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On the Exponential Decay of Strongly Interacting Cold Atoms from a Double-Well Potential

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Abstract In this article, we study an exponential decay for the gas of bosons with strong repulsive delta interactions from a double-well potential. We consider an exactly solvable model comprising an infinite wall and two Dirac delta barriers. We explore its features both within the exact method and with the resonance expansion approach. The study reveals the effect of the splitting barrier on the decay rate in dependence on the number of particles. Among other things, we find that the effect of the splitting barrier on the decay rate is most pronounced in systems with odd particle numbers. During exponential decay, the spatial correlations in an internal region are well captured by the "radiating state".

1 Introduction

Over the past few years, there has been a growing interest in understanding the decay properties of unstable quantum states [1–8]. In particular, recent progress in fabricating systems of interacting particles has inspired the theoretical community to study the decay properties of unstable many-particle states [9–14]. A simple model to study the decay process of many-particle states is the system of bosons with infinitely strong delta-contact interactions, i.e., the so-called Tonks–Girardeau (TG) gas [15]. Considerable effort has been made already to understand the decay properties of such systems. Among other works, the relevant for exponential and long-time decays were presented in [12,13], respectively. A recent paper [14] went even further and explained the decay mechanism of TG gases at intermediate stages of the time evolution (between exponential and long-time regimes).

In the present paper, we provide a deeper insight into the exponential decay of TG gases from the doublewell trap. As a model, we consider the potential in the form [14]

$$V(x) = \begin{cases} \infty, & \text{if } x \le -L\\ \alpha \delta(x) + \eta \delta(x - L) & \text{otherwise.} \end{cases}$$
(1)

Note that the model is a modification of the celebrated *Winter* model [3], to which a Delta barrier at x = 0 was added, see Fig. 1. The remainder of this article is structured as follows. Section 2 discusses the theoretical tools for studying the time evolution of the decaying TG gas and focuses on the results. Section 3 presents some concluding remarks.

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Fig. 1 Ilustrative diagram of double-well structure in Eq. (1)

2 Results

The scenario we consider is typical of controllable studies of the tunnelling phenomena in modern experiments. In the case studied, the system is initially prepared (t < 0) in the ground state of the TG gas in a hard-wall split trap ($\eta = \infty$). At t = 0, the strength of the right barrier is changed to a finite value of η . As a result, the initial state is no longer stationary and begins to evaluate in time. According to Bose-Fermi mapping [15], the time-dependent TG wave function is given by

$$\Psi(x_1, x_2, \dots, x_N, t) = \prod_{k < l} \operatorname{sgn}(x_k - x_l) \frac{1}{\sqrt{N!}} \det_{i,j}^N [\phi_i(x_j, t)],$$
(2)

where the one-particle state $\phi_k(x, t)$ is governed by the Schrödinger equation,

$$I\hbar\frac{\partial\phi_k(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\phi_k(x,t),\tag{3}$$

with the initial condition as the bound-state eigenfunction of the hard-wall split trap, $\varphi_k(x)$, that is, $\phi_k(x,t)|_{t=0} = \varphi_k(x)$. From here we set $L = \hbar = m = 1$ so that the spatial coordinates, time coordinates, and energies are measured in units of L and mL^2/\hbar , and in $\hbar^2/(mL^2)$, respectively. The system under consideration has a nice feature where both eigenfunctions of the hard-wall split trap, $\varphi_k(x)$ and continuum wave functions ($\eta < \infty$) $\psi_p(x)$ (normalised to a delta Dirac distribution) can be obtained in closed analytical forms. For further detail, we refer readers to the papers [14, 16], in which the relevant formulas are reported. Thanks to those, the solutions to Eq. (3) can be condensed in a Fourier series as follows:

$$\phi_k(x,t) = \int_0^\infty c_k(p)\psi_p(x)e^{-\frac{ltp^2}{2}}dp,$$
(4)

where $c_k(p)$,

$$c_k(p) = \int_{-1}^1 \varphi_k(x) \psi_p(x) dx,$$
(5)

is given in closed analytical form [14]. Nonetheless, numerical computations are required to evaluate the integrals in Eq. (4). We conduct our analysis in terms of the non-escape probability,

$$P^{(N)}(t) = \int_{\Delta^N} |\Psi(x_1, x_2, \dots, x_N, t)|^2 dx_1 \dots dx_N,$$
(6)

 $\Delta^N = [-1, 1]^N$, which informs us of the probability that N bosons remain in the internal region Δ at time t. For the TG wavefunction in Eq. (2), the non-escape probability can be reduced to the matrix form [12,13] $P^{(N)}(t) = \det_{k,l}^N [P_{kl}(t)]$ with the entries $P_{kl}(t) = \int_{-1}^1 \phi_k^*(x, t)\phi_l(x, t)dx$. The diagonal elements $P_{kk}(t)$ are nothing but the non-escape probabilities of the one-particle states. We denote $P_k(t) = P_{kk}(t)$. Within the resonance expansion method [4], the one-particle state that experiences an exponential decay follows the approximation: $\psi_k(x, t) \approx M_k(x)e^{-\Gamma_k t/2 - I\varepsilon_k t}$ with $\Gamma_k = -\text{Im}\{p_k^2\}$, $\varepsilon_k = (\text{Im}\{p_k\}^2 - \text{Re}\{p_k\}^2)/2$, and $M_k(x) = 2\pi I res_{p_k} \{c_k(p)\psi_p(x)\}$, where p_k are the roots of the denominator in the integrand in Eq. (4) on the fourth quadrant of the complex *p*-plane (proper poles) and $res_{p_k}\{f\}$ stands for the residue of *f* at a pole p_k . It should be mentioned that numerical calculations are needed to obtain the values of p_k . The corresponding non-escape probability is $P_k(t) \approx m_k e^{-\Gamma_k t}$, where $m_k = \int_{-1}^1 |M_k(x)|^2 dx$. If $m_k \approx 1$ ($P_k(0) \approx 1$), then the exponential decay starts at about t = 0 and the state $|\psi_k(t)\rangle$ can well be approximated in the region Δ by the so-called "radiating state": $|\psi_k(t)\rangle \approx e^{-\Gamma_k t/2 - I\varepsilon_k t}|\psi_k(0)\rangle$. When taking this approach, the time-dependent TG wavefunction in the region Δ^N takes the form:

$$\Psi(x_1, x_2, \dots, x_N, t) \approx e^{-\Gamma^{(N)} t/2 - I\varepsilon^{(N)} t} \prod_{k < l} \operatorname{sgn}(x_k - x_l) \frac{1}{\sqrt{N!}} \operatorname{det}_{i,j}^N[\phi_i(x_j, 0)],$$
(7)

with $\Gamma^{(N)} = \sum_{k=1}^{N} \Gamma_k$, $\varepsilon^{(N)} = \sum_{k=1}^{N} \varepsilon_k$. Consequently, its validity is expected to hold when $m_k \approx 1$ for k = 1, ..., N ($M^{(N)} = \prod_{k=1}^{N} m_k \approx 1$). The corresponding N-particle non-escape probability is:

$$P^{(N)}(t) \approx e^{-\Gamma^{(N)}t}.$$
(8)

Now, we focus on examining when the above approximations are satisfied and what follows from their applicability.

Our results are summarised in Fig. 2. Figure 2a shows the behaviour of $M^{(N)}$ as a function of N. We can observe a local minimum with a value slightly smaller than one. As a result, there is a value where $N = N_c$ (greater than where the minimum occurs) such that $M^{(N_c)} \approx 1$. Starting from $N = N_c$ the deviation of $M^{(N)}$ from 1 rapidly increases as N increases. This suggests that the point $N = N_c$ can be viewed as a transition point to the regime in which the non-escape probability begins to diverge significantly from its approximation in Eq. (8). To clarify this, Fig. 2b offers a comparison of the results obtained from Eq. (8) with the results of the *exact* numerical calculations, where to support the presentation only case $\eta = 10$, $\alpha = 0$ is shown. As the results indicate, the period in which the decay is consistent with the "radiating state" shrinks with increasing N. When N exceeds the critical value $N_c = 10$ (see Fig. 2a), the decay of the N-particle state switches to the non-exponential regime at a very small t value. When the splitting barrier is present an effect of the parity of a number of particles appears. This is demonstrated in Fig. 2c which displays the behaviour of a relative change defined as $\gamma^{(N)} = (\Gamma^{(N)} - \Gamma^{(N)}_{\alpha=0}) : \Gamma^{(N)}_{\alpha=0}$, where $\Gamma^{(N)}_{\alpha=0}$ represents the decay rate for a system without the splitting barrier. We conclude that the change in the decay process caused by the addition of the splitting barrier is most pronounced for systems with odd numbers of particles and in the small N regime, i. e. where $\gamma^{(N)}$ exhibits its most rapid variation. When there is instead an even number of particles, the decay rate becomes almost insensitive to changes in α . It is worth mentioning that the coherence of the initial state depends on α in the opposite way [16]. That is to say, it strongly depends on α only when N is even. To determine whether Eq. (7) can capture the correlation in the internal region, we tested its ability to reproduce a function n(x, t) defined as $P^{(N)}(t) = \int_{-1}^{1} n(x, t) dx$,

$$n(x,t) = \int_{\Delta^{N-1}} |\Psi(x, x_2, \dots, x_N, t)|^2 dx_2 \dots dx_N.$$
(9)

If n(x, t) is divided by the corresponding value of $P^{(N)}(t)$, then the resulting quantity can be interpreted as the probability density of finding a particle, provided that all the particles remain in the region Δ . The correctness of the "radiating state" in reproducing the spatial correlation is confirmed in Fig. 2d, where the results for n(x, t) obtained using it and the *exact* wave function are compared. The results imply that during exponential decay, spatial correlations in the internal region are indeed consistent with those in the initial state. More details regarding the correlation in the initial state (i.e. in the TG ground state in the hard-wall split trap) can be found in [16].



Fig. 2 Results for different quantities as discussed in the text obtained for some transparent control parameter values. **a** Results for $M^{(N)}$ as a function of N. **b** N-particle non-escape probability, where the continuous lines represent the results of Eq. (8) and the markers the results of *exact* calculations. For clarity, the dashed vertical lines mark the points where the deviations from the exponential decay begin in the cases of N = 11 and N = 15. **c** The behaviour of the relative change $\gamma^{(N)}$. **d** Results for n(x, t), obtained for N = 3 particles at t = 10 from the numerically *exact* wave function (markers) and its approximation in Eq. (7) (continuous lines)

3 Conclusions

We have studied the exponential decay of N bosons with strong delta interactions from the double-well structure. Using the resonance expansion approach, we have analysed the effect of the splitting barrier on the decay rate of the N-particle system in dependence on N. We have found that the decay rate of an initial state with an odd number of particles is strongly affected by the splitting barrier, in contrast to its coherence which is insensitive to changes in the height of the splitting barrier. Our results have shown that in the exponential regime the "radiating state" effectively reproduces the spatial correlations in the internal region.

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