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Erratum

Strong Dunford-Pettis sets and spaces of operators (Monatsh. Math. 144, 275–284 (2005))

By

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Abstract. In a recent paper, Ghenciu and Lewis studied strong Dunford-Pettis sets and made the following two assertions:

(1) The Banach space X^* contains a nonrelatively compact strong Dunford-Pettis set if and only if $\ell_\infty \hookrightarrow X^*$.

(2) If $c_0 \hookrightarrow Y$ and H is a complemented subspace of X so that H^* is a strong Dunford-Pettis space, then $W(X, Y)$ is not complemented in $L(X, Y)$.

While the statements are correct, the proofs are flawed. The difficulty with the proofs is discussed, and a fundamental result of Elton is used to establish a simple lemma which leads to quick proofs of both (1) and (2).

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Each of X and Y will be a real Banach space. The continuous linear dual of X will be denoted by X^*. The space of all continuous linear transformations (= operators) from X to Y will be denoted by $L(X, Y)$ and the space of weakly compact operators will be denoted by $W(X, Y)$. The Banach space X is said to have the Dunford-Pettis property provided that every weakly compact operator with domain X is completely continuous, and a bounded subset B of X is a Dunford-Pettis subset of X if $L(B)$ is relatively compact in Y whenever $L : X \to Y$ is a weakly compact operator. Note that every Dunford-Pettis set is weakly precompact; i.e., if (x_n) is a sequence in the Dunford-Pettis set B, then (x_n) has a weakly Cauchy subsequence [1], [5]. The bounded subset B of X is called a strong Dunford-Pettis set provided that H is a Dunford-Pettis subset of $[H]$ whenever H is a non-empty subset of $\mathrm{aco}(B)$. (Note that there was a typographical error in the definition of a strong Dunford-Pettis set in [6]; aco was omitted from the definition.) If (x_n) is a basic sequence in X, then the associated sequence of coefficient functionals (= a.s.c.f.) which is defined on $[(x_n)]$ will be denoted by

(x_n^*). Specifically, we note that (e_n) will denote the unit vector basis in c_0 and (e_n^*) will denote the unit vector basis in ℓ_1. See [4] or [7] for undefined concepts or notation.

In Lemma 2.2 of [6], Ghenciu and Lewis showed that if (x_n, f_n^*) is a bibasic sequence in $X \times X^*$, $x_i^* = f_i^*|_{[(x_n)]}$ for all i, and $(x_n^*) \sim (e_n^*)$, then $(x_n) \sim (e_n)$ and $[(x_n)]^*$ contains a complemented copy of ℓ_1. If (y_n) is a seminormalized basic sequence *and* a Dunford-Pettis sequence in *both* X and $[(y_n)]$, then the weak precompactness of Dunford-Pettis sets and the characterization of Dunford-Pettis sets in [3] and [1] can be combined with Rosenthal's ℓ_1 theorem to show that if (f_n^*) is a sequence of Hahn-Banach extensions of the a.s.c.f. (y_n^*) to all of X, then there are subsequences $(y_{n_i}^*)$ and $(f_{n_i}^*)$ so that $(y_{n_i}^*) \sim (e_i^*) \sim (f_{n_i}^*)$. Note specifically that $(y_{n_i}^*)$ is defined on all of $[(y_n)_{n=1}^{\infty}]$ and not just on $S :=$ the closed linear span of the subsequence (y_{n_i}). Further, if $w_j^* = y_{n_j}^*|_S$ for each j, then (w_j^*) may fail to be equivalent to (e_j^*) unless the original basic sequence (y_n) is unconditional; e.g., see [7], Proposition 17.3 and Example 17.2. The proofs of both Theorem 2.1 and 3.3(i) of [6] required that (w_j^*) be equivalent to (e_j^*). The following lemma rectifies both problems.

Lemma. *If (x_n) is a seminormalized and weakly null basic sequence in X so that $\{y_i : i \in \mathbf{N}\}$ is a Dunford-Pettis subset of $[(y_i)]$ for each subsequence (y_i) of (x_n), then there is a subsequence (y_i) of (x_n) so that $(y_i) \sim (e_i)$.*

Proof. Let $z_n = x_n/||x_n||$ for each n. Suppose no subsequence of (z_n) is equivalent to (e_n). Apply the construction (due to J. Elton) on p. 28 of [3] to obtain a subsequence (w_n) of (z_n) so that the a.s.c.f. (w_n^*) defined on $W = [(w_n)]$ is weakly null. Since $\{w_n : n \in \mathbf{N}\}$ is a Dunford-Pettis subset of W, we obtain the contradiction $1 = w_n^*(w_n) \to 0$. Thus some subsequence of (x_n) must be equivalent to (e_n).

We now show that this lemma can be used to easily complete the proofs of (1) and (2). No additional results concerning strong Dunford-Pettis sets in [6] need to be modified.

Note first that $\{e_n : n \in \mathbf{N}\}$ is strong Dunford-Pettis set. In fact, the unit ball of any Banach space which does not contain ℓ_1 and has the hereditary Dunford-Pettis property is a strong Dunford-Pettis set.

Now suppose that K is a strong Dunford-Pettis subset of X^* and that K is not relatively compact. Use the precompactness of K, the non-relative compactness of K, and the Bessaga-Pelczynski selection principle [4] to select a seminormalized and weakly null basic sequence (x_n) in $K - K$ which satisfies the hypotheses of the lemma. Thus $c_0 \hookrightarrow X^*$. Consequently, $\ell_\infty \hookrightarrow X^*$, and ℓ_1 is complemented in X. This completes the proof of (1).

Next we suppose that $c_0 \hookrightarrow Y$, H is an infinite dimensional and complemented subspace of X and every relatively weakly compact subset of H^* is a strong Dunford-Pettis set. Since H is a complemented in X, the assumption that $W(X,Y)$ is complemented in $L(X,Y)$ implies that $W(H,Y)$ is complemented in $L(H,Y)$ and showing that $W(H,Y)$ is not complemented in $L(H,Y)$ will complete the proof of (2). The Josefson-Nissenzweig theorem produces a w^*-null and normalized sequence (h_n^*) in H^*. Clearly no subsequence of (h_n^*) is norm Cauchy. Now suppose that (u_n^*) is a weakly Cauchy subsequence of (h_n^*), and let $(v_n^*) = (u_n^* - u_{n+1}^*)$. Then (v_n^*) is a seminormalized and weakly null strong Dunford-Pettis sequence. The Bessaga-Pelczynski selection principle along with the preceding lemma places c_0- and therefore ℓ_∞ – in H^*. Thus ℓ_1 is complemented in H, and Theorem 3 of [2] ensures that $W(H, Y)$ is not complemented in $L(H, Y)$.

References

[1] Andrews K (1979) Dunford-Pettis sets in the space of Bochner integrable functions. Math Ann **241**: 35–41

[2] Bator E, Lewis P (2002) Complemented spaces of operators. Bull Polish Acad Sci Math **50**: 413–416

[3] Diestel J (1980) A survey of results related to the Dunford-Pettis property. Contemporary Math **2**: 15–60

[4] Diestel J (1984) Sequences and Series in Banach Spaces. Berlin Heidelberg New York: Springer

[5] Emmanuele G (1992) Banach spaces in which Dunford-Pettis sets are relatively compact. Arch Math **58**: 477–485

[6] Ghenciu I, Lewis P (2005) Strong Dunford-Pettis sets and spaces of operators. Monatsh Math **144**: 275–284

[7] Singer I (1970) Bases in Banach spaces I. Berlin Heidelberg New York: Springer

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