Rock Mech Rock Eng (2012) 45:995–999
DOI 10.1007/s00603-012-0278-2

ISRM SUGGESTED METHOD

Drucker–Prager Criterion

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Published online: 18 July 2012

List of Symbols

λ	Drucker–Prager material constant
κ	Drucker–Prager material constant
J_2	Second invariant of the stress deviator tensor
I'_1	First invariant of the effective stress tensor
σ'_1	Major principal effective stress
σ'_2	Intermediate principal effective stress
σ'_3	Minor principal effective stress
τ_{oct}	Octahedral shear stress
σ'_{oct}	Octahedral effective normal stress
C_0	Uniaxial compressive strength
T_0	Uniaxial tensile strength
θ	Lode angle
b	$MSDP_u$ parameter that defines the shape of the criterion in the π-plane (usually, $b \cong 0.75$)
a_1	$MSDP_u$ parameter
a_2	$MSDP_u$ parameter
ϕ	Angle of internal friction
c	Cohesion

1 Description

The Drucker–Prager failure criterion is a three-dimensional pressure-dependent model to estimate the stress state at which the rock reaches its ultimate strength. The criterion is based on the assumption that the octahedral shear stress at failure depends linearly on the octahedral normal stress through material constants.

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2 Background

The Drucker–Prager failure criterion was established as a generalization of the Mohr–Coulomb criterion for soils (Drucker and Prager 1952). It can be expressed as:

$$\sqrt{J_2} = \lambda I'_1 + \kappa \tag{1}$$

where λ and κ are material constants, J_2 is the second invariant of the stress deviator tensor and I'_1 is the first invariant of the stress tensor, and are defined as follows:

$$\begin{aligned} I'_1 &= \sigma'_1 + \sigma'_2 + \sigma'_3 \\ J_2 &= \frac{1}{6}\left[\left(\sigma'_1 - \sigma'_2\right)^2 + \left(\sigma'_1 - \sigma'_3\right)^2 + \left(\sigma'_3 - \sigma'_1\right)^2\right] \end{aligned} \tag{2}$$

σ_1', σ_2', and σ_3', are the principal effective stresses.

The criterion, when expressed in terms of octahedral shear stress, τ_{oct}, and octahedral normal stress, σ_{oct}', takes the form:

$$\tau_{oct} = \sqrt{\frac{2}{3}}\left(3\,\lambda\,\sigma'_{oct} + \kappa\right) \tag{3}$$

where $\sigma'_{oct} = 1/3\ I_1'$ and $\tau_{oct} = \sqrt{\frac{2}{3} J_2}$. The Drucker–Prager criterion can thus be considered as a particular case of Nadai's criterion that states that the mechanical strength of brittle materials takes the form $\tau_{oct} = f(\sigma'_{oct})$, where f is a monotonically increasing function (Nadai 1950; Addis and Wu 1993; Chang and Haimson 2000; Yu 2002). It can be also considered as an extension of the Von Mises failure criterion, which is recovered when $\lambda = 0$.

The original Drucker–Prager criterion has been modified to incorporate tension cut off or a cap model (e.g. Lubarda et al. 1996), which allows yield under hydrostatic pressure. Extended Drucker–Prager models have been proposed where the criterion is expressed in linear (i.e. the original criterion), general exponent, or hyperbolic form (e.g. Pariseau 1972 or Hadjigeorgiou et al. 1998).

The modified Drucker–Prager criterion includes the generalized Priest criterion (GP) (Priest 2005), which is discussed in detail in this issue, and the $MSDP_u$ (Mises–Schleicher and Drucker–Prager unified) criterion. The $MSDP_u$ has been proposed to approximate the short-term laboratory strength of low-porosity rocks (Aubertin and Simon 1996; Aubertin et al. 1999; Li et al. 2000) and provides for a non-circular surface in the π-plane, which allows for different strength values in triaxial compression and extension. The $MSDP_u$ criterion is expressed as:

$$\sqrt{J_2} = b\sqrt{\frac{\alpha^2\left(I_1^2 - 2\,a_1 I_1\right) + a_2^2}{b^2 + (1 - b^2)\sin^2(45^\circ - 1.5\,\theta)}} \tag{4a}$$

$$\alpha = \frac{2\sin\phi}{\sqrt{3}\,(3 - \sin\phi)} \tag{4b}$$

$$a_1 = \frac{1}{2}(C_o - T_o) - \frac{C_o^2 - \left(\frac{T_o}{b}\right)^2}{6\,\alpha^2\,(C_o + T_o)} \tag{4c}$$

$$a_2 = \sqrt{\left[\frac{C_o + \frac{T_o}{b^2}}{3\,(C_o + T_o)} - \alpha^2\right] C_o T_o} \tag{4d}$$

where C_o and T_o are the uniaxial compression and tension strengths, respectively; ϕ is the internal friction angle of the rock, θ is the Lode angle and b is a parameter that defines the shape of the criterion in the π-plane (usually, $b \cong 0.75$).

3 Formulation

The original criterion, i.e. Eq. (1), describes a right-circular cone in the stress space when $\lambda > 0$, or a right circular cylinder when $\lambda = 0$; hence the intersection with the π-plane is a circle (Fig. 1).

The parameters λ and κ can be determined from triaxial tests by plotting the results in the I'_1 and $\sqrt{J_2}$ space. Alternatively, the parameters can be obtained from standard compression triaxial tests and can be expressed in terms of internal friction angle and cohesion intercept (Colmenares and Zoback 2001, 2002; Yi et al. 2005, 2006):

$$\lambda = \frac{2\sin\phi}{\sqrt{3}\,(3 - \sin\phi)} \tag{5a}$$

$$\kappa = \frac{6\,c\cos\phi}{\sqrt{3}\,(3 - \sin\phi)} \tag{5b}$$

where c and ϕ are the cohesion intercept and internal friction angle of the rock, respectively. The Drucker–Prager failure cone is circumscribed to the Mohr–Coulomb hexagonal pyramid. There is also the option of obtaining the values of λ and κ that match results from triaxial extension tests. The failure cone passes through the interior vertices of the pyramid, resulting in the middle cone shown in Fig. 1. As a result, and considering only triaxial loading conditions, the circumscribed cone overestimates strength when the stress field evolves from triaxial compression ($\sigma'_1 > \sigma'_2 = \sigma'_3$) to triaxial extension ($\sigma'_1 = \sigma'_2 > \sigma'_3$), and the middle cone underestimates strength, with increasing errors, as the stress state moves from triaxial extension to triaxial compression.

For plane strain, assuming that the dilation angle of the rock is equal to the internal friction angle, i.e. an associated flow rule (inscribed cone in Fig. 1):

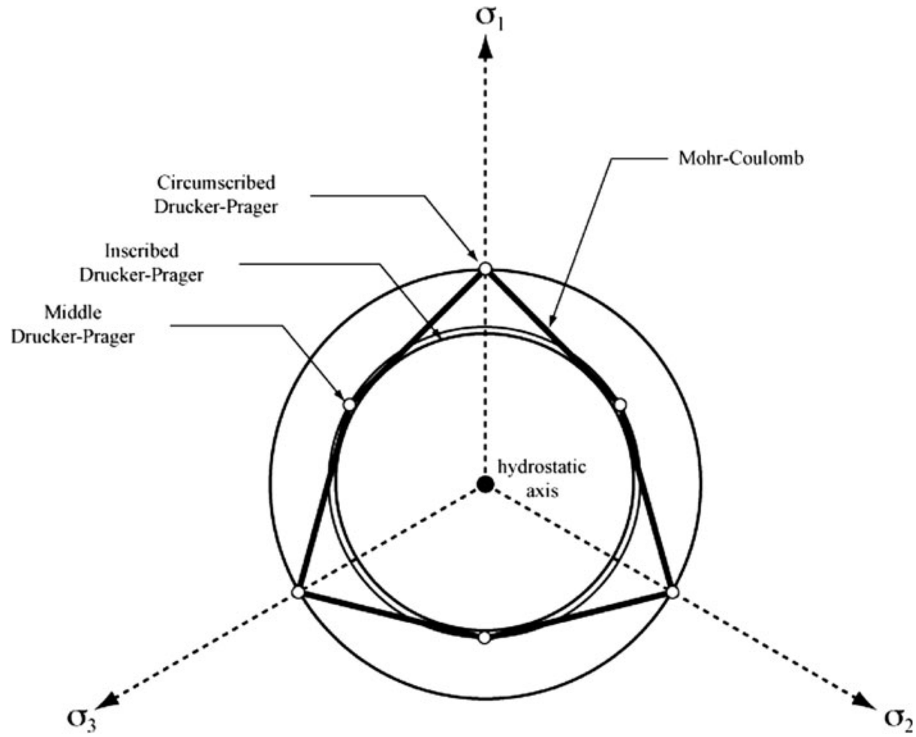
$$\lambda = \frac{\tan\phi}{\sqrt{9 + 12\tan^2\phi}} \tag{6a}$$

$$\kappa = \frac{3c}{\sqrt{9 + 12\tan^2\phi}}. \tag{6b}$$

4 Experimental Data

The difficulties of the Drucker–Prager criterion in predicting polyaxial strength data of intact rock have been documented in the technical literature. It was perhaps Mogi (1967) who first recognized the inability of the criterion to match experimental observations when plotted in the τ_{oct}–σ_{oct}' space, as the data showed different results in triaxial compression than in triaxial extension. Later, Vermeer and De Borst (1984) indicated that the Drucker–Prager approximation was useful for stiff clays with low friction angles but not for sand, rock or concrete. Comparisons between laboratory results and predictions from the criterion have consistently shown that Drucker–Prager criterion tends to overestimate the strength of rock. This was the conclusion reached by Colmenares and Zoback (2002) when they compared the suitability of the criterion with the strength of the following five rocks, obtained from laboratory results reported by others: KTB amphibolite (laboratory results obtained from Chang and Haimson 2000), Dunham dolomite (Mogi 1971), Solnhofen limestone (Mogi 1971), Shirahama sandstone (Takahashi and Koide 1989) and Yuubari shale (Takahashi and Koide 1989). Colmenares and Zoback (2002) observed that Drucker–Prager yielded errors larger than other criteria including

Fig. 1 Drucker–Prager and Mohr–Coulomb failure criteria in stress space



Mohr–Coulomb, Hoek–Brown, Modified Lade, Modified Wiebols and Cook, Mogi (1967) and (1971). Similar conclusions were reached by Al-Ajmi and Zimmerman (2005, 2006) who added to the Colmenares and Zoback (2002) rock database laboratory results from Mizuho trachyte (Mogi 1971), coarse-grained dense marble (Michelis 1985, 1987) and Westerly granite (Haimson and Chang 2000).

The shortcomings of the Drucker–Prager failure criterion in reproducing polyaxial laboratory experiments are illustrated in Fig. 2, which is a plot of laboratory strength tests on Dunham dolomite (Mogi 1971).

Figure 2a shows the strength of the rock in π-stress plane for tests where I'_1 ranges between 800 and 1,000 MPa, together with the corresponding failure envelopes of Mohr–Coulomb and Drucker–Prager inscribed and circumscribed. The Mohr–Coulomb and Drucker–Prager parameters are obtained from triaxial compression tests results, i.e. $\sigma'_1 > \sigma'_2 = \sigma'_3$. As expected, the figure shows a good match between results and predictions of Mohr–Coulomb and Drucker–Prager around the triaxial compression stresses. The errors, however, increase as the differences between σ'_2 and σ'_3 increase. These errors are highlighted in Fig. 2b, which is a plot of two sets of results, each at a different confining stress, $\sigma'_3 = 25$ and 105 MPa, and for different intermediate principal stresses, σ'_2.

In Fig. 2b, Mohr–Coulomb plots as a horizontal line for each value of the minor principal stress σ'_3, as the criterion does not depend on the intermediate principal stress. The predictions match results for the triaxial compression tests results, i.e. for $\sigma'_2 = \sigma'_3$, but the errors increase as σ'_2 increases. A similar trend is observed for the predictions from Drucker–Prager, but with a much larger increase of the errors as the intermediate principal stress σ'_2 increases. This is because in Drucker–Prager the contribution of σ'_2 to strength is the same as that of σ'_3, while in Mohr–Coulomb there is no contribution. The final result is that Mohr–Coulomb underestimates the strength of the rock with increasing intermediate principal stress and Drucker–Prager overestimates it.

Statistical and theoretical considerations also show that the Drucker–Prager criterion provides inaccurate predictions of rock strength and tends to overestimate the magnitude of σ'_1 at failure. Pariseau (2007) proposed the use of the Euclidean or distance norm, defined as the square root of the sum of the squares of the differences between estimated and maximum shear stress at failure, to evaluate different criteria, including Drucker–Prager. Laboratory data from a sandstone (results obtained from Pariseau 2007), norite (Pariseau 2007), Indiana limestone (Schwartz 1964) and Dunham dolomite (Mogi 1971) were used for the comparisons. The Drucker–Prager criterion resulted in the worst predictions, revealing increasing errors with increasing confining pressure. Theoretical considerations by Ewy (1999) and Priest (2010) highlighted the disproportionate sensitivity of the criterion on the intermediate principal stress σ'_2, resulting in an overestimation of the rock strength.

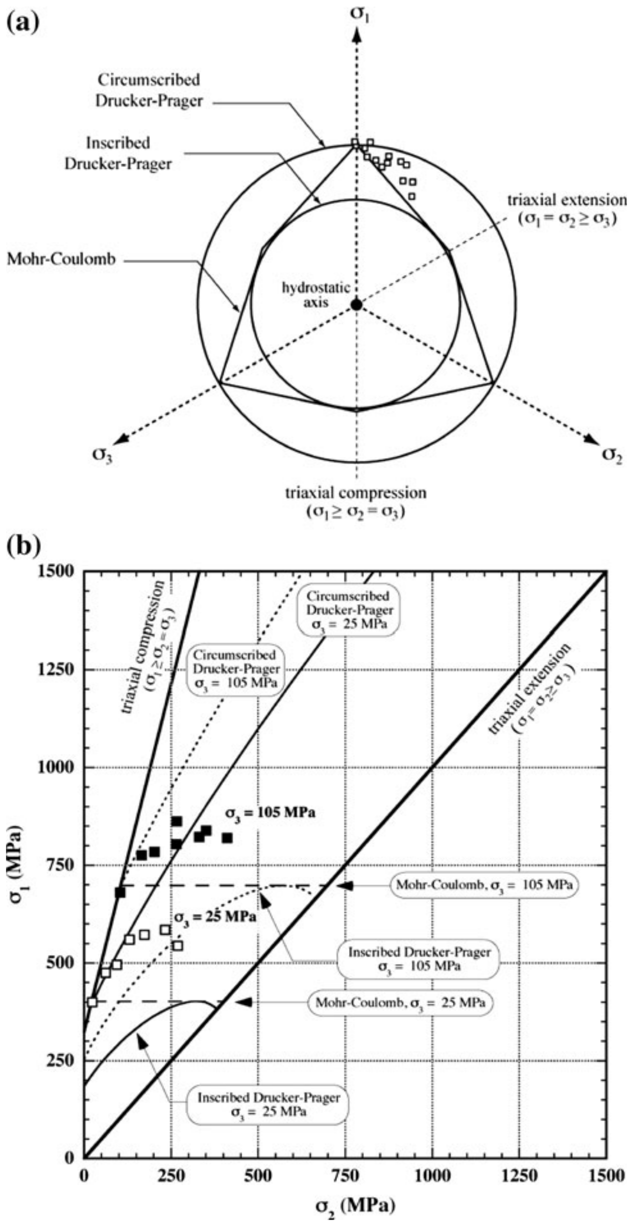


Fig. 2 Polyaxial compression tests results of Dunham dolomite (Mogi 1971). **a** π-Plane and **b** predictions from Mohr–Coulomb and Drucker–Prager failure criteria

5 Advantages and Limitations

The advantages of the Drucker–Prager criterion are its simplicity and its smooth and, with the exception of some of the modified criteria, symmetric failure surface in the stress-space, which facilitate its implementation into numerical codes (Cividini 1993). The criterion gives as much weight to σ'_2 as it does to σ'_1 and σ'_3. While it is certain that σ'_2 has a strengthening effect, it is not as profound as that predicted by Drucker–Prager. The main limitation of the criterion is that it tends to overestimate rock strength for general stress states (Ewy 1999) and produces significant errors in triaxial extension, i.e. $\sigma'_1 = \sigma'_2 > \sigma'_3$. In addition, while the parameters of the criterion can be chosen to match the uniaxial tensile strength of the rock through Eqs. (4), (5) or (6), the criterion does not produce accurate predictions when one or more principal stresses are tensile.

6 Recommendation

Comparisons between laboratory results and predictions from the Drucker–Prager failure criterion consistently show that the criterion tends to overestimate the strength of intact rock. This is because the strengthening effect of σ'_2 is the same as that of σ'_3 in the criterion, which is not supported by laboratory observations. Because the criterion parameters are typically obtained from triaxial tests results, where the intermediate and the minor principal stresses are identical, i.e. $\sigma'_2 = \sigma'_3$, the errors between predictions and results rapidly increase as the values of σ'_2 differ from σ'_3. The Drucker–Prager failure criterion is easy to use and implement in numerical models, but due to the potentially large errors that can occur in estimating intact rock strength, its use should be limited to a narrow range of stresses in the vicinity of the intermediate and minor principal stresses from which the parameters of the criterion are obtained.

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