthonormal sequence $(f_k)_{k\in\mathbb{N}}$. More precisely, if E is a closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ with the canonical Schauder basis $(\langle\cdot, f_k\rangle f_k)_{k\in\mathbb{N}}$, then*

$$E \cong \lambda^\infty(|f_k|_q)$$

as Fréchet *-algebras and the isomorphism is given by $\langle\cdot, f_k\rangle f_k \mapsto e_k$ for $k \in \mathbb{N}$.

Conversely, if $(f_k)_{k\in\mathbb{N}}$ is an s-complete orthonormal sequence, then $\lambda^\infty(|f_k|_q)$ is isomorphic as a Fréchet *-algebra to some closed maximal commutative *-subalgebra of $\mathcal{L}(s', s)$.

Proof In order to prove the second statement, take an arbitrary s-complete orthonormal sequence $(f_k)_{k\in\mathbb{N}}$. By Proposition 5.1 and the remark above our Corollary, $\mathrm{alg}(\{\langle\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$ is maximal commutative and from the first statement it follows that it is isomorphic as a Fréchet *-algebra to $\lambda^\infty(|f_k|_q)$. □

It is also worth pointing out the following result.

Proposition 5.6 *Every commutative (not necessary closed) *-subalgebra of $\mathcal{L}(s', s)$ is contained in some maximal commutative *-subalgebra of $\mathcal{L}(s', s)$.*

Proof Let E be a commutative *-subalgebra of $\mathcal{L}(s', s)$. Clearly,

$$\mathcal{X} := \{\widetilde{E} : \widetilde{E} \text{ commutative}^* \text{-subalgebra of } \mathcal{L}(s', s) \text{ and } E \subset \widetilde{E}\}$$

with the inclusion relation is a partially ordered set. Consider a nonempty chain $\mathcal{C}$ in $\mathcal{X}$ and let $E_{\mathcal{C}} := \bigcup_{F\in\mathcal{C}} F$. It is easy to check that $E_{\mathcal{C}} \in \mathcal{X}$, and, of course, $E_{\mathcal{C}}$ is an upper bound of $\mathcal{C}$. Hence, by the Kuratowski–Zorn lemma, $\mathcal{X}$ has a maximal element; let us call it M. By the continuity of the algebra operations, $\overline{M}^{\mathcal{L}(s',s)}$ is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$, hence from the maximality of M, we have $M = \overline{M}^{\mathcal{L}(s',s)}$, i.e. M is a (closed) maximal commutative *-subalgebra of $\mathcal{L}(s', s)$ containing E. □

6 Closed commutative *-subalgebras of $\mathcal{L}(s', s)$ with the property (Ω)

The main purpose of the last section is to prove that a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of s if and only if it is isomorphic as a Fréchet space to some complemented subspace of s (Theorem 6.2), i.e. if it has the so-called property (Ω).

Definition 6.1 A Fréchet space E with a fundamental sequence $(||\cdot||_q)_{q\in\mathbb{N}_0}$ of seminorms has the *property* (Ω) if the following condition holds:

$$\forall p\ \exists q\ \forall r\ \exists\theta\in(0,1)\ \exists C>0\ \forall y\in E' \quad ||y||'_q\leq C||y||'^{1-\theta}_p||y||'^{\theta}_r,$$

where E' is the topological dual of E and $||y||'_p:=\sup\{|y(x)|:||x||_p\leq 1\}$.

The property (Ω) (together with the property (DN)) plays a crucial role in the theory of nuclear Fréchet spaces (for details, see [13, Ch. 29]).

Recall that a subspace F of a Fréchet space E is called *complemented* (in E) if there is a continuous projection $\pi: E\to E$ with $\operatorname{im}\pi=F$. Since every subspace of $\mathcal{L}(s',s)$ has the property (DN) (and, by [3, Proposition 3.2], the norm $||\cdot||_{\ell_2\to\ell_2}$ is already a dominating norm) [13, Proposition 31.7] implies that a closed *-subalgebra of $\mathcal{L}(s',s)$ is isomorphic to a complemented subspace of s if and only if it has the property (Ω). The class of complemented subspaces of s is still not well-understood (e.g. we do not know whether every such subspace has a Schauder basis—the Pełczyński problem) and, on the other hand, the class of closed *-subalgebras of s has a simple description (see Corollary 4.4). The following theorem implies that, when restricting to the family of closed commutative *-subalgebras of $\mathcal{L}(s',s)$, these two classes of Fréchet spaces coincide.

Theorem 6.2 *Let E be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s',s)$ and let $(\sum_{j\in\mathcal{N}_k}\langle\cdot,f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis. Then the following assertions are equivalent:*

(i) *E is isomorphic as a Fréchet *-algebra to some closed *-subalgebra of s;*
(ii) *E is isomorphic as a Fréchet space to some complemented subspace of s;*
(iii) *E has the property (Ω);*
(iv) $\exists p\ \forall q\ \exists r\ \exists C>0\ \forall k \quad \max_{j\in\mathcal{N}_k}|f_j|_q\leq C\max_{j\in\mathcal{N}_k}|f_j|^r_p$.

In order to prove Theorem 6.2, we will need Lemmas 6.3, 6.4 and Propositions 6.5, 6.6.

The following result is a consequence of nuclearity of closed commutative *-subalgebras of $\mathcal{L}(s',s)$.

Lemma 6.3 *Let $(f_k)_{k\in\mathbb{N}}\subset s$ be an orthonormal sequence and let $(\mathcal{N}_k)_{k\in\mathbb{N}}$ be a family of finite pairwise disjoint subsets of natural numbers. For $r\in\mathbb{N}_0$ let $\sigma_r:\mathbb{N}\to\mathbb{N}$ be a bijection such that the sequence $(\max_{j\in\mathcal{N}_{\sigma_r(k)}}|f_j|_r)_{k\in\mathbb{N}}$ is non-decreasing. Then there is $r_0\in\mathbb{N}$ such that*

$$\lim_{k\to\infty}\frac{k}{\max_{j\in\mathcal{N}_{\sigma_r(k)}}|f_j|_r}=0$$

for all $r\geq r_0$.

Proof By Corollary 5.4, $\lambda^\infty(\max_{j\in\mathcal{N}_k}|f_j|_q)$ is a nuclear space. Hence, by the Grothendieck–Pietsch theorem (see e.g. [13, Theorem 28.15]), for every $q\in\mathbb{N}_0$ there is $r\in\mathbb{N}_0$ such that

$$\sum_{k=1}^{\infty}\frac{\max_{j\in\mathcal{N}_k}|f_j|_q}{\max_{j\in\mathcal{N}_k}|f_j|_r}<\infty.$$

In particular (for $q=0$), there is r_0 such that for $r\geq r_0$ we have

$$\sum_{k=1}^{\infty}\frac{1}{\max_{j\in\mathcal{N}_{\sigma_r(k)}}|f_j|_r}=\sum_{k=1}^{\infty}\frac{1}{\max_{j\in\mathcal{N}_k}|f_j|_r}<\infty.$$

Since the sequence $(\max_{j\in\mathcal{N}_{\sigma_r(k)}}|f_j|_r)_{k\in\mathbb{N}}$ is non-decreasing, the conclusion follows from the elementary theory of number series. □

Lemma 6.4 *Let $(a_k)_{k\in\mathbb{N}}\subset[1,\infty)$ be a non-decreasing sequence such that $a_k\geq 2k$ for k big enough. Then there exist a strictly increasing sequence $(b_k)_{k\in\mathbb{N}}$ of natural numbers and $C>0$ such that*

$$\frac{1}{C}a_k\leq b_k\leq Ca_k^2$$

for every $k\in\mathbb{N}$.

Proof Let $k_0\in\mathbb{N}$ be such that $a_k\geq 2k$ for $k>k_0$ and choose $C\in\mathbb{N}$ so that

$$\frac{1}{C}a_k\leq k\leq Ca_k^2$$

for $k\in\mathcal{N}_0:=\{1,\ldots,k_0\}$. Denote also $\mathcal{N}_1:=\{k\in\mathbb{N}:a_k=a_{k_0+1}\}$ and, recursively, $\mathcal{N}_{j+1}:=\{k\in\mathbb{N}:a_k=a_{\max\mathcal{N}_j+1}\}$. Clearly, $\mathcal{N}_j$ are finite, pairwise disjoint, $\bigcup_{j\in\mathbb{N}_0}\mathcal{N}_j=\mathbb{N}$ and $k<l$ for $k\in\mathcal{N}_j$, $l\in\mathcal{N}_{j+1}$.

Let $b_k:=k$ for $k\in\mathcal{N}_0$ and let

$$b_{m_j+l-1}:=C\lceil\max\{a_{m_j-1}^2,a_{m_j}\}\rceil+l$$

for $j\in\mathbb{N}$ and $1\leq l\leq|\mathcal{N}_j|$, where $m_j:=\min\mathcal{N}_j$ and $\lceil x\rceil:=\min\{n\in\mathbb{Z}:n\geq x\}$ stands for the ceiling of $x\in\mathbb{R}$. We will show inductively that $(b_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers such that

$$\frac{1}{C}a_k\leq b_k\leq Ca_k^2\tag{3}$$

for every $k\in\mathbb{N}$.

Clearly, the condition (3) holds for $k\in\mathcal{N}_0$. Assume that $(b_k)_{k\in\mathcal{N}_0\cup\cdots\cup\mathcal{N}_j}$ is a strictly increasing sequence of natural numbers for which the condition (3) holds. For simplicity, denote $m := \min \mathcal{N}_{j+1}$. By the inductive assumption, we obtain $b_{m-1} \leq Ca_{m-1}^2$, hence

$$b_m - b_{m-1} \geq C\lceil \max\{a_{m-1}^2, a_m\}\rceil + 1 - Ca_{m-1}^2 \geq Ca_{m-1}^2 + 1 - Ca_{m-1}^2 \geq 1$$

so $b_{m-1} < b_m$, and, clearly, $b_m < b_{m+1} < \cdots < b_{\max \mathcal{N}_{j+1}}$.

Fix $1 \leq l \leq |N_{j+1}|$. We have

$$b_{m+l-1} \geq Ca_m = Ca_{m+l-1} \geq \frac{1}{C}a_{m+l-1}$$

so the first inequality in (3) holds for $k \in \mathcal{N}_{j+1}$. Next, by assumption, we get

$$a_{m+l-1} \geq 2(m + l - 1), \tag{4}$$

whence

$$l \leq a_{m-l+1} - m + 1. \tag{5}$$

Consider two cases. If $a_m \geq a_{m-1}^2$, then, from (5)

$$\begin{aligned} b_{m-l+1} &= C\lceil a_m\rceil + l = C\lceil a_{m+l-1}\rceil + l \leq 2Ca_{m+l-1} + a_{m+l-1} - m + 1 \\ &\leq (2C + 1)a_{m+l-1} \leq Ca_{m+l-1}^2, \end{aligned}$$

where the last inequality holds because $C \geq 1$ and, from (4), we have

$$a_{m-l+1} \geq 2(m + l - 1) \geq 2m \geq 2(k_0 + 1) \geq 4.$$

Finally, if $a_{m-1}^2 > a_m$, then, from (4), we obtain (note that, by the definition of $\mathcal{N}_j$ and $\mathcal{N}_{j+1}$, we have $a_{m-1} < a_m$)

$$\begin{aligned} b_{m-l+1} &= C\lceil a_{m-1}^2\rceil + l \\ &\leq C\lceil (a_m - 1)^2\rceil + l \\ &= C\lceil a_m^2 - 2a_m + 1\rceil + l \\ &\leq C(a_m^2 - 2a_m + 2) + l \\ &\leq Ca_m^2 - 2Ca_m + 2C + Cl \\ &= Ca_{m+l-1}^2 - C(2a_{m+l-1} - 2 - l) \\ &\leq Ca_{m+l-1}^2 - C(4(m + l - 1) - 2 - l) \\ &= Ca_{m+l-1}^2 - C(4m + 3l - 6) \leq Ca_{m+l-1}^2. \end{aligned}$$

Hence we have shown that the second inequality in (3) holds for $k \in \mathcal{N}_{j+1}$, and the proof is complete. □

Proposition 6.5 *Let E be an infinite-dimensional closed commutative *-subalgebra of $\mathcal{L}(s', s)$ and let $(\sum_{j\in\mathcal{N}_k}\langle\cdot, f_j\rangle f_j)_{k\in\mathbb{N}}$ be its canonical Schauder basis. Moreover, let $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers and let F be the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$. Then the following assertions are equivalent:*

(i) *E is isomorphic to F as a Fréchet *-algebra;*
(ii) *$\lambda^\infty(\max_{j\in\mathcal{N}_k}|f_j|_q)\cong\lambda^\infty(n_k^q)$ as Fréchet *-algebras;*
(iii) *there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ such that $\lambda^\infty(\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_q)=\lambda^\infty(n_k^q)$ as Fréchet *-algebras;*
(iv) *there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ such that $\lambda^\infty(\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_q)=\lambda^\infty(n_k^q)$ as sets;*
(v) *there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ such that*
  (α) $\forall q\in\mathbb{N}_0\ \exists r\in\mathbb{N}_0\ \exists C>0\ \forall k\in\mathbb{N}\quad \max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_q\le Cn_k^r$,
  (β) $\forall r'\in\mathbb{N}_0\ \exists q'\in\mathbb{N}_0\ \exists C'>0\ \forall k\in\mathbb{N}\quad n_k^{r'}\le C'\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{q'}$.

Proof This is an immediate consequence of Proposition 4.2 and Corollary 5.4. □

In view of Corollary 4.4, every closed *-subalgebra of s is isomorphic as a Fréchet *-algebra to $\lambda^\infty(n_k^q)$ (i.e. the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$) for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$, hence Proposition 6.5 characterizes closed commutative *-subalgebras of $\mathcal{L}(s', s)$ which are isomorphic as Fréchet *-algebras to some *-subalgebra of s.

The property (DN) for the space s gives us the following inequality.

Proposition 6.6 *For every $p, r\in\mathbb{N}_0$ there is $q\in\mathbb{N}_0$ such that for all $\xi\in s$ with $||\xi||_{\ell_2}=1$ the following inequality holds*

$$|\xi|_p^r\le|\xi|_q.$$

Proof Take $p, r\in\mathbb{N}_0$ and let $j\in\mathbb{N}_0$ be such that $r\le 2^j$. Applying iteratively (j-times) the inequality from Proposition 3.2 to $\xi\in s$ with $||\xi||_{\ell_2}=1$ we get

$$|\xi|_p^r\le|\xi|_p^{2^j}\le|\xi|_{2^jp},$$

and thus the required inequality holds for $q=2^jp$. □

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. (i)⇒(ii): By Corollary 4.4, each closed *-subalgebra of s is a complemented subspace of s.

(ii)⇔(iii): See e.g. [13, Proposition31.7].

(iii)⇒(iv): By Corollary 5.4 and nuclearity (see e.g. [13, Proposition28.16]),

$$E\cong\lambda^\infty\left(\max_{j\in\mathcal{N}_k}|f_j|_q\right)=\lambda^1\left(\max_{j\in\mathcal{N}_k}|f_j|_q\right)$$

as Fréchet *-algebras. Hence, by [21,22, Proposition 5.3], the property (Ω) yields

$$\forall l\ \exists m\ \forall n\ \exists t\ \exists C>0\ \forall k \quad \max_{j\in\mathcal{N}_k}|f_j|_l^t \max_{j\in\mathcal{N}_k}|f_j|_n \le C\max_{j\in\mathcal{N}_k}|f_j|_m^{t+1}.$$

In particular, taking $l=0$, we get (iv).

(iv)⇒(i): Take p from the condition (iv). By Lemma 6.3(ii), there is $p_1 \ge p$ and a bijection $\sigma\colon \mathbb{N}\to\mathbb{N}$ such that $(\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1})_{k\in\mathbb{N}}$ is non-decreasing and $\lim_{k\to\infty}\frac{k}{\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}}=0$. Consequently, for k big enough

$$\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}\ge 2k,$$

and therefore, by Lemma 6.4, there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ and $C_1>0$ such that

$$\frac{1}{C_1}\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}\le n_k\le C_1\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}^2 \tag{6}$$

for every $k\in\mathbb{N}$. Now, by the conditions (iv) and (6), we get that for all q there is r and $C_2:=CC_1^r$ such that

$$\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_q\le C\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}^r\le C_2 n_k^r$$

for all $k\in\mathbb{N}$, so the condition (α) from Proposition 6.5(v) holds. Finally, by (6) and Proposition 6.6 we obtain that for all r' there is q' and $C_3:=C_1^{r'}$ such that

$$n_k^{r'}\le C_3\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{p_1}^{2r'}\le C_3\max_{j\in\mathcal{N}_{\sigma(k)}}|f_j|_{q'}$$

for every $k\in\mathbb{N}$. Hence the condition (β) from Proposition 6.5(v) is satisfied, and therefore, by Proposition 6.5, E is isomorphic as a Fréchet *-algebra to the closed *-subalgebra of s generated by $\{e_{n_k}\}_{k\in\mathbb{N}}$. □

Now we shall give an example of some class of closed commutative *-subalgebras of $\mathcal{L}(s',s)$ which are isomorphic to closed *-subalgebras of s.

Example 6.7 Let $\mathbb{H}_1:=[1]$. We define recursively *Hadamard matrices*

$$\mathbb{H}_{2^n}:=\begin{bmatrix}\mathbb{H}_{2^{n-1}} & \mathbb{H}_{2^{n-1}}\\ \mathbb{H}_{2^{n-1}} & -\mathbb{H}_{2^{n-1}}\end{bmatrix}$$

for $n\in\mathbb{N}$. Then the matrices $\widehat{\mathbb{H}}_{2^n}:=2^{-\frac{n}{2}}\mathbb{H}_{2^n}$ are unitary, and thus their rows form an orthonormal system of 2^n vectors. Now fix an arbitrary sequence $(d_n)_{n\in\mathbb{N}}\subset\mathbb{N}_0$ and define

$$U := \begin{bmatrix} \widehat{\mathbb{H}}_{2^{d_1}} & 0 & 0 & \dots \\ 0 & \widehat{\mathbb{H}}_{2^{d_2}} & 0 & \dots \\ 0 & 0 & \widehat{\mathbb{H}}_{2^{d_3}} & \\ \vdots & \vdots & & \ddots \end{bmatrix}.$$

Let f_k denote the k-th row of the matrix U. Then $(f_k)_{k\in\mathbb{N}}$ is an orthonormal basis of ℓ_2 and clearly each f_k belongs to s. We will show that the closed (maximal) commutative *-subalgebra $\operatorname{alg}(\{\langle\cdot,\cdot, f_k\rangle f_k\}_{k\in\mathbb{N}})$ of $\mathcal{L}(s', s)$ is isomorphic to some closed *-subalgebra of s. By Theorem 6.2, it is enough to prove that

$$\exists p\ \forall q\ \exists r\ \exists C > 0\ \forall k \quad |f_k|_{\infty,q} \le C|f_k|^r_{\infty,p}. \tag{7}$$

Fix $q \in \mathbb{N}_0$, $k \in \mathbb{N}$ and find $n \in \mathbb{N}$ such that $2^{d_1} + \dots + 2^{d_{n-1}} < k \le 2^{d_1} + \dots + 2^{d_n}$. Then

$$\frac{|f_k|_{\infty,q}}{|f_k|^{2q}_{\infty,1}} = \frac{2^{-\frac{d_n}{2}}(2^{d_1} + \dots + 2^{d_n})^q}{2^{-d_n q}(2^{d_1} + \dots + 2^{d_n})^{2q}} = 2^{d_n(q-1/2)}(2^{d_1} + \dots + 2^{d_n})^{-q} \le 1$$

and thus the condition (7) holds with $p = C = 1$ and $r = 2q$.

The next theorem solves in negative [3, Open Problem 4.13]. In contrast to the algebra s, all of whose closed *-subalgebras are complemented subspaces of s (Corollary 4.4), Theorems 6.2 and 6.9 imply that there is a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ which is not complemented in $\mathcal{L}(s', s)$ (otherwise it would have the property (Ω), see [13, Proposition 31.7]). In the proof we will use the following identity.

Lemma 6.8 *For every increasing sequence $(\alpha_j)_{j\in\mathbb{N}} \subset (0, \infty)$ and every $p \in \mathbb{N}$ we have*

$$\sup_{j\in\mathbb{N}} \left(\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i \right) = \prod_{i=1}^{p} \alpha_i.$$

Proof For $j \ge p + 1$ we get

$$\frac{\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i}{\prod_{i=1}^{p} \alpha_i} = \alpha_j^{p-j+1} \cdot \prod_{i=p+1}^{j-1} \alpha_i = \frac{\prod_{i=p+1}^{j-1} \alpha_i}{\alpha_j^{j-p-1}} \le 1$$

and, similarly, for $j \le p - 1$ we obtain

$$\frac{\alpha_j^{p-j+1} \cdot \prod_{i=1}^{j-1} \alpha_i}{\prod_{i=1}^{p} \alpha_i} = \frac{\alpha_j^{p-j+1}}{\prod_{i=j}^{p} \alpha_i} \le 1.$$

Since $\alpha_p^{p-p+1} \cdot \prod_{i=1}^{p-1} \alpha_i = \prod_{i=1}^{p} \alpha_i$, the supremum is attained for $j = p$, and we are done. □

Theorem 6.9 *There is a closed commutative* **-subalgebra of* $\mathcal{L}(s', s)$ *which is not isomorphic to any closed* **-subalgebra of* s.

Proof Let m_k be the k-th prime number, $N_{k,1} := m_k$, $N_{k,j+1} := m_k^{N_{k,j}}$ for $j, k \in \mathbb{N}$. Define $a_{k,1} := c_k$ and

$$a_{k,j} := c_k \frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}$$

for $j \geq 2$, where the sequence $(c_k)_{k\in\mathbb{N}}$ is choosen so that $||(a_{k,j})_{j\in\mathbb{N}}||_{\ell_2} = 1$, i.e.

$$c_k := \left(\sum_{j=1}^{\infty} \left(\frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}} \right)^2 \right)^{-1/2}.$$

The numbers c_k make sense, because, by Lemma 6.8,

$$\begin{aligned}
\sum_{j=1}^{\infty} \left(\frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}} \right)^2 &= \sum_{j=1}^{\infty} \left(N_{k,j}^{-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i} \right)^2 \\
&= \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} \left(N_{k,j}^{1-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i} \right)^2 \\
&\leq \sup_{j\in\mathbb{N}} \left(N_{k,j}^{1-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i} \right)^2 \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} \\
&= N_{k,1}^2 \sum_{j=1}^{\infty} \frac{1}{N_{k,j}^2} < N_{k,1}^2 \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.
\end{aligned}$$

Finally, define an orthonormal sequence $(f_k)_{k\in\mathbb{N}}$ by

$$f_k := \sum_{j=1}^{\infty} a_{k,j} e_{N_{k,j}}.$$

We will show that $\mathrm{alg}(\{\langle \cdot, f_k \rangle f_k\}_{k\in\mathbb{N}})$ is a closed *-subalgebra of $\mathcal{L}(s', s)$ which is not isomorphic as an algebra to any closed *-subalgebra of s. By Theorem 6.2 and nuclearity, it is enough to show that each f_k belongs to s and for every $p, r \in \mathbb{N}$ the following condition holds

$$\lim_{k\to\infty} \frac{|f_k|_{\infty,p+1}}{|f_k|_{\infty,p}^r} = \infty,$$

where $|\xi|_{\infty,q} := \sup_{j\in\mathbb{N}} |\xi_j| j^q$.

Note first that $|f_k|_{\infty,p} = a_{k,p}N_{k,p}^p$. In fact, by Lemma 6.8, we get

$$
\begin{aligned}
|f_k|_{\infty,p} &= \sup_{j\in\mathbb{N}} a_{k,j}N_{k,j}^p = c_k \sup_{j\in\mathbb{N}}\left(N_{k,j}^p \cdot \frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}\right) \\
&= c_k \sup_{j\in\mathbb{N}}\left(N_{k,j}^{p-j+1} \cdot \prod_{i=1}^{j-1} N_{k,i}\right) \\
&= c_k \prod_{i=1}^{p} N_{k,i} = c_k N_{k,p}^p \cdot \frac{\prod_{i=1}^{p-1} N_{k,i}}{N_{k,p}^{p-1}} = a_{k,p}N_{k,p}^p.
\end{aligned}
$$

In particular, $f_k \in s$ for $k \in \mathbb{N}$. Next, for $j, k \in \mathbb{N}$, we have

$$
\frac{a_{k,j+1}N_{k,j+1}^j}{a_{k,j}} = \frac{c_k N_{k,j+1}^j \cdot \frac{\prod_{i=1}^{j} N_{k,i}}{N_{k,j+1}^j}}{c_k \frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}} = \frac{\prod_{i=1}^{j} N_{k,i}}{\frac{\prod_{i=1}^{j-1} N_{k,i}}{N_{k,j}^{j-1}}} = N_{k,j}^j.
$$

Moreover, for every $j, r \in \mathbb{N}$ we get

$$
\frac{N_{k,j+1}}{N_{k,j}^r} = \frac{m_k^{N_{k,j}}}{N_{k,j}^r} \geq \frac{2^{N_{k,j}}}{N_{k,j}^r} \xrightarrow[k\to\infty]{} \infty,
$$

and clearly $a_{k,j} \leq 1$ for $j, k \in \mathbb{N}$. Hence, for $p, r \in \mathbb{N}$ we obtain

$$
\begin{aligned}
\frac{|f_k|_{\infty,p+1}}{|f_k|_{\infty,p}^r} &= \frac{a_{k,p+1}N_{k,p+1}^{p+1}}{a_{k,p}^r N_{k,p}^{pr}} = \frac{a_{k,p+1}N_{k,p+1}^p}{a_{k,p}} \cdot \frac{1}{a_{k,p}^{r-1}} \cdot \frac{N_{k,p+1}}{N_{k,p}^{pr}} \\
&= N_{k,p}^p \cdot \frac{1}{a_{k,p}^{r-1}} \cdot \frac{N_{k,p+1}}{N_{k,p}^{pr}} \geq \frac{N_{k,p+1}}{N_{k,p}^{pr}} \xrightarrow[k\to\infty]{} \infty,
\end{aligned}
$$

which is the desired conclusion. □

We end this section with two consequences of Theorem 6.2.

For a monotonically increasing sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ in $[0,\infty)$ such that $\lim_{j\to\infty}\alpha_j = \infty$ we define the power series space of infinite type

$$
\Lambda_\infty(\alpha) := \{(\xi_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}} \colon \sum_{k=1}^{\infty} |\xi_k|^2 e^{2q\alpha_k} < \infty \quad \text{for all}\, q \in \mathbb{N}_0\}.
$$

Corollary 6.10 *Let E be a closed commutative *-subalgebra of $\mathcal{L}(s', s)$ isomorphic as Fréchet space to $\Lambda_\infty(\alpha)$. Then E is isomorphic to $\Lambda_\infty(\alpha)$ as a Fréchet *-algebra.*

Proof Let $(P_k)_{k\in\mathbb{N}}$ be the canonical Schauder basis of E. In view of Proposition 4.2, we should show that there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ such that

(α) $\forall q\in\mathbb{N}_0\ \exists r\in\mathbb{N}_0\ \exists C>0\ \forall k\in\mathbb{N}\quad ||P_{\sigma(k)}||_q\le Ce^{r\alpha_k}$,
(β) $\forall r'\in\mathbb{N}_0\ \exists q'\in\mathbb{N}_0\ \exists C'>0\ \forall k\in\mathbb{N}\quad e^{r'\alpha_k}\le C'||P_{\sigma(k)}||_{q'}$.

By Theorem 6.2, E is isomorphic as a Fréchet *-algebra to some infinite-dimensional closed *-subalgebra of s, and thus by Corollary 4.4, E is isomorphic as a Fréchet *-algebra to $\lambda^\infty(n_k^q)$ for some strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$. Hence, by Proposition 4.2, there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ such that

$$\forall q\in\mathbb{N}_0\ \exists r\in\mathbb{N}_0\ \exists C>0\ \forall k\in\mathbb{N}\quad ||P_{\sigma(k)}||_q\le Cn_k^r, \tag{8}$$

$$\forall r'\in\mathbb{N}_0\ \exists q'\in\mathbb{N}_0\ \exists C'>0\ \forall k\in\mathbb{N}\quad n_k^{r'}\le C'||P_{\sigma(k)}||_{q'}. \tag{9}$$

Since $\lambda^\infty(n_k^q)=\Lambda_\infty(\log n_k)$, it follows from [13, Theorem 29.1] that there is $q\in\mathbb{N}$ and k_0 such that for $k\ge k_0$

$$\frac{1}{q}\alpha_k\le\log n_k\le q\alpha_k.$$

Consequently, there is $q\in\mathbb{N}$ and $D>0$ such that

$$e^{\alpha_k}\le Dn_k^q\quad\text{and}\quad n_k\le De^{q\alpha_k}$$

for all $k\in\mathbb{N}$. Now (8) and (9) yield the desired conclusion. □

By the theorem of Crone and Robinson [5] it follows that all bases of the space s are quasi-equivalent, i.e. given any two bases $(f_k)_{k\in\mathbb{N}}$ and $(g_k)_{k\in\mathbb{N}}$ of s, there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ and a sequence $(c_k)_{k\in\mathbb{N}}$ of non-zero scalars such that the operator $T:s\to s$ defined by $Te_k=c_kf_{\sigma(k)}$ is a Fréchet space isomorphism. Our last result shows that in the case of bases of s which form an orthonormal sequence of ℓ_2, the sequence $(c_k)_{k\in\mathbb{N}}$ can always be taken constant and equal to 1.

Corollary 6.11 *For every Schauder basis $(f_k)_{k\in\mathbb{N}}$ of the space s which is at the same time an orthonormal sequence of ℓ_2 there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ such that $T:s\to s$ defined by $Te_k:=f_{\sigma(k)}$, $k\in\mathbb{N}$, is a Fréchet space isomorphism.*

Proof Clearly, the closed *-subalgebra E of $\mathcal{L}(s',s)$ generated by the sequence of one-dimensional projections $(\langle\cdot,f_k\rangle f_k)_{k\in\mathbb{N}}$ is isomorphic as a Fréchet space to s. Hence, by Corollaries 5.5 and 6.10, $\lambda^\infty(|f_k|_q)\cong E\cong s$ as Fréchet *-algebras. Now, by Proposition 4.2, there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ such that

$$\forall q\in\mathbb{N}_0\ \exists r\in\mathbb{N}_0\ \exists C>0\ \forall k\in\mathbb{N}\quad |f_{\sigma(k)}|_r\le Ck^r,$$

$$\forall r'\in\mathbb{N}_0\ \exists q'\in\mathbb{N}_0\ \exists C'>0\ \forall k\in\mathbb{N}\quad k^{r'}\le C'|f_{\sigma(k)}|_{q'}.$$

This shows that the map $T:s\to s$ which sends e_k to $f_{\sigma(k)}$, $k\in\mathbb{N}$, defines an automorphism of the Fréchet space s. □

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