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MATHEMATIK IN FORSCHUNG UND ANWENDUNG - MATHEMATICAL RESEARCH AND APPLICATIONS

A simple method for perfect packing of squares of sidelengths $n^{-1/2-\epsilon}$

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Abstract A simple method for perfect packing a square by squares of sidelengths $1, 2^{-t}, 3^{-t}, 4^{-t}, ...$ is presented for $1/2 < t \leq 17/32$.

Keywords Packing · Perfect packing · Square

1 Definitions

There are many questions about packing. In this note, we will describe one problem of perfect packing.

Let Q_n be a square, for $n = 1, 2, ...,$ and let R be a rectangle. We say that the squares $Q_1, Q_2, ...$ can be *packed* into R if it is possible to apply translations and rotations to the sets Q_n so that the resulting translated and rotated squares are contained in R and have mutually disjoint interiors. If the area of R is equal to the sum of areas of the squares, then the packing is *perfect*.

Example 1 Three squares of sidelength $1/2$, three squares of sidelength $1/4$, three squares of sidelength $1/8$, ... (i.e., three squares of sidelength 2^{-n} for $n = 1, 2, 3, ...$), of the sum of areas equal to $\frac{3}{4} + \frac{3}{16} + \frac{3}{64} + ... = 3 \cdot \sum_{n=1}^{\infty} \frac{1}{4^n} = 1$, can be packed into the square I of sidelength 1 (see Fig. 1). The sum of areas of squares equals the area of I, so the packing is perfect.

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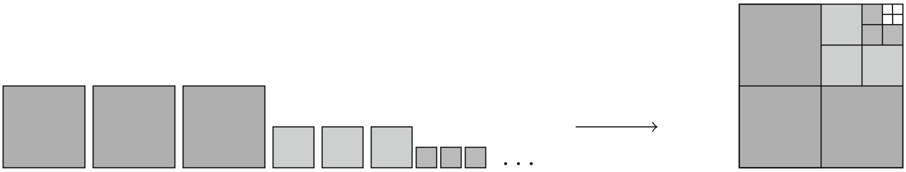

Fig. 1 Example 1, perfect packing

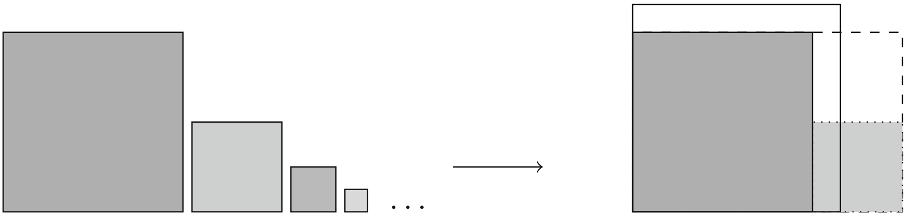

Fig. 2 Example 2

Example 2 One square of sidelength 1, one square of sidelength $1/2$, one square of sidelength $1/4$, ... (i.e., squares of sidelength 2^{-n} for $n = 0, 1, 2, ...$), of the sum of areas equal to $1 + \frac{1}{4} + \frac{1}{16} + ... = \frac{4}{3}$, cannot be packed into the square of sidelength $\frac{2\sqrt{3}}{3}$; the reason is that $1 + \frac{1}{2} > \frac{2\sqrt{3}}{3}$ (see Fig. 2). Moreover, the squares cannot be packed into any rectangle of area 4/3; the smallest rectangle ($1 \times 3/2$) into which a square of sidelength 1 can be packed together with a square of sidelength 1/2 is of area greater than 4/3. Consequently, the squares cannot be perfectly packed into any rectangle.

Given a rectangle R, by the *width* $w(R)$ we mean the smaller of the two sidelengths; the other sidelength $h(R)$ of R is called the *height*. We will write $R = w(R) \times h(R)$. Clearly, if R is a square, then $w(R) = h(R)$.

2 Packing of squares of harmonic sidelengths

In 1966 Moser [9] posed the following well known problem (see also problem LM6 in [10]): find the smallest $\varepsilon \geq 0$ such that the squares of sidelengths $1/2, 1/3, 1/4, ...$ (see Fig. 3) can be packed into a rectangle of area $\frac{1}{6}\pi^2 - 1 + \varepsilon$ (the sum of areas of the squares equals $\frac{1}{6}\pi^2 - 1$). Obviously, if $\varepsilon = 0$, then we get the perfect packing.

This problem is still open. Only some upper bounds are known for ε:

- Meir and Moser [8] showed that the squares can be packed into a square of sidelength 5/6 (consequently, $\varepsilon < 1/20$). Obviously, this is the smallest possible square; to pack a square of sidelength 1/2 together with a square of sidelength 1/3, a square of sidelength at least 1/2+1/3=5/6 is needed.
- Jennings ([5]) proved that $\varepsilon < 1/127$.
- Ball [1] showed that $\varepsilon < 1/198$.

Fig. 3 Squares of harmonic sidelengths

- Paulhus [11] obtained the very impressive bound $\varepsilon \leq 1/1244918662$. However, Joós (see [6]) pointed out that the proof given in the article is incorrect. In [3] it is showed that the Paulhus' lemma can be refolmulated so that the the upper bound $\varepsilon \leq 1/1244918662$ remains valid.

The packing method presented in [11] is very easy. Squares of sidelengths 1/2,1/3,1/4, ... are packed into a rectangle $R_1 = 1/2 \times (\pi^2/3 - 2)$. Difficulties arise only in estimating the effectiveness of this method.

Paulhus' method [11].

1. The first square is packed into a corner of R_1. After packing, the rectangle $V_2 = 1/2 \times (\pi^2/3 - 2 - 1/2)$ remains uncovered. We take $\mathcal{R}_2 = \{V_2\}$.
2. The square of sidelength 1/3 is packed into a corner of V_2. After packing, the uncovered part of V_2 is divided into $U_3 \cup V_3$, where $U_3 = (1/2 - 1/3) \times 1/3$ and $V_3 = (h(V_2) - 1/3) \times 1/2$; we take $\mathcal{R}_3 = \{U_3, V_3\}$.
3. Assume that $n > 3$, that the squares of sidelength $1/2$,$1/3$, ..., $1/(n-1)$ are packed into R_1 and that the family $\mathcal{R}_{n-1}$ is defined. We choose the rectangle with the smallest width from $\mathcal{R}_{n-1}$ into which the square of sidelength $1/n$ can be packed. Denote this rectangle by R. We pack the square into a corner of R. After packing, we divide the uncovered part of R into $U_n \cup V_n$, where U_n is the rectangle of sidelengths $1/n$ and $w(R) - 1/n$ and where V_n is the rectangle of sidelengths $w(R)$ and $h(R) - 1/n$; it is possible that U_n is an empty set. We take $\mathcal{R}_n = (\mathcal{R}_{n-1} \setminus \{R\}) \cup \{U_n, V_n\}$.

Fig. 4 illustrates the initial stage of the packing process. The first square is packed into a corner of R_1. The second square (of sidelength 1/3) is packed into a corner of the uncovered area. The family $\mathcal{R}_3$ consists of two rectangles: U_3 and V_3. Since $1/4 > 1/2 - 1/3$, the square of sidelength 1/4 cannot be packed into U_3 (the width $w(U_3) = 1/6$). It is packed into a corner of V_3 of width $w(V_3) = \pi^2/3 - 2 - 1/2 - 1/3 < 1/2$ and height $h(V_3) = 1/2$. After packing, the uncovered part of V_3 is divided into rectangles V_4 and U_4. Now the family $\mathcal{R}_3$ consists of three rectangles: U_3, U_4 and V_4. Since $w(U_3) < 1/5 < w(U_4) < w(V_4)$, the square of sidelength 1/5 is packed into U_4. From among four rectangles in $\mathcal{R}_5$, the rectangle U_3 is the one with the smallest width $(w(U_3) = 1/6)$ into which the square of sidelength 1/6 can be packed.

Paulhus used computer calculations and checked that at least 10^9 squares can be packed into R_1. It is highly unlikely that this method would produce a perfect packing. However, it is not known how many squares can be packed, i.e., which square stops the packing process with this method.

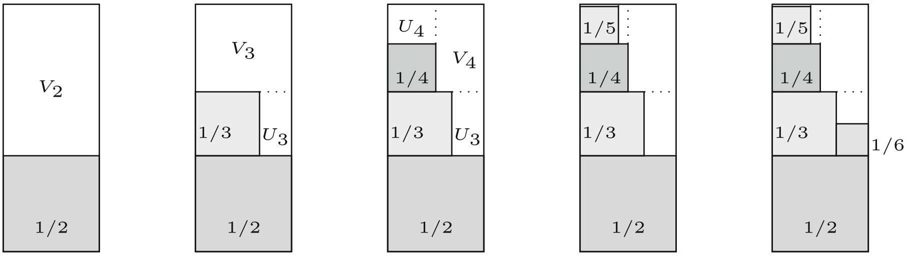

Fig. 4 Paulhus' method

3 Generalization

Let S_n^t be a square of sidelength n^{-t} for $n = 1, 2, ...$ If $t \le 1/2$, then the total area of the squares is equal to $\sum_{n=1}^{\infty} \frac{1}{n^{2t}}$ and the series is divergent. However, if $t > 1/2$, then the sum of areas of the squares is finite. Therefore, one can ask whether $S_1^t, S_2^t, ...$ (for $t > 1/2$) can be packed perfectly into a rectangle. Obviously, for $t = 1$ we get Moser's original question.

Note that $\sum_{n=1}^{\infty} \frac{1}{n^{2t}} = \zeta(2t)$, where $\zeta(s)$ is the Riemann zeta function.

Some results for packing are known for $t < 1$. Chalcraft [2] showed that $S_1^t, S_2^t, S_3^t, ...$ can be packed perfectly into a square for all t in the range [0.5964,0.6]. Joos [7] checked that these squares can be also packed perfectly for all t in the range [$\log_3 2$,2/3] ($\log_3 2 \approx 0.63$). Wästlund [13] proved that $S_1^t, S_2^t, S_3^t, ...$ can be packed into a finite collection of squares of the same area as the sum of areas of the squares, provided $1/2 < t < 2/3$. In [4] it is showed that for all t in the range (1/2,2/3], the squares $S_1^t, S_2^t, S_3^t, ...$ can be packed perfectly into a single square. Tao [12] proved that for any $1/2 < t < 1$, and any n_0 that is sufficiently large depending on t, the squares $S_{n_0}^t, S_{n_0+1}^t, ...$ can be packed perfectly into a square. Unfortunately, existing packing methods and proofs are not very easy. This note presents a simple method for perfect packing, but only for t slightly greater than 1/2. In particular, the packing method is not effective for $t = 1$ (for packing of squares of harmonic sidelength).

4 Perfect packing of squares

Let t be a fixed number from the interval (1/2,17/32] and let S be a square of area $\sum_{n=1}^{\infty} \frac{1}{n^{2t}}$. We will write S_m instead of S_m^t. The idea of the packing method is as follows. For each $n \ge 2$, the empty space in S, i.e., the part of S not covered by packed squares $S_1, ..., S_{n-1}$, will be divided into $2n - 1$ rectangles. Then, as in the Paulhus' method, S_n will be packed into a corner of one of these rectangles.

A rectangle R is *m-big*, provided that $w(R) \ge 2m^{-t}$.

A rectangle R is *basic*, provided that $w(R) \le h(R) \le 2w(R)$.

Obviously, each m-big rectangle is also n-big for $n > m$. Moreover, each basic rectangle is n-big for sufficiently large value of n.

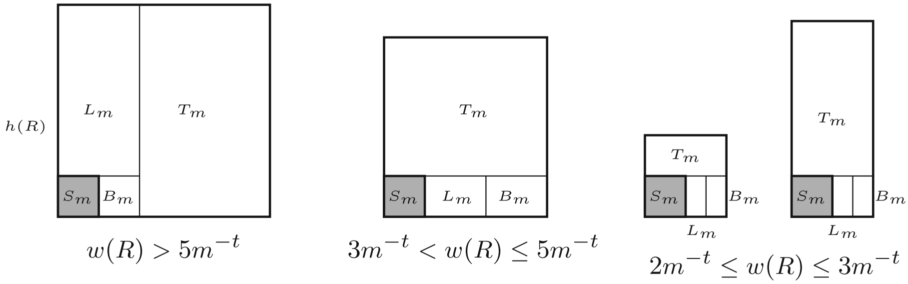

Fig. 5 $w(R) \geq 2m^{-t}$, $R = S_m \cup L_m \cup B_m \cup T_m$

Lemma 1 *Let m be a positive integer and let R be an m-big rectangle. Then R can be divided into four parts: S_m and three rectangles that are either basic or m-big.*

Proof *Case 1, when $w(R) > 5m^{-t}$.* The rectangle R is divided into: S_m, B_m, L_m and T_m (see Fig. 5, left), where $B_m = m^{-t} \times m^{-t}$, $L_m = (2m^{-t}) \times (h(R) - m^{-t})$ and $T_m = (w(R) - 2m^{-t}) \times h(R)$. Clearly, B_m is basic. Moreover, L_m and T_m are m-big. It is possible that T_m is m-big and basic at the same time.

Case 2, when $w(R) \leq 5m^{-t}$. Let B_m and L_m be rectangles of sidelengths m^{-t} and $(w(R) - m^{-t})/2$. Obviously, B_m and L_m are basic.

By $h(R) \geq w(R) \geq 2m^{-t}$ we get $h(R) - m^{-t} \geq w(R) - m^{-t} \geq \frac{1}{2}w(R)$. Let T_m be a rectangle of sidelengths $w(R)$ and $h(R) - m^{-t}$. Observe that T_m is either m-big (provided $h(R) - m^{-t} \geq 2m^{-t}$) or basic (provided $h(R) - m^{-t} \leq 2w(R)$). It is possible that T_m is m-big and basic at the same time. The rectangle R is divided into S_m, L_m, B_m and T_m (see Fig. 5, middle and right).

Packing method.

1. The first square is packed into a corner of S. After packing S_1, the uncovered part of S is divided into $L_1 \cup B_1 \cup T_1$ (as in the proof of Lemma 1) and we take $\mathcal{R}_1 = \{L_1, B_1, T_1\}$.
2. Assume that $n > 1$, that the squares $S_1, ..., S_{n-1}$ are packed into S and that the family $\mathcal{R}_{n-1}$ is defined. We choose one of n-big rectangles from $\mathcal{R}_{n-1}$ in any way. Denote this rectangle by R. We pack S_n into a corner of R. After packing S_n we divide the uncovered part of R into $L_n \cup B_n \cup T_n$ (as in the proof of Lemma 1) and we take $\mathcal{R}_n = (\mathcal{R}_{n-1} \setminus \{R\}) \cup \{L_n, B_n, T_n\}$.

Fig. 6 illustrates the initial stage of the packing process. The first square is packed into a corner of S. We have four possibilities; for example, we pack S_1 into the lower left corner. Since $w(L_1) \geq 2 \cdot 2^{-t}$ as well as $w(T_1) \geq 2 \cdot 2^{-t}$, both rectangles L_1 and T_1 are 2-big. We have eight possibilities for packing S_2: either in one of the corners of L_1 or in one of the corners of T_1. We choose one of them.

Clearly, $\mathcal{R}_{n-1}$ contains $2n - 1$ rectangles with mutually disjoint interiors, for any $n \geq 2$. Each rectangle from $\mathcal{R}_{n-1}$ is either n-big or basic.

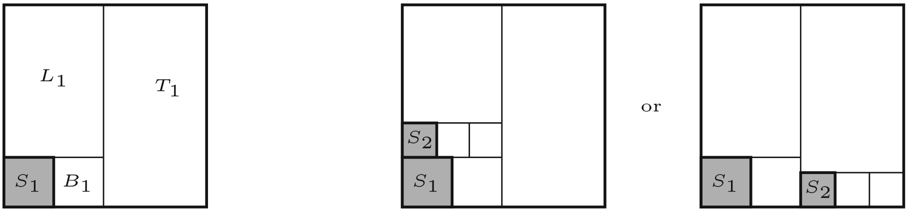

Fig. 6 Packing method

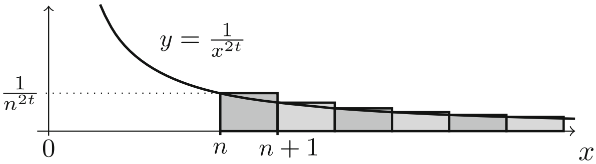

Fig. 7 Sum of areas of rectangles

Theorem 1 For each t in the range $1/2 < t \leq 17/32$, the squares S_n^t can be packed perfectly into the square S.

Proof Let t be a fixed number from the interval (1/2,17/32]. The area of S is equal to $\sum_{i=1}^{\infty} \frac{1}{i^{2t}} = \zeta(2t) \geq \zeta(17/16) > 16$. Consequently, $w(S) > 2 \cdot \frac{1}{1^t}$, i.e., S is 1-big. We pack $S_1^t, S_2^t, \ldots$ into S by our method. To prove Theorem 1 it suffices to show that for any n there is an n-big rectangle in $\mathcal{R}_{n-1}$ (into which S_n^t can be packed).

First we estimate the sum of areas of rectangles in $\mathcal{R}_{n-1}$, i.e., the area of the uncovered part of S after packing S_{n-1}^t. This value is equal to the sum of areas of unpacked squares $S_n^t, S_{n+1}^t, \ldots$ (which is equal to the sum of areas of rectangles of sidelengths 1 and $\frac{1}{i^{2t}}$, for $i = n, n+1, n+2, \ldots$), i.e., is equal to (see Fig. 7)

$$\frac{1}{n^{2t}} + \frac{1}{(n+1)^{2t}} + \ldots > \int_n^{+\infty} \frac{1}{x^{2t}} \mathrm{d}x = \frac{1}{2t-1} n^{1-2t} \geq \frac{1}{2 \cdot \frac{17}{32} - 1} n^{1-2t} = 16 n^{1-2t}.$$

Assume that there is an integer n such that S_n^t cannot be packed into S by our method. This implies that there is no n-big rectangle in $\mathcal{R}_{n-1}$. Then all rectangles in $\mathcal{R}_{n-1}$ are basic and the width of each such rectangle is smaller than $2n^{-t}$. The area of each such rectangle is smaller than $(2n^{-t}) \cdot 2(2n^{-t}) = 8n^{-2t}$. Since $2n-1$ rectangles are in $\mathcal{R}_{n-1}$, it follows that the total area of rectangles in $\mathcal{R}_{n-1}$ is smaller than $(2n-1) \cdot 8n^{-2t} < 16n^{1-2t}$, which is a contradiction.

Consequently, $S_1^t, S_2^t, \ldots$ can be packed into S.

Remark 1 The same packing method permits a perfect packing of $S_1^t, S_2^t, \ldots$ into any rectangle R of area $\zeta(2t)$, provided $w(R) \geq 2$ and $1/2 < t \leq 17/32$.

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