Bull Braz Math Soc, New Series (2023) 54:58
https://doi.org/10.1007/s00574-023-00374-4

Necessary Codimension One Components of the Projection of the Jacobian Blow-Up

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Received: 12 April 2023 / Accepted: 16 October 2023 / Published online: 6 November 2023

Abstract
For a complex analytic function, the exceptional divisor of the jacobian blow-up is of great importance. In this short paper, we show what a lemma from the thesis of Lazarsfeld tells one about the structure of the projections of this exceptional divisor into the base space.

Keywords Hypersurface singularities · Jacobian blow up · Vanishing cycles · a_f Thom condition

Mathematics Subject Classification 32S25 · 32S05 · 32S30 · 32S50

1 introduction

Let $\mathcal{U}$ be an open subset of $\mathbb{C}^{n+1}$ and let $f : (\mathcal{U}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a nowhere locally constant complex analytic function. Near the origin, the critical locus Σf of f is contained in the hypersurface $V(f)$ defined by f; we assume that $\mathcal{U}$ is chosen small enough so that this is true throughout $\mathcal{U}$. We use $\mathbf{z} := (z_0, \dots, z_n)$ for the coordinates on $\mathbb{C}^{n+1}$ and so on $\mathcal{U}$.

We let $\pi : \mathrm{Bl}_{j(f)}\mathcal{U} \to \mathcal{U}$ be the projection map of the blow-up of $\mathcal{U}$ along the jacobian ideal

$$j(f) := \left\langle \frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n} \right\rangle,$$

where $\mathrm{Bl}_{j(f)}\mathcal{U} \subseteq \mathcal{U} \times \mathbb{P}^n$. Let $E = \pi^{-1}(\Sigma f)$ denote the exceptional divisor, which is purely n-dimensional. Let $W_0, W_1, \dots, W_r$ denote the distinct irreducible components of E over $\mathbf{0}$, i.e., the irreducible components W of E such that $\mathbf{0} \in \pi(W)$.

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Now let us identify the $\mathcal{U} \times \mathbb{P}^n$ which contains the jacobian blow-up with the projectivized cotangent space $\mathbb{P}(T^*\mathcal{U})$. Under this identification, $\mathrm{Bl}_{j(f)}\mathcal{U}$ is equal to the projectivized closure of the relative conormal of f, that is,

$$\mathrm{Bl}_{j(f)}\mathcal{U} = \mathbb{P}\big(\overline{T^*_{f|\mathcal{U}\backslash\Sigma f}\mathcal{U}}\big).$$

For $0 \leq k \leq r$, we let Y_k denote the irreducible analytic set $\pi(W_k)$. Then the fact that E is purely n-dimensional, combined with the existence of an a_f stratification, tells us that W_k is equal to the closure of the conormal space of the regular part Y_k° of Y_k, that is, $W_k = \mathbb{P}\big(\overline{T^*_{Y_k^\circ}\mathcal{U}}\big)$. We refer to the Y_k as **the Thom varieties of f** at $\mathbf{0}$.

Clearly, the Thom varieties are important for understanding limiting relative conormals and the a_f condition. In addition, the result of (Lê and Mebkhout 1983, Thèoréme 3.3) (which uses (Kashiwara 1978, Theorem 6.3.1)) or our generalization in (Massey 2000, Theorem 2.10) tells us that the exceptional divisor E is equal to the projectivized characteristic cycle of the sheaf of vanishing cycles of $\mathbb{C}^\bullet_{\mathcal{U}}$ (or $\mathbb{Z}^\bullet_{\mathcal{U}}$) along f (note that Lê and Mebkhout (1983) and Kashiwara (1978) are in the language of $\mathcal{D}$-modules). Thus, the Thom varieties are also closely related to the topology of the Milnor fibers of f at points in Σf.

While we prove a more general result in Theorem 2.3, a special case, Corollary 2.5, is much easier to state. See (Lê and Massey 2006, Definition 1.1) or Definition 2.4 of this paper for the precise definition of a simple μ-constant family.

Corollary *Suppose that Σf is smooth at $\mathbf{0}$. Then, at $\mathbf{0}$, either*

(1) *there is a Thom variety of f of codimension 1 in Σf, or*
(2) *Σf itself is the only Thom variety, and f defines a family of isolated singularities with constant Milnor number, that is, defines a simple μ-constant family.*

So, if Σf is smooth and there are any proper sub-Thom varieties in Σf, then there must be one of codimension 1. We find this somewhat surprising.

The crux of the proof of our theorem lies in a lemma from the Ph.D. thesis (Lazarsfeld 1980) of Lazarsfeld, which we recall in the next section.

2 The Lemma and the Theorem

We now state Lazarsfeld's lemma (Lazarsfeld 1980, Lemma 2.3), with some changes in notation; this also appears in Lazarsfeld's book (Lazarsfeld 2004, Example 3.3.20).

Lemma 2.1 (Lazarsfeld) *Let Z be an irreducible normal variety of dimension $n+1$, and let $X \subseteq Z$ be a subvariety which is locally defined (set-theoretically) by $n+1-e$ equations. Fix an irreducible component V of X. Then, for all $x \in V \cap \overline{X\backslash V}$, $\dim_x \big(V \cap \overline{X\backslash V}\big) \geq e-1$.*

The proof of our theorem below uses the above lemma in a crucial way. Our proof also uses our notation and results on relative polar varieties and Lê varieties (these are the analytic sets underlying the relative polar cycles and Lê cycles) as presented in

Massey (1995). We quickly give the definitions here. Recall that $\mathbf{z} = (z_0, \ldots, z_n)$ are our coordinates on $\mathcal{U}$. The following is a combination, on the level of analytic sets, of (Massey 1995, Definition 1.3, Definition 1.11, and Theorem 1.28).

Definition 2.2 Let $0 \leq e \leq n+1$. Let

$$X^e_{f,\mathbf{z}} := V\left(\frac{\partial f}{\partial z_e}, \ldots, \frac{\partial f}{\partial z_n}\right),$$

where we mean that $X^{n+1}_{f,\mathbf{z}} = \mathcal{U}$. Note that all components of $X^e_{f,z}$ must have dimension $\geq e$.

Let $\Gamma^e_{f,z}$ to be the union of the irreducible components of X which are **not** contained in Σf (and so, $\Gamma^0_{f,z} = \emptyset$). It is trivial that

$$X^e_{f,z} = \Gamma^e_{f,z} \cup \Sigma f.$$

Now let

$$Y^{e-1}_{f,z} := \Gamma^e_{f,z} \cap V\left(\frac{\partial f}{\partial z_{e-1}}\right).$$

It is easy to see that, for $1 \leq e \leq n$, the union of the irreducible components of $Y^{e-1}_{f,z}$ which are **not** contained in Σf is precisely $\Gamma^{e-1}_{f,z}$.

We define the analytic set $\Lambda^{e-1}_{f,\mathbf{z}}$ to be the union of the irreducible components of $Y^{e-1}_{f,z}$ which **are** contained in Σf.

Given f and a point $\mathbf{p} \in V(f)$, for a generic linear choice of coordinates (which we still denote by $\mathbf{z}$), in a neighborhood of $\mathbf{p}$, for all k, $\Gamma^k_{f,\mathbf{z}}$ and $\Lambda^k_{f,\mathbf{z}}$ are purely k-dimensional (which vacuously allows for them to be empty). Thus we refer to them as **the k-dimensional relative polar and Lê varieties**.

We should mention that the next theorem and its proof are closely related to (Massey 1995, Proposition 1.31). However, the hypotheses of that proposition are confusing, the result depends on a coordinate choice, the conclusion is not as general as it could be, and the proof omits important details.

Theorem 2.3 (Thom Going Down) *Let V be an irreducible smooth component of Σf at $\mathbf{0}$. Fix $e \leq \dim V$. Suppose that, at $\mathbf{0}$, for all $j \geq e$, the Thom varieties of f of dimension j, which are contained in V, are smooth. Then, at $\mathbf{0}$, one of the following must hold:*

(1) $\dim_{\mathbf{0}} V \cap \overline{\Sigma f \backslash V} \geq e-1$, *that is, V intersected with the union of the other irreducible components of Σf has dimension at least $e-1$, or*
(2) *there is a Thom variety of f of dimension $e-1$ inside V, or*
(3) $V = \Sigma f$, *and there are no Thom varieties of f of dimension less than e.*

Proof Throughout, we work at the origin, i.e., consider the germ of the situation at the origin. We use our notation from Definition 2.2; however, for convenience, we write simply X in place of $X^e_{f,\mathbf{z}}$.

We will need the characterization of Lê varieties/cycles given in (Massey 1995, Corollary 10.15), which tells us that the Lê variety of dimension j is the union of the j-dimensional absolute polar varieties of the Thom varieties of dimension $\geq j$. The reader may see (Massey 1994, Theorem 7.5) for the details. (This follows immediately from (Massey 1994, Theorem 7.5) if one uses that the exceptional divisor E of the jacobian blow-up is equal to the projectivized characteristic cycle of the sheaf of vanishing cycles of $\mathbb{C}^\bullet_{\mathcal{U}}$ along f. However, to see the result without passing through the vanishing cycles, see (Massey 2003, Theorem 2.20 and Proposition 1.9).)

By our smoothness assumption on the Thom varieties, we may also assume that our coordinates are chosen so that, for all Thom varieties $W \subseteq V$ such that $\dim W \geq e$, the restriction of the maps $(z_0, \dots, z_m)$ to W have no critical points for $m < \dim W$, i.e., $V(z_0, \dots, z_m)$ transversely intersects W for $m < \dim W$. In terms of absolute polar varieties (for our notation and definition, see (Massey 1994, Definition 7.1)), this means that, for all Thom varieties $W \subseteq V$ such that $e \leq \dim W$, the absolute polar varieties (as sets) $\Gamma^m_{\mathbf{z}}(W)$ are empty for $m < \dim W$. Note that $\Gamma^{\dim W}_{\mathbf{z}}(W) = W$.

Note that an irreducible component of Σf of dimension $< e$ can not be an irreducible component of X (its dimension is too small), but rather must be contained in an irreducible component of $\Gamma^e_{f,\mathbf{z}}$. However, since our coordinates are such that $\Gamma^e_{f,\mathbf{z}}$ is purely e-dimensional, every component of Σf of dimension $\geq e$ must be an irreducible component of X. In particular, V is an irreducible component of X.

We now apply Lazarsfeld's Lemma. It tells us that, if $\mathbf{0} \in V \cap \left(\Gamma^e_{f,\mathbf{z}} \cup \overline{\Sigma f \backslash V}\right)$, then either

$$\dim_{\mathbf{0}} V \cap \Gamma^e_{f,\mathbf{z}} \geq e - 1 \qquad \text{or} \qquad \dim_{\mathbf{0}} V \cap \overline{\Sigma f \backslash V} \geq e - 1.$$

Suppose that we are not in case (1) of the theorem, i.e., suppose that $\dim_{\mathbf{0}} V \cap \overline{\Sigma f \backslash V} < e - 1$. Then either $\dim_{\mathbf{0}} V \cap \Gamma^e_{f,\mathbf{z}} \geq e - 1$, or $\mathbf{0} \notin \Gamma^e_{f,\mathbf{z}}$ and $\mathbf{0} \notin \overline{\Sigma f \backslash V}$. We claim that these correspond to cases (2) and (3), respectively.

Case 2:

Suppose that $\dim_{\mathbf{0}} V \cap \overline{\Sigma f \backslash V} < e - 1$ and $\dim_{\mathbf{0}} V \cap \Gamma^e_{f,\mathbf{z}} \geq e - 1$.

By (Massey 1995, Proposition 1.15), as sets,

$$V \cap \Gamma^e_{f,\mathbf{z}} = V \cap \bigcup_{j \leq e-1} \Lambda^j_{f,\mathbf{z}},$$

where $\Lambda^j_{f,\mathbf{z}}$ is the purely j-dimensional Lê cycle. Thus, $\dim_{\mathbf{0}} V \cap \Gamma^e_{f,\mathbf{z}} \geq e-1$ implies that V contains an $(e-1)$-dimensional irreducible component Y of $\Lambda^{e-1}_{f,\mathbf{z}}$ at the origin. By (Massey 1995, Corollary 10.15 and/or Theorem 10.18), Y is a component of an absolute polar variety of a Thom variety T (which necessarily must have dimension $\geq e - 1$) of f; however, since we are assuming that $\dim_{\mathbf{0}} V \cap \overline{\Sigma f \backslash V} < e - 1$, T cannot be contained in another irreducible component of Σf, but rather must be contained in V. But by our hypotheses, the Thom varieties in V of dimension $> e - 1$

are smooth and have no absolute polar varieties of dimension $(e-1)$. Thus T must be $(e-1)$-dimensional and so $Y = T$, and we have the conclusion of Case 2.
Case 3:

Now suppose that $\mathbf{0} \notin \Gamma^e_{f,\mathbf{z}}$ and $\mathbf{0} \notin \overline{\Sigma f \backslash V}$. First, $\mathbf{0} \notin \overline{\Sigma f \backslash V}$ immediately implies that $V = \Sigma f$ at the origin. As we saw in Case 2, but using that $V = \Sigma f$, we have

$$V \cap \Gamma^e_{f,\mathbf{z}} = \bigcup_{j \le e-1} \Lambda^j_{f,\mathbf{z}},$$

and, as we are assuming that $\mathbf{0} \notin \Gamma^e_{f,\mathbf{z}}$, this implies that, at the origin, $\Lambda^j_{f,\mathbf{z}} = \emptyset$ for all $j \le e-1$. But $\Lambda^j_{f,\mathbf{z}}$ includes any Thom variety of dimension j, and so there are none for $j \le e-1$. Therefore, we have the conclusion of Case 3. □

We wish to consider families of singularities. The following definition is (Lê and Massey 2006, Definition 1.1).

Definition 2.4 Let $s := \dim_{\mathbf{0}} \Sigma f$. Suppose we have $G : \mathcal{U} \to \mathbb{C}^s$, the restriction to $\mathcal{U}$ of a linear surjection from $\mathbb{C}^{n+1}$. If $\mathbf{q} \in \mathcal{U}$, we define $f_{\mathbf{q}} := f_{|_{G^{-1}(G(\mathbf{q}))}}$.

We say that $f_{\mathbf{q}}$ is a simple μ-constant family at the origin if and only if there exists such a G such that

(1) $f_{\mathbf{0}}$ has an isolated critical point at the origin,
(2) Σf is smooth at the origin,
(3) $G_{|_{\Sigma f}}$ has a regular point at the origin,
(4) and, for all $\mathbf{q} \in \Sigma f$ close to the origin, the Milnor number $\mu_{\mathbf{q}}(f_{\mathbf{q}})$ is independent of $\mathbf{q}$.

Letting $e = \dim V$ in Theorem 2.3, we obtain the corollary from the introduction:

Corollary 2.5 *Suppose that Σf is smooth at $\mathbf{0}$. Then, at $\mathbf{0}$, either*

(1) *there is a Thom variety of f of codimension 1 in Σf, or*
(2) *Σf itself is the only Thom variety, and f defines a family of isolated singularities with constant Milnor number, that is, a simple μ-constant family as given in Definition 1.1 of Lê and Massey (2006).*

Proof Let $e = \dim V$ in Theorem 2.3. Then we obtain essentially the whole corollary. If Σf is smooth, then it is irreducible, and Case 1 from the theorem cannot occur. Cases 2 and 3 from the theorem correspond to Cases 1 and 2, respectively, of the corollary. The only thing that requires further proof is that Case 3 of the theorem implies that f defines a family of isolated singularities with constant Milnor number.

However, Case 3 of Theorem 2.3 is the case where $\mathbf{0} \notin \Gamma^e_{f,\mathbf{z}}$, that is, $\Gamma^e_{f,\mathbf{z}}$ is empty near the origin or, with its cycle structure, is 0. Then one applies the equivalence of Conditions 3 and 5 from Theorem 2.3 of Lê and Massey (2006) to conclude that f defines a family of isolated singularities with constant Milnor number (a simple μ-constant family). □

We can use Theorem 2.3 to prove a version of itself which refers to super-Thom varieties rather than sub-Thom varieties.

Theorem 2.6 (Thom Going Up) *Let T be an r-dimensional Thom variety of f at $\mathbf{0}$. Let $V \supseteq T$ be an irreducible component of Σf at $\mathbf{0}$. Then, at $\mathbf{0}$, one of the following must hold:*

(1) $T = V$, *or*
(2) $T \subseteq V \cap \overline{\Sigma f \backslash V}$, *or*
(3) *there exists a Thom variety $T' \subseteq V$ of f at $\mathbf{0}$ such that $T \subseteq \Sigma T'$, or*
(4) *there exists a Thom variety $T' \subseteq V$ of f at $\mathbf{0}$ such* $\dim T' = r + 1$ *and $T \subseteq T'$.*

Proof Suppose that we are not in Cases 1, 2, or 3. Then, let

$$X := T \backslash \left(\overline{\Sigma f \backslash V} \cup \bigcup_{T'} \Sigma T' \cup \bigcup_{T'' \not\supseteq T} T'' \right),$$

where the unions are over all Thom varieties T' and T'' contained in V, and $T'' \not\supseteq T$. Since we are not in Cases 2 or 3, X is an open, dense subset of T. Let $x \in X$. Then, $x \notin \overline{\Sigma f \backslash V}$ and, at x, every Thom variety $T' \subseteq V$ contains T and is smooth at x.

We apply Theorem 2.3 at x in place of $\mathbf{0}$. Let e be the smallest dimension of a Thom variety $T' \subseteq V$ at x such that T' properly contains T; there is such an e since we are not in Case 1, i.e., V itself is a Thom variety in V which properly contains T. Then we must be in Case 2 of Theorem 2.3, and there must be a Thom variety $\widetilde{T}$ of dimension $(e-1)$ in V at x. But this $\widetilde{T}$ must contain T (by the choice of x), and we would have a contradiction of the definition of e unless $\widetilde{T}$ does not **properly** contain T. Thus we must have $\widetilde{T} = T$ at x and $e = r + 1$. Since this is true for x in an open, dense subset of T, the conclusion of Case 4 follows. □

3 Examples

Example 3.1 Suppose that the irreducible components of Σf at the origin are a line L and a plane P. Is it possible that L and P are the only Thom varieties of f at $\mathbf{0}$? The answer is "no", and one might suspect that that is because $\{\mathbf{0}\}$ must also be a Thom variety. However, Theorem 2.3 with $e = 2$ tells us that, in fact, the plane P must contain a 1-dimensional Thom variety.

Let us look at a specific example. Let $f = w^2 + xyz^2$. Then,

$$\Sigma f = V\big(2w, yz^2, xz^2, 2xyz\big) = V(w, z) \cup V(w, y, x) = P \cup L.$$

Of course, P and L are Thom varieties, but Theorem 2.3 tells us that there must be a 1-dimensional Thom variety contained in P.

The reader is invited to calculate the blow-up of the jacobian ideal to show that the other Thom varieties are, in fact, $V(w, z, x)$, $V(w, z, y)$ and $\{\mathbf{0}\}$.

Example 3.2 Is the smoothness requirement in Theorem 2.3 and Corollary 2.5 really necessary? Yes. Consider $f = w^2 + (x^2 + y^2 + z^2)^2$. Then,

$$\begin{aligned}\Sigma f &= V\big(2w,\ 4(x^2 + y^2 + z^2)x,\ 4(x^2 + y^2 + z^2)y,\ 4(x^2 + y^2 + z^2)z\big)\\ &= V(w, x^2 + y^2 + z^2).\end{aligned}$$

Now $V(w, x^2 + y^2 + z^2)$ is a Thom variety. However, by symmetry, there cannot be a 1-dimensional Thom variety inside $V(w, x^2 + y^2 + z^2)$ and yet, by direct calculation, one can show that $\{\mathbf{0}\}$ is a Thom variety.

Thus, if Σf is irreducible, but not smooth, the conclusions of Corollary 2.5 need not hold.

Funding Open access funding provided by Northeastern University Library

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