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Common Singularities of Commuting Vector Fields

Leonardo Biliotti[1] · **Oluwagbenga Joshua Windare**[1]

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Abstract

We study the singularities of commuting vector fields of a real submanifold of a Kähler manifold Z.

Keywords Momentum map · Reductive Lie group

Mathematics Subject Classification 53D20 · 14L24

1 Introduction

Let (Z, ω) be a connected Kähler manifold with a holomorphic action of a complex reductive group $U^{\mathbb{C}}$, where $U^{\mathbb{C}}$ is the complexification of a compact connected Lie group U with Lie algebra $\mathfrak{u}$. We also assume ω is U-invariant and that there is a U-equivariant momentum map $\mu : Z \to \mathfrak{u}^*$. By definition, for any $\xi \in \mathfrak{u}$ and $z \in Z$, $d\mu^\xi = i_{\xi_Z}\omega$, where $\mu^\xi(z) := \mu(z)(\xi)$ and ξ_Z denotes the fundamental vector field induced on Z by the action of U, i.e.,

$$\xi_Z(z) := \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)z$$

(see, for example, Kirwan 1984 for more details on the momentum map). Since U is compact we may identify $\mathfrak{u} \cong \mathfrak{u}^*$ by means of a $\mathrm{Ad}(U)$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{u}$. Hence, we consider a momentum map as a $\mathfrak{u}$-valued map, i.e., $\mu : Z \to \mathfrak{u}$.

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✉ Leonardo Biliotti
leonardo.biliotti@unipr.it

Oluwagbenga Joshua Windare
oluwagbengajoshua.windare@unipr.it

[1] Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma, Parma, Italy

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Recently, the momentum map has been generalized to the following settings (Heinzner and Schwarz 2007; Heinzner et al. 2008).

We say that a subgroup G of $U^{\mathbb{C}}$ is compatible if G is closed and the Cartan decomposition $U^{\mathbb{C}} = U\exp(\mathrm{i}\mathfrak{u})$ induces a Cartan decomposition of G. This means that the map $K \times \mathfrak{p} \to G$, $(k, \beta) \mapsto k\exp(\beta)$ is a diffeomorphism where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap \mathrm{i}\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G. In particular K is a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Using $\langle\cdot,\cdot\rangle$, we define an $\mathrm{Ad}(U)$-invariant scalar product on $\mathrm{i}\mathfrak{u}$ requiring multiplication by i to be an isometry between $\mathfrak{u}$ and $\mathrm{i}\mathfrak{u}$. The G-gradient map $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$ associated with μ is the orthogonal projection of $\mathrm{i}\mu$ onto $\mathfrak{p}$. If $\beta \in \mathfrak{p}$ then

$$\mu_{\mathfrak{p}}^{\beta}(z) := \langle \mu_{\mathfrak{p}}(z), \beta\rangle = \langle \mathrm{i}\mu(z), \beta\rangle = \langle \mu(z), -\mathrm{i}\beta\rangle = \mu^{-\mathrm{i}\beta}(z),$$

for any $\beta \in \mathfrak{p}$ and $z \in Z$. In this paper, a G-invariant compact connected locally closed real submanifold X of Z is fixed and the restriction of $\mu_{\mathfrak{p}}$ to X is also denoted by $\mu_{\mathfrak{p}}$. Then $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ is a K-equivariant map such that $\mathrm{grad}\mu_{\mathfrak{p}}^{\beta} = \beta_X$, where the gradient is computed with respect to the induced Riemannian metric on X denoted by $(\cdot,\cdot)$. By the linearization Theorem (Heinzner et al. 2008; Sjamaar 1998), $\mu_{\mathfrak{p}}^{\beta}$ is a Morse–Bott function (Biliotti et al. 2013; Heinzner et al. 2008) and the limit

$$\varphi_{\infty}^{\beta}(x) := \lim_{t\to+\infty} \exp(t\beta)x,$$

exists and belongs to $X^{\beta} := \{z \in X : \beta_X(z) = 0\}$ for any $x \in X$. The linearization theorem (Heinzner et al. 2008; Sjamaar 1998) also proves that any connected component of X^{β} is an embedded submanifold, see for instance (Biliotti et al. 2013; Heinzner et al. 2008).

Let $C_1, \ldots, C_k$ be the connected components of X^{β}. Let $W_i := \{x \in X : \lim_{t\mapsto+\infty} \exp(t\beta)x \in C_i\}$. Then $\mu_{\mathfrak{p}}^{\beta}(C_i) = c_i$ and applying again the linearization theorem (Heinzner et al. 2008; Sjamaar 1998), the submanifold C_i is a connected component of $(\mu_{\mathfrak{p}}^{\beta})^{-1}(c_i)$. One of the most important Theorem of Morse theory proves that W_i is an embedded submanifold, which is called *unstable manifold* of the critical submanifold C_i, and $\varphi_{\infty}^{\beta} : W_i \longrightarrow C_i$ is smooth (Bott 1954).

Let T be a torus of U. This means that T is a connected compact Abelian subgroup of U (Adams 1969). By a Theorem of Koszul, (Duistermaat and Kolk 2000), the connected components of $Z^T := \{x \in Z : T\cdot x = x\}$ are embedded Kähler submanifolds of Z. Let $\mathfrak{t}$ be the Lie algebra of T. It is well-known that the set

$$\left\{\beta \in \mathfrak{t} : \overline{\exp(\mathbb{R}\beta)} = T\right\},$$

is dense in $\mathfrak{t}$, see for instance (Adams 1969). Hence,

$$Z^T = Z^{T^{\mathbb{C}}} = \{p \in Z : \beta_Z(p) = 0\}, \tag{1}$$

for some $\beta \in \mathfrak{t}$. This means Z^T is the set of the singularities of the vector field β_Z, i,e., the zero of the vector field β_Z. Moreover, Z^T is the image of the gradient flow φ_∞^β defined by μ^β.

In this paper, we investigate the fixed point set of the action of an Abelian compatible subgroup of $U^{\mathbb{C}}$ acting on a real submanifold of Z.

Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra and $A = \exp(\mathfrak{a})$. Notice that A is automatically closed in G and hence compatible, since $\{e\} \times \mathfrak{a}$ is closed in $K \times \mathfrak{p}$. Then the A-gradient map on X is given by $\mu_\mathfrak{a} = \pi_\mathfrak{a} \circ \mu_\mathfrak{p}$, where $\pi_\mathfrak{a} : \mathfrak{p} \longrightarrow \mathfrak{a}$ denotes the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{a}$. Since A is Abelian, then by Lemma 2.2 below for any $p \in X$ the stabilizer $A_p = \{a \in A : ap = p\} = \exp(\mathfrak{a}_p)$, where $\mathfrak{a}_p$ is the Lie algebra of A_p. Therefore $X^A = \{p \in X : A \cdot p = p\} = \{p \in X : \beta_X(p) = 0, \forall \beta \in \mathfrak{a}\}$. Hence, if $\alpha_1, \ldots, \alpha_n$ is a basis of $\mathfrak{a}$ then X^A is the set of the common singularities of the commuting vector fields $(\alpha_1)_X, \ldots, (\alpha_n)_X$. Our first main result is the following

Theorem 1.1 *The set* $\left\{\beta \in \mathfrak{a} : X^\beta = X^A\right\}$ *is dense in* $\mathfrak{a}$.

Hence X^A is the set of the singularities of a vector field β_X for some $\beta \in \mathfrak{a}$ and so the critical points of the Morse–Bott function $\mu_\mathfrak{p}^\beta$.

We point out that X^A contains a lot of information of the geometry of both the A gradient map and the G gradient map. Indeed, for any $x \in X$, $\mu_\mathfrak{a}(A \cdot x)$ is an open convex subset of $\mu_\mathfrak{a}(x) + \mathfrak{a}_x$ and $\overline{\mu_\mathfrak{a}(A \cdot x)} = \mathrm{conv}(\mu_\mathfrak{a}(X^A \cap \overline{A \cdot x}))$, see Atiyah (1982), Biliotti and Ghigi (2018) and Heinzner and Schützdeller (2010), where $\mathrm{conv}(\cdot)$ denotes the convex hull of $(\cdot)$. In particular $\mu_\mathfrak{a}(X^A)$ is a finite set and $\mathrm{conv}(\mu_\mathfrak{a}(X)) = \mathrm{conv}(\mu_\mathfrak{a}(X^A))$ and so a polytope. Moreover, if $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subalgebra, then $\mathrm{conv}(\mu_\mathfrak{p}(X))$ is given by $K\mathrm{conv}(\mu_\mathfrak{a}(X))$ (Biliotti et al. 2016).

The second main result proves the existence of $\beta \in \mathfrak{a}$ such that the limit map associated with the gradient flow of $\mu_\mathfrak{p}^\beta$ defines a map from X onto X^A. Hence, the set X^A is the image of the gradient flow of the Morse–Bott function $\mu_\mathfrak{p}^\beta$ for some $\beta \in \mathfrak{a}$.

Let $\alpha_1, \ldots, \alpha_n \in \mathfrak{a}$ be a basis of $\mathfrak{a}$. Then $\varphi_\infty^{\alpha_n} \circ \cdots \circ \varphi_\infty^{\alpha_1}$ defines a map from the manifold X onto $X^{\alpha_1} \cap \cdots \cap X^{\alpha_n} = X^A$.

Theorem 1.2 *Let* $\alpha_1, \ldots, \alpha_n \in \mathfrak{a}$ *be a basis of* $\mathfrak{a}$. *There exists* $\delta > 0$ *such that for any* $0 < \epsilon_2, \ldots, \epsilon_n < \delta$ *we have* $\varphi_\infty^{\alpha_1+\epsilon_2\alpha_2+\cdots+\epsilon_n\alpha_n} = \varphi_\infty^{\alpha_n} \circ \cdots \circ \varphi_\infty^{\alpha_1}$.

2 Proof of the Main Results

Suppose $X \subset Z$ is G-invariant compact connected real submanifold of Z with the gradient map $\mu_\mathfrak{p} : X \to \mathfrak{p}$. If $x \in X$ then $G_x = \{g \in G : gx = x\}$ denotes the stabilizer of G at x. If G_x acts on a manifold S, then $G \times^{G_x} S$ denotes the associated bundle with principal bundle $G \to G/G_x$ defined as the quotient of $G \times S$ by the G_x-action $h(g, s) = (gh^{-1}, hs)$. We recall the Slice Theorem, see Heinzner et al. (2008) for details.

Theorem 2.1 (Slice Theorem (Heinzner et al. 2008, Thm. 3.1; Sjamaar 1998) *If* $x \in X$ *and* $\mu_\mathfrak{p}(x) = 0$, *there are a* G_x*-invariant decomposition* $T_x X = \mathfrak{g} \cdot x \oplus W$, *open*

G_x-invariant neighborhood S of $0 \in W$, a G-stable open neighborhood Ω of $x \in X$ and a G-equivariant diffeomorphism $\Psi : G \times^{G_x} S \to \Omega$ where $\Psi([e, 0]) = x$.

Corollary 2.1.1 *If $x \in X$ and $\mu_{\mathfrak{p}}(x) = \beta$, there are a $(G^\beta)_x$-invariant decomposition $T_x X = \mathfrak{g}^\beta \cdot x \oplus W$, open $(G^\beta)_x$-invariant neighborhood S of $0 \subset W$, a G^β-stable open neighborhood Ω of $x \in X$ and a G^β-equivariant diffeomorphism $\Psi : G^\beta \times^{(G^\beta)_x} S \to \Omega$ where $\Psi([e, 0]) = x$.*

This follows applying the previous theorem to the action of G^β on X. Indeed, it is well known that $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$ is compatible (Biliotti et al. 2013, Lemma 2.7, p.584) and the orthogonal projection of $\mathrm{i}\mu$ onto $\mathfrak{p}^\beta$ is the G^β-gradient map $\mu_{\mathfrak{p}^\beta}$ associated with μ (Heinzner et al. 2008). The group G^β is also compatible with the Cartan decomposition of $(U^{\mathbb{C}})^\beta = (U^{\mathbb{C}})^{\mathrm{i}\beta} = (U^{\mathrm{i}\beta})^{\mathbb{C}}$ and $\mathrm{i}\beta$ is fixed by the $U^{\mathrm{i}\beta}$-action on $\mathfrak{u}^{\mathrm{i}\beta}$. A momentum map of the $(U^{\mathbb{C}})^{\mathrm{i}\beta}$-action on Z is given by $\widehat{\mu_{\mathfrak{u}^{\mathrm{i}\beta}}(z)} = \pi_{\mathfrak{u}^{\mathrm{i}\beta}} \circ \mu + \mathrm{i}\beta$, where $\pi_{\mathfrak{u}^{\mathrm{i}\beta}}$ is the orthogonal projection of $\mathfrak{u}$ onto $\mathfrak{u}^{\mathrm{i}\beta}$, i.e., $U^{\mathrm{i}\beta}$-shifted momentum map by an element of the center of $\mathfrak{u}^{\mathrm{i}\beta}$. Then, the associated G^β-gradient map with $\widehat{\mu_{\mathfrak{u}^{\mathrm{i}\beta}}(z)}$ is given by $\widehat{\mu_{\mathfrak{p}^\beta}} := \mu_{\mathfrak{p}^\beta} - \beta$ and so $\widehat{\mu_{\mathfrak{p}^\beta}}(x) = 0$. Now, the result follows by Theorem 2.1. In particular, if G is commutative, then we have a Slice Theorem for G at every point of X, see Heinzner et al. (2008, p.169) and Sjamaar (1998) for more details.

If $\beta \in \mathfrak{p}$, then β_X is a vector field on X, i.e. a section of the bundle TX. For $x \in X$, the differential is a map $T_x X \to T_{\beta_X(x)}(TX)$. If $\beta_X(x) = 0$, there is a canonical splitting $T_{\beta_X(x)}(TX) = T_x X \oplus T_x X$. Accordingly, the differential of β_X, regarded as a section of TX, splits into a horizontal and a vertical part. The horizontal part is the identity map. We denote the vertical part by $\mathrm{d}\beta_X(x)$. The linear map $\mathrm{d}\beta_X(x) \in \mathrm{End}(T_x X)$ is indeed the so-called intrinsic differential of β_X, regarded as a section in the tangent bundle TX, at the vanishing point x. Let $\{\varphi_t = \exp(t\beta)\}$ be the flow of β_X. There is a corresponding flow on TX. Since $\varphi_t(x) = x$, the flow on TX preserves $T_x X$ and there it is given by $d\varphi_t(x) \in \mathrm{Gl}(T_x X)$. Thus we get a linear $\mathbb{R}$-action on $T_x X$ given by $\mathbb{R} \times T_x X \longrightarrow T_x X$, $(t, v) \mapsto d\varphi_t(x)(v)$. The flow of the vector field β_X defines an action of $\mathbb{R}$ on X, i.e., $\mathbb{R} \times X \longrightarrow X$, $(t, x) \mapsto \exp(t\beta)x$.

Corollary 2.1.2 *If $\beta \in \mathfrak{p}$ and $x \in X$ is a critical point of $\mu_{\mathfrak{p}}^\beta$, then there are open $\mathbb{R}$-invariant neighborhoods $S \subset T_x X$ and $\Omega \subset X$ and an $\mathbb{R}$-equivariant diffeomorphism $\Psi : S \to \Omega$, such that $0 \in S$, $x \in \Omega$, $\Psi(0) = x$. (Here $t \in \mathbb{R}$ acts as $d\varphi_t(x)$ on S and as φ_t on Ω.)*

Proof Since $\exp : \mathfrak{p} \longrightarrow G$ is a diffeomorphism onto the image, the subgroup $H := \exp(\mathbb{R}\beta)$ is closed and so it is compatible. Hence, it is enough to apply the previous corollary to the H-action on X and the value at x of the corresponding gradient map.□

Lemma 2.2 *Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra and let $A = \exp(\mathfrak{a})$ which is closed and compatible. If $x \in X$, then A_x is compatible, i.e., $A_x = \exp(\mathfrak{a}_x)$.*

Proof If $a \in A_x$, then $a = \exp(\beta)$ for a $\beta \in \mathfrak{a}$. Let $f(t) = \langle \mu_{\mathfrak{a}}(\exp(t\beta)x), \beta \rangle$. Then $f(1) = \langle \mu_{\mathfrak{a}}(\exp(\beta)x), \beta \rangle = \langle \mu_{\mathfrak{a}}(ax), \beta \rangle = \langle \mu_{\mathfrak{a}}(x), \beta \rangle = f(0)$ and $f'(t) = \| \beta_X(\exp(t\beta)x) \|^2 \geq 0$. This implies $\beta_X(x) = 0$ and so $\beta \in \mathfrak{a}_x$, proving $A_x = \exp(\mathfrak{a}_x)$. □

Let $\alpha, \beta \in \mathfrak{p}$ be such that $[\alpha, \beta] = 0$ and let $\mathfrak{a}$ be the vector space in $\mathfrak{p}$ generated by α and β. By the above Lemma, it follows that $X^A = X^\beta \cap X^\beta$, where $A = \exp(\mathfrak{a})$, which is closed and compatible due to the fact that the exponential map is a diffeomorphism restricted on $\mathfrak{p}$.

Lemma 2.3 *Let $\beta, \alpha \in \mathfrak{p}$ be such that $[\beta, \alpha] = 0$. If X is compact, then there exists $\delta > 0$ such that for any $\epsilon \in (0, \delta)$, $X^{\beta+\epsilon\alpha} = X^\beta \cap X^\alpha$.*

Proof Let $\epsilon > 0$ and let $A = \exp(\mathfrak{a})$, where $\mathfrak{a} = \text{span}(\alpha, \beta)$. Let X^A denote the fixed point set of A, i.e., $X^A = \{z \in X : A \cdot x = x\}$. By Lemma 2.2, $X^A = X^\beta \cap X^\alpha$. Corollary 2.1.2 applies for A and $H = \exp(\mathbb{R}(\alpha + \epsilon\beta))$. Therefore $X^\beta \cap X^\alpha$ and $X^{\alpha+\epsilon\beta}$ are compact submanifolds satisfying $X^\beta \cap X^\alpha \subseteq X^{\alpha+\epsilon\beta}$.

Let C be a connected component of $X^\alpha \cap X^\beta$. C is a compact connected submanifold of X and so it is arcwise connected. If $x \in C$ then $C \subseteq C'$, where C' is the connected component of $X^{\alpha+\epsilon\beta}$ containing x. On the other hand, if L is a connected component of $X^{\alpha+\epsilon\beta}$ then L is A-stable and so there exists a A-gradient map (Heinzner et al. 2008). Since L is compact the norm square A-gradient map has a maximum. By Heinzner et al. (2008, Corollary 6.12) L has a fixed point of A. This implies that L contains a connected component of $X^\alpha \cap X^\beta$. Summing up, we have proved that the number of the connected components of $X^\alpha \cap X^\beta$ is greater than or equal to the number of connected components of $X^{\alpha+\epsilon\beta}$ and any connected component of $X^{\alpha+\epsilon\beta}$ contains at least a connected component of $X^\alpha \cap X^\beta$.

Let $C_1, \ldots, C_m$ be the connected components of $X^\alpha \cap X^\beta$. Let C_i' denote the connected component of $X^{\alpha+\epsilon\beta}$ containing C_i, for $i = 1, \ldots, m$. We point out that C_k' would coincide with C_j' for $k \neq i$. We shall prove that there exists $\delta > 0$ such that for any $\epsilon < \delta$ the connected components $C_1', \ldots, C_m'$ are pairwise disjoints and $C_i = C_i'$ for $i = 1, \ldots, m$.

Let $x_i \in C_i$. Since x_i is fixed by A, Corollary 2.1.2 implies there exists A-invariant open subsets Ω of $x_i \in X$ and S of $0 \in T_{x_i}X$ and a A-equivariant diffeomorphism $\varphi : S \to \Omega$ such that $\varphi(0) = x_i$, $d\varphi_0 = id_{T_{x_i}X}$. Since the $\mathbb{R}$-action on S is linear and φ is $\mathbb{R}$-equivariant, we may assume that $S = \Omega = \mathbb{R}^n$ by means of φ, α, β are symmetric matrices of order n satisfying $[\alpha, \beta] = 0$. Moreover, $T_{x_i}X^{\alpha+\epsilon\beta} = \text{Ker}\,(\alpha + \epsilon\beta)$ and $T_{x_i}X^\alpha \cap T_{x_i}X^\beta = \text{Ker}\,\alpha \cap \text{Ker}\,\beta$.

The matrices α and β are simultaneously diagonalizable. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$ such that $\alpha e_k = a_k e_k$ and $\beta e_k = b_k e_k$ for $k = 1, \ldots, n$. Let $J = \{1 \leq k \leq n : a_k b_k \neq 0\}$. Pick $\delta_i = \min\{\frac{|a_k|}{|b_k|} : k \in J\}$. Now, $(\alpha + \epsilon\beta)e_k = 0$ if and only if $a_k + \epsilon b_k = 0$. If $a_k \neq 0$, then $b_k \neq 0$ and vice-versa. If $\epsilon < \delta_i$ then $(\alpha + \epsilon\beta)e_k = 0$, if and only if $a_k = b_k = 0$. Therefore, $\text{Ker}\,(\alpha + \epsilon\beta) = \text{Ker}\,\alpha \cap \text{Ker}\,\beta$. This implies $T_{x_i}C_i = T_{x_i}C_i'$. Although δ_i depends on x_i, since $C_i \subseteq C_i'$ and both are compact submanifolds it follows that $C_i = C_i'$. Pick $\delta = \min(\delta_1, \ldots, \delta_k)$. Then for any $\epsilon < \delta$ we have $C_i = C_i'$ for $i = 1, \ldots, m$. In particular $C_1', \ldots, C_m'$ are pairwise disjoints. Since the number of the connected components of $X^{\alpha+\epsilon\beta}$ is less than or equal to the number of connected components of $X^\alpha \cap X^\beta$, it follows that for any $\epsilon < \delta$ both $X^\alpha \cap X^\beta$ and $X^{\alpha+\epsilon\beta}$ have the same connected components and so $X^\alpha \cap X^\beta = X^{\alpha+\epsilon\beta}$ concluding the proof. □

Theorem 2.4 *Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra and let $A = \exp(\mathfrak{a})$. Then the set*

$$\left\{\alpha \in \mathfrak{a} : X^A = X^\alpha\right\}$$

is dense.

Proof Let $\alpha_1, \dots, \alpha_n$ be a basis of $\mathfrak{a}$. Then

$$X^A = X^{\alpha_1} \cap \cdots \cap X^{\alpha_n}.$$

By the above Lemma, there exists $\delta > 0$ such that for any $\epsilon_2, \dots, \epsilon_n < \delta$, we have

$$X^A = X^{\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n} \tag{2}$$

Let $\alpha \in \mathfrak{a}$ different form 0. It is well known that there exists $\alpha_2, \dots \alpha_n \in \mathfrak{a}$ such that $\alpha, \alpha_2, \dots, \alpha_n$ is a basis of $\mathfrak{a}$. By (2), for any neighborhood U of α, there exists $\beta \in U$ such that $X^A = X^\beta$, concluding the proof. □

The following lemma is proved in Biliotti and Windare (2023), see also Bruasse and Teleman (2005, pag. 1036).

Lemma 2.5 *Let $x \in X$ and $\beta, \alpha \in \mathfrak{p}$ be such that $[\beta, \alpha] = 0$. Set $y := \lim_{t\to\infty} \exp(t\beta)x$ and $z := \lim_{t\to\infty} \exp(t\alpha)y$. Let δ be as in Lemma 2.3. Then for $0 < \epsilon < \delta$,*

$$\lim_{t\to\infty} \exp(t(\beta + \epsilon\alpha))x = z.$$

As a consequence of the above lemma, we get the following result.

Theorem 2.6 *Let $\alpha_1, \dots, \alpha_n$ be a basis of $\mathfrak{a}$. Let $x \in X$. Set $x_1 := \lim_{t\to\infty} \exp(t\alpha_1)x$ and $x_i = \lim_{t\to\infty} \exp(t\alpha_i)x_{i-1}$ for $i = 2, \dots, n$. Then there exists $\delta > 0$ such that for $0 < \epsilon_2, \dots, \epsilon_n < \delta$, we have*

$$\lim_{t\to\infty} \exp(t(\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n))x = x_n,$$

for any $x \in X$. In particular, $\varphi_\infty^{\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n} = \varphi_\infty^{\alpha_n} \circ \cdots \circ \varphi_\infty^{\alpha_1}$.

Proof By Theorem 2.4, there exists $\delta > 0$ such that for any $0 < \epsilon_2, \dots, \epsilon_n < \delta$, we have

$$X^A = X^{\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon\alpha_n}.$$

Let $A = \exp(\mathfrak{a})$. Let $z \in X^A$. By Corollary 2.1.2, there exists A-invariant open subsets $\Omega \subset X$ and $S \subset T_z X$ and a A-equivariant diffeomorphism $\varphi : S \to \Omega$ such that $0 \in S$, $z \in \Omega$, $\varphi(0) = z$, $d\varphi_0 = id_{T_z X}$. Let $x \in X$. Set $x_1 := \lim_{t\to\infty} \exp(t\alpha_1)x$ and $x_i = \lim_{t\to\infty} \exp(t\alpha)x_{i-1}$ for $i = 2, \dots, n$. If $x_n \in \Omega$, keeping in mind that

Ω is A-invariant, it follows that $x_1, \ldots, x_n \in \Omega$. If one reads carefully the proof of Lemma 2.3, then $\delta > 0$ works whenever that $y \in \Omega$. The same argument applies in this case. Hence there exists δ such that for any $0 < \epsilon_2, \ldots, \epsilon_n < \delta$, we have

$$\lim_{t \mapsto} \exp(t(\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon\alpha_n))x = x_n$$

whenever x_1, and so $x_1, \ldots, x_n$, belongs to Ω. By compactness of X^A there exist open subsets $\Omega_1, \ldots, \Omega_k$ satisfying the above property and such that

$$X^A \subseteq \Omega_1 \cup \cdots \cup \Omega_k.$$

Let $\delta_1, \ldots .\delta_k$ as before. Pick $\delta = \min(\delta_1, \ldots, \delta_k)$. Let $x \in X$. Set $x_1 := \lim_{t \to \infty} \exp(t\alpha_1)x$ and $x_i = \lim_{t \to \infty} \exp(t\alpha_i)x_{i-1}$ for $i = 2, \ldots, n$. Since $x_1 \in \Omega_j$ for some $j = 1, \ldots, k$, it follows that for any $0 < \epsilon_2, \ldots, \epsilon_n < \delta$ we have

$$\lim_{t \mapsto +\infty} \exp(t(\alpha_1 + \epsilon_2\alpha_2 + \cdots + \epsilon_n\alpha_n))x = x_n.$$

This holds for any $x \in X$, concluding the proof. □

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