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Sinc integrals revisited

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Abstract The purpose of this paper is the evaluation of the Fourier transform of powers of the sinc function multiplied by monomials, also in the cases when log terms arise. Such evaluations appear only rarely in the literature. Some old sources are hardly available. Because of notations not in use today, several original works are difficult to read. We apply an approach by J. H. Michell in a variant of G. H. Hardy to integrals over sinc powers and their Fourier transforms. Moreover, the connection of such integrals with B-splines is accentuated.

Keywords Integrals of Riemann type · Explicit integration · MSC Classification 26A42

1 Introduction

There are numerous methods to evaluate the so-called Dirichlet integral

$$\int_0^{+\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}. \tag{1}$$

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In calculus lectures, one normally uses differentiation of the Laplace transform $F(s) = \int_0^{+\infty} e^{-st} \operatorname{sinc}(t)\, dt$ under the integral sign. It can be justified that $F'(s) = -\int_0^{+\infty} e^{-st} \sin(t)\, dt$. It is a nice exercise to evaluate $F'(s) = -1/\left(1+s^2\right)$ by applying standard procedures of integration. Taking its anti-derivative $F(s) = \pi/2$–arctan(s), satisfying $\lim_{s\to+\infty} F(s) = 0$, yields the value $F(0) = \pi/2$. Another common evaluation is via complex integration and residue theorem.

In 1909, G. H. Hardy [16] listed 7 proofs of the identity (1) and examined them with regard to their simplicity. He remarked that "Practically all methods of evaluating any definite integral depend ultimately upon the inversion of two or more operations of procedure to a limit." In his opinion, such inversions "constitute what we may call the difficulty of the problem". Hardy based his system of marking primarily upon them. Besides these marks he added "marks of artificiality, complexity, etc. The proof obtaining least marks is to be regarded as the simplest and best". Hardy's list comprises proofs given by Berry [3] and Nanson [21]. He added a further proof by Bromwich [7, §173, last line of Ex. 1] which "he had forgotten". A couple of years later, in 1916, Hardy [17] extended his list by three more proofs of (1), given by Dixon, Bromwich and Whipple. Remarkable is a proof by J. H. Michell which was reported by Nanson [21] accompanying his own proof. Hardy claimed to have "used a proof, in teaching, which is in principle substantially the same [...] though slightly more simple in details and arrangement, viz.:

$$\begin{aligned}\int_0^{\infty} \frac{\sin x}{x} dx &= \frac{1}{2i}\int_0^{\infty} dx \int_0^{\pi} e^{i\left(t+xe^{it}\right)} dt = \frac{1}{2i}\int_0^{\pi} e^{it} dt \int_0^{\infty} e^{ixe^{it}} dx \\ &= \frac{1}{2}\int_0^{\pi} dt = \frac{\pi}{2}.\end{aligned}$$

The successive steps of the proof can of course be stated in a form free from i by merely taking the real part of the integrand." Here it was used that $\frac{\sin x}{x} = -\frac{1}{2}\int_1^{-1} e^{ixz} dz = -\frac{i}{2}\int_0^{\pi} e^{ixe^{it}} e^{it} dt$, where the path in the complex plane, parametrized by $z = e^{it}$, joins 1 to -1, when t increases through real values from 0 to π. Hardy gave a justification of the inversion by the estimate

$$\left|\int_0^{\pi} e^{it} dt \int_X^{\infty} e^{ixe^{it}} dx\right| \le \pi \frac{1-e^{-X}}{X}.$$

Recently, Hardy's version of Michell's proof was rediscovered in [6].

In a first step, we apply this approach to integrals over the powers of the sinc function $\operatorname{sinc}(t) := (\sin t)/t$,

$$I_n := \int_0^{+\infty} \left(\frac{\sin t}{t}\right)^n dt \qquad (n \in \mathbb{N}). \tag{2}$$

There are several derivations of its values

$$I_n = \frac{\pi}{2^n\,(n-1)\,!}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}(n-2k)^{n-1}. \tag{3}$$

The problem appears as an exercise in [26, Ex. 13, p. 123]. See also [2, 8, 12]. The numerators and denominators of the rational multiples of π in (3) are listed as A049330 and A049331, respectively, in the On-Line Encyclopedia of Integer Sequences [24]. Borwein and Borwein [4] studied the more general integrals

$$\int_0^{+\infty}\prod_{k=1}^{n}\frac{\sin a_k t}{t}dt \qquad (n\in\mathbb{N}),$$

for real numbers $a_1, ..., a_n$ satisfying certain conditions and derived (3) as a special case. Further proofs can be found in [13] and [1] (see also [5]; further publications related to the matter are [15, 18, 20, 25]).

We show that the method of Michell and Hardy can be applied to evaluate the Fourier transform,

$$I_{n,m}(\xi) := \int_0^{+\infty}\frac{\sin^n t}{t^{n-m}}e^{i\xi t}dt \qquad (\xi\in\mathbb{R}) \tag{4}$$

for integers n, m with $0 \le m < n$. We highlight the relation to B-splines. Note that, for even m, $\int_0^{+\infty}\frac{\sin^n t}{t^{n-m}}\cos(\xi t)\,dt$ and, for odd m, $\int_0^{+\infty}\frac{\sin^n t}{t^{n-m}}\sin(\xi t)\,dt$ can easily be obtained by inverse Fourier transform (see Eqs. (17) and (18)). The remaining cases are not easy to find in the literature. Therefore, they are explicitly evaluated in Sect. 4.

Integrals of this type were already published, in 1855, by Lobatschewskji [19] who studied $\int_0^{+\infty}\left(\frac{\sin t}{t}\right)^n e^{-at}dt$, for $a > 0$. Because of the notation not in use today, the original work is difficult to read. In 1860, Schlömilch [23] evaluated the integrals $\int_0^{+\infty}\frac{\sin^p t}{t^q}dt$, even for real values of $q > 0$. To this end he took advantage of the formulas

$$\int_0^{+\infty}\frac{\sin\beta t}{t^q}dt = \frac{\pi\beta^{q-1}}{2\Gamma(q)\sin(q\pi/2)} \qquad (0<q<2),$$
$$\int_0^{+\infty}\frac{\cos\beta t}{t^q}dt = \frac{\pi\beta^{q-1}}{2\Gamma(q)\cos(q\pi/2)} \qquad (0<q<1)$$

[23, Eqs. 16 and 17]. Schlömilch [23, Eqs. 18, 20, 21] presented explicit formulas for $\int_0^{+\infty}\frac{\sin^p t}{t^q}dt$, apart of the case that p is an even integer and $0 < q \le 1$, in which the integrals $\int_0^{+\infty}\frac{\sin^p t}{t^q}dt$ are divergent [23, Eq. 19].

We note the Laplace transform of the powers of the absolute value of the sinc function, i.e., $|\sin(t)/t|^p$, for exponents p with $\Re(p) \geq 0$, was recently studied by Glasser [14]. In the special case that p is an even number, he derived a closed expression as a finite sum involving log and arctan functions [14, Eq. (1.13)].

In the special case $\xi = 0$ of Eq. (4) we obtain formulas for the integrals

$$I_{n,m} := I_{n,m}(0) = \int_0^{+\infty} \frac{\sin^n t}{t^{n-m}} dt \qquad (0 \leq m < n), \tag{5}$$

which were derived in 1877 by J. Wolstenholme in the hardly accessible publication [27]. He calculated the integrals (5) by reduction to the integrals (1) and the Frullani-type integrals

$$\int_0^{+\infty} \frac{\cos(at) - \cos(bt)}{t} dt = \log \frac{b}{a} \qquad (a, b > 0)$$

(see [11, 1014]). A splendid source of such calculations is the fundamental treatise on integrals by Edwards [11]. Recursive formulas for the indefinite integrals of type (5) can be found in the first volume [11, 265]. Many interesting calculations are contained in the second volume [11, 1023ff]. Some of these methods are used in Sect. 5.

Complete formulas for the Fourier transform $I_{n,m}(\xi)$ in the special case $m = 0$ appear in Oberhettinger's table book [22, 5.12 on p. 20, 5.13 on p. 21, and 5.15, 5.16 on p. 134].

Historical remarks on the Fourier transform (4) and its connection to B-splines can be found in [9, see Eqs. (2.1) and (2.4)].

The integrals (2), (5), (4) connected with the sinc function, i.e., $I_n = \int_0^{+\infty} (\text{sinc} t)^n dt$, $I_{n,m}(0) = \int_0^{+\infty} t^m (\text{sinc} t)^n dt$ and the Fourier transform $I_{n,m}(\xi) = \int_0^{+\infty} t^m (\text{sinc} t)^n e^{i\xi t} dt$ have well-known values, for suitable integers n, m. As already mentioned, the purpose of the present article is to extend and apply Michell's and Hardy's approach to the above mentioned integrals. All evaluations are explicitly given in terms of B-splines and their derivatives. These formulas are well-known. It appears that in several cases log-terms arise. However, proofs of the corresponding formulas are rarely to find in standard textbooks. Their proofs appear mostly in old sources which are difficult to read and not easy to access.

2 Theoretical background and notation

The unnormalized sinc function, also called sinus cardinalis, is defined by

$$\text{sinc}(t) := \begin{cases} \dfrac{\sin t}{t} & \text{for } t \neq 0, \\ 1 & \text{for } t = 0. \end{cases}$$

With the value $\text{sinc}(0) := \lim_{t \to 0} \text{sinc}(t) = 1$, sinc is continuous on the whole real axis. In order to avoid confusion, we note that there are different notations. In digital signal processing and information theory, the normalized sinc function is commonly defined by $\operatorname{sinc}(t) := (\sin \pi t) / (\pi t)$ which will not be used in the following. The sinc function plays an important role in pure mathematics as well as in physics and engineering. When separating variables of the Helmholtz equation in spherical coordinates, the spherical Bessel functions j_n are solutions of the radial equation, where $j_n(x) = (-x)^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \operatorname{sinc}(x)$. In particular, the zeroth spherical Bessel function $j_0(x) = \operatorname{sinc}(x)$ is the unnormalized sinc function. A further application in physics is diffraction from a slit. The Fraunhofer diffraction from a slit is the Fourier transform of a rectangular function (see (8)), which is a sinc function (see (9)). The irradiance (the radiant flux received by a surface per unit area) is then given in terms of sinc^2.

The term "B-spline" was coined by Isaac Jacob Schoenberg and is short for basis spline. For a given set of distinct knots $t_0 < t_1 < \cdots < t_n$, there are, up to a scaling factor, unique splines $B_{i,n}(x)$ of order n with compact support $[t_i, t_{i+n}]$. Choosing the scaling factors such that $\sum_i B_{i,n}(x) = 1$ for $t_0 < x < t_n$, the resulting $B_{i,n}(x)$ are called B-splines. They can easily be determined by the Cox–de Boor recursion formula [10, p. 90, Eqs. (14) and (15)]. A direct representation is given by

$$B_{i,n}(x) = (t_n - t_0)\,[t_0, ..., t_n]\,(\cdot - x)_+^{n-1}, \tag{6}$$

where the n-th divided difference $[t_0, ..., t_n]$ applies to the variable at the placeholder. Here x_+^r denotes the truncated power function, defined, for $r \in \mathbb{N}$, by $x_+^r = x^r\,(x \geq 0)$ and $x_+^r = 0\,(x < 0)$. In the case $r = 0$, one defines $x_+^0 = 1\,(x > 0)$, $0_+^0 = 1/2$ and $x_+^0 = 0\,(x < 0)$. Note that the function $f(x) = (x - x_0)_+^r$ is a piecewise polynomial with one break, at x_0, and is continuous at x_0 in the case $r > 0$, while, for $r = 0$, it has a jump across x_0, of size 1. Since $f'(x) = r(x - x_0)_+^{r-1}$, we see that f has $r - 1$ continuous derivatives, with a jump in the rth derivative across x_0, of size $r!$.

Recalling the formula $[t_0, ..., t_n]\,f = \sum_{k=0}^n f(t_k) \prod_{j=0, j\neq k}^n \left(t_k - t_j\right)^{-1}$, formula (6) provides the explicit form

$$B_{i,n}(x) = (t_n - t_0) \sum_{k=0}^{n} (t_k - x)_+^{n-1} \prod_{\substack{j=0,\\ j\neq k}}^{n} \left(t_k - t_j\right)^{-1}. \tag{7}$$

Cardinal B-splines have knots that are equidistant from each other. If $t_{i+1} - t_i = h > 0$, for all i, the $B_{i,n}(x)$ are just shifted copies of each other.

The close connection between sinc integrals and B-splines has its origin in the fact that the Fourier transform of the sinc function is, up to scaling factors, the rectangular function,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\operatorname{sinc}\left(\frac{t}{2}\right)e^{itx}dt=\operatorname{rect}(x):=\begin{cases}1, & |x|<1/2,\\ 1/2, & |x|=1/2,\\ 0, & |x|>1/2.\end{cases}\tag{8}$$

The inverse transform

$$\int_{-\infty}^{+\infty}\operatorname{rect}(t)\,e^{-itx}dx=\operatorname{sinc}\left(\frac{t}{2}\right)\tag{9}$$

can easily be verified. The rectangular function is just the central B-spline M_1 of first order. Note that in the theory of splines M_1 can be defined such that $M_1(x)\neq\operatorname{rect}(x)$, for $|x|=1/2$. The central B-splines are the cardinal B-splines, which arise if we choose the knots $t_i=-n/2+i$ $(i=0,...,n)$. They are the Fourier transforms of the powers of the sinc function,

$$M_n(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\operatorname{sinc}\frac{t}{2}\right)^n e^{itx}dt.\tag{10}$$

In the special case $n=1$, this formula reduces to (8), i.e., $M_1(x)=\operatorname{rect}(x)$. Direct computation verifies that

$$M_n(x)=\int_{x-1/2}^{x+1/2}M_{n-1}(t)\,dt,$$

where M_n has compact support $[-n/2,n/2]$ with $M_n(x)>0$, for $-n/2<x<n/2$. Noting that $\int_{-\infty}^{+\infty}M_1(t)\,dt=1$, mathematical induction shows that $\int_{-\infty}^{+\infty}M_n(t)\,dt=1$, for all $n\in\mathbb{N}$. Application of (7) shows that the integrals (10) possess the explicit representation

$$M_n(x)=\frac{1}{(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(x+\frac{n}{2}-k\right)_+^{n-1}.$$

Thus, the rectangular function can be written in the form $\operatorname{rect}(x)=M_1(x)=\left(x+\frac{1}{2}\right)_+^0-\left(x-\frac{1}{2}\right)_+^0$.

The integer translates of the function $M_n(x)$ form a basis in the sense that every spline function $S_n(x)$ of order n, namely a function which has a continuous derivative of order $n-2$ on the real axis, and reduces on each interval

$(k - n/2, k + 1 - n/2)$, $k \in \mathbb{Z}$, to a polynomial of degree $n - 1$, can be uniquely represented in the form

$$S_n(x) = \sum_{k=-\infty}^{\infty} c_k M_n(x - k)$$

with appropriate constant coefficients c_k. Conversely, any such series represents a spline of order n (see [9, p. 139]).

For our purposes it appears to be more convenient to consider the cardinal B-splines with knots $t_i = i$ $(i = 0, ..., n)$, i.e.,

$$M_n\left(x - \frac{n}{2}\right) =: B_{n-1}(x) .$$

Then,

$$B_{n-1}(x) = n\,[0, 1, ..., n]\,(\cdot - x)_+^{n-1} = \frac{1}{(n-1)\,!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)_+^{n-1}.$$

Eq. (8) can be rewritten in the form

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\, e^{ixt} dt = \pi \cdot B_0\left(\frac{x+1}{2}\right),$$

where

$$B_0(x) = x_+^0 - (x-1)_+^0 = \begin{cases} 1, & 0 < x < 1, \\ 1/2, & x \in \{0,1\}, \\ 0, & x \notin [0,1] . \end{cases}$$

More generally, Eq. (10) can be rewritten in the form

$$\int_{-\infty}^{\infty} (\operatorname{sinc}(t))^n e^{ixt} dt = \pi \cdot M_n\left(\frac{x}{2}\right) = \pi \cdot B_{n-1}\left(\frac{x+n}{2}\right).$$

We emphasize the fact that the subsequent sections do not take advantage of the results presented in this section. In order to keep the exposition self-contained, we shall use in the following only elementary theorems like Taylor formula and the Riemann–Lebesgue lemma.

3 The evaluation of I_n

The aim of this section is a short proof of (3) using only elementary methods by reducing the integrals (2) to (1). The tools used are the forward differences with step 1,

$$\Delta^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k) \tag{11}$$

and the Taylor formula with remainder in integral form,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \int_0^x \frac{f^{(n)}(z)}{(n-1)!} (x-z)^{n-1} dz.$$

Since $\Delta^n p(x) = 0$, for each polynomial p of degree at most $n-1$, the Taylor formula and Eq. (11) yield

$$\Delta^n f(0) = \int_0^n f^{(n)}(z) B_{n-1}(z) dz, \tag{12}$$

where

$$B_{n-1}(z) := \frac{1}{(n-1)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (k-z)_+^{n-1}. \tag{13}$$

Now we are in position to prove (3).

Proof of Formula (3) For $t > 0$, we have, by (12),

$$\begin{aligned} \left(\frac{\sin t}{t}\right)^n &= \left(\frac{e^{-it}}{2it}\right)^n \left(e^{2it}-1\right)^n = \left(\frac{e^{-it}}{2it}\right)^n \Delta^n g_t(0) \\ &= \frac{1}{2} \int_{-n}^{n} e^{itz} B_{n-1}\left(\frac{z+n}{2}\right) dz, \end{aligned}$$

where $g_t(z) := e^{2itz}$. Integrating with respect to t, we obtain, for $x > 0$,

$$\begin{aligned} \int_0^x \left(\frac{\sin t}{t}\right)^n dt &= \frac{1}{2} \int_{-n}^{n} \frac{e^{ixz}-1}{iz} B_{n-1}\left(\frac{z+n}{2}\right) dz \\ &= \frac{1}{2} \int_{-n}^{n} \frac{\sin(xz)}{z} B_{n-1}\left(\frac{z+n}{2}\right) dz, \end{aligned}$$

because the integral has a real value. Observe that

$$B_{n-1}\left(\frac{z+n}{2}\right) - B_{n-1}\left(\frac{n}{2}\right) = zh\left(z\right),$$

where h is a bounded function and $h\left(0\right) = B'_{n-1}\left(\frac{n}{2}\right)/2$. Hence,

$$\int_0^x \left(\frac{\sin t}{t}\right)^n dt = \frac{1}{2}\int_{-n}^n \sin\left(xz\right) h\left(z\right) dz + \frac{1}{2} B_{n-1}\left(\frac{n}{2}\right) \int_{-n}^n \frac{\sin\left(xz\right)}{z} dz.$$

Passing to the limit $x \to +\infty$, the first integral tends to zero, by the Riemann–Lebesgue lemma, and a change of variable leads to

$$I_n = \frac{1}{2} B_{n-1}\left(\frac{n}{2}\right) \lim_{x\to\infty} \int_{-nx}^{nx} \frac{\sin z}{z} dz = I_1 B_{n-1}\left(\frac{n}{2}\right),$$

which in view of (13) proves (3).

Remark 1 Note that the above proof derives, as a by-product, the well-known representation of the powers of $\operatorname{sinc}\left(t\right) := \left(\sin t\right)/t$, as a finite Fourier transform

$$\left(\frac{\sin t}{t}\right)^n = \frac{1}{2}\int_{-n}^n e^{itz} B_{n-1}\left(\frac{z+n}{2}\right) dz. \tag{14}$$

Remark 2 An alternative approach is as follows. Note that $\Delta^n g_t\left(0\right) = n!\left[0, 1, ..., n\right]$
$g_t\left(0\right)$. For real numbers $z_0, ..., z_n$ satisfying $z_0 < \cdots < z_n$, the divided differences $\left[z_0, ..., z_n\right] f$ possess the Peano form

$$\left[z_0, ..., z_n\right] f = \frac{1}{n!}\int_{z_0}^{z_n} f^{(n)}\left(z\right) B_{n-1}\left(z\right) dz,$$

where the Peano kernel B_{n-1} is a (cardinal) B-spline of degree $n-1$ for the data points $z_0, ..., z_n$, normalized such that

$$\int_0^n B_{n-1}\left(z\right) dz = 1.$$

In particular, for $z_k = k$ $\left(k = 0, ..., n\right)$,

$$B_{n-1}\left(z\right) = \left(-1\right)^n n \cdot \left[0, 1, ..., n\right]\left(z - \cdot\right)_+^{n-1} = n \sum_{k=0}^n \frac{\left(-1\right)^k}{k!\left(n-k\right)!}\left(z-k\right)_+^{n-1}, \tag{15}$$

which is equivalent to Eq. (13). This can be seen by an easy manipulation noting that $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P(k)=0$, for each polynomial P of degree less than n (see Lemma 2).

Using integration by parts repeatedly and noting that the B-spline of degree $n-1$ satisfies $B_{n-1}^{(k)}(0)=B_{n-1}^{(k)}(n)=0$, for $k=0,...,n-2$, yield, for $m\le n-1$,

$$\int_{-n}^{n} e^{itz}B_{n-1}\left(\frac{z+n}{2}\right)dz=\left(\frac{-1}{2it}\right)^m\int_{-n}^{n} e^{itz}B_{n-1}^{(m)}\left(\frac{z+n}{2}\right)dz.$$

Hence, Eq. (14) implies the finite Fourier transform

$$\frac{\sin^n t}{t^{n-m}}=\frac{1}{2}\left(\frac{i}{2}\right)^m\int_{-n}^{n} B_{n-1}^{(m)}\left(\frac{z+n}{2}\right)e^{itz}dz.$$

For $n-m\ge 2$, inverse Fourier transform yields

$$\frac{i^m}{2^{m+1}}B_{n-1}^{(m)}\left(\frac{\xi+n}{2}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^n t}{t^{n-m}}e^{-i\xi t}dt. \tag{16}$$

If m is even, we infer that

$$\int_0^{\infty}\frac{\sin^n t}{t^{n-m}}\cos(\xi t)\,dt=(-1)^{m/2}\frac{\pi}{2^{m+1}}B_{n-1}^{(m)}\left(\frac{\xi+n}{2}\right). \tag{17}$$

In particular, for $\xi=0$,

$$I_{n,m}=\int_0^{\infty}\frac{\sin^n t}{t^{n-m}}dt=(-1)^{m/2}\frac{\pi}{2^{m+1}}B_{n-1}^{(m)}\left(\frac{n}{2}\right).$$

Differentiating Eq. (13) m times, we obtain the more explicit expression

$$I_{n,m}=(-1)^{m/2}\frac{\pi}{2^n(n-m-1)!}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\binom{n}{k}(n-2k)^{n-m-1}.$$

In the special case $m=0$, this formula reduces to Eq. (3).

If m is odd, we conclude that

$$\int_0^{\infty}\frac{\sin^n t}{t^{n-m}}\sin(\xi t)\,dt=(-1)^{(m+1)/2}\frac{\pi}{2^{m+1}}B_{n-1}^{(m)}\left(\frac{\xi+n}{2}\right). \tag{18}$$

The corresponding integrals (17), for odd m, and (18), for even m, will be evaluated in the next section.

We close this section with some consequences. These formulas deliver several interesting relations between the sinc integrals and the B-splines. For instance, we put $\xi=2$ in Eq. (17). Using $\cos(2t)=1-2\sin^2 t$, we obtain, for even m and $n\ge m+2$,

$$I_{n,m} - 2I_{n+2,m+2} = (-1)^{m/2} \frac{\pi}{2^{m+1}} B_{n-1}^{(m)}\left(\frac{n}{2}+1\right),$$

which can be considered as a representation of a derivative value of the B-spline in terms of integrals. It follows that

$$I_{n+2k,m+2k} - 2I_{n+2k+2,m+2k+2} = (-1)^{m/2+k} \frac{\pi}{2^{m+2k+1}} B_{n+2k-1}^{(m+2k)}\left(\frac{n}{2}+k+1\right).$$

Multiplying this equation with 2^k and summing up, leads to

$$\sum_{k=0}^{K}\left(2^k I_{n+2k,m+2k} - 2^{k+1} I_{n+2k+2,m+2k+2}\right) = \sum_{k=0}^{K}(-1)^{m/2+k} \frac{\pi}{2^{m+k+1}} B_{n+2k-1}^{(m+2k)}\left(\frac{n}{2}+k+1\right).$$

Finally, we infer that

$$I_{n,m} - 2^{K+1} I_{n+2(K+1),m+2(K+1)} = (-1)^{m/2} \frac{\pi}{2^{m+1}} \sum_{k=0}^{K} \frac{(-1)^k}{2^k} B_{n+2k-1}^{(m+2k)}\left(\frac{n}{2}+k+1\right).$$

4 More general reduction formulas

In this section we demonstrate that the approach of the Sect. 3 can be applied also to the integrals (5). More generally, we consider, for integers n, m with $0 \le m < n$, the Fourier transform (4), i.e.,

$$I_{n,m}(\xi) = \int_0^{+\infty} \frac{\sin^n t}{t^{n-m}} e^{i\xi t}\, dt \qquad (\xi \in \mathbb{R}).$$

For information about earlier appearances of the emerging formulas we refer to the introductory Sect. 1.

First we state the result in the special case $m = 0$.

Theorem 1 For integers $n \ge 2$ and $\xi \in \mathbb{R}$, the value of $I_{n,0}(\xi)$ is given by

$$\int_0^{+\infty} \frac{\sin^n t}{t^n} e^{i\xi t}\, dt = \frac{\pi}{2} B_{n-1}\left(\frac{\xi+n}{2}\right) + \frac{i}{2^n (n-1)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\xi - n + 2k)^{n-1} \log|\xi - n + 2k|.$$

In the latter sum and in the following, the expression $w^p \log|w|$ is to be read as zero if $w = 0$ and $p > 0$.

Remark 3 Since $B_{n-1}(z) = 0$, for all real $z \notin (0, n)$, we infer that the Fourier transform of the function sinc^n has purely imaginary values outside of the interval $(-n, n)$.

Remark 4 The symmetry $B_{n-1}\left(\frac{n+\xi}{2}\right) = B_{n-1}\left(\frac{n-\xi}{2}\right)$ and the identity

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\xi - n + 2k)^{n-1} \log|\xi - n + 2k| =$$
$$-\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (-\xi - n + 2k)^{n-1} \log|-\xi - n + 2k|,$$

which can be verified by reversing the order of summation, reflect the fact that $I_{n,m}(-\xi)$ is the complex conjugate of $I_{n,m}(\xi)$.

Finally, we give the evaluation of $I_{n,m}(\xi)$, for arbitrary $m \geq 0$.

Theorem 2 For nonnegative integers n, m with $n - m \geq 2$ and $\xi \in \mathbb{R}$, it holds

$$I_{n,m}(\xi) = \frac{(-i)^m \pi}{2^{m+1}} B_{n-1}^{(m)}\left(\frac{\xi + n}{2}\right)$$
$$+ \frac{(-1)^m i^{m+1}}{2^n (n-m-1)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\xi - n + 2k)^{n-m-1} \log|\xi - n + 2k|.$$

We list resulting evaluations for some real integrals. In the case of even m, we recover the cosine transform (17) and

$$\int_0^{+\infty} \frac{\sin^n t}{t^{n-m}} \sin(\xi t) dt = \frac{(-1)^{m/2}}{2^n (n-m-1)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\xi - n + 2k)^{n-m-1} \log|\xi - n + 2k|.$$

In the case of odd m, we recover the sine transform (18) and

$$\int_0^{+\infty} \frac{\sin^n t}{t^{n-m}} \cos(\xi t) dt = \frac{(-1)^{(m-1)/2}}{2^n (n-m-1)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\xi - n + 2k)^{n-m-1} \log|\xi - n + 2k|.$$

In particular, for $\xi = 0$,

$$\int_0^{+\infty} \frac{\sin^n t}{t^{n-m}} dt = \frac{(-1)^{(m-1)/2}}{2^n (n-m-1)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (2k - n)^{n-m-1} \log|2k - n|.$$

For the proof of the theorems we need three auxiliary results.

Lemma 1 *For nonnegative integers* n, m, *there are certain real numbers* c_m, *such that, for* $x > 0$,

$$\left(\frac{d}{dx}\right)^m x^n \log x = x^{n-m}\left(m!\binom{n}{m}\log x + c_m\right).$$

The proof by mathematical induction with respect to m is left to the reader.

Lemma 2 *Let* n *be a positive integer. For each polynomial* P *of degree less than* n, *it holds*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k) = 0.$$

Proof A polynomial P of degree less than n in the variable k can be written as a linear combination of binomial coefficients $\binom{k}{r}$ with $0 \le r < n$. Then the assertion follows from $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k}{r} = \binom{n}{r}\sum_{k=r}^{n}(-1)^k\binom{n-r}{k-r} = 0$ because $n > r$.

Lemma 3 *For* $0 < a < b$, *it holds*

$$\lim_{x\to+\infty}\int_a^b \frac{e^{ixz}-1}{z}dz = -\lim_{x\to+\infty}\int_{-b}^{-a}\frac{e^{ixz}-1}{z}dz = -\log\frac{b}{a}$$

and

$$\lim_{x\to+\infty}\int_{-a}^{b}\frac{e^{ixz}-1}{iz}dz = \pi + i\log\frac{b}{a}.$$

Proof Let $x > 0$. We start with the latter formula:

$$\int_{-a}^{b}\frac{e^{ixz}-1}{iz}dz = \int_{-ax}^{bx}\frac{e^{iz}-1}{iz}dz = \int_{-ax}^{bx}\frac{\sin z}{z}dz + i\int_{-ax}^{bx}\frac{1-\cos z}{z}dz.$$

For sufficiently small $\delta > 0$, we have

$$\begin{aligned}\int_{-ax}^{bx}\frac{1-\cos z}{z}dz &= \int_{-ax}^{-\delta}\frac{1-\cos z}{z}dz + \int_{-\delta}^{\delta}\frac{1-\cos z}{z}dz + \int_{\delta}^{bx}\frac{1-\cos z}{z}dz \\ &= -\int_{\delta}^{ax}\frac{1-\cos z}{z}dz + 0 + \int_{\delta}^{bx}\frac{1-\cos z}{z}dz \\ &= -\log\frac{ax}{\delta} + \log\frac{bx}{\delta} - \int_{ax}^{bx}\frac{\cos z}{z}dz \to \log\frac{b}{a} \qquad (x\to+\infty).\end{aligned}$$

The other formulas follow in a similar manner.

Now we are in position to prove the theorems.

Proof of Theorem 1 We have to prove that

$$\begin{aligned} I_{n,0}(\xi) &= \frac{\pi}{2} B_{n-1}\left(\frac{\xi+n}{2}\right) \\ &- \frac{i}{2^n (n-1)!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\xi - n + 2k)^{n-1} \log|\xi - n + 2k| . \end{aligned} \tag{19}$$

A direct consequence of Eq. (14) is

$$\left(\frac{\sin t}{t}\right)^n e^{i\xi t} = \frac{1}{2}\int_{-n+\xi}^{n+\xi} e^{itz} B_{n-1}\left(\frac{z-\xi+n}{2}\right) dz.$$

Integrating with respect to t, we obtain, for $x > 0$,

$$\int_0^x \frac{\sin^n t}{t^n} e^{i\xi t} dt = \frac{1}{2}\int_{-n+\xi}^{n+\xi} \frac{e^{ixz}-1}{iz} B_{n-1}\left(\frac{z-\xi+n}{2}\right) dz.$$

Using the representation (15) we obtain

$$\begin{aligned} &\int_0^x \frac{\sin^n t}{t^n} e^{i\xi t} dt = \\ &\frac{1}{2^n (n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_{2k-n+\xi}^{n+\xi} \frac{e^{ixz}-1}{iz} (z - \xi + n - 2k)^{n-1} dz. \end{aligned}$$

Note that, for $k = n$, the integration interval has length zero. By the binomial formula, we have

$$\begin{aligned} &\int_{2k-n+\xi}^{n+\xi} \frac{e^{ixz}-1}{iz} (z - \xi + n - 2k)^{n-1} dz = (-\xi + n - 2k)^{n-1} \int_{2k-n+\xi}^{n+\xi} \frac{e^{ixz}-1}{iz} dz \\ &+ \sum_{j=1}^{n-1} \binom{n-1}{j} (-\xi + n - 2k)^{n-1-j} \int_{2k-n+\xi}^{n+\xi} \frac{e^{ixz}-1}{i} z^{j-1} dz. \end{aligned}$$

By the Riemann–Lebesgue lemma, we obtain

$$\lim_{x\to+\infty} \int_{2k-n+\xi}^{n+\xi} \frac{e^{ixz}-1}{i} z^{j-1} dz = i \int_{2k-n+\xi}^{n+\xi} z^{j-1} dz = P_{n,\xi,j}(k),$$

where $P_{n,\xi,j}$ is a polynomial of degree at most j in the variable k. Therefore, by Lemma 2, $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(-\xi+n-2k)^{n-1-j}P_{n,\xi,j}(k)=0$, for $j=1,...,n-1$. Hence,

$$I_{n,0}(\xi)=\frac{1}{2^n(n-1)!}\lim_{x\to+\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(-\xi+n-2k)^{n-1}\int_{2k-n+\xi}^{n+\xi}\frac{e^{ixz}-1}{iz}dz. \quad (20)$$

Now we study the limits $\lim_{x\to+\infty}\int_{2k-n+\xi}^{n+\xi}\frac{e^{ixz}-1}{iz}dz$, for $0\le k<n$. The desired formula for $I_{n,0}(\xi)$ can be deduced from Lemma 3. For each real $\xi\notin(-n,n)$, we have $2k-n+\xi<n+\xi<0$ or $0<2k-n+\xi<n+\xi$, such that

$$\lim_{x\to+\infty}\int_{2k-n+\xi}^{n+\xi}\frac{e^{ixz}-1}{iz}dz=-\frac{1}{i}\log\left|\frac{n+\xi}{2k-n+\xi}\right|$$

implies, by Eq. (20),

$$I_{n,0}(\xi)=\frac{i}{2^n(n-1)!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\xi-n+2k)^{n-1}\log|\xi-n+2k|.$$

Here we used that $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\xi-n+2k)^{n-1}\log|n+\xi|=0$, by Lemma 2. Next let us consider the case $-n\le\xi\le n$. We can exclude the case when the lower integration limit $2k-n+\xi$ takes the value zero. If $2k=n-\xi$, the integral $\int_{2k-n+\xi}^{n+\xi}\frac{e^{ixz}-1}{iz}dz$ has factor zero and does not occur in Eq. (20). In the case $-n<\xi\le n$ the limit $\lim_{x\to+\infty}\int_{2k-n+\xi}^{n+\xi}\frac{e^{ixz}-1}{iz}dz$ depends on the conditions $2k>n-\xi$ or $2k<n-\xi$. By Eq. (20) and Lemma 3, we infer that

$$\begin{aligned}I_{n,0}(\xi)=&\frac{-i}{2^n(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(-\xi+n-2k)^{n-1}\log|\xi-n+2k|\\&+\frac{\pi}{2^n(n-1)!}\sum_{2k<n-\xi}(-1)^k\binom{n}{k}(-\xi+n-2k)^{n-1}.\end{aligned}$$

Note that, by Lemma 2 and Eq. (13), the latter sum is equal to

$$\begin{aligned}-\sum_{2k>n-\xi}(-1)^k\binom{n}{k}(-\xi+n-2k)^{n-1}&=2^{n-1}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\left(k-\frac{n-\xi}{2}\right)_+^{n-1}\\&=2^{n-1}(n-1)!B_{n-1}\left(\frac{n-\xi}{2}\right).\end{aligned}$$

The remaining case $\xi=-n$ can be deduced by taking advantage of the fact that $I_{n,0}(-\xi)$ is the complex conjugate of $I_{n,0}(\xi)$, as was mentioned in Remark 4.

Instead of this, we treat the case $\xi = -n$ by direct calculation. For $\xi = -n$, Eq. (20) reads

$$I_{n,0}(-n) = \frac{1}{2^n (n-1)!} \lim_{x \to +\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-2k)^{n-1} \int_{2k-2n}^{0} \frac{e^{ixz}-1}{iz} dz.$$

Since $(2n-2k)^{n-1} = 0$, for $k = n$, and $2k - 2n \leq -2$, for $0 \leq k < n$, Lemma 2 and Lemma 3 imply that

$$\begin{aligned} & I_{n,0}(-n) \\ & = \frac{1}{2^n (n-1)!} \lim_{x \to +\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-2k)^{n-1} \int_{2k-2n}^{-2} \frac{e^{ixz}-1}{iz} dz \\ & = \frac{1}{2^n (n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-2k)^{n-1} \frac{1}{i} \log\left(\frac{2n-2k}{2}\right), \end{aligned}$$

which is the desired value of $I_{n,0}(\xi)$, for $\xi = -n$, since $B_{n-1}(n) = 0$.

***Proof of Theorem* 2** When we differentiate the formula for $I_{n,0}(\xi)$, which is given in Theorem 1, m times with respect to ξ, the general case follows after application of Lemma 1 and Lemma 2.

5 A further approach

In this section the method presented and developed in [11, Art. 1031] is used. If p and q are positive integers and $2 \leq q \leq p$, the integral

$$I_{p,p-q} = \int_0^{+\infty} \frac{\sin^p t}{t^q} dt$$

can be investigated by a method which does not entail the successive calculation of previous results of the same form leading up to this integral, as was done in [11, Art. 1023]. Taking advantage of the identity

$$\int_0^{+\infty} z^{q-1} e^{-tz} dz = \frac{\Gamma(q)}{t^q} \qquad (\Re q > 0, \quad \Re t > 0)$$

we obtain

$$I_{p,p-q} = \frac{1}{\Gamma(q)} \int_0^{+\infty} \int_0^{+\infty} z^{q-1} e^{-tz} (\sin^p t) \, dz dt.$$

For $p \geq 2$, repeated application of the recursive formula

$$\int_0^{+\infty} e^{-at} \sin^p t \, dt = \frac{p\,(p-1)}{p^2+a^2} \int_0^{+\infty} e^{-at} \left(\sin^{p-2} t\right) dt \qquad (a>0)$$

leads to

$$I_{p,p-q} = \frac{p!}{\Gamma(q)} \int_0^{+\infty} \frac{z^{q-1}}{z\left(z^2+2^2\right)\left(z^2+4^2\right)\cdots\left(z^2+p^2\right)} dz \quad \text{if } p \text{ is even,}$$

and

$$I_{p,p-q} = \frac{p!}{\Gamma(q)} \int_0^{+\infty} \frac{z^{q-1}}{\left(z^2+1^2\right)\left(z^2+3^2\right)\left(z^2+5^2\right)\cdots\left(z^2+p^2\right)} dz \quad \text{if } p \text{ is odd.}$$

These integrals can be evaluated by partial fraction decomposition (see [11, Arts. 162 to 165]). In the two cases p, q both even or p, q both odd the emerging integrals are of the form $\int_0^{+\infty} \frac{1}{z^2+k^2} dz = \frac{\pi}{2k}$, but in the remaining cases the integrals are logarithmic.

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