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On a generalization of the Euler totient function

Jerzy Kaczorowski

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Abstract For a general polynomial Euler product $F(s)$ we define the associated Euler totient function $\varphi(n, F)$ and study its asymptotic properties. We prove that for $F(s)$ belonging to certain subclass of the Selberg class of L-functions, the error term in the asymptotic formula for the sum of $\varphi(n, F)$ over positive integers $n \le x$ behaves typically as a linear function of x. We show also that for the Riemann zeta function the square mean value of the error term in question is minimal among all polynomial Euler products from the Selberg class, and that this property uniquely characterizes $\zeta(s)$.

Keywords Euler totient function · Square mean value · Selberg class · Polynomial Euler products · Converse theorems · Riemann zeta function

Mathematics Subject Classification Primary 11N37; Secondary 11M41 · 11M06

1 Introduction and statement of results

By a polynomial Euler product we mean a function $F(s)$ of a complex variable $s = \sigma + it$ which for $\sigma > 1$ is defined by an absolutely convergent product of the form

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J. Kaczorowski (✉)
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, 61-614 Poznan, Poland
e-mail: kjerzy@amu.edu.pl

J. Kaczorowski
Institute of Mathematics of the Polish Academy of Sciences, 00-956 Warsaw, Poland

$$F(s)=\prod_p F_p(s)=\prod_p\prod_{j=1}^{d}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1}, \tag{1.1}$$

where p runs over primes and $|\alpha_j(p)|\le 1$ for all p and $1\le j\le d$. We assume that d is chosen as small as possible, i.e. that there exists at least one prime number p_0 such that

$$\prod_{j=1}^{d}\alpha_j(p_0)\neq 0.$$

Then d is called the *Euler degree* of F. Note that the L-functions from number theory including the Riemann zeta function, Dirichlet L-functions, Dedekind zeta and Hecke L-functions of the algebraic number fields, as well as the (normalized) L-functions of the holomorphic modular form and, conjecturally, all general automorphic L-functions are polynomial Euler products.

For F in (1.1) we define the associated Euler totient function as follows ($n\in\mathbb{N}$)

$$\varphi(n,F)=n\prod_{p|n}F_p(1)^{-1}.$$

We see that the classical Euler φ-function corresponds to the Riemann zeta function i.e. $\varphi(n)=\varphi(n,\zeta)$. Euler totient function twisted by a primitive Dirichlet character $\varphi(n,\chi)$ as considered in a recent paper [1] corresponds to the case where $F(s)$ is the Dirichlet L-function $L(s,\chi)$.

Let

$$\gamma(p)=p\left(1-\frac{1}{F_p(1)}\right) \tag{1.2}$$

and

$$C(F)=\frac{1}{2}\prod_p\left(1-\frac{\gamma(p)}{p^2}\right). \tag{1.3}$$

Theorem 1.1 *For a polynomial Euler product F of degree d and $x\ge 1$ we have*

$$\sum_{n\le x}\varphi(n,F)=C(F)x^2+O(x(\log(2x))^d).$$

Let us put

$$E(x,F)=\sum_{n\le x}\varphi(n,F)-C(F)x^2.$$

A deeper analysis of $E(x,F)$ can be performed assuming more on the analytic nature of $F(s)$. A convenient framework for such an analysis is provided by the theory of the Selberg class $\mathcal{S}$. We refer to [2–5] for the basic definitions and mention only that $\mathcal{S}$ consists of the Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{a_F(n)}{n^s}$$

which are absolutely convergent for $\sigma > 1$, have meromorphic continuation to the whole complex plane $\mathbb{C}$ with the only possible pole at $s = 1$, satisfy a general functional equation of the Riemann type, and have a very general Euler product expansion.

Let $\mathcal{S}_0$ denote the set of all L-functions $F(s)$ from $\mathcal{S}$ which are polynomial Euler products and such that $F(s) \neq 0$ for

$$\sigma > 1-\frac{c_0(F)}{\log(|t|+10)} \quad (s=\sigma+it,\ -\infty<t<\infty), \tag{1.4}$$

where $c_0(F)$ denotes a positive constant depending on F. Note that it is expected that every $F \in \mathcal{S}$ has an Euler product of this type and satisfies the General Riemann Hypothesis (GRH) i.e. $F(s) \neq 0$ for $\sigma > 1/2$. In particular, we expect that $\mathcal{S}_0 = \mathcal{S}$.

Theorem 1.2 *For $F \in \mathcal{S}_0$ and $x \geq 1$ we have*

$$\int_1^x |E(\xi,F)|^2\,d\xi = \beta(F)x^3 + O(x^3\exp(-c\sqrt{\log x})), \tag{1.5}$$

where

$$\beta(F)=\frac{1}{6\pi^2}\prod_p\left(1+\frac{|\gamma(p)-1|^2}{p^2-1}\right) \tag{1.6}$$

and c denotes a positive constant depending on F.

Let us remark that in the case of the classical Euler φ-function Chowla [6] proved that for $x \geq 2$

$$\int_1^x \left|\sum_{n\leq\xi}\varphi(n)-\frac{3}{\pi^2}\xi^2\right|^2 d\xi = \frac{1}{6\pi^2}x^3+O\left(\frac{x^3}{\log^4 x}\right).$$

This immediately follows from (1.5) since $C(\zeta) = 3/\pi^2$ and $\beta(\zeta) = 1/(6\pi^2)$.

Assuming larger zero-free regions one can obtain sharper estimates of the remainder term in (1.5). In the extremal case we have the following result.

Theorem 1.3 *Under the GRH, the remainder term in* (1.5) *can be replaced by $O(x^{14/5+\varepsilon})$ with an arbitrary $\varepsilon > 0$.*

Theorems 1.2 and 1.3 show that typically $|E(x,F)|$ behave like a linear function of x. More explicitly, most of the time $|E(x,F)|$ is close to $\sqrt{3\beta(F)}x$. In particular, we have the following omega estimate.

Corollary 1.4 *For $x \to \infty$ we have $E(x,F) = \Omega(x)$.*

Recalling (1.6) we can formulate the following extremal property of the Riemann zeta function.

Corollary 1.5 *For every polynomial Euler product F we have*

$$\beta(F) \geq \beta(\zeta) = 1/(6\pi^2),$$

where ζ is the Riemann zeta function.

This leads to the following converse theorem characterizing $\zeta(s)$ as the only polynomial Euler product from the Selberg class with the minimal square mean of $|E(x, F)|$.

In general, by a "converse theorem" we mean a statement identifying a known L-function by its analytic properties. The most classical example of such result is the well known Hamburger theorem saying roughly that the Riemann zeta function is up to a multiplicative constant the only ordinary Dirichlet series satisfying the functional equation of $\zeta(s)$. The theory of the Selberg class is a natural place for discussing general converse theorems. Some of them refer directly to the Riemann zeta function. We have the following results.

1. Riemann zeta function is the only L-function from the Selberg class of degree and conductor equal to 1 [7].
2. Riemann zeta function is the only L-function from the Selberg class of degree 1 which is not entire [7].
3. Riemann zeta function is the only L-function F from the Selberg class with "easy" Dirichlet coefficients meaning that $a_F(n) = \phi(\log n)$ for certain entire function $\phi(z)$ of order 1 and a finite type *i.e.*, satisfying $\phi(z) \ll e^{\alpha|z|}$ for certain positive α and every complex z [8].
4. Riemann zeta function is the only L-function from the Selberg class for which the series

$$\sum_{n=1}^{\infty} \frac{a_F(n) - 1}{n^s}$$

converges for $\sigma > 1/4 - \delta$ for certain positive δ ([9], see also [10]).

We add to this list the following result.

Theorem 1.6 *Let $F \in \mathcal{S}$ be a polynomial Euler product and $\beta(F) = 1/(6\pi^2)$. Then $F(s) = \zeta(s)$.*

The main tool used in the proof of this theorem is the following strong multiplicity one result which is a generalization of the main theorem from [11]. Note that the first multiplicity one theorem for Selberg's class was proved by M. Ram Murty and V. Kumar Murty, see [12].

Theorem 1.7 *Suppose $F, G \in \mathcal{S}$ are two polynomial Euler products such that*

$$a_F(p) = a_G(p) + O\left(\frac{1}{p^\theta}\right)$$

for certain $\theta > 1/2$ and all primes p. Then $F = G$.

As an immediate consequence we have the following result.

Corollary 1.8 *Suppose $F, G \in \mathcal{S}$ are two polynomial Euler products such that $F_p(1) = G_p(1)$ for almost all primes p. Then $F = G$.*

General notation. By c, possibly with a subscript, we denote a generic positive constant which may depend on F and other parameters but is independent of x and y. Its numerical value is not the same in each occurrence, so that we may write for instance $x^c \log x \ll x^c$ as $x \to \infty$. The same convention applies to ε which denotes a generic positive (small) real number. We denote by p_n the n-th prime number, and by $\omega(n)$ the number of distinct prime divisors of n. We shall use the following well-known estimate

$$\omega(n) \ll \frac{\log n}{\log\log n} \quad \text{for all} \quad n \geq 3. \tag{1.7}$$

Moreover, by $\tau_d(n)$ we denote the familiar divisor function of order d, so that $\zeta^d(s) = \sum_{n=1}^{\infty} \tau_d(n) n^{-s}$ for $\sigma > 1$. In particular $\tau_1(n) = 1$ for all n. We shall use the following well-known facts about $\tau_d(n)$. For $x \geq 1$ we have

$$\sum_{n \leq x} \tau_d(n) \ll x(\log(2x))^{d-1}, \tag{1.8}$$

$$\sum_{n \leq x} \frac{\tau_d(n)}{n} \ll (\log(2x))^d, \tag{1.9}$$

and for $x^{\frac{d-1}{d+1}} \leq h \leq x$

$$\sum_{x \leq n \leq x+h} \tau_d(n) \ll h(\log(2x))^{d-1}. \tag{1.10}$$

Moreover,

$$\sum_{n \leq x} \tau_d^2(n) \ll x(\log(2x))^{d^2-1}. \tag{1.11}$$

Estimates (1.8)–(1.10) follow from [13], Theorem 12.2. whereas (1.11) can be easily proved using (1.9), (1.8) and the submultiplicativity of $\tau_d(n)$.

We denote by $\{x\}$ and $[x]$ the fractional and the integer part of a real number x respectively so that $x = [x] + \{x\}$, $[x] \in \mathbb{Z}$ and $0 \leq \{x\} < 1$. Moreover, $||x||$ denotes the distance from x to the nearest integer. Finally, we use the following common notation for the complex exponential function $e(x) = \exp(2\pi i x)$.

2 Lemmas related to general polynomial Euler products

In this section F will always denote a polynomial Euler product of degree d.

Lemma 2.1 *We have*

$$\varphi(n, F) \ll n(\log\log n)^d.$$

Proof We have

$$|\varphi(n, F)| \le n \prod_{p|n} \prod_{j=1}^{d} \left(1 + \frac{|\alpha_j(p)|}{p}\right) = n \exp\left(d \sum_{p|n} \log\left(1 + \frac{1}{p}\right)\right)$$
$$\le n \exp\left(d \sum_{p|n} \frac{1}{p}\right) \ll n \exp(d \log\log p_{\omega(n)}) \ll n(\log\log n)^d,$$

and the lemma follows.

Let

$$\alpha(n) = \mu(n) \prod_{p|n} \gamma(p), \tag{2.1}$$

where $\gamma(p)$ is defined by (1.2).

Lemma 2.2 *The series*

$$\sum_{n=1}^{\infty} \frac{\varphi(n, F)}{n^s}$$

converges absolutely for $\sigma > 2$ *and in this half-plane we have*

$$\sum_{n=1}^{\infty} \frac{\varphi(n, F)}{n^s} = \zeta(s-1) \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}, \tag{2.2}$$

where $\zeta(s)$ *denotes the Riemann zeta function and coefficients* $\alpha(n)$ *are defined by* (2.1). *In particular,*

$$\varphi(n, F) = n \sum_{m|n} \frac{\alpha(m)}{m}.$$

Proof Absolute convergence of the series immediately follows from Lemma 2.1. In order to show (2.2) observe that Dirichlet series of the both sides of this identity have multiplicative coefficients. It suffices therefore to check equality of the local factors. We have

$$\sum_{k=0}^{\infty} \frac{\varphi(p^k, F)}{p^{ks}} = 1 + \frac{1}{F_p(1)} \sum_{k=1}^{\infty} \frac{1}{p^{k(s-1)}}$$

$$= 1 + \frac{1}{F_p(1)} \frac{1}{p^{s-1} - 1}$$
$$= \zeta_p(s-1)\left(1 - \frac{\gamma(p)}{p^s}\right)$$

and the lemma follows.

Let us observe that $\alpha(n) \ll n^\varepsilon$ for every positive ε. Hence the series

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s}$$

absolutely converges for $\sigma > 1$.

Lemma 2.3 *For $\sigma > 1$ we have*

$$\sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = \frac{H(s)}{F(s)}, \tag{2.3}$$

where $H(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ converges absolutely for $\sigma > 1/2$. Moreover, as n runs over square-free positive integers we have

$$h(n) \ll \frac{1}{n} \exp\left(c \frac{\log n}{\log\log(n+2)}\right). \tag{2.4}$$

In particular for such n, $h(n)$ is bounded.

Proof For $\sigma > 1$ we have

$$F(s) \sum_{n=1}^{\infty} \frac{\alpha(n)}{n^s} = \prod_p \left(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \cdots\right)\left(1 - \frac{\gamma(p)}{p^s}\right)$$
$$= \prod_p \left(1 + \frac{a(p) - \gamma(p)}{p^s} + \sum_{k=2}^{\infty} \frac{a(p^k) - a(p^{k-1})\gamma(p)}{p^{ks}}\right)$$
$$= \prod_p \left(1 + \sum_{k=1}^{\infty} h(p^k) p^{ks}\right). \tag{2.5}$$

Since

$$h(p) = a(p) - \gamma(p) = a(p) - p\left(1 - \prod_{j=1}^{d}\left(1 - \frac{\alpha_j(p)}{p}\right)\right) = \sum_{j=1}^{d-1} \frac{A_j(p)}{p^j},$$

where

$$|A_j(p)| \le \binom{d}{j+1} \qquad (1 \le j \le d-1)$$

we see that $h(p) \ll 1/p$. Hence for a square-free n we have

$$|h(n)| = \prod_{p|n} |h(p)| \le \frac{1}{n} c^{\omega(n)},$$

and (2.4) follows form (1.7). Moreover, for $k \ge 2$

$$a(p^k) - a(p^{k-1})\gamma(p) \ll p^{k\varepsilon}$$

for every positive ε. Hence the product in (2.5) absolutely converges for $\sigma > 1/2$ and the lemma follows.

Lemma 2.4 *Let $\alpha(n)$ be defined by* (2.1). *Then*

$$|\alpha(n)| \le \tau_{d+1}(n). \tag{2.6}$$

Moreover, for $x \ge 1$

$$\sum_{n \le x} |\alpha(n)| \ll x(\log(2x))^d \tag{2.7}$$

and

$$\sum_{n \le x} \frac{|\alpha(n)|}{n} \ll (\log(2x))^d. \tag{2.8}$$

Proof For $\sigma > 1$ we have

$$\frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^s} = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_j(p)}{p^s}\right)$$

and hence $|\mu_F(n)| \le \tau_d(n)$. Moreover, observing that $\alpha(n) = 0$ unless n is squarefree, and using (2.3) we obtain

$$|\alpha(n)| \le |\mu(n)| \sum_{m|n} |\mu_F(m)| |h\left(\frac{n}{m}\right)| \ll \sum_{m|n} \tau_d(m) = \tau_{d+1}(n)$$

since, according to (2.4), $h(n)$ is bounded when n runs over squarefree integers. This shows (2.6). Moreover, we have

$$\sum_{n \le x} |\alpha(n)| \le \sum_{n \le x} |\mu(n)| \sum_{m|n} \tau_d(m) \left|h\left(\frac{n}{m}\right)\right| \le \sum_{m \le x} \tau_d(m) \sum_{n \le x/m} |\mu(n)||h(n)|$$
$$\ll x \sum_{m \le x} \frac{\tau_d(m)}{m} \ll x(\log(2x))^d.$$

Similarly,

$$\sum_{n\le x}\frac{|\alpha(n)|}{n}\le\sum_{n\le x}\frac{1}{n}\sum_{m|n}\tau_d(m)\left|h\left(\frac{n}{m}\right)\right|\le\sum_{m\le x}\frac{\tau_d(m)}{m}\sum_{n\le x/m}\frac{|h(n)|}{n}$$
$$\ll\sum_{m\le x}\frac{\tau_d(m)}{m}\ll(\log(2x))^d$$

since, according to Lemma 2.3, we have

$$\sum_{n\le x}\frac{|h(n)|}{n}\ll 1$$

for every $x\ge 1$. The proof is complete.

For $F\in\mathcal{S}_0$ and real $x\ge 1$ we put

$$f(x,\alpha)=\sum_{n\le x}\frac{\alpha(n)}{n}\mathbf{s}\left(\frac{x}{n}\right), \tag{2.9}$$

where

$$\mathbf{s}(x)=\begin{cases}\frac{1}{2}-\{x\} & \text{if } x\notin\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2.5 *For $x\ge 1$, $x\notin\mathbb{Z}$, we have*

$$\sum_{n\le x}\varphi(n,F)=\frac{x^2}{2}\sum_{n\le x}\frac{\alpha(n)}{n^2}+xf(x,\alpha)-R(x,\alpha),$$

where

$$R(x,\alpha)=\frac{1}{2}\sum_{n\le x}\alpha(n)\left\{\frac{x}{n}\right\}\left(1-\left\{\frac{x}{n}\right\}\right). \tag{2.10}$$

Proof By Lemma 2.2 we have

$$\sum_{n\le x}\varphi(n,F)=\sum_{n\le x}\sum_{m|n}\frac{n}{m}\alpha(m)=\frac{1}{2}\sum_{m\le x}\alpha(m)\left[\frac{x}{m}\right]\left(\left[\frac{x}{m}\right]+1\right)$$
$$=\frac{1}{2}\sum_{m\le x}\alpha(m)\left(\frac{x}{m}-\left\{\frac{x}{m}\right\}\right)\left(\frac{x}{m}+1-\left\{\frac{x}{m}\right\}\right).$$

The lemma now follows after multiplying expressions in brackets and suitably rearranging terms.

3 Proof of Theorem 1.1

Using (2.7) and partial summation we see that

$$\sum_{n>x} \frac{\alpha(n)}{n^2} \ll \frac{(\log(2x))^d}{x}.$$

Hence using multiplicativity of $\alpha(n)$ and recalling (1.3) we have

$$\sum_{n\le x} \frac{\alpha(n)}{n^2} = 2C(F) + O\left(\frac{(\log(2x))^d}{x}\right).$$

Consequently, by Lemma 2.5 we have

$$E(x, F) \ll x \sum_{n\le x} \frac{|\alpha(n)|}{n} + \sum_{n\le x} |\alpha(n)| + x(\log(2x))^d \ll x(\log(2x))^d$$

by (2.8) and (2.7).

4 Lemmas on sums involving divisor function

Recall that for a real number x we denote by $||x||$ the distance from x to the nearest integer.

Lemma 4.1 *For any integers $a, q \ge 1$ and any real number $V \ge 1$ we have*

$$\sum_{\substack{V\le n\le 2V\\ q\nmid an}} \frac{1}{\left\|\frac{an}{q}\right\|} \ll (V+q)\log q. \tag{4.1}$$

Proof This lemma is known, and in fact implicitly contained in [14] (see the proof of Hilfsatz 6). Nevertheless, we give a short proof for sake of completeness. Let us observe that we can assume without the loss of generality that $(a, q) = 1$, and put $N = [V/q]$, $M = [2V/q]$. Then the sum on the right-hand side of (4.1) is at most equal to

$$\sum_{k=N}^{M} \sum_{kq<n<(k+1)q} \frac{1}{\left\|\frac{an}{q}\right\|} \le (M-N+1)\sum_{k=1}^{q-1} \frac{q}{k} \ll \left(\frac{V}{q}+1\right) q\log q,$$

as required.

For real x, y such that $1 \le y \le x$ and a positive integer d let

$$S_d^-(x, y) = \sum_{l\ge 1} \frac{1}{l} \sum_{m\le y} \sum_{n\le y} \frac{\tau_d(m)\tau_d(n)}{mn} \sum_{\substack{k\ge 1\\ km\ne ln}} \frac{1}{k} \min\left(x, \frac{mn}{|km-ln|}\right). \tag{4.2}$$

Moreover, for real $U, V \geq 1$ such that $1 \leq U \leq V \leq y$ let

$$S_d^-(x, y, U, V) = \sum_{l \geq 1} \frac{1}{l} \sum_{U \leq m \leq 2U} \sum_{V \leq n \leq 2V} \frac{\tau_d(m)\tau_d(n)}{mn} \sum_{\substack{k \geq 1 \\ km \neq ln}} \frac{1}{k} \min\left(x, \frac{mn}{|km - ln|}\right).$$

Sums lake these but without coefficients $\tau_d(n)$ where considered in [14]. As we shall see the presence of coefficients invites new difficulties and requires additional reasonings.

Lemma 4.2 *Suppose that* $y \geq x^{1-\varepsilon}$ *for certain* $0 < \varepsilon < \frac{2}{5(d+1)}$. *Then there exists a positive constant* $A = A(d)$ *such that*

$$S_d^-(x, y) \ll (xy)^{1/2} (\log(2x))^A,$$

with an implied constant depending on ε.

Proof We split the range of summation over n and m into $\ll \log(2x)$ localized subranges of the form $U \leq m \leq 2U$ and $V \leq n \leq 2V$. Hence if $S_d^-(x, y, U, V) \ll (xy)^{1/2} (\log(2x))^B$ then $S_d^-(x, y) \ll (xy)^{1/2} (\log(2x))^A$ with $A = B + 2$. Using $\tau_d(n) \ll n^\varepsilon$ and Lemma 4.1 we see that the part of $S_d^-(x, y, U, V)$ with $l \geq x^\varepsilon$ contributes at most

$$\begin{aligned}
&\frac{x^\varepsilon}{U} \sum_{l \geq x^\varepsilon} \frac{1}{l} \sum_{U \leq m \leq 2U} \sum_{V \leq n \leq 2V} \sum_{\substack{k \geq 1 \\ k \neq \frac{ln}{m}}} \frac{1}{k\left(k - \frac{ln}{m}\right)} \\
&\ll \frac{x^\varepsilon}{V} \sum_{l \geq x^\varepsilon} \frac{1}{l^2} \sum_{U \leq m \leq 2U} \sum_{\substack{V \leq n \leq 2V \\ m \nmid ln}} \frac{1}{\left|\left|\frac{ln}{m}\right|\right|} \\
&\ll x^\varepsilon U (\log(2x)) \sum_{l \geq x^\varepsilon} \frac{1}{l^2} \ll U \log(2x) \ll y \log(2x).
\end{aligned}$$

Similarly, contribution of terms with $k \geq x^\varepsilon$ is $\ll y \log(2x)$. Moreover, if $U \leq x^{1-2\varepsilon}$ then

$$S_d^-(x, y, U, V) \ll \frac{x^\varepsilon}{V} \sum_{l \geq 1} \frac{1}{l^2} \sum_{U \leq m \leq 2U} \sum_{\substack{V \leq n \leq 2V \\ m \nmid ln}} \frac{1}{\left|\left|\frac{ln}{m}\right|\right|} \ll x^\varepsilon U \log(2x) \ll y \log(2x)$$

since $x^\varepsilon U \le x^{1-\varepsilon} \le y$. Hence we can assume that $U \ge x^{1-2\varepsilon}$. We have

$$\begin{aligned} S_d^-(x, y, U, V) &\ll \sum_{l \le x^\varepsilon} \frac{1}{l} \sum_{\substack{U \le m \le 2U \\ \left|\left|\frac{ln}{m}\right|\right| \le \frac{V}{x}}} \sum_{V \le n \le 2V} \frac{\tau_d(m)\tau_d(n)}{mn} x \log(2x) \\ &\quad + \frac{1}{V} \sum_{l \le x^\varepsilon} \frac{1}{l^2} \sum_{\substack{U \le m \le 2U \\ \left|\left|\frac{ln}{m}\right|\right| > \frac{V}{x}}} \sum_{V \le n \le 2V} \frac{\tau_d(m)\tau_d(n)}{\left|\left|\frac{ln}{m}\right|\right|} + y \log(2x) \\ &= S_1 + S_2 + y \log(2x), \end{aligned} \tag{4.3}$$

say.

We have

$$S_1 \ll \frac{x \log(2x)}{UV} \sum_{l \le x^\varepsilon} \frac{1}{l} \sum_{\substack{U \le m \le 2U \\ \left|\left|\frac{ln}{m}\right|\right| \le \frac{V}{x}}} \sum_{V \le n \le 2V} \tau_d(m)\tau_d(n)$$

and we split the sum over n into $\ll lV/U$ subranges of lengths $\ll UV/(lx)$ consisting of consecutive integers. Since $UV/(lx) \ge U^2 x^{-1-\varepsilon} \ge x^{1-5\varepsilon} > x^{(d-1)/(d+1)}$ we see that according to (1.9) the sum of $\tau_d(n)$ over every such subrange is

$$\ll \frac{UV}{lx} (\log(2x))^{d-1}.$$

Consequently,

$$\begin{aligned} S_1 &\ll \frac{x \log(2x)}{UV} \sum_{l \le x^\varepsilon} \frac{1}{l} \sum_{U \le m \le 2U} \tau_d(m) \frac{lV}{U} \frac{UV}{lx} (\log(2x))^{d-1} \\ &\ll V(\log(2x))^{2d} \le y(\log(2x))^{2d}. \end{aligned} \tag{4.4}$$

We have

$$S_2 \ll \frac{x^{1/2}}{V^{3/2}} \sum_{l \le x^\varepsilon} \frac{1}{l^2} \sum_{U \le m \le 2U} \sum_{\substack{V \le n \le 2V \\ m \nmid ln}} \frac{\tau_d(m)\tau_d(n)}{\left|\left|\frac{ln}{m}\right|\right|^{1/2}}$$

and hence applying Cauchy-Schwarz inequality and then Lemma 4.1 and (1.11) we have

$$S_2 \ll \frac{x^{1/2}}{V^{3/2}} \sum_{l \le x^\varepsilon} \frac{1}{l^2} \sum_{V \le n \le 2V} \tau_d^2(n) \left(\sum_{U \le m \le 2U} \sum_{\substack{V \le n \le 2V \\ m \nmid ln}} \frac{1}{\left|\left|\frac{ln}{m}\right|\right|} \right)^{1/2}$$

$$\ll \frac{x^{1/2}}{V^{3/2}} V(\log(2x))^{d^2-1}(UV\log(2x))^{1/2}$$
$$\ll (xy)^{1/2}(\log(2x))^{d^2-\frac{1}{2}}. \tag{4.5}$$

Gathering (4.3), (4.4) and (4.5) we obtain the assertion of the lemma with $A(d) = \max(2d, d^2 - \frac{1}{2}) + 2$.

Lemma 4.3 *For every* $\varepsilon > 0$ *we have*

$$S_d^-(x, y) \ll y^{1+\varepsilon}.$$

Proof We proceed as in the proof of Lemma 4.2. Using $\tau_d(n) \ll n^\varepsilon$ and Lemma 4.1 we see that

$$S_d^-(x, y) \ll y^\varepsilon \sum_{l\geq 1} \frac{1}{l^2} \sum_{1\leq m\leq y} \frac{1}{m} \sum_{\substack{m\leq n\leq y \\ m \nmid ln}} \frac{1}{\left|\left|\frac{ln}{m}\right|\right|} \ll y^{1+\varepsilon}.$$

The proof is complete.

For real $y \geq 1$ and a positive integer d let

$$S_d^+(y) = \sum\sum_{1\leq m\leq n\leq y} \tau_d(m)\tau_d(n) \sum_{l\geq 1}\sum_{k\geq 1} \frac{1}{kl(km + ln)}. \tag{4.6}$$

Lemma 4.4 *For* $y \geq 1$ *we have*

$$S_d^+(y) \ll y(\log(2y))^{2d}.$$

Proof We have

$$\sum_{k\geq 1}\sum_{l\geq 1} \frac{1}{kl(km + ln)} = \frac{1}{m} \sum_{l\geq 1} \frac{1}{l} \sum_{k\geq 1} \frac{1}{k\left(k + \frac{ln}{m}\right)}$$
$$\ll \frac{1}{m} \sum_{l\geq 1} \frac{1}{l} \frac{1}{\frac{ln}{m}} \log \frac{ln}{m} \ll \frac{1}{n} \log(2y).$$

Consequently, using (1.9) and (1.10) we obtain

$$S_d^+(y) \ll \sum\sum_{1\leq m\leq n\leq y} \tau_d(m) \frac{\tau_d(n)}{n} \log(2y) \ll y(\log(2y))^{2d}$$

and the lemma follows.

5 Lemmas on sums involving $\alpha(n)$

In this section we assume that $F \in \mathcal{S}_0$, and $\alpha(n)$, $f(x,\alpha)$ and $R(x,\alpha)$ are the associated functions defined by (2.1), (2.9) and (2.10) respectively.

Lemma 5.1 *There exists a positive constant $c_0 = c_0(F)$ such that for $x \geq 1$ we have*

$$\sum_{n\leq x} \alpha(n) \ll x \exp(-c_0\sqrt{\log x}), \tag{5.1}$$

and for every $\sigma \geq 1$

$$\sum_{n\geq x} \frac{\alpha(n)}{n^s} \ll x^{1-\sigma} \exp(-c_0\sqrt{\log x}). \tag{5.2}$$

Under the GRH we have

$$\sum_{n\leq x} \alpha(n) \ll x^{\frac{1}{2}+\varepsilon},$$

and

$$\sum_{n\geq x} \frac{\alpha(n)}{n^s} \ll x^{\frac{1}{2}-\sigma+\varepsilon} \tag{5.3}$$

for every $\sigma > 1/2$ and $\varepsilon > 0$.

This lemma can be proved using Lemma 2.3 and the standard complex integration method. In the proof of (5.1) and (5.2) zero-free region (1.4) is used. Details are skipped.

Lemma 5.2 *For $1 \leq y \leq x$ we have*

$$f(x,\alpha) = \sum_{n\leq y} \frac{\alpha(n)}{n}\mathbf{s}\left(\frac{x}{n}\right) + \rho(x,y),$$

where

$$\rho(x,y) \ll \frac{x}{y}\exp(-c_0\sqrt{\log y}).$$

Under the GRH we have

$$\rho(x,y) \ll xy^{-\frac{3}{2}+\varepsilon}$$

for every positive ε.

Proof We split interval $[y, x]$ into $\ll x/y$ subintervals $I = [a, b)$ where $\mathbf{s}\left(\frac{x}{n}\right)$ is monotonic as a function of n. For every such interval I we have by partial summation and Lemma 5.1

$$\sum_{n\in I} \frac{\alpha(n)}{n} \mathbf{s}\left(\frac{x}{n}\right) \ll \max_{a\le t\le b} \left| \sum_{a\le n\le t} \frac{\alpha(n)}{n} \right| \ll \begin{cases} \exp(-c_0\sqrt{\log y}) & \text{unconditionally,} \\ y^{-\frac{1}{2}+\varepsilon} & \text{under the GRH,} \end{cases}$$

and the lemma follows.

Lemma 5.3 *There exists a positive constant $c_1 = c_1(F)$ such that for $x \ge 1$ we have*

$$R(x, \alpha) \ll x \exp(-c_1\sqrt{\log x}).$$

Under the GRH we have

$$R(x, \alpha) \ll x^{\frac{3}{4}+\varepsilon}.$$

Proof We split the sum on the right hand side of (2.10) into two parts, one over $n \le y$ and the second over $y < n \le x$, where $1 \le y \le x$ is a free parameter to be chosen later on. The second part is estimated similarly as in the proof of Lemma 5.2 using partial summation by

$$\frac{x}{y} \max_{y\le a<b\le x} \left| \sum_{a\le n\le b} \alpha(n) \right| \ll \begin{cases} \frac{x^2}{y} \exp(-c_0\sqrt{\log y}) & \text{unconditionally,} \\ \frac{x^{\frac{3}{2}+\varepsilon}}{y} & \text{under the GRH.} \end{cases}$$

The first part is estimated trivially using Lemma 2.4 and (1.10) by

$$\sum_{n\le y} \tau_{d+1}(n) \ll y(\log(2y))^d.$$

Now we put

$$y = \begin{cases} x \exp(-\frac{c_0}{2}\sqrt{\log x}) & \text{unconditionally,} \\ x^{\frac{3}{4}} & \text{under the GRH,} \end{cases}$$

and the lemma follows.

Remark The exponent in the conditional part of Lemma 5.3 can be improved. We decide to prove a weaker result because of it's simplicity and since it suffices for our purposes.

Lemma 5.4 *We have*

$$\sum_{\substack{k,l\ge 1 \\ km=ln}} \sum \frac{1}{kl} = \frac{\pi^2}{6} \frac{(m, n)^2}{mn}. \tag{5.4}$$

This is Hilfsatz 3 in [14].

Lemma 5.5 *We have*

$$\sum\sum_{m,n\geq 1}\frac{\alpha(m)\overline{\alpha(n)}}{(mn)^2}(m,n)^2 = 36\beta(F).$$

Proof By Lemma 5 in [6] we have

$$\sum\sum_{m,n\geq 1}\frac{\alpha(m)\overline{\alpha(n)}}{(mn)^2}(m,n)^2 = \frac{6}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left|\sum_{d|n}\alpha(d)\right|^2. \tag{5.5}$$

Note that in [6], the above formula was proved for a general arithmetic function $\alpha(n)$ such that $\alpha(n) = O(1)$, but from the provided proof it is evident that the last condition can be relaxed to $\alpha(n) \ll n^{\theta}$ with any fixed $\theta < 1/2$. In particular it applies for our $\alpha(n)$ since $\alpha(n) \ll n^{\varepsilon}$ for every positive ε. Since on the right hand side of (5.5) we sum values of a multiplicative function, and the series converges absolutely, we can replace it by the corresponding Euler product. More explicitly, the right hand side of (5.5) equals

$$\frac{6}{\pi^2}\prod_p\left(1+\sum_{k=1}^{\infty}\frac{1}{p^{2k}}\left|\sum_{d|p^k}\alpha(d)\right|^2\right) = \frac{6}{\pi^2}\prod_p\left(1+\frac{|\gamma(p)-1|^2}{p^2-1}\right)$$

and the lemma follows.

6 Proof of Theorem 1.2

Let us put $y = x\exp(-c\sqrt{\log x})$. Using Lemmas 2.5, 5.2, 5.3 and (5.2) with $s = 2$ we can write for $x/2 \leq \xi \leq x$, $\xi \notin \mathbb{Z}$, $x \geq 2$

$$E(\xi, F) = \xi\Sigma_y(\xi) + O(x\exp(-c\sqrt{\log x}))$$

where

$$\Sigma_y(\xi) = \sum_{n\leq y}\frac{\alpha(n)}{n}\mathbf{s}\left(\frac{\xi}{n}\right). \tag{6.1}$$

Hence using the Cauchy–Schwarz inequality we obtain

$$\int_{x/2}^{x}|E(\xi, F)|^2\,d\xi = I(x) + O((I(x)^{1/2}x^{3/2} + x^3)\exp(-c\sqrt{\log x})),$$

where

$$I(x) = \int_{x/2}^{x} \xi^2 \left|\Sigma_y(\xi)\right|^2 \, d\xi. \tag{6.2}$$

Integrating by parts we get

$$I(x) = x^2 J(x) - 2 \int_{x/2}^{x} \xi J(\xi)\, d\xi, \tag{6.3}$$

where

$$J(\xi) = \int_{x/2}^{\xi} \left|\Sigma_y(u)\right|^2 \, du. \tag{6.4}$$

Squaring out and using the following Fourier expansion

$$\mathbf{s}(u) = \frac{1}{2\pi i} \sum_{k\neq 0} \frac{1}{k} e(ku),$$

we obtain

$$J(\xi) = \frac{1}{4\pi^2} \sum_{m\leq y} \sum_{n\leq y} \frac{\overline{\alpha(m)}\alpha(n)}{mn} \sum_{k\neq 0} \sum_{l\neq 0} \frac{1}{kl} \int_{x/2}^{\xi} e\left(\left(\frac{k}{n} - \frac{l}{m}\right) u\right) du$$

The part of the sum with $km = ln$ equals

$$\begin{aligned} &\frac{1}{2\pi^2}\left(\xi - \frac{x}{2}\right) \sum_{m\leq y} \sum_{n\leq y} \frac{\overline{\alpha(m)}\alpha(n)}{mn} \sum\sum_{\substack{k,l\geq 1\\ km=ln}} \frac{1}{kl} \\ &= \frac{1}{12}\left(\xi - \frac{x}{2}\right) \sum_{m\leq y} \sum_{n\leq y} \frac{\overline{\alpha(m)}\alpha(n)}{m^2 n^2} (m,n)^2, \end{aligned}$$

according to Lemma 5.4. Recalling (2.6), (4.2) and (4.6) we find that the remaining terms contribute at most

$$\sum_{m\leq y} \sum_{n\leq y} \frac{\tau_{d+1}(m)\tau_{d+1}(n)}{mn} \sum\sum_{\substack{k,l\neq 0\\ km\neq ln}} \frac{1}{|kl|} \min\left(x, \frac{mn}{|km - ln|}\right).$$

The part of the last sum with $kl < 0$ equals $2S^{+}_{d+1}(x, y)$, whereas the part with $kl > 0$ equals $2S^{-}_{d+1}(x, y)$. Consequently, we have

$$J(\xi) = \frac{1}{12}\left(\xi - \frac{x}{2}\right) \sum_{m\le y} \sum_{n\le y} \frac{\overline{\alpha(m)}\alpha(n)}{m^2n^2}(m, n)^2 + O(S^{-}_{d+1}(x, y) + S^{+}_{d+1}(y)). \quad (6.5)$$

Now we drop restrictions $m \le y$ and $n \le y$ in the main term. This induces an error of size

$$\ll x\left(\sum_{m\le y}\sum_{n>y} + \sum_{m>y}\sum_{n>y}\right)\frac{\tau_{d+1}(m)\tau_{d+1}(n)}{m^2n^2}(m, n)^2$$

$$\ll x\sum_{d\ge1} d^2\left(\sum_{\substack{m\le y, n>y\\(m,n)=d}}\sum + \sum_{\substack{m>y, n>y\\(m,n)=d}}\sum\right)\frac{1}{(mn)^{2-\varepsilon}}$$

$$\ll x\sum_{d\ge1}\frac{1}{d^{2-\varepsilon}}\left(\sum_{\substack{m\le y/d, n>y/d\\(m,n)=1}}\sum + \sum_{\substack{m>y/d, n>y/d\\(m,n)=1}}\sum\right)\frac{1}{(mn)^{2-\varepsilon}}$$

$$\ll xy^{-1+\varepsilon}\sum_{d\le y}\frac{1}{d} + x\sum_{d>y}\frac{1}{d^{2-\varepsilon}} \ll xy^{-1+\varepsilon} \ll x^{\varepsilon}.$$

Hence applying Lemmas 5.5, 4.2 and 4.4 we obtain

$$J(\xi) = 3\beta(F)(\xi - \frac{x}{2}) + O(x\exp(-c\sqrt{\log x})).$$

We insert this into (6.3) and after some elementary manipulations we obtain

$$I(x) = \frac{7}{8}\beta(F)x^3 + O(x^3\exp(-c\sqrt{\log x})).$$

Let now $N = [\log x/\log 2]$. Then

$$\int_1^x |E(\xi, F)|^2\,d\xi = \sum_{j=0}^{N} I\left(\frac{x}{2^j}\right) + O(1) = \beta(F)x^3 + O(x^3\exp(-c\sqrt{\log x}))$$

and the result follows.

7 Proof of Theorem 1.3

Using Lemmas 2.5, 5.2, 5.3 and (5.3) with $s = 2$ we can write for $x/2 \le \xi \le x$, $\xi \notin \mathbb{Z}$, $x \ge 2$, $1 \le y \le x^{5/6}$

$$E(\xi, F) = \xi\,\Sigma_y(\xi) + O(x^2y^{-\frac{3}{2}+\varepsilon}),$$

where $\Sigma_y(\xi)$ is defined in (6.1). Consequently,

$$\int_{x/2}^{x} |E(\xi, F)|^2 \, d\xi = I(x) + O(I(x)^{1/2} x^{5/2} y^{-(3/2)+\varepsilon} + x^5 y^{-3+\varepsilon}), \tag{7.1}$$

where $I(x)$ is defined in (6.2). As before we have $I(x) = x^2 J(x) - 2\int_{x/2}^{x} \xi J(\xi)\, d\xi$, where $J(\xi)$ is defined in (6.4). Inserting Fourier expansion of $\mathbf{s}(u)$ and integrating term by term we see that (6.5) still holds, but under the GRH we have sharper estimates of the remainder term. In fact, according to Lemmas 4.3 and 4.4 we have

$$S_{d+1}^{-}(x, y) + S_{d+1}^{+} \ll y^{1+\varepsilon}.$$

Moreover, as in the proof of Theorem 1.2 we have

$$x\left(\sum_{m \le y}\sum_{n>y} + \sum_{m>y}\sum_{n>y}\right) \frac{\tau_{d+1}(m)\tau_{d+1}(n)}{m^2 n^2}(m, n)^2 \ll x y^{-1+\varepsilon}.$$

Hence

$$J(\xi) = 3\beta(F)\left(\xi - \frac{x}{2}\right) + O(x y^{-1+\varepsilon} + y^{1+\varepsilon}),$$

and consequently

$$I(x) = \frac{7}{8}\beta(F) x^3 + O(x^3 y^{-1+\varepsilon} + x^2 y^{1+\varepsilon}).$$

In particular $I(x) \ll x^3$. Recalling (7.1) we obtain

$$\begin{aligned}\int_{x/2}^{x} |E(\xi, F)|^2 \, d\xi &= I(x) + O(x^4 y^{\frac{3}{2}+\varepsilon} + x^5 y^{-3+\varepsilon}) \\ &= \frac{7}{8}\beta(F) x^3 + O(x^3 y^{-1+\varepsilon} + x^2 y^{1+\varepsilon} + x^4 y^{\frac{3}{2}+\varepsilon} + x^5 y^{-3+\varepsilon}).\end{aligned}$$

Choosing $y = x^{4/5}$ we get

$$\int_{x/2}^{x} |E(\xi, F)|^2 \, d\xi = \frac{7}{8}\beta(F) x^3 + O(x^{\frac{14}{5}+\varepsilon}),$$

and the result follows as in the proof of Theorem 1.2 by summing integrals over intervals of the form $[x2^{-j-1}, x2^{-j}]$. The proof is complete.

8 Proof of Theorem 1.7

We modify the proof of Theorem 1 from [11]. Since required modifications are rather small we shall be very brief. We consider the quotient $H(s) = F(s)/G(s)$. With obvious notation for $\sigma > 1$ we have

$$H(s) = \prod_p \sum_{m=0}^{\infty} \frac{a_F \star a_G^{-1}(p^m)}{p^{ms}}.$$

Since

$$a_F \star a_G^{-1}(p) = O(\frac{1}{p^\theta}),$$
$$a_F \star a_G^{-1}(p^2) = a_F(p^2) - a_G(p^2) + O\left(\frac{1}{p^\theta}\right)$$

and

$$a_F \star a_G^{-1}(p^m) \ll p^{m\varepsilon}$$

for every $\varepsilon > 0$, we can write

$$H(s) = P(s) f(2s),$$

where $P(s)$ is holomorphic and non-vanishing for $\sigma > \max(\frac{1}{3}, 1-\theta)$, and

$$f(s) = \prod_p \left(1 - \frac{c(p)}{p^s}\right)^{-1},$$
$$c(p) = a_F(p^2) - a_G(p^2).$$

Now we can practically copy the proof of Theorem 1 from [11]. Since $c(p) \ll 1$ we can apply Lemma 1 from [11]. We conclude that H(s) has at most a finite number of poles and zeros on the critical line $\sigma = 1/2$. Hence denoting by $\gamma_F(s)$ and $\gamma_G(s)$ the gamma-factors of the functional equations of $F(s)$ and $G(s)$ respectively, we see that there exists a rational function $R(s)$ satisfying

$$R(s) = \eta \overline{R(1-\overline{s})}$$

for certain $\eta = \pm 1$ and all $s \in \mathbb{C}$ and such that the function

$$K(s) = R(s) H(s) \frac{\gamma_F(s)}{\gamma_G(s)}$$

is holomorphic, non-vanishing for $\sigma \geq 1/2$, and satisfies $K(s) = \vartheta \overline{K(1-\overline{s})}$ for certain $|\vartheta| = 1$. Thus $K(s)$ is a non-vanishing entire function of order 1, and hence by Hadamard's theorem we have

$$K(s) = e^{as+b}, \quad a, b \in \mathbb{C}.$$

Hence

$$\frac{F(s)}{G(s)} = \frac{e^{as+b}}{R(s)} \frac{\gamma_F(s)}{\gamma_G(s)}$$

and by Stirling's formula we get

$$\frac{F(2+it)}{G(2+it)} = ce^{\alpha t} t^{\beta} e^{i\gamma t \log t} e^{i\delta t} \left(1 + O\left(\frac{1}{t}\right)\right), \quad t \to \infty, \tag{8.1}$$

with $c \in \mathbb{C}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. But the left-hand side is almost periodic, thus $\alpha = \beta = \gamma = 0$. Therefore (8.1) becomes

$$e^{-\delta t} \frac{F(2+it)}{G(2+it)} = c + o(1), \quad t \to \infty.$$

The left-hand side of the last equality is almost periodic and tends to a limit as $t \to \infty$, and hence it must be constant. Hence

$$e^{\delta(2-s)} F(s) = cG(s) \quad (s = 2 + it),$$

and by the uniqueness principle for generalized Dirichlet series we have $\delta = 0$. Moreover, since $a_F(1) = a_G(1) = 1$ we have also $c = 1$, and the result follows by analytic continuation.

9 Proof of Corollary 1.8 and Theorem 1.6

Observe that for a polynomial Euler product F we have

$$F(1) = 1 + \frac{a_F(p)}{p} + O\left(\frac{1}{p^2}\right).$$

Hence $F_p(1) = G_p(1)$ for almost all primes p implies

$$a_F(p) = a_G(p) + O\left(\frac{1}{p}\right).$$

So Corollary 1.8 immediately follows from Theorem 1.7.

Let us now prove Theorem 1.6. It readily follows from (1.6) that equality $\beta(F) = \beta(\zeta)$ implies $\gamma(p) = 1$ for all primes p. Recalling (1.2) we see therefore that

$$1 - \frac{1}{F_p(1)} = \frac{1}{p}$$

for all p, and consequently

$$F_p(1) = \left(1 - \frac{1}{p}\right)^{-1} = \zeta_p(1).$$

Hence the result follows from Corollary 1.8.

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