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Extraction of Compton Form Factors in Scalar QED

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Abstract When deeply virtual Compton scattering is used as a tool to study the structure of hadrons in an exclusive process, one way to analyze this process is to express the amplitudes in terms of generalized parton distributions (GPDs). The definition of the latter quantities requires a special kinematics, that cannot always be realized in experiments. Therefore, one may use the expression of the scattering amplitudes in terms of the invariant Compton form factors (CFFs) as a stepping stone to finding the GPDs. In a simple case we illustrate the influences of making approximations in the extraction of CFFs on the values obtained.

1 Introduction

In scalar QED (sQED) the number of Coulomb form factors (CFFs) is known to be five [1–4]. In the case where the incoming photon is virtual, namely produced by electron scattering, while the final photon is real, the physical amplitudes depend on only three of them [5]. The connection between the quark structure of the target hadron is usually expressed in terms of generalized parton distributions (GPDs). In a convenient kinematics, the CFFs can be found as integrals over GPDs, see for instance Refs. [1,2,6].

We study the CFF formulation in a simple exactly solvable model and discuss in this model the forward scattering limit and the deeply-virtual Compton scattering (DVCS) limit, where the incoming-photon virtuality is large compared to the mass scales and the incoming and emitted photon are (almost) collinear.

In virtual Compton scattering (see Fig. 1) the physical amplitudes can be written as the contraction of a tensor operator with the photon polarization vectors. It is important to use the most general form of that tensor operator consistent with electro-magnetic gauge invariance. We sketch a well-known construction of this tensor [3,4] and discuss the tensor obtained using it.

2 Tensor Formulation

We write the physical amplitudes as contractions of a tensor with the polarization vectors of the photons:

$$A(h', h) = \epsilon^*(q'; h')_\mu T^{\mu\nu} \epsilon(q; h)_\nu. \tag{1}$$

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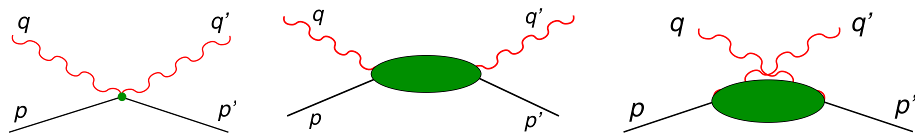

Fig. 1 The tree-level diagrams

The tensor is written in terms of scalars (CFFs) and basis tensors. The choice of basis tensors is motivated in Ref. [5]. Here we only give the results.

The basis tensors are written in terms of the momenta of the photons, q and q', and the sum of the momenta of the target $\bar{P} = p + p'$. (The unprimed momenta refer to the initial state, the primed ones to the final state.) To make the basis tensors transverse to the photon momenta a two-sided projector is applied that is transverse to q and q':

$$\tilde{g}^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q'^\nu}{q \cdot q'} \tag{2}$$

This procedure provides the following transverse momenta:

$$\tilde{q}^\nu_{\mathrm{R}} = q_n \, \tilde{g}^{n\nu}, \quad \tilde{P}^\mu_{\mathrm{L}} = \tilde{g}^{\mu m} \bar{P}_m, \quad \tilde{P}^\nu_{\mathrm{R}} = \bar{P}_n \, \tilde{g}^{n\nu}. \tag{3}$$

Then one may write $T^{\mu\nu}$ in the following form:

$$T^{\mu\nu} = \mathcal{H}_0 \, \tilde{g}^{\mu\nu} + \mathcal{H}_1 \, \tilde{P}^\mu_{\mathrm{L}} \tilde{P}^\nu_{\mathrm{R}} + \mathcal{H}_2 \, \tilde{P}^\mu_{\mathrm{L}} \tilde{q}^\nu_{\mathrm{R}} \tag{4}$$

This form is valid if one of the photons is real, namely $q'^2 = 0$, and the number of independent physical amplitudes reduces to three, say $A(1,1)$, $A(1,0)$, and $A(1,-1)$. One could write a more general tensor consisting of five independent parts, but the additional pieces would be orthogonal to the polarization vectors of the real photon and not contribute to the amplitudes.

The tree-level DVCS amplitude corresponds to the CFFs

$$\mathcal{H}_0 = -2, \quad \mathcal{H}_1 = \left(\frac{1}{s - M^2} + \frac{1}{u - M^2} \right), \quad \mathcal{H}_2 = 0, \tag{5}$$

where $s = (p+q)^2$, $u = (p - q')^2$ and we use in what follows $q^2 = -Q^2$. Thus, in this approximation to the dynamics, only two out of three CFFs contribute. The tree-level amplitude has the same number of CFFs whatever the kinematics. Here they are simple functions of the Mandelstam variables, but they will be much more complicated if one goes beyond the lowest order in perturbation theory.

We note that $\mathcal{H}_1$ is of relative order $1/Q^2$ compared to $\mathcal{H}_0$ at large Q, reflecting the fact that the s- and u-channel diagrams involve a propagator, while the seagull does not.

3 Kinematics

We shall work in the hadronic center-of-mass frame and align the z-axis with the incoming momenta. Then the momenta are given by

$$\begin{aligned} p^\mu &= (E_{\mathrm{C}}, 0, 0, -q_{\mathrm{C}}), \quad q^\mu = (q^0_{\mathrm{C}}, 0, 0, q_{\mathrm{C}}), \\ p'^\mu &= (E'_{\mathrm{C}}, -q'_{\mathrm{C}} \sin\theta, 0, -q'_{\mathrm{C}} \cos\theta), \quad q'^\mu = (q'_{\mathrm{C}}, q'_{\mathrm{C}} \sin\theta, 0, q'_{\mathrm{C}} \cos\theta). \end{aligned} \tag{6}$$

The scattering angle is denoted by θ. To make the connection with the variables used in deep-inelastic processes, we write the components in terms of the Mandelstam invariant s and the Bjorken variable x_{Bj} defined by

$$s = (p+q)^2, \quad x_{\mathrm{Bj}} = \frac{Q^2}{2p \cdot q} \quad \longleftrightarrow \quad s = M^2 + \frac{1 - x_{\mathrm{Bj}}}{x_{\mathrm{Bj}}} Q^2, \quad x_{\mathrm{Bj}} = \frac{Q^2}{s + Q^2 - M^2}, \tag{7}$$

which demonstrates that $x_{\mathrm{Bj}} \to 0$ for vanishing Q unless $s \to M^2$ and that s scales as Q^2 for $0 < x_{\mathrm{Bj}} < 1$.

After substituting these expressions for s one finds

$$q_{\mathrm{C}}^0 = \frac{(1-2x_{\mathrm{Bj}})Q^2}{2\sqrt{x_{\mathrm{Bj}}^2 M^2 + x_{\mathrm{Bj}}(1-x_{\mathrm{Bj}})Q^2}},$$
$$q_{\mathrm{C}} = \frac{\sqrt{4M^2Q^2 + \frac{Q^4}{x_{\mathrm{Bj}}^2}}}{2\sqrt{M^2 + \frac{1-x_{\mathrm{Bj}}}{x_{\mathrm{Bj}}}Q^2}}, \quad E_{\mathrm{C}} = \frac{Q^2 + 2x_{\mathrm{Bj}}M^2}{2\sqrt{x_{\mathrm{Bj}}^2 M^2 + x_{\mathrm{Bj}}(1-x_{\mathrm{Bj}})Q^2}}, \tag{8}$$
$$q'_{\mathrm{C}} = \frac{(1-x_{\mathrm{Bj}})Q^2}{2\sqrt{x_{\mathrm{Bj}}^2 M^2 + x_{\mathrm{Bj}}(1-x_{\mathrm{Bj}})Q^2}}, \quad E'_{\mathrm{C}} = \frac{2M^2 + \frac{1-x_{\mathrm{Bj}}}{x_{\mathrm{Bj}}}Q^2}{2\sqrt{M^2 + \frac{1-x_{\mathrm{Bj}}}{x_{\mathrm{Bj}}}Q^2}}.$$

The large-Q limits are easily obtained

$$q_C \to \frac{Q}{2\sqrt{x_{\mathrm{Bj}}(1-x_{\mathrm{Bj}})}}, \quad E_C \to \frac{Q}{2\sqrt{x_{\mathrm{Bj}}(1-x_{\mathrm{Bj}})}}, \quad q'_C \to \frac{\sqrt{1-x_{\mathrm{Bj}}}}{2\sqrt{x_{\mathrm{Bj}}}}Q, \quad E'_C \to \frac{\sqrt{1-x_{\mathrm{Bj}}}}{2\sqrt{x_{\mathrm{Bj}}}}Q. \tag{9}$$

The $Q \to 0$ limits are gotten in a straight forward way from the original formulas Eq. (6) and calculating s using these expressions. The result is

$$q_C = \frac{s-M^2}{2\sqrt{s}}, \quad E_C = \frac{s+M^2}{2\sqrt{s}}, \quad q'_C = \frac{s-M^2}{2\sqrt{s}}, \quad E'_C = \frac{s+M^2}{2\sqrt{s}}. \tag{10}$$

Note that in this limit s is not necessarily equal to M^2, because from Eq. (7) we see that the factor Q^2/x_{Bj} is identical to $2p \cdot q$ which does not necessarily vanish when $Q \to 0$.

Because GPDs are understood to be defined in the limit $t \to 0$, it is interesting to study this limit both for large Q and for small Q. We first write the general formula

$$t = -\frac{M^4 - (2s - Q^2)M^2 + s(s+Q^2) - (s-M^2)\sqrt{M^4 - 2M^2(s-Q^2) + (s+Q^2)^2}\cos\theta}{2s}. \tag{11}$$

The limit for large Q and fixed x_{Bj} is

$$t \to -\frac{1-\cos\theta}{2x_{\mathrm{Bj}}}Q^2 + \frac{1 - 2x_{\mathrm{Bj}} - \left(1 - 2x_{\mathrm{Bj}} + 2x_{\mathrm{Bj}}^2\right)\cos\theta}{2(1-x_{\mathrm{Bj}})}M^2 + \mathcal{O}\left(M^4/Q^2\right), \tag{12}$$

where we retained the term independent of Q to demonstrate that if $\theta \to 0$, t goes to zero up to corrections of $\mathcal{O}(M^2)$, thus it does not strictly vanish in the forward limit. If the experimental set-up limits the scattering angle from below to some value θ_{lim}, t remains of order Q^2. So, for moderate values of Q^2 and $x_{\mathbf{Bj}}$, which are found in the planned experiments at JLab [7], the ratio $|t|/Q^2$ has a finite minimum value, which should be taken into account when connecting the CFFs to GPDs.

4 Amplitudes

To obtain the amplitudes, we need the polarization vectors. They are in our kinematics

$$\epsilon^\mu(q', \pm 1) = \frac{1}{\sqrt{2}}(0, \mp\cos\theta, i, \pm\sin\theta), \quad \epsilon^\mu(q, \pm 1) = \frac{1}{\sqrt{2}}(0, \mp 1, i, 0), \quad \epsilon^\mu(q, 0) = \frac{1}{\sqrt{-Q^2}}\left(-q_{\mathrm{C}}, 0, 0, q_{\mathrm{C}}^0\right). \tag{13}$$

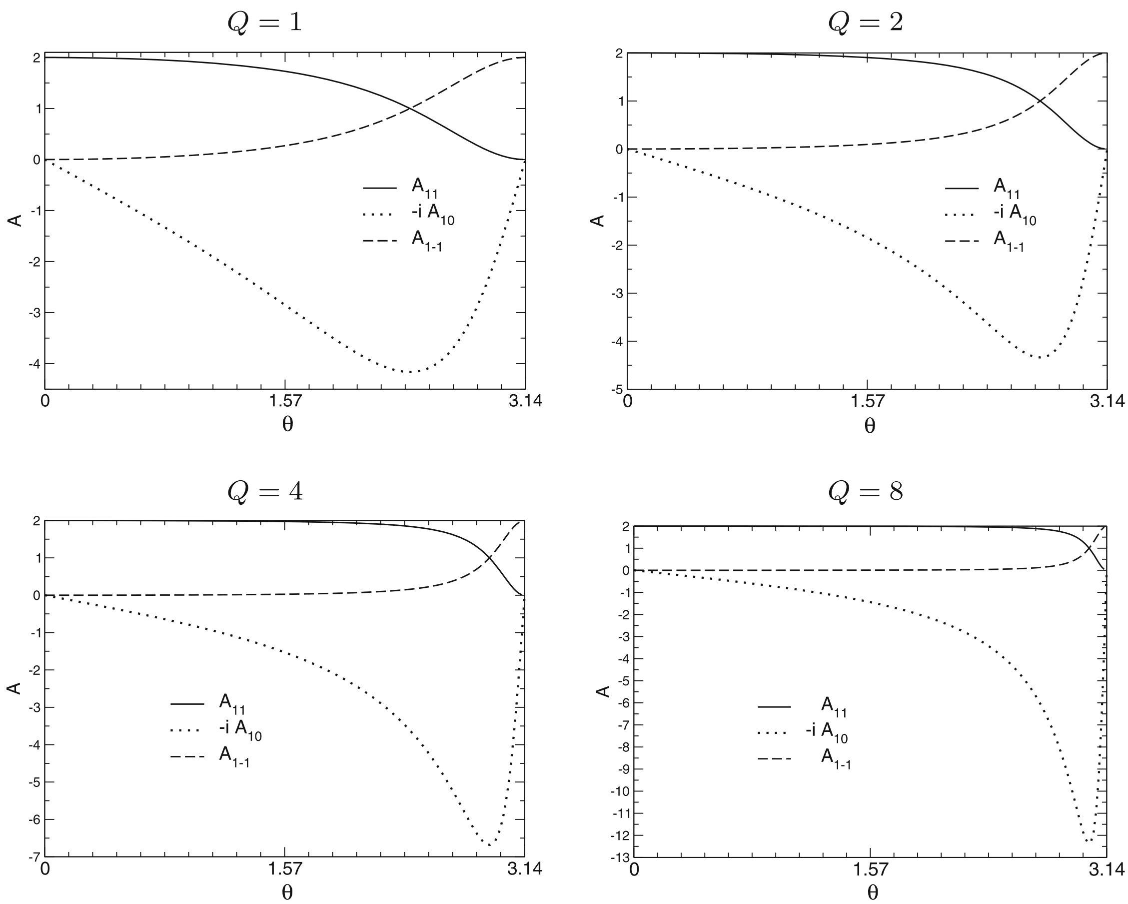

Fig. 2 The tree-level amplitudes for $Q = 1, 2, 4,$ and 8 GeV

The three independent amplitudes (the ones with photon helicity -1 in the final state are not independent) are in the tree-level case which are found to be (their numerical values are shown in Fig. 2)

$$
\begin{aligned}
A(q', 1; q, \pm 1) &= \frac{Q^2 + 2x_{\rm Bj}M^2 \pm Q\sqrt{Q^2 + 4x_{\rm Bj}^2 M^2}}{2[(1-x_{\rm Bj})Q^2 + x_{\rm Bj}M^2]} \frac{(1-x_{\rm Bj})Q^2(1 \pm \cos\theta)}{Q^2 + x_{\rm Bj}t}, \\
A(q', 1; q, 0) &= i\frac{Q^2 + 4x_{\rm Bj}M^2}{Q\sqrt{2x_{\rm Bj}[(1-x_{\rm Bj})Q^2 + x_{\rm Bj}M^2]}} \frac{x_{\rm Bj}Q^2 \sin\theta}{Q^2 + x_{\rm Bj}t}.
\end{aligned}
\tag{14}
$$

The quantity t in these results is the Mandelstam variable, thus these formulas show that the amplitudes do not change much if t is set to zero when Q^2 is much larger than t. Still, we shall demonstrate that the extraction of the CFFs from the amplitudes will be sensitive even to small differences between $t = 0$ and the true value of t.

At tree level, there is an additional symmetry, namely $A(1, 1) + A(1, -1) = -\mathcal{H}_0$. If one tries to minimize the value of t by making θ small, one sees that $A(1, 1)$ becomes proportional to θ^2 and $A(1, 0)$ to θ, which hinders an accurate extraction of the CFFs from the amplitudes. In particular one finds

$$
\begin{aligned}
\mathcal{H}_1 &\to -\frac{2x_{\rm Bj}^3}{1-x_{\rm Bj}} \frac{A(1,1)}{\theta^2} + {\rm i}\frac{\sqrt{2x_{\rm Bj}(1-x_{\rm Bj})}}{1-x_{\rm Bj}} \frac{A(1,0)}{\theta} - \frac{x_{\rm Bj}^2}{2}\mathcal{H}_0 \\
\mathcal{H}_2 &\to -\frac{2x_{\rm Bj}^2(2-x_{\rm Bj})}{1-x_{\rm Bj}} \frac{A(1,1)}{\theta^2} + {\rm i}x_{\rm Bj}\sqrt{2x_{\rm Bj}(1-x_{\rm Bj})} \frac{A(1,0)}{\theta} + \frac{x_{\rm Bj}^2}{2}\mathcal{H}_0.
\end{aligned}
\tag{15}
$$

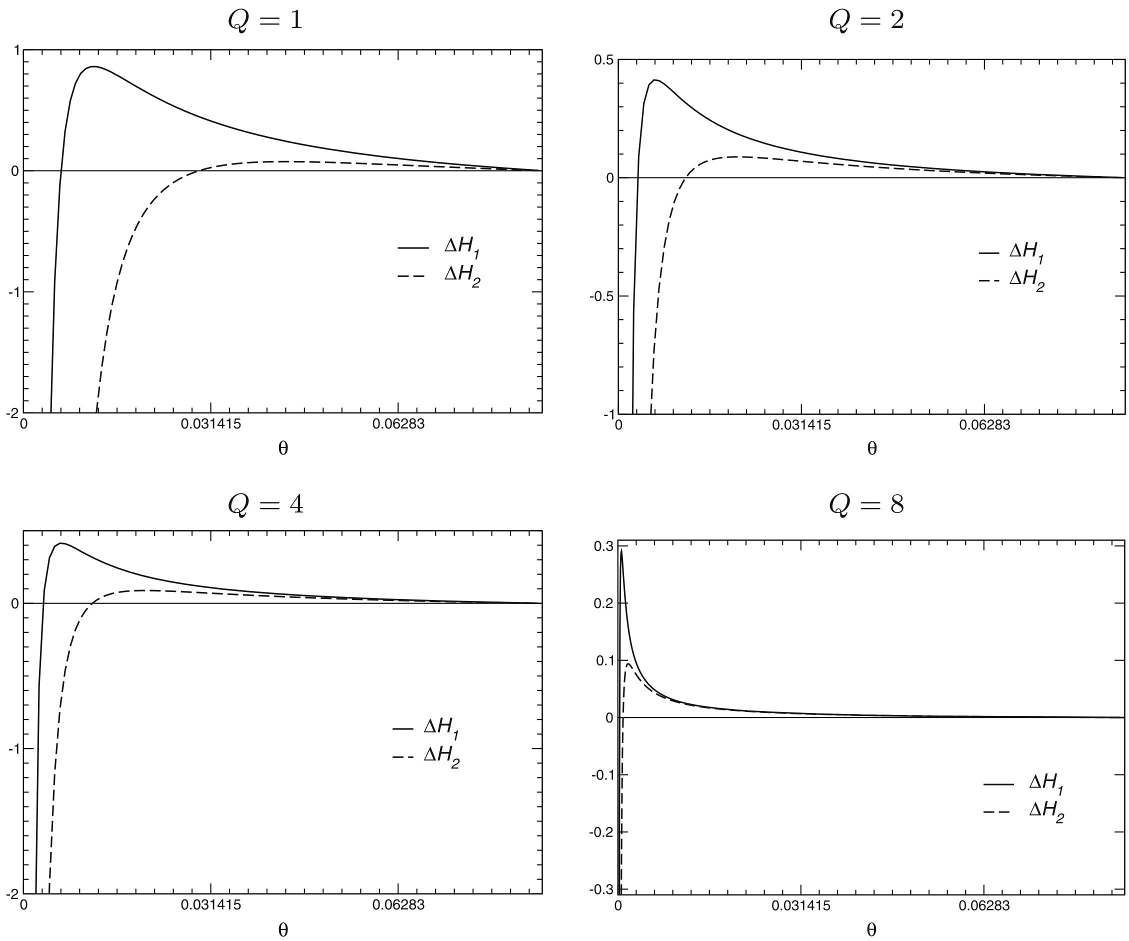

Fig. 3 The inversion of the tree-level amplitudes for $Q = 1, 2, 4,$ and 8 GeV

Because $A(1, 1)/\theta^2$ and $iA(1, 0)/\theta$ are finite for $\theta \to 0$, the limit $\theta \to 0$ exists, but it means that $\mathcal{H}_1$ and $\mathcal{H}_2$ must be determined from the angular dependence of the differential cross-section data and cannot be disentangled in the forward limit. Indeed, if $\theta = 0$, the general sQED Compton tensor reduces to a form with only two independent CFFs, namely $\mathcal{H}_0$ and $\mathcal{H}'_1 = 2x_{\mathrm{Bj}}^2\mathcal{H}_0 + (1 - x_{\mathrm{Bj}})\mathcal{H}_1 - x_{\mathrm{Bj}}(2 - x_{\mathrm{Bj}})\mathcal{H}_2$, which means that $\mathcal{H}_1$ and $\mathcal{H}_2$ cannot *separately* be determined in forward scattering.

5 Extraction

To find out how important is the fact that the experimental set up does not allow for a kinematics that has $t = 0$, we did the following exercise. Construct *simulated "data"* by calculating the amplitudes at tree level for a small but finite value of the scattering angle and analyze them for the CFFs. Because at tree level with a real photon in the final state only two CFFs occur, Eq. (5), we will find that $\mathcal{H}_2$ must vanish if the analysis is done correctly.

In sQED, for instance Compton scattering on a ^{4}He target, the calculation of the amplitudes is simple enough to allow for an exact algebraic inversion of the relation of the amplitudes to the CFFs even if one does not take the small scattering-angle or large Q limits. Should one study Compton scattering off a spin-1/2 target such an inversion will become very complicated and one may have to resort to a least-square minimization technique to find the CFFs from the amplitudes. Moreover, it will be difficult to do an experiment where all independent scattering amplitudes can be determined. Here we consider the ideal case.

In the spirit of assuming t to be small, we calculate the amplitudes for a scattering angle $\theta_S = \pi/36 = 5°$ and analyze them for the CFFs using the exact inversion formulas. We insert in these formulas values for θ

between 0 and θ_S. In Fig. 3 we plot the differences between the input values of the CFFs $\mathcal{H}_1$ given in Eq. (5) and the one extracted as a function of θ. Also given are the values of $\mathcal{H}_2$ obtained in the extraction. If a finite value is found this is *completely spurious*, because $\mathcal{H}_2$ vanishes at tree level.

Some remarks. The dominant CFF is $\mathcal{H}_0 = -2$. Thus the differences between the input and the values obtained using exact inversion should be compared to that magnitude. Secondly, one can see that if the analyzing angle θ is equal to θ_S, the differences vanish, which is a consistency check on our procedure. Finally, we note that for increasing values of Q, the differences between the input and output decrease for fixed θ_S, but the inversion formulas run into a singularity near $\theta = 0$ which prevents the use of $\theta = 0$ for an accurate determination of the CFFs. These singularities are clearly seen in Eq. (15) as poles at $\theta = 0$ if the amplitudes are calculated for a fixed θ_S.

6 Conclusion

Given the instability of the extrapolation of the analysis of Compton amplitudes in sQED to scattering angle zero, a straight forward determination of the three Compton amplitudes from deeply-virtual Compton scattering off scalar targets will not be accurate if $\theta = 0$ is taken in the analysis. Moreover, considering the fact that the data will show some experimental uncertainties, the only trustworthy way to extract the CFFs must take into account that $t \neq 0$ and $\theta_S \neq 0$.

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