Few-Body Syst (2024) 65:41
https://doi.org/10.1007/s00601-024-01915-3

A. M. Gasparyan · E. Epelbaum · N. Jacobi · Y. Komissarova

Examples and Counterexamples of Renormalizability of an EFT in the Nonperturbative Regime

Received: 24 January 2024 / Accepted: 19 March 2024
© The Author(s) 2024

Abstract An important feature of an effective field theory is its renormalizability, which implies that one can apply a certain power counting to renormalized quantities and perform a systematic expansion of the calculated observables in terms of some small parameter. When nonperturbative effects become relevant, the requirement of renormalizability imposes nontrivial constraints on a choice of the effective interaction and the renormalization scheme. We discuss several instructive examples and counterexamples of renormalizability to illustrate potential issues one has to deal with in realistic calculations such as nuclear chiral effective field theory.

1 Introduction

Renormalizability of an effective field theory (EFT) is an ability to replace the bare parameters of the effective Lagrangian in terms of renormalized ones and express observable quantities such as the scattering amplitude in terms of those renormalized parameters.

Since any EFT implies some power counting and an expansion in terms of some soft scales, the power counting should be also formulated in terms of the renormalized quantities.

The procedure of renormalization requires an introduction of counter terms that absorb divergent and power-counting-breaking contributions. The contact terms have the form of (quasi-)local contact interactions so that the bare low-energy constants (LECs) C_i corresponding to such contact interactions can be split into the renormalized ones, C_i^r, and the counter terms as follows

$$C_i = C_i^r + \delta C_i. \tag{1}$$

The renormalization procedure becomes especially challenging when nonperturbative effects are important, so that one has to handle an infinite number of divergent or/and power-counting-breaking contributions. One encounters such a situation, e.g., in nuclear chiral EFT when following the scheme of Weinberg [1,2] or its modifications and extensions, see Refs. [3–7] for reviews. Typically in such approaches, one relies on an implicit renormalization trying to find a numeric solution for bare LECs. However, without being able to guarantee the assumed power counting for higher-order contributions, one cannot obtain a fully trustable estimate of theoretical uncertainties.

Explicit proofs of renormalizability within nuclear chiral EFT were obtained in Ref. [8] for the nucleon-nucleon amplitude at next-to-leading order in the chiral expansion. In the subsequent publication [9], the proof was extended to incorporate the nonperturbative effects. In those works the scheme with a finite cutoff, i.e., with

A. M. Gasparyan, E. Epelbaum, N. Jacobi and Y. Komissarova have equally contributed to this work.

A. M. Gasparyan (✉)· E. Epelbaum · N. Jacobi · Y. Komissarova
Fakultät für Physik und Astronomie, Institut für Theoretische Physik II, Ruhr-Universität Bochum, 44780 Bochum, Germany
E-mail: ashot.gasparyan@rub.de

Published online: 12 April 2024

a cutoff of the order of the chiral expansion breakdown scale $\Lambda \sim \Lambda_b$, was considered. In the case of nuclear chiral EFT the breakdown scale can be taken to be of the order of the ρ-meson mass, $\Lambda_b \sim 600 - 700$ MeV. For the present study, however, a particular value of Λ_b is not important.

As a byproduct of the study of the renormalizability of the finite cutoff scheme, certain limitations for alternative approaches with an infinite cutoff ($\Lambda \gg \Lambda_b$), see, e.g., Refs., [10–14], have been demonstrated [15]. The limitations are related to the "exceptional" cutoffs, i.e., the cutoff values where the renormalization fails due to the inconsistency of the renormalization conditions.

In this work we illustrate how the renormalization procedure works and how it could fail by considering two examples: one with a fully local interaction and one with a fully nonlocal separable interaction. We argue that the property of renormalizability is not necessarily fulfilled in an arbitrary quantum mechanical approach and is a rather unique feature of schemes based on an EFT.

2 Scattering Amplitude at Next-to-Leading Order

Our examples are motivated by the nucleon-nucleon system studied in Refs. [8,9]. We consider the scattering amplitude of two particles of mass m_N at leading order (LO) and at next-to-leading order (NLO). The expansion parameter Q is given by the ratio q/Λ_b, where the soft scale q is either the external on-shell momentum p_{on} or the pion mass M_π. The interaction is characterized by the LO and NLO potentials (of order $O(Q^0)$ and $O(Q^2)$) V_0 and V_2 both containing the long-range and short-range parts:

$$V_0(p', p) = V_{0,\text{L}}(p', p) + V_{0,\text{S}}(p', p), \qquad V_2(p', p) = V_{2,\text{L}}(p', p) + V_{2,\text{S}}(p', p), \tag{2}$$

where, for simplicity, we consider the single-channel S-wave scattering. According to the adopted power counting of the Weinberg type the LO potential has to be iterated an infinite number of times, whereas the NLO potential is treated perturbatively, so that the LO amplitude T_0 and the unrenormalized NLO amplitude T_2 can be represented by the series:

$$T_0 = \sum_{n=0}^{\infty} T_0^{[n]}, \qquad T_0^{[n]} = V_0 (G V_0)^n, \tag{3}$$

$$T_2 = \sum_{m,n=0}^{\infty} T_2^{[m,n]}, \qquad T_2^{[m,n]} = (V_0 G)^m V_2 (G V_0)^n, \tag{4}$$

where G is the noninteracting Green's function.

Summing up the series for the LO potential, we obtain

$$T_0 = V_0\, (\mathbb{1} - G V_0)^{-1}, \tag{5}$$

$$T_2 = (\mathbb{1} - V_0 G)^{-1}\, V_2\, (\mathbb{1} - G V_0)^{-1}. \tag{6}$$

The LO off-shell amplitude satisfies the partial-wave Lippmann-Schwinger equation $T_0 = V_0 + V_0 G T_0$, or explicitly:

$$T_0(p', p; p_{\text{on}}) = V_0(p', p) + \int \frac{p''^2 dp''}{(2\pi)^3} V_0(p', p'') G(p''; p_{\text{on}}) T_0(p'', p; p_{\text{on}}),$$
$$G(p''; p_{\text{on}}) = \frac{m_N}{p_{\text{on}}^2 - p''^2 + i\epsilon}. \tag{7}$$

The potentials V_0 and (if necessary) V_2 are regulated with a cutoff $\Lambda \sim \Lambda_b$ to make all relevant amplitudes finite. This leads to the appearance of positive powers of Λ in the amplitude stemminig from the integrals over off-shell momenta, specifically from the regions characterized by $p \sim \Lambda$. In the LO amplitude all of them are compensated by the corresponding negative powers of the hard scale $\Lambda_V \sim \Lambda_b$ coming from the normalization of the LO potential, see Ref. [8]. However, in the NLO amplitude not all positive powers of Λ gets compensated, thereby generating power-counting-breaking contributions. To restore the power counting, a counter term in the form of a momentum independent contact interaction (already present in the LO potential) is introduced in the NLO potential. Its value can be fixed, e.g., by the following renormalization condition for the renormalized on-shell NLO amplitude $\mathbb{R}(T_2)$:

$$\mathbb{R}(T_2)(p_{\text{on}} = 0) = 0. \tag{8}$$

3 Fully Local Potential

The first example we consider is just a particular case of the nucleon-nucleon chiral EFT with a finite cutoff where the interaction is chosen in the form of a fully local potential $V(\vec{p}\,', \vec{p}) = V(\vec{q} = \vec{p}\,' - \vec{p})$, or when transformed to coordinate space,

$$V(\vec{r}\,', \vec{r}\,) = V(r)\delta(\vec{r} - \vec{r}\,'). \tag{9}$$

In principle, the results of Ref. [8,9] are directly applicable in this case. Nevertheless, it is instructive to utilize the locality property of the interaction and to redo the analysis for the NLO amplitude in coordinate space, which is given by the convolution integral

$$T_2(p_{\rm on}) = \int r^2 dr\, V_2(r)\psi^{(+)}_{p_{\rm on}}(r\,)^2 = \frac{(4\pi)^2}{f(p_{\rm on})^2}\int dr\, V_2(r)\phi_{p_{\rm on}}(r\,)^2. \tag{10}$$

Here, $\psi^{(+)}_{p_{\rm on}}$ is the scattering wave function, which in turn is represented as a ratio of the regular solution of the Schrödinger equation $\phi_{p_{\rm on}}$, and the Jost function (Fredholm determinant) $f(p_{\rm on})^2$. The function $\phi_{p_{\rm on}}$ can be expressed as a convergent series in the LO potential V_0:

$$\phi_{p_{\rm on}} = \sum_{n=0}^{\infty}\phi^{(n)}_{p_{\rm on}}, \qquad \phi^{(n+1)}_{p_{\rm on}}(r) = \frac{m_N}{p_{\rm on}}\int_0^r dr'\, \sin[p_{\rm on}(r - r')]V_0(r')\phi^{(n)}_{p_{\rm on}}(r'), \tag{11}$$

with a similar series for the Jost function [16], which allows one to perform a perturbative analysis of $T_2(p_{\rm on})$ even when the Born series in V_0 diverges. A similar decomposition for $T_2(p_{\rm on})$ can be constructed in momentum space for a more general interaction [9]. The advantage of using coordinate representation is that the series in Eq. (11) converges much faster (as $1/n!$) than follows from the most general assessment in momentum space. Moreover, the renormalization can be performed in a more straightforward way, because the amplitude T_2 in Eq. (10) is given by a single integral over r, and the LO potential V_0 entering Eq. (11) contributes only at distances not exceeding r. Therefore, the power counting breaking contributions can be unambiguously identified with the region of small r, i.e. $r \lesssim 1/\Lambda$. This is in contrast to the analysis in momentum space, where the nontrivial sector decomposition of the momentum integrals together with multiple subtractions in all nested subdiagrams have to be performed [8].

In particular, the renormalization of the short-range NLO potential of the form $V_{2,\rm S} \sim \vec{q}\,^2/\Lambda_b^2$ is elementary as the integral in Eq. (10) shrinks to $r = 0$. The analysis of a more general case including the long-range subleading interactions is in progress.

4 Fully Nonlocal Potential

In this section we analyze the opposite example, where both long- and short-range parts of the interaction have the form of a nonlocal separable potential.

In this study, we discuss a toy model that has common features with nucleon-nucleon chiral EFT, for which, however, the standard renormalization procedure fails. More examples of this kind will be studied in a separate publication.

For simplicity, we consider a purely short-range LO potential and a purely long-range NLO potential regulated with some form factor $F_\Lambda(p)$:

$$V_0 = \frac{C_0}{m_N\Lambda_V}F_\Lambda(p')F_\Lambda(p), \tag{12}$$

$$V_2 = \frac{C_2}{m_N\Lambda_V}\frac{p'^2 + p^2}{\Lambda_b^2}\frac{p'^2 p^2}{(M_\pi^2 + p'^2)(M_\pi^2 + p^2)}F_\Lambda(p')F_\Lambda(p). \tag{13}$$

The latter mimics the two-pion-exchange potential in nuclear chiral EFT.

To observe the problems with renormalization, it is sufficient to estimate a single perturbative iteration of V_0 in the on-shell NLO amplitude:

$$V_0 G V_2 \sim \frac{C_0 C_2}{m_N \Lambda_V} \frac{\Lambda}{\Lambda_V} \frac{\Lambda^2}{\Lambda_b^2} \frac{p_{\text{on}}^2}{(M_\pi^2 + p_{\text{on}}^2)} \sim O(Q^0). \tag{14}$$

It contributes at order $O(Q^0)$ and not $O(Q^2)$ and, therefore, violates the power counting. Moreover, such a contribution has a long-range structure and cannot be absorbed by a renormalization of the LO contact interaction. If we introduced another long-range term into the LO potential to compensate for the power-counting-breaking terms, we would have to do this for all higher orders in advance, which is obviously impossible. This signals a failure of renormalization.

It is instructive to trace the origin of such a phenomenon. In the conventional chiral EFT, the long-range part of the nucleon-nucleon potential (at least unregularized, i.e. at small momenta) is local as it is represented by multiple pion exchanges. For the short-range part, there is no such restriction, and it is given by a polynomial in momenta regulated in a rather arbitrary way. As long as the potential fulfills the above criteria, one can deduce the following inequalities for the subtraction remainders [8]

$$\begin{aligned} \left|V(p', p) - V(p', 0)\right| &\leq \left|\frac{p}{p'}\right| \times (\ldots) \text{ if } |p'| > |p|, \\ \left|V(p', p) - V(0, p)\right| &\leq \left|\frac{p'}{p}\right| \times (\ldots) \text{ if } |p| > |p'|, \end{aligned} \tag{15}$$

and similarly for higher order subtractions,

$$\begin{aligned} \left|V(p', p) - \sum_{i=0}^{n} \frac{\partial^i V(p', p)}{i!(\partial p)^i}\bigg|_{p=0} p^i\right| &\leq \left|\frac{p}{p'}\right|^{n+1} \times (\ldots) \text{ if } |p'| > |p|, \\ \left|V(p', p) - \sum_{i=0}^{n} \frac{\partial^i V(p', p)}{i!(\partial p')^i}\bigg|_{p'=0} (p')^i\right| &\leq \left|\frac{p'}{p}\right|^{n+1} \times (\ldots) \text{ if } |p| > |p'|. \end{aligned} \tag{16}$$

Thanks to Eqs. (15), (16), subtractions corresponding to counter terms suppress the contributions from the off-shell momenta of order $p \sim \Lambda$.

Inequalities (15), (16) do, however, not hold for the considered counterexample with the nonlocal long-range potential, which explains the nonrenormalizability of the scheme. Thus we conclude that the interaction obtained from the chiral effective Lagrangian is unique in the sense that it guarantees to a large extent renormalizability of a theory, which is not the case for an arbitrary quantum mechanical potential.

5 Conclusion

We have analyzed two examples motivated by nuclear chiral EFT to demonstrate the importance of explicit renormalization of a theory, especially when going beyond leading order in the EFT expansion. Explicit renormalization enables one to control a consistent power counting for observable quantities by absorbing power-counting-breaking terms into redefinition of lower-order contact interactions.

First, we have chosen the interaction potential in a completely local form. In that case, switching to coordinate space allows one to make the analysis of renormalizability more simple and straightforward. The convergence of the series for various parts of the NLO amplitude in terms of the LO potential becomes much more rapid as compared to the general momentum space analysis.

In the second example, both long- and short-range parts of the potential have been taken in a nonlocal separable form. Such an interaction is at odds with the one obtained, e.g., in nuclear chiral EFT, where the (unregulated) long-range part of the effective potential is always local (apart from polynomial nonlocalities from relativistic corrections). Nevertheless, some typical features of nuclear chiral EFT are preserved. We have shown that one can construct a model that is not renormalizable at NLO in the standard sense, i.e., one cannot modify the LO interaction and absorb the power-counting-breaking terms. Obviously, this destroys predictive power of the theory within the considered example.

Our findings indicate a characteristic property of interactions based on an EFT Lagrangian of being renormalizable in contrast to an arbitrary quantum-mechanical phenomenological scheme, where such a property is not a priori guaranteed.

Acknowledgements This research was supported by ERC AdG NuclearTheory (Grant No. 885150), by the MKW NRW under the funding code NW21-024-A, and by the EU Horizon 2020 research and innovation programme (STRONG-2020, grant agreement No. 824093).

Author Contributions All authors contributed equally to this work.

Data availability No datasets were generated or analysed during the current study.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. S. Weinberg, Nuclear forces from chiral Lagrangians. Phys. Lett. B **251**, 288–292 (1990). https://doi.org/10.1016/0370-2693(90)90938-3
2. S. Weinberg, Effective chiral Lagrangians for nucleon—pion interactions and nuclear forces. Nucl. Phys. B **363**, 3–18 (1991). https://doi.org/10.1016/0550-3213(91)90231-L
3. P.F. Bedaque, U. Kolck, Effective field theory for few nucleon systems. Ann. Rev. Nucl. Part. Sci. **52**, 339–396 (2002). https://doi.org/10.1146/annurev.nucl.52.050102.090637. arXiv:nucl-th/0203055 [nucl-th]
4. E. Epelbaum, H.-W. Hammer, U.-G. Meißner, Modern theory of nuclear forces. Rev. Mod. Phys. **81**, 1773–1825 (2009). https://doi.org/10.1103/RevModPhys.81.1773. arXiv:0811.1338 [nucl-th]
5. R. Machleidt, D.R. Entem, Chiral effective field theory and nuclear forces. Phys. Rept. **503**, 1–75 (2011). https://doi.org/10.1016/j.physrep.2011.02.001. arXiv:1105.2919 [nucl-th]
6. E. Epelbaum, H. Krebs, P. Reinert, High-precision nuclear forces from chiral EFT: state-of-the-art, challenges and outlook. Front. Phys. **8**, 98 (2020). https://doi.org/10.3389/fphy.2020.00098. arXiv:1911.11875 [nucl-th]
7. H.-W. Hammer, S. König, U. Kolck, Nuclear effective field theory: status and perspectives. Rev. Mod. Phys. **92**(2), 025004 (2020). https://doi.org/10.1103/RevModPhys.92.025004. arXiv:1906.12122 [nucl-th]
8. A.M. Gasparyan, E. Epelbaum, Nucleon-nucleon interaction in chiral effective field theory with a finite cutoff: explicit perturbative renormalization at next-to-leading order. Phys. Rev. C **105**(2), 024001 (2022). https://doi.org/10.1103/PhysRevC.105.024001. arXiv:2110.15302 [nucl-th]
9. A.M. Gasparyan, E. Epelbaum, Renormalization of nuclear chiral effective field theory with nonperturbative leading-order interactions. Phys. Rev. C **107**(4), 044002 (2023). https://doi.org/10.1103/PhysRevC.107.044002. arXiv:2301.13032 [nucl-th]
10. A. Nogga, R.G.E. Timmermans, U. Kolck, Renormalization of one-pion exchange and power counting. Phys. Rev. C **72**, 054006 (2005). https://doi.org/10.1103/PhysRevC.72.054006. arXiv:nucl-th/0506005 [nucl-th]
11. B. Long, C.J. Yang, Renormalizing chiral nuclear forces: a case study of 3P0. Phys. Rev. C **84**, 057001 (2011). https://doi.org/10.1103/PhysRevC.84.057001. arXiv:1108.0985 [nucl-th]
12. B. Long, C.J. Yang, Renormalizing chiral nuclear forces: triplet channels. Phys. Rev. C **85**, 034002 (2012). https://doi.org/10.1103/PhysRevC.85.034002. arXiv:1111.3993 [nucl-th]
13. M.P. Valderrama, Perturbative renormalizability of chiral two pion exchange in nucleon-nucleon scattering. Phys. Rev. C **83**, 024003 (2011). https://doi.org/10.1103/PhysRevC.83.024003. arXiv:0912.0699 [nucl-th]
14. M. Pavon Valderrama, Perturbative renormalizability of chiral two pion exchange in nucleon-nucleon scattering: P- and D-waves. Phys. Rev. C **84**, 064002 (2011). https://doi.org/10.1103/PhysRevC.84.064002. arXiv:1108.0872 [nucl-th]
15. A.M. Gasparyan, E. Epelbaum, "Renormalization-group-invariant effective field theory" for few-nucleon systems is cutoff dependent. Phys. Rev. C **107**(3), 034001 (2023). https://doi.org/10.1103/PhysRevC.107.034001. arXiv:2210.16225 [nucl-th]
16. R.G. Newton, *Scattering Theory of Waves and Particles* (Springer, Berlin, 1982)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.