Bull Braz Math Soc, New Series (2023) 54:27
https://doi.org/10.1007/s00574-023-00344-w

Analytic Semiroots for Plane Branches and Singular Foliations

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Received: 29 July 2022 / Accepted: 26 April 2023 / Published online: 8 May 2023

Abstract

The analytic moduli of equisingular plane branches has the semimodule of differential values as the most relevant system of discrete invariants. Focusing in the case of cusps, the minimal system of generators of this semimodule is reached by the differential values attached to the differential 1-forms of the so-called standard bases. We can complete a standard basis to an extended one by adding a last differential 1-form that has the considered cusp as invariant branch and the "correct" divisorial order. The elements of such extended standard bases have the "cuspidal" divisor as a "totally dicritical divisor" and hence they define packages of plane branches that are equisingular to the initial one. These are the analytic semiroots. In this paper we prove that the extended standard bases are well structured from this geometrical and foliated viewpoint, in the sense that the semimodules of differential values of the branches in the dicritical packages are described just by a truncation of the list of generators of the initial semimodule at the corresponding differential value. In particular they have all the same semimodule of differential values.

The authors are supported by the Spanish research project PID2019-105621GB-I00/AEI/10.13039/501100011033 funded by the Agencia Estatal de Investigación—Ministerio de Ciencia e Innovación. David Senovilla-Sanz is also supported by a predoctoral contract "Concepción Arenal" of the Universidad de Cantabria.

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Keywords Analytic invariants · Equisingularity · Semimodule · Cusp · Standard basis · Differential values · Dicritical foliation · Analytic semiroots

Mathematics Subject Classification 14H15 · 14H20 · 32S65 · 32S15 · 32S05

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1 Introduction

The analytic classification of plane branches starts with Zariski (2006), who pointed the importance of the differential values in this problem. The semimodule of differential values was extensively described by Delorme (1978), although the complete analytic classification is due to Hefez and Hernandes (2011).

Geometrically, the "most interesting" differential values are viewed as the contact $\nu_{\mathcal{C}}(\omega)$ of a given branch $\mathcal{C}$ with the foliations defined by differential 1-forms ω without common factors in the coefficients. From the moduli view-point, the semimodule of differential values Λ is interpreted as the "discrete structure" supporting the continuous part of the moduli. More precisely, the semimodule Λ has a well defined basis $\{\lambda_j\}_{j=-1}^{s}$; so, it is reasonable to fix our attention in the differential forms that produce precisely the elements of the basis as differential values: these are the elements of the standard bases (for more details, see Hefez and Hernandes 2001, 2007).

In this paper we focus in the case of cusps, that is, branches with a single Puiseux pair (n, m). Our objective is to describe the cusps close to a cusp $\mathcal{C}$, in terms of a given standard basis $\mathcal{H}$ and the dicritical foliated behaviour of the elements of $\mathcal{H}$ in the final divisor E of the reduction of singularities of $\mathcal{C}$. Let us precise this.

We consider a cusp $\mathcal{C}$ with Puiseux pair (n, m). In view of Zariski Equisingularity Theory, we know that the semigroup $\Gamma = n\mathbb{Z}_{\geq 0} + m\mathbb{Z}_{\geq 0}$ of $\mathcal{C}$ is an equivalent data of the equisingularity class of $\mathcal{C}$. The differential values define a semimodule $\Lambda^{\mathcal{C}}$ over Γ, that will have a strictly increasing basis

$$\lambda_{-1} = n, \lambda_0 = m, \lambda_1, \ldots, \lambda_s,$$

to be the minimal one such that $\Lambda^{\mathcal{C}} = \cup_{j=-1}^{s}(\lambda_j + \Gamma)$. By definition, an *extended standard basis* is a list of 1-forms

$$\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_{s+1},$$

such that $\nu_{\mathcal{C}}(\omega_i) = \lambda_i$ for $i = -1, 0, 1, \ldots, s$ and $\mathcal{C}$ is an invariant branch of ω_{s+1}, that is $\nu_{\mathcal{C}}(\omega_{s+1}) = \infty$, with some restrictions on the weighted order of ω_{s+1}.

Associated to the final divisor E given by $\mathcal{C}$, we have a *divisorial order* $\nu_E(\omega)$ defined for functions and 1-forms. In adapted coordinates it is the weighted monomial order that assigns the weight $an + bm$ to the monomial $x^a y^b$. Both the differential values and the divisorial orders act "like" valuations and we have that $\nu_E(\omega) \leq \nu_{\mathcal{C}}(\omega)$. For the case of a function we have that if $\nu_E(df) < nm$, then there is no resonance in the sense that $\nu_E(df) = \nu_{\mathcal{C}}(df)$. Thus, the "new differential values" in $\Lambda^{\mathcal{C}}$ will correspond to resonant 1-forms ω such that $\nu_E(\omega) < \nu_{\mathcal{C}}(\omega)$.

The structure of the semimodule $\Lambda^{\mathcal{C}}$ is well known (see Delorme 1978; Alberich-Carramiñana et al. 2022; Almirón and Moyano-Fernández 2021); anyway, we provide complete proofs using another approach in the appendices of the paper. The key elements are the *axes* u_i, and the *critical orders* t_i, defined by

$$u_{i+1} = \min(\Lambda_{i-1} \cap (\lambda_i + \Gamma)), \quad t_{i+1} = t_i + u_{i+1} - \lambda_i,$$

starting at $u_0 = n$ and $t_{-1} = n$, $t_0 = m$, where $\Lambda_{i-1}^{\mathcal{C}} = \cup_{j=-1}^{i-1}(\lambda_j + \Gamma)$. The axes are defined for $i = 0, 1, \ldots, s+1$ and the critical orders for $i = -1, 0, \ldots, s+1$. We know that the semimodule is *increasing* in the sense that $\lambda_i > u_i$ for $i = 1, 2, \ldots, s$ and the elements of any extended standard basis are characterized by the following properties

(1) $\nu_E(\omega_i) = t_i$ and $\nu_{\mathcal{C}}(\omega_i) \notin \Lambda_{i-1}^{\mathcal{C}}$, for $i = -1, 0, \ldots, s$.
(2) $\nu_E(\omega_{s+1}) = t_{s+1}$ and $\nu_{\mathcal{C}}(\omega_{s+1}) = \infty$.

Of course, the above properties assure that $\nu_{\mathcal{C}}(\omega_i) = \lambda_i$.

From the geometrical viewpoint, for each $i = 1, 2, \ldots, s+1$, the elements ω_i of an extended standard basis are what we call *basic and resonant*. This property implies that the transform $\tilde{\omega}_i$ of the 1-form ω_i by the morphism π of reduction of singularities of $\mathcal{C}$ has two remarkable properties:

(a) The greatest common divisor of the coefficients of $\tilde{\omega}_i$ defines a normal crossings divisor at the points of E contained in the exceptional divisor of the morphism π.
(b) The divisor E is dicritical (not invariant) for the foliation given by $\tilde{\omega}_i = 0$. Moreover, this foliation is nonsingular and it has normal crossings with the exceptional divisor of π at the points of E.

As a consequence of this, given an extended standard basis, we find a *dicritical package* $\{\mathcal{C}^i_P\}$ of cusps for each $i = 1, 2, \ldots, s+1$ parameterized by the points $P \in E$ that are not corners of the exceptional divisor (that is, elements of $\mathbb{C}^*$). Each $\mathcal{C}^i_P$ corresponds to the invariant curve of $\tilde{\omega}_i = 0$ through the point P. In particular, if P_0 is the infinitely near point of $\mathcal{C}$ at E, we have that $\mathcal{C}^{s+1}_{P_0} = \mathcal{C}$. In a terminology inspired in Equisingularity Theory and Reduction of Singularities (see for instance Abhyankar and Moh 1973a, b; Wall 2004; Seidenberg 1968 for the case of foliations), we could say that $\{\mathcal{C}^i_{P_0}\}$ are the *specific analytic semiroots* and that $\{\mathcal{C}^i_P\}$ are the *general analytic semiroots* of $\mathcal{C}$ associated to the given extended standard basis.

The property of E to be dicritical for the 1-forms ω_i has been suggested to us by M. E. Hernandes. We have a work in progress with him in this direction (Corral et al. 2023).

The main objective of this paper is to describe the semimodule and extended standard bases of the analytic semiroots. The statement is the following one:

Theorem 1.1 *Let $\Lambda^{\mathcal{C}} = \cup_{j=-1}^{s}(\lambda_i + \Gamma)$ be the semimodule of differential values and consider an extended standard basis*

$$\omega_{-1} = dx, \omega_0 = dy, \omega_1, \ldots, \omega_{s+1}$$

of the cusp $\mathcal{C}$. Take an index $i \in \{1, 2, \ldots, s+1\}$ and an analytic semiroot $\mathcal{C}^i_P$ of $\mathcal{C}$ associated to the given extended standard basis. Then the semimodule of differential values of $\mathcal{C}^i_P$ is precisely $\Lambda^{\mathcal{C}}_{i-1}$ and

$$\omega_{-1} = dx, \omega_0 = dy, \omega_1, \ldots, \omega_i$$

is a extended standard basis for $\mathcal{C}^i_P$.

The proof of this result uses as a main tool Delorme's decomposition of the elements of a standard basis. In the appendices, we provide proofs, using a different approach to the one of Delorme, of the structure results for the semimodule of differential values and of Delorme's decomposition.

Let us remark that it is possible to have curves of the dicritical package of the elements ω_j, when $j \geq 2$, of an extended standard basis that are not analytically equivalent, although they have the same semimodule of differential values. This occurs for instance if we compute a standard basis for the curve

$$t \mapsto (t^7, t^{17} + t^{30} + t^{33} + t^{36}).$$

as shown in Example 8.13. A natural question arises about "how many" analytic classes may be obtained in this way.

2 Cusps and Cuspidal Divisors

We are interested in the analytic moduli of branches with only one Puiseux pair, the *analytic cusps*. The last divisor of the minimal reduction of singularities of an analytic cusp is what we call a *cuspidal divisor*. As we shall see below, the study of the analytic moduli may be done through a fixed cuspidal divisor.

2.1 Cuspidal Sequences of Blowing-ups

Our ambient space is a two-dimensional germ of nonsingular complex analytic space (M_0, P_0). We are going to consider a specific type of finite sequences of blowing-ups centered at points, that we call *cuspidal sequences of blowing-ups* and we introduce below.

First of all, let us establish some notations concerning a nonempty finite sequence of blowing-ups centered at points

$$\mathcal{S} = \{\pi_k : (M_k, K_k) \to (M_{k-1}, K_{k-1});\ k = 1, 2, \ldots, N\},$$

starting at $(M_0, P_0) = (M_0, K_0)$. For any $k = 1, 2, \ldots, N$, the center of π_k is denoted by P_{k-1}, note that $P_{k-1} \in K_{k-1}$. We denote the intermediary morphisms as $\sigma_k : (M_k, K_k) \to (M_0, P_0)$ and $\rho_k : (M_N, K_N) \to (M_k, K_k)$, where

$$\sigma_k = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_k, \quad \rho_k = \pi_{k+1} \circ \pi_{k+2} \circ \cdots \circ \pi_N.$$

We denote the exceptional divisor of π_k as $E_k^k = \pi_k^{-1}(P_{k-1})$. By induction, for any $1 \leq j < k$ we denote by $E_j^k \subset M_k$ the strict transform of E_j^{k-1} by π_k. In this way we have that

$$K_k = \sigma_k^{-1}(P_0) = E_1^k \cup E_2^k \cup \cdots \cup E_k^k.$$

For any $P \in K_k$, we define $e(P) = \#\{j;\ P \in E_j^k\}$. Note that $e(P) \in \{1, 2\}$. If $e(P) = 1$, we say that P is a *free point* and if $e(P) = 2$ we say that it is a *corner point*. Note that all the points in $E_1^1 = K_1$ are free points. The last divisor E_N^N will be denoted $E = E_N^N$. We will also denote $M = M_N$, $K = K_N$ and $\pi = \sigma_N : (M, K) \to (M_0, P_0)$.

Definition 2.1 Following usual Hironaka's terminology, we say that the sequence $\mathcal{S}$ is a *bamboo* if $P_k \in E_k^k$ for any $k = 1, 2, \ldots, N-1$. We say that $\mathcal{S}$ is a *cuspidal sequence* if it is a bamboo and $e(P_{k-1}) \leq e(P_k)$, for any $2 \leq k \leq N-1$. The last divisor E of a cuspidal sequence is called a *cuspidal divisor*.

Remark 2.2 In the frame of Algebraic Geometry, the cuspidal divisor E corresponds to a valuation ν_E of the field of rational functions and it determines completely the cuspidal sequence, once the starting ambient space is fixed. We will work with this valuation, but we present it in a direct way.

Given a cuspidal sequence $\mathcal{S}$ with $N \geq 2$, there is well defined *index of freeness* f with $1 \leq f \leq N-1$ such that $P_1, P_2, \ldots, P_f$ are free points and $P_{f+1}, P_{f+2}, \ldots, P_{N-1}$ are corner points. If $N = 1$ we put $f = 0$. A nonsingular branch $(Y, P_0) \subset (M_0, P_0)$ has *maximal contact with* $\mathcal{S}$ if and only if P_k is an infinitely near point of (Y, P_0) for each $k = 1, 2, \ldots, P_f$.

Remark 2.3 For any cuspidal sequence $\mathcal{S}$ there is at least one nonsingular branch (Y, P_0) having maximal contact with $\mathcal{S}$. Moreover, if (Y, P_0) has maximal contact with $\mathcal{S}$ and (Y', P_0) is another nonsingular branch, we have that (Y', P_0) has maximal contact with $\mathcal{S}$ if and only if $i_{P_0}(Y, Y') \geq f + 1$, where $i_{P_0}(Y, Y')$ stands for the intersection multiplicity.

We define intermediate cuspidal sequences of a cuspidal sequence $\mathcal{S}$ as follows. Given an index $0 \leq j \leq N-1$, the *intermediate* jth-*cuspidal sequence* $\mathcal{S}^{(j)}$ *of* $\mathcal{S}$ is the sequence of length $N - j$, starting at (M_j, P_j) such that the blowing ups

$$\pi_k^{(j)} : (M_{k+j}, K_k^{(j)}) \to (M_{k+j-1}, K_{k-1}^{(j)}), \quad k = 1, 2, \ldots, N - j$$

are obtained by restriction from π_{k+j}, where we put $K_0^{(j)} = \{P_j\}$ and $K_k^{(j)} \subset K_{k+j}$ is the image inverse of P_j by $\pi_{j+1} \circ \pi_{j+2} \circ \cdots \circ \pi_{j+k}$.

Remark 2.4 Note that the (k, i)-divisor of $\mathcal{S}^{(j)}$ corresponds to the $(k+j, i+j)$ divisor of $\mathcal{S}$. In particular the last divisors of $\mathcal{S}^{(j)}$ and $\mathcal{S}$ are both equal to E.

The *Puiseux pair* (n, m) of $\mathcal{S}$ is defined by an inductive process that corresponds to Euclides' algorithm as follows. If $N = 1$, we put $(n, m) = (1, 1)$. If $N > 1$, we consider the intermediate cuspidal sequence $\mathcal{S}^{(1)}$ starting at (M_1, P_1) that is supposed to have Puiseux pair (n_1, m_1). Then

(1) If $f \geq 2$, we have that $f_1 = f - 1$ and we put $(n, m) = (n_1, m_1 + n_1)$.
(2) If $f = 1$, we put $(n, m) = (m_1, n_1 + m_1)$.

We see that $1 \leq n \leq m$ and n, m are without common factor. Note also that $f \geq 2$ if and only if $m \geq 2n$. Moreover, if $f = 1$ and $N \geq 2$, we have that $2 \leq n < m < 2n$.

Proposition 2.5 *Consider* $1 \leq n \leq m$ *without common factor and a nonsingular branch* $(Y, P_0) \subset (M_0, P_0)$. *There is a unique cuspidal sequence* $\mathcal{S}$ *starting at* (M_0, P_0) *having maximal contact with* (Y, P_0) *and such that* (n, m) *is the Puiseux pair of* $\mathcal{S}$.

Proof If $n = m = 1$, the only possibility is that $N = 1$ and then $\mathcal{S}$ consists in the blowing-up of P_0. Let us proceed by induction on $n + m$ and assume that $n + m > 2$. We necessarily have that $N \geq 2$, the first blowing-up π_1 is centered in P_0 and P_1 is the infinitely near point of Y in E_1^1.

Assume first that $2n \leq m$. We apply induction to (Y_1, P_1) with respect to the pair n', m' where $n' = n, m' = m - n$ and we obtain a cuspidal sequence $\mathcal{S}'$ over (M_1, P_1) of length N' with the required properties. We construct $\mathcal{S}$ of length $N = N' + 1$ by taking π_k centered at the point P'_{k-2}, for $k = 2, 3, \ldots, N' + 1$.

In the case that $n \leq m < 2n$ we consider the branch $(Y_1', P_1) = (E_1^1, P_1)$, we apply induction to (Y_1', P_1) with respect to the pair n', m' where $n' = m - n$, $m' = n$ and we obtain a cuspidal sequence $\mathcal{S}'$ over (M_1, P_1) of length N'. We construct $\mathcal{S}$ os length $N = N' + 1$ as before.

The uniqueness of $\mathcal{S}$ follows by an inductive invoking of the uniqueness after one blowing-up. □

We denote by $\mathcal{S}_Y^{n,m}$ the sequence obtained in Proposition 2.5. Recall that Y' has maximal contact with $\mathcal{S}_Y^{n,m}$ if and only if $i_{P_0}(Y, Y') \geq f + 1$, and hence in this case we have that $\mathcal{S}_Y^{n,m} = \mathcal{S}_{Y'}^{n,m}$. Note also that given a cuspidal sequence $\mathcal{S}$ there is a nonsingular branch (Y, P_0) and a Puiseux pair (n, m) in such a way that $\mathcal{S} = \mathcal{S}_Y^{n,m}$.

2.2 Coordinates Adapted to a Cuspidal Sequence of Blowing-ups

Consider a cuspidal sequence $\mathcal{S}$ over (M_0, P_0). A system (x, y) of local coordinates at P_0 is *adapted to* $\mathcal{S}$ if and only if $y = 0$ has maximal contact with $\mathcal{S}$. In particular, we have that $\mathcal{S} = \mathcal{S}_{y=0}^{n,m}$, where (n, m) is the Puiseux pair of $\mathcal{S}$.

The blowing-ups of $\mathcal{S}$ have a monomial expression in terms of adapted coordinates as we see below. Assume that $\mathcal{S} = \mathcal{S}_{y=0}^{n,m}$, with $N \geq 2$. Let us describe a local coordinate system (x_1, y_1) at P_1 and a pair (n_1, m_1):

- If $f \geq 2$, we know that $2n \leq m$ and we put

$$n_1 = n, \quad m_1 = m - n, \quad x = x_1, \quad y = x_1 y_1.$$

- If $f = 1$, we have that $2n > m > n \geq 2$ and we put

$$n_1 = m - n, \quad m_1 = n, \quad y = x_1 y_1, \quad x = y_1.$$

The reader can verify that (x_1, y_1) is a coordinate system adapted to $\mathcal{S}^{(1)}$ and that (n_1, m_1) is its Puiseux pair. In this way, we have local coordinates x_j, y_j at each P_j, for $0 \leq j \leq N - 1$.

Once we have an adapted coordinate system (x, y), we denote (H_0, P_0) the normal crossings germ given by $xy = 0$. Define $H_j = \sigma_j^{-1}(H_0)$, then the germ of H_j at P_j is given by $x_j y_j = 0$, for any $0 \leq j \leq N-1$. We can also consider $H = \pi^{-1}(H_0) \subset M$; it is a normal crossings divisor of (M, K) containing K.

2.3 Cuspidal Analytic Module

Consider a cuspidal sequence $\mathcal{S}$ with Puiseux pair (n, m) with $2 \leq n$. Let E be the last divisor of $\mathcal{S}$. We say that an analytic branch $(\mathcal{C}, P_0) \subset (M_0, P_0)$ is an *E-cusp*, or a *$\mathcal{S}$-cusp* if the strict transform of $(\mathcal{C}, P_0)$ under the sequence of blowing-ups π is nonsingular and cuts transversely E at a free point. Let us denote by $\text{Cusps}(E) = \text{Cusps}(\mathcal{S})$ the family of E-cusps.

Each element of $\text{Cusps}(\mathcal{S})$ is equisingular to the irreducible cusp $y^n - x^m = 0$, where (n, m) is the Puiseux pair of $\mathcal{S}$. Moreover, we have the following result

Proposition 2.6 *Consider a cuspidal sequence $\mathcal{S}$ with Puiseux pair (n, m) and last divisor E. Let $(\mathcal{C}, P_0)$ be a branch of (M_0, P_0) equisingular to the irreducible cusp $y^n - x^m = 0$. There is an E-cusp analytically equivalent to $(\mathcal{C}, P_0)$.*

Proof Choose a local coordinate system (x, y) adapted to $\mathcal{S}$.

If $n = 1$, the branch $(\mathcal{C}, P_0)$ is nonsingular. Then, there is an automorphism $\phi : (M_0, P_0) \to (M_0, P_0)$ such that $\phi(\mathcal{C}) = (y = 0)$. We are done since in this case $y = 0$ is an E-cusp.

Assume that $2 \leq n < m$. In view of the classical arguments of Hironaka (see for instance Aroca et al. 2018, p. 105), there is a nonsingular branch (Z, P_0) having maximal contact with $(\mathcal{C}, P_0)$, that is with the property that

$$i_{P_0}(Z, \mathcal{C}) = m.$$

Take an automorphism $\phi : (M_0, P_0) \to (M_0, P_0)$ such that $\phi(Z) = (y = 0)$. We have that $(\phi(\mathcal{C}), P_0)$ is an E-cusp. □

According to the above result, the analytic moduli of the family of branches equisingular to the irreducible cusp $y^n - x^m = 0$ is faithfully represented by the analytic moduli of the family Cusps$(\mathcal{S})$.

Along the rest of this paper, we consider a fixed cuspidal sequence $\mathcal{S}$ where (n, m) is its Puiseux pair and E is the last divisor. Recall also that the composition of all the blowing-ups of $\mathcal{S}$ is denoted by

$$\pi : (M, K) \to (M_0, P_0).$$

We also choose a local coordinate system (x, y) adapted to $\mathcal{S}$.

3 Divisorial Order

Consider a holomorphic function h in (M, K) defined globally in $E \subset K$, the *divisorial order $\nu_E(h)$ of h* is obtained as follows. Take a point $P \in E$ and choose a reduced local equation $u = 0$ of the germ (E, P), then

$$\nu_E(h) = \max\{a \in \mathbb{Z};\ u^{-a}h \in \mathcal{O}_{M,P}\}.$$

This definition does not depend on the chosen point $P \in E$ nor on the local reduced equation of E. Take a point P_j, with $j \in \{0, 1, \ldots, N-1\}$ and a germ of holomorphic function $h \in \mathcal{O}_{M_j, P_j}$. Then $\rho_j^* h$ is a germ of function in (M, K) globally defined in E. We define the *divisorial order* $\nu_E(h)$ by $\nu_E(h) = \nu_E(\rho_j^* h)$.

Proposition 3.1 *Consider a germ $h \in \mathcal{O}_{M_0, P_0}$ that we write as*

$$h = \sum_{\alpha,\beta} h_{\alpha,\beta} x^\alpha y^\beta, \quad h_{\alpha\beta} \in \mathbb{C}.$$

Then $\nu_E(h) = \min\{n\alpha + m\beta;\ h_{\alpha,\beta} \neq 0\}$.

Proof If $n = m = 1$ we have a single blowing-up and we recover the usual multiplicity, that we visualize in E as $\nu_E(h)$. Let us work by induction on $n + m$ and assume that $n+m \geq 2$. We remark that $\nu_E(h) = \nu_E(\pi_1^* h)$. Consider the first intermediate sequence $\mathcal{S}^{(1)}$ of $\mathcal{S}$, with adapted coordinates (x_1, y_1). Recalling how we obtain intermediate coordinate systems, we conclude that

$$\pi_1^* h = \sum_{\alpha,\beta} h_{\alpha,\beta} x_1^{\alpha+\beta} y_1^{\beta};\ \text{if } f \geq 2;\ \text{here } n_1 = n, m_1 = m - n,$$

$$\pi_1^* h = \sum_{\alpha,\beta} h_{\alpha,\beta} x_1^{\beta} y_1^{\alpha+\beta};\ \text{if } f = 1;\ \text{here } n_1 = n - m, m_1 = n.$$

We end by applying induction hypothesis. □

3.1 Divisorial Order of a Differential Form

Recall that we denote

$$H_0 = (xy = 0) \subset M_0, \quad H_j = \sigma_j^{-1}(H_0) \subset M_j$$

and that H_j is locally given at P_j by $x_i y_j = 0$ for $0 \leq j \leq N - 1$. We also consider $H_N = H = \pi^{-1}(H_0) \subset M$. Each H_j is a normal crossings divisor in (M_j, K_j), containing K_j.

Take a point $Q \in K_j$, not necessarily equal to P_j, in particular we consider also the case $j = N$. Select local coordinates (u, v) such that $(u = 0) \subset H_j \subset (uv = 0)$, then we have that either $H_j = (u = 0)$ or $H_j = (uv = 0)$ locally at Q. The $\mathcal{O}_{M_j,Q}$-module $\Omega^1_{M_j,Q}[\log H_j]$ of germs of H_j*-logarithmic 1-forms* is the rank two free $\mathcal{O}_{M_j,Q}$-module generated by

$$\begin{array}{ll} du/u, dv & \text{if } H_j = (u = 0), \\ du/u, dv/v & \text{if } H_j = (uv = 0). \end{array}$$

Note that $\Omega^1_{M_j,Q} \subset \Omega^1_{M_j,Q}[\log H_j]$. Indeed, a differential 1-form $\omega = adu + bdv$ may be written as

$$\omega = ua\frac{du}{u} + bdv = ua\frac{du}{u} + vb\frac{dv}{v}.$$

Now, let us consider a 1-form $\omega \in \Omega^1_M[\log H]$ defined in the whole divisor E (we suppose that the reader recognizes the sheaf nature of $\Omega^1_M[\log H]$). Select a point $Q \in E$ and a local reduced equation $u = 0$ of E at Q. We define the *divisorial order* $\nu_E(\omega)$ by

$$\nu_E(\omega) = \max\{\ell \in \mathbb{Z};\ u^{-\ell}\omega \in \Omega^1_{M,Q}[\log H]\}.$$

The definition is independent of $Q \in E$ and of the reduced local equation of E.

Remark 3.2 Let $\omega \in \Omega^1_M[\log E]$ be globally defined on E as before. Since E is one of the irreducible components of H, we have that

$$\Omega^1_M[\log E] \subset \Omega^1_M[\log H].$$

Let us choose a reduced local equation $u = 0$ of E at a point $Q \in E$ as before. A direct computation shows that

$$\nu_E(\omega) = \max\{\ell \in \mathbb{Z};\ u^{-\ell}\omega \in \Omega^1_{M,Q}[\log E]\}. \tag{1}$$

This remark shows that the divisorial order, applied to 1-forms $\omega \in \Omega^1_M[\log E]$ is independent of the choice of the adapted coordinate system that defines H_0. Anyway, this is only a remark for the case $n = 1$, since when $n \geq 2$ the divisor H at the points of E is itself independent of the adapted coordinate system.

Definition 3.3 For any $\omega \in \Omega^1_{M_j,P_j}$, the *divisorial order* $\nu_E(\omega)$ is defined by $\nu_E(\omega) = \nu_E(\rho_j^*\omega)$.

Proposition 3.4 *Consider a differential 1-form* $\omega = adx + bdy \in \Omega^1_{M_0,P_0}$*, that we can write as*

$$\omega = xa(dx/x) + yb(dy/y) \in \Omega^1_{M_0,P_0}[\log H_0].$$

Then, we have that $\nu_E(\omega) = \min\{\nu_E(xa), \nu_E(yb)\}$.

Proof Write $\omega = f(dx/x) + g(dy/y)$. We proceed by induction on N. If $N = 1$ we have that $E = (x' = 0)$ where $x = x'$, $y = x'y'$ and

$$\pi_1^*\omega = (f + g)(dx'/x') + g(dy'/y').$$

Then $\nu_E(\omega) = \min\{\nu_E(f + g), \nu_E(g)\} = \min\{\nu_E(f), \nu_E(g)\}$ and we are done. If $N \geq 2$, we have that

$$\nu_E(\omega) = \nu_E(\pi^*\omega) = \nu_E(\rho_1^*(\pi_1^*\omega)) = \nu_E(\pi_1^*\omega).$$

By induction hypothesis, we have

$$\nu_E(\pi_1^*\omega) = \min\{\nu_E(f + g), \nu_E(g)\} = \min\{\nu_E(f), \nu_E(g)\}$$

and we are done as before. □

Corollary 3.5 *If* $f \in \mathcal{O}_{M_0,P_0}$ *and* $\omega = df$*, then* $\nu_E(\omega) = \nu_E(f)$.

Proof It is enough to write $df = x(\partial f/\partial x)(dx/x) + y(\partial f/\partial y)(dy/y)$, recalling Euler's identity $gP = xP_x + yP_y$ for degree g homogeneous polynomials. □

3.2 Weighted Initial Parts

Consider a nonzero germ $h \in \mathcal{O}_{M_0,P_0}$, that we write as $h = \sum_{\alpha,\beta} h_{\alpha\beta} x^\alpha y^\beta$. Suppose that $q \leq \nu_E(h)$. We define the *weighted initial part* $\mathrm{In}^q_{n,m;x,y}(h)$ by

$$\mathrm{In}^q_{n,m;x,y}(h) = \sum_{n\alpha+m\beta=q} h_{\alpha\beta} x^\alpha y^\beta.$$

Note that $\mathrm{In}^q_{n,m;x,y}(h) = 0$ if and only if $q < \nu_E(h)$. Anyway, we can write

$$h = \mathrm{In}^q_{n,m;x,y}(h) + \tilde{h}, \quad \nu_E(\tilde{h}) > q.$$

This definition extends to logarithmic differential 1-forms $\omega \in \Omega^1_{M_0,P_0}[\log(xy = 0)]$ as follows. Take $q \leq \nu_E(\omega)$. Write $\omega = f(dx/x) + g(dy/y)$. We define

$$\mathrm{In}^q_{n,m;x,y}(\omega) = \mathrm{In}^q_{n,m;x,y}(f)(dx/x) + \mathrm{In}^q_{n,m;x,y}(g)(dy/y).$$

As before, we have $\omega = \mathrm{In}^q_{n,m;x,y}(\omega) + \tilde{\omega}$, with $\nu_E(\tilde{\omega}) > q$.

Remark 3.6 The definition of initial part we have presented should be made in terms of graduated rings and modules to be free of coordinates. Anyway, this “coordinate-based” definition is enough for our purposes.

Proposition 3.7 *Assume that $N > 1$, take $\omega \in \Omega^1_{M_0,P_0}[\log(xy = 0)]$ and $q \in \mathbb{Z}_{\geq 0}$ with $q \leq \nu_E(\omega)$. If $W = \mathrm{In}^q_{n,m;x,y}(\omega)$, then $\pi_1^*(W) = \mathrm{In}^q_{n_1,m_1;x_1,y_1}(\pi_1^*\omega)$.*

Proof Left to the reader. □

4 Total Cuspidal Dicriticalness

This section is devoted to characterize the 1-forms $\omega \in \Omega^1_{M_0,P_0}$ whose transform $\pi^*\omega$ defines a foliation that is transversal to E and has normal crossings with K at each point of E. These 1-forms are the so-called pre-basic and resonant 1-forms. We detect these properties in terms of resonances of the initial part. The initial part is visible in the Newton polygon as the contribution of the 1-form to a single vertex (a, b), under the condition that the Newton polygon is contained in the particular region $R^{n,m}(a, b)$.

4.1 Reduced Divisorial Order and Basic Forms

Let us consider a nonnull differential 1-form $\omega \in \Omega^1_{M_0,P_0}$. Let $V_\omega = x^a y^b$ be the monomial defined by the property that $\omega = V_\omega \eta$, where $\eta \in \Omega^1_{M_0,P_0}[\log(xy = 0)]$ is a logarithmic form that cannot be divided by any nonconstant monomial. We define the *reduced divisorial order* $\mathrm{rdo}_E(\omega)$ to be $\mathrm{rdo}_E(\omega) = \nu_E(\eta)$.

Definition 4.1 We say that $\omega \in \Omega^1_{M_0,P_0}$ is a *basic 1-form* if and only if its reduced divisorial order satisfies that $\mathrm{rdo}_E(\omega) < nm$.

Proposition 4.2 *Assume that $N \geq 2$ and take $\omega \in \Omega^1_{M_0,P_0}$. If ω is a basic 1-form, then $\pi_1^*\omega$ is also a basic 1-form.*

Proof Put $p = \mathrm{rdo}_E(\omega) = \nu_E(\eta) < nm$. Recall that $\nu_E(\eta) = \nu_E(\pi_1^*\eta)$. Since monomials are well behaved under π_1, it is enough to show that there are $c, d \geq 0$ such that $\pi_1^*\eta = x_1^c y_1^d \eta'$, with $\nu_E(\eta') < n_1 m_1$. Write

$$\eta = \sum_{\alpha,\beta} x^\alpha y^\beta \eta_{\alpha\beta}, \quad \eta_{\alpha\beta} = \mu_{\alpha\beta}\frac{dx}{x} + \zeta_{\alpha\beta}\frac{dy}{y}, \quad (\mu_{\alpha\beta}, \zeta_{\alpha\beta}) \in \mathbb{C}^2.$$

Recall that $p = \min\{n\alpha + m\beta;\ \eta_{\alpha\beta} \neq 0\}$. Put $r = \min\{\alpha + \beta;\ \eta_{\alpha\beta} \neq 0\}$. We have two cases: $f = 1$ and $f \geq 2$, where f is the index of freeness.

Assume first that $f \geq 2$ and hence $2n \leq m$. In this situation, we have that $x = x_1$, $y = x_1 y_1$, $n_1 = n$, $m_1 = m - n \geq n$ and

$$\pi_1^*(\eta) = x_1^r \eta', \quad \eta' = \sum_{\alpha,\beta} x_1^{\alpha+\beta-r} y_1^\beta \eta'_{\alpha\beta}, \quad \eta'_{\alpha\beta} = \left(\mu_{\alpha\beta} + \zeta_{\alpha\beta}\right)\frac{dx_1}{x_1} + \zeta_{\alpha\beta}\frac{dy_1}{y_1}.$$

Note that $\eta'_{\alpha\beta} \neq 0$ if and only if $\eta_{\alpha\beta} \neq 0$. Hence

$$\begin{aligned} \nu_E(\eta') &= \min\{n_1(\alpha + \beta - r) + m_1\beta;\ \eta_{\alpha\beta} \neq 0\} \\ &= \min\{n(\alpha + \beta - r) + (m - n)\beta;\ \eta_{\alpha\beta} \neq 0\} \\ &= \min\{n\alpha + m\beta - nr;\ \eta_{\alpha\beta} \neq 0\} = p - nr. \end{aligned}$$

We have to verify that $p - nr < n_1 m_1$, where $n_1 m_1 = n(m - n) = nm - n^2$. If $r \geq n$, we are done, since by hypothesis we have that $p < nm$. Assume that $r < n$. There are $\tilde{\alpha}, \tilde{\beta}$ with $\eta_{\tilde{\alpha}\tilde{\beta}} \neq 0$ such that $\tilde{\alpha} + \tilde{\beta} = r$. Then

$$\begin{aligned} p - nr &\leq n\tilde{\alpha} + m\tilde{\beta} - nr = n(\tilde{\alpha} + \tilde{\beta}) + (m - n)\tilde{\beta} - nr \\ &= (m - n)\tilde{\beta} < (m - n)n, \end{aligned}$$

since $\tilde{\beta} \leq r < n$.

Assume that $f = 1$ and thus $n < m < 2n$. We have $x = y_1$, $y = x_1 y_1$, $n_1 = m - n < n$, $m_1 = n$ and

$$\pi_1^*(\eta) = y_1^r \eta'', \quad \eta'' = \sum_{\alpha,\beta} x_1^\beta y_1^{\alpha+\beta-r} \eta''_{\alpha\beta}, \quad \eta''_{\alpha\beta} = \zeta_{\alpha\beta}\frac{dx_1}{x_1} + \left(\mu_{\alpha\beta} + \zeta_{\alpha\beta}\right)\frac{dy_1}{y_1}.$$

As before, we have that $\eta''_{\alpha\beta} \neq 0$ if and only if $\eta_{\alpha\beta} \neq 0$. Hence

$$\begin{aligned}\nu_E(\eta'') &= \min\{n_1\beta + m_1(\alpha+\beta-r);\ \eta_{\alpha\beta} \neq 0\}\\ &= \min\{(m-n)\beta + n(\alpha+\beta-r);\ \eta_{\alpha\beta} \neq 0\}\\ &= \min\{m\beta + n\alpha - nr;\ \eta_{\alpha\beta} \neq 0\} = p - nr.\end{aligned}$$

We verify that $p - nr < n_1 m_1$ exactly as before. □

4.2 Resonant Basic Forms

Let $\omega \in \Omega^1_{M_0,P_0}$ be a basic 1-form with $p = \mathrm{rdo}_E(\omega)$. This means that there is $\eta \in \Omega^1_{M_0,P_0}[\log(xy=0)]$ and $a, b \geq 0$ such that $\omega = x^a y^b \eta$

$$\omega = x^a y^b \eta, \quad \eta \in \Omega^1_{M_0,P_0}[\log(xy=0)],$$

where $p = \nu_E(\eta) < nm$. The initial part of ω may be written

$$\mathrm{In}^{p+na+mb}_{n,m;x,y}(\omega) = x^a y^b W, \quad W = \mathrm{In}^p_{n,m;x,y}(\eta).$$

Note that there is exactly one pair $(c, d) \in \mathbb{Z}^2_{\geq 0}$ such that $cn + dm = p$. Then we have that

$$W = x^c y^d \left\{\mu \frac{dx}{x} + \zeta \frac{dy}{y}\right\}.$$

We say that ω is *resonant* if and only if $n\mu + m\zeta = 0$.

We have the following result that follows directly from the computations in the proof of Proposition 4.2.

Corollary 4.3 *Assume that $N \geq 2$. A basic differential 1-form $\omega \in \Omega^1_{M_0,P_0}$ is resonant if and only if $\pi_1^*\omega$ is resonant.*

4.3 Pre-Basic Forms

Let us introduce a slightly more general class of 1-forms that we call *pre-basic forms*. Given a 1-form

$$\omega = \sum_{\alpha,\beta} c_{\alpha\beta} x^\alpha y^\beta \omega_{\alpha\beta}, \quad \omega_{\alpha\beta} = \left\{\mu_{\alpha\beta}\frac{dx}{x} + \zeta_{\alpha\beta}\frac{dy}{y}\right\}, \tag{2}$$

the *cloud of points* $\mathrm{Cl}(\omega; x, y)$ is $\mathrm{Cl}(\omega; x, y) = \{(\alpha, \beta);\ \omega_{\alpha\beta} \neq 0\}$ and the *Newton Polygon* $\mathcal{N}(\omega; x, y)$ is the positive convex hull of $\mathrm{Cl}(\omega; x, y)$ in $\mathbb{R}^2_{\geq 0}$.

Consider a pair (n, m) with $1 \leq n \leq m$ such that n, m have no common factor. There are unique $b, d \in \mathbb{Z}_{\geq 0}$ such that $dn - bm = 1$ with the property that $0 \leq b < n$ and $0 < d \leq m$. We call (b, d) the *co-pair* of (n, m).

Remark 4.4 Suppose that $1 \leq n \leq m$ are without common factor and take b, d such that $dn - bm = 1$. If $0 \leq b < n$, we have that $0 < d \leq m$ and then (b, d) is the co-pair of (n, m). In the same way, if $0 < d \leq m$, we have that $0 \leq b < n$ and then (b, d) is the co-pair of (n, m).

Definition 4.5 Given a pair $1 \leq n \leq m$ without common factor, we define the region $R^{n,m}$ by $R^{n,m} = H_-^{n,m} \cap H_+^{n,m}$, where

$$H_-^{n,m} = \{(\alpha, \beta) \in \mathbb{R}^2;\ (n-b)\alpha + (m-d)\beta \geq 0\},$$
$$H_+^{n,m} = \{(\alpha, \beta) \in \mathbb{R}^2;\ b\alpha + d\beta \geq 0\},$$

and (b, d) is the co-pair of (n, m).

Remark 4.6 If $n = m = 1$, the co-pair of $(1, 1)$ is $(b, d) = (0, 1)$. Then

$$H_-^{1,1} = \{(\alpha, \beta);\ \alpha \geq 0\}, \quad H_+^{1,1} = \{(\alpha, \beta);\ \beta \geq 0\}.$$

Thus, we have that $R^{1,1}$ is the quadrant $R^{1,1} = \mathbb{R}^2_{\geq 0}$.

Remark 4.7 The slopes $-(n-b)/(m-d)$ and $-b/d$ satisfy that

$$-(n-b)/(m-d) < -n/m < -b/d.$$

Indeed, we have $-n/m < -b/d \Leftrightarrow -dn < -mb = -dn + 1$. On the other hand

$$-(n-b)/(m-d) < -n/m \Leftrightarrow m(n-b) > n(m-d) \Leftrightarrow bm < dn = bm + 1.$$

We conclude that $R^{n,m}$ is a positively convex region of $\mathbb{R}^2$ such that $(0, 0)$ is its only vertex and we have that

$$R^{n,m} \cap \{(\alpha, \beta) \in \mathbb{R}^2;\ n\alpha + m\beta = 0\} = \{(0, 0)\}.$$

Given a point $(a, b) \in \mathbb{R}^2_{\geq 0}$, we define $R^{n,m}(a, b)$ by $R^{n,m}(a, b) = R^{n,m} + (a, b)$.

Definition 4.8 We say that $\omega \in \Omega^1_{M_0, P_0}$ is *pre-basic* if and only if there is a point $(a, b) \in \mathrm{Cl}(\omega; x, y)$ such that $\mathrm{Cl}(\omega; x, y) \subset R^{n,m}(a, b)$.

Remark 4.9 Note that ω is pre-basic if and only if $(a, b) \in \mathcal{N}(\omega; x, y)$ and $\mathcal{N}(\omega; x, y) \subset R^{n,m}(a, b)$.

If ω is pre-basic, we have that

$$\mathrm{Cl}(\omega; x, y) \cap \{(\alpha, \beta) \in \mathbb{R}^2;\ n\alpha + m\beta = \nu_E(\omega)\} = \{(a, b)\}.$$

Thus, the initial part W of ω has the form

$$W = x^a y^b \left\{ \mu_{ab} \frac{dx}{x} + \zeta_{ab} \frac{dy}{y} \right\}. \tag{3}$$

As for basic forms, we say that ω is *resonant* if and only if $n\mu_{ab} + m\zeta_{ab} = 0$.

Lemma 4.10 *Assume that $1 \leq n < m$, where n, m are without common factor and let (b, d) be the co-pair of (n, m). Let us put $(n_1, m_1) = (n, m - n)$, if $m \geq 2n$ and $(n_1, m_1) = (m - n, n)$, if $m < 2n$. Then, the co-pair (b_1, d_1) of (n_1, m_1) is given by $(b_1, d_1) = (b, d - b)$, if $m \geq 2$, and by $(b_1, d_1) = (m - n - d + b, n - b)$, if $m < 2n$. Moreover, we have that $\Psi(R^{n,m}) = R^{n_1,m_1}$, where Ψ is the linear automorphism of $\mathbb{R}^2$ given by $\Psi(\alpha, \beta) = (\alpha + \beta, \beta)$, if $m \geq 2n$, and $\Psi(\alpha, \beta) = (\beta, \alpha + \beta)$, if $m < 2n$.*

Proof Let us show the first statement. If $m \geq 2n$, we have that

$$d_1 n_1 - b_1 m_1 = (d - b)n - b(m - n) = 1.$$

Moreover, since $0 \leq b_1 = b < n_1 = n$ we conclude that (b_1, d_1) is the co-pair of (n_1, m_1), in view of Remark 4.4. If $m < 2n$, we have

$$d_1 n_1 - b_1 m_1 = (n - b)(m - n) - (m - n - d + b)n = 1.$$

We know that $0 \leq b < n$, hence $0 < d_1 = n - b \leq m_1 = n$; by Remark 4.4, we deduce that (b_1, d_1) is the co-pair of (n_1, m_1).

Let us show the second statement. Consider $(\alpha, \beta) \in \mathbb{R}^2$ and put $(\alpha_1, \beta_1) = \Psi(\alpha, \beta)$.

Case $m \geq 2n$. In order to prove that $\Psi(R^{n,m}) = R^{n_1,m_1}$ it is enough to see that

$$(\alpha, \beta) \in H_-^{n,m} \Leftrightarrow (\alpha_1, \beta_1) \in H_-^{n_1,m_1} \text{ and } (\alpha, \beta) \in H_+^{n,m} \Leftrightarrow (\alpha_1, \beta_1) \in H_+^{n_1,m_1}.$$

We verify these properties as follows:

$$\begin{aligned}
(\alpha_1, \beta_1) \in H_-^{n_1,m_1} &\Leftrightarrow (n_1 - b_1)\alpha_1 + (m_1 - d_1)\beta_1 \geq 0 \\
&\Leftrightarrow (n - b)(\alpha + \beta) + (m - n - d + b)\beta \geq 0 \\
&\Leftrightarrow (n - b)\alpha + (m - d)\beta \geq 0 \Leftrightarrow (\alpha, \beta) \in H_-^{n,m}. \\
(\alpha_1, \beta_1) \in H_+^{n_1,m_1} &\Leftrightarrow b_1\alpha_1 + d_1\beta_1 \geq 0 \\
&\Leftrightarrow b(\alpha + \beta) + (d - b)\beta \geq 0 \\
&\Leftrightarrow b\alpha + d\beta \geq 0 \Leftrightarrow (\alpha, \beta) \in H_+^{n,m}.
\end{aligned}$$

Case $m < 2n$. In this case, we have that

$$(\alpha, \beta) \in H_+^{n,m} \Leftrightarrow (\alpha_1, \beta_1) \in H_-^{n_1,m_1} \tag{4}$$

$$(\alpha, \beta) \in H_-^{n,m} \Leftrightarrow (\alpha_1, \beta_1) \in H_+^{n_1,m_1}. \tag{5}$$

and this also implies that $\Psi(R^{n,m}) = R^{n_1,m_1}$. We verify the properties in Eqs. (4) and (5) as follows:

$$\begin{aligned}(\alpha_1, \beta_1) \in H_-^{n_1,m_1} &\Leftrightarrow (n_1 - b_1)\alpha_1 + (m_1 - d_1)\beta_1 \geq 0\\ &\Leftrightarrow (m - n - (m - n - d + b))\beta + (n - n + b)(\alpha + \beta) \geq 0\\ &\Leftrightarrow d\beta + b\alpha \geq 0 \Leftrightarrow (\alpha, \beta) \in H_+^{n,m}.\\ (\alpha_1, \beta_1) \in H_+^{n_1,m_1} &\Leftrightarrow b_1\alpha_1 + d_1\beta_1 \geq 0\\ &\Leftrightarrow (m - n - d + b)\beta + (n - b)(\alpha + \beta) \geq 0\\ &\Leftrightarrow (m - d)\beta + (n - b)\alpha \geq 0 \Leftrightarrow (\alpha, \beta) \in H_-^{n,m}.\end{aligned}$$

The proof is ended. □

Proposition 4.11 *Assume that $N \geq 2$. For any $\omega \in \Omega^1_{M_0,P_0}$, we have*

(1) *ω is pre-basic if and only if $\pi_1^*\omega$ is pre-basic.*
(2) *ω is pre-basic and resonant if and only if $\pi_1^*\omega$ is pre-basic and resonant.*

Proof We consider two cases as in the statement of Lemma 4.10, the case $m \geq 2n$ and $m < 2n$ and we define the linear automorphism Ψ accordingly to these cases, as well as the Puiseux pair (n_1, m_1). A monomial by monomial computation shows that

$$\mathrm{Cl}(\pi_1^*\omega; x_1, y_1) = \Psi(\mathrm{Cl}(\omega; x, y)). \tag{6}$$

In view of Lemma 4.10, we have that

$$\Psi(R^{n,m}(a, b)) = R^{n_1,m_1}(\Psi(a, b)). \tag{7}$$

Statement (1) is now a direct consequence of Eqs. (6) and (7). Property (2) is left to the reader. □

Proposition 4.12 *Take a differential 1-form $\omega \in \Omega^1_{M_0,P_0}$. We have*

(1) *If $N = 1$, then ω is pre-basic if and only if it is basic.*
(2) *If ω is basic then it is pre-basic.*
(3) *If ω is basic and resonant then it is pre-basic and resonant.*

Proof If $N = 1$, we have $n = m = 1$ and $R^{1,1}(a, b) = \mathbb{R}^2_{\geq 0} + (a, b)$. Then being basic is the same property of being pre-basic: the Newton Polygon has a single vertex.

Assume now that ω is basic. In view of the stability result in Proposition 4.2, we have that $\tilde{\omega}$ is basic, where $\tilde{\omega}$ is the pull-back of ω in the last center P_{N-1} of the cuspidal sequence. By the previous argument we have that $\tilde{\omega}$ is pre-basic. Now we apply Proposition 4.11 to conclude that ω is pre-basic.

The resonance for pre-basic 1-forms that are basic ones is the same property as for basic 1-forms. □

4.4 Totally *E*-dicritical Forms

Consider a 1-form $\omega \in \Omega^1_M$ defined around the divisor E. Recall that we have a normal crossings divisor H such that $H \supset E$, coming from our choice of adapted coordinates, although if $n \geq 2$ the divisor H around E is intrinsically defined and it coincides with K. We say that ω is *totally E-dicritical with respect to H* if for any point $P \in E$ there are local coordinates u, v such that $E = (u = 0)$, $H \subset (uv = 0)$ and ω has the form

$$\omega = u^a v^b dv,$$

where $b = 0$ when $H = (u = 0)$. Note that ω defines a non-singular foliation around E, this foliation has normal crossings with H and E is transversal to the leaves.

Proposition 4.13 *For any $\omega \in \Omega^1_{M_0, P_0}$, the following properties are equivalent:*

(1) *$\pi^*\omega$ is totally E-dicritical with respect to H.*
(2) *The 1-form ω is pre-basic and resonant.*

Proof In view of the stability of the property "pre-basic and resonant" under the blowing-ups of $\mathcal{S}$ given in Proposition 4.11, it is enough to consider the case when $N = 1$. In this case we have a single blowing-up, $H_0 = (xy = 0)$ and the property for $\pi^*\omega$ of being totally E-dicritical with respect to H is equivalent to say that

$$\omega = h(x, y)x^a y^b \left[\left\{ \frac{dx}{x} - \frac{dy}{y} \right\} + \sum_{\alpha+\beta \geq 1} x^\alpha y^\beta \left\{ \mu_{\alpha\beta} \frac{dx}{x} + \zeta_{\alpha\beta} \frac{dy}{y} \right\} \right], \quad a, b \geq 1,$$

where $h(0, 0) \neq 0$. That is, the 1-form ω is pre-basic and resonant. □

Remark 4.14 If $n \geq 2$ the axes $x'y' = 0$ around P_{N-1} coincide with the germ of $K_{N-1} = \sigma^{-1}_{N-1}(P_0)$ at P_{N-1}. In this situation, the property of being basic and resonant does not depend on the chosen adapted coordinate system.

Definition 4.15 Given a resonant pre-basic 1-form ω, we say that a branch $(\mathcal{C}, 0)$ in (M_0, P_0) is a *ω-cusp* if and only if it is invariant by ω and the strict transform of $(\mathcal{C}, P_0)$ by π cuts E at a free point.

Let us note that each free point of E defines a ω-cusp and conversely, in view of the fact that $\pi^*\omega$ is totally E-dicritical with respect to H.

One of the results in this paper is that any element of $\text{Cusps}(\mathcal{S})$ is a ω-cusp for certain resonant basic ω and hence can be included in the corresponding "dicritical package".

5 Differential Values of a Cusp

Let us consider a branch $(\mathcal{C}, P_0) \subset (M_0, P_0)$ belonging to $\text{Cusps}(E)$. It has a Puiseux expansion of the form

$$(x, y) = \phi(t) = (t^n, \alpha t^m + t^{m+1}\xi(t)), \ \alpha \neq 0.$$

defined by the fact that for any germ $h \in \mathcal{O}_{M_0,P_0}$ we have that $(\mathcal{C}, P_0) \subset (h = 0)$ if and only if $h \circ \phi = 0$. We recall that the intersection multiplicity of $(\mathcal{C}, P_0)$ with a germ h is given by

$$\mathrm{i}_{P_0}(\mathcal{C}, h) = \mathrm{order}_t(h \circ \phi).$$

We also denote $\nu_{\mathcal{C}}(h) = \mathrm{i}_{P_0}(\mathcal{C}, h)$. The *semigroup* Γ of $\mathcal{C}$ is defined by

$$\Gamma \cup \{\infty\} = \{\nu_{\mathcal{C}}(h);\ \ h \in \mathcal{O}_{M_0,P_0}\}.$$

As stated in Zariski's Equisingularity Theory, this semigroup depends only on the equisingularity class (or topological class) of $\mathcal{C}$. In our case, we know that all the elements in $\mathrm{Cusps}(E)$ are equisingular to the cusp $y^n - x^m = 0$. Hence Γ does not depend on the particular choice of $\mathcal{C} \in \mathrm{Cusps}(E)$. More precisely, we know that Γ is the subsemigroup of $\mathbb{Z}_{\geq 0}$ generated by n, m. That is

$$\Gamma = \{an + bm;\ \ a, b \in \mathbb{Z}_{\geq 0}\}.$$

An important feature of Γ is the existence of its conductor $c_\Gamma = (n-1)(m-1)$, which is the smallest element $c_\Gamma \in \Gamma$ such that any non-negative integer greater or equal to c_Γ is contained in Γ. In a more algebraic way, the conductor ideal (t^{c_Γ}) is contained in the image of the morphism

$$\phi^\# : \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}, \quad f \mapsto f \circ \phi.$$

On the other hand, as it was pointed by Zariski, *the differential values of* $\mathcal{C}$ may strongly depend on the analytic class of $\mathcal{C}$. In fact, they are the main discrete invariants in the analytic classification of branches (see Hefez and Hernandes 2011).

Given a differential 1-form $\omega \in \Omega^1_{M_0,P_0}$ with $\omega = gdx + hdy$. If we write $\phi(t) = (x(t), y(t))$, we have that $\phi^*(\omega) = (g(\phi(t))x'(t) + h(\phi(t))y'(t))dt$. We put $a(t) = t(g(\phi(t))x'(t) + h(\phi(t))y'(t))$, hence

$$\phi^*(\omega) = a(t)\frac{dt}{t}$$

and we define the *differential value* $\nu_{\mathcal{C}}(\omega)$ by $\nu_{\mathcal{C}}(\omega) = \mathrm{order}_t(a(t))$.

We know that $(\mathcal{C}, P_0)$ is an *invariant branch of* ω if and only if $\phi^*(\omega) = 0$ and hence $\nu_{\mathcal{C}}(\omega) = \infty$. The *semimodule* $\Lambda^{\mathcal{C}}$ *of the differential values* is defined by

$$\Lambda^{\mathcal{C}} = \{\nu_{\mathcal{C}}(\omega);\quad \omega \in \Omega^1_{M_0,P_0}, \nu_{\mathcal{C}}(\omega) \neq \infty\} \subset \mathbb{Z}_{\geq 0}.$$

It is a Γ-semimodule in the sense that

$$p \in \Gamma, q \in \Lambda^{\mathcal{C}} \Rightarrow p + q \in \Lambda^{\mathcal{C}}.$$

Remark 5.1 Note that $\nu_{\mathcal{C}}(\omega) \geq 1$ for any $\omega \in \Omega^1_{M_0,P_0}$. Anyway, we have the important property that $\Gamma \subset \{0\} \cup \Lambda^{\mathcal{C}}$. If $\Lambda^{\mathcal{C}} \cup \{0\} = \Gamma$, we say that $\mathcal{C}$ is quasi-homogeneous and it is analytically equivalent to the cusp $y^n - x^m = 0$. Otherwise, if λ_1 is the minimum of $\Lambda^{\mathcal{C}} \setminus \Gamma$, we know that $\lambda_1 - n$ is the Zariski invariant, the first nontrivial analytic invariant. This invariant was introduced by Zariski (1966).

Remark 5.2 Let us note that $\nu_E(\omega) \in \Gamma$, for any $\omega \in \Omega^1_{M_0,P_0}$.

5.1 Divisorial Order and Differential Values

In view of the definition of the differential values, for any $\omega \in \Omega^1_{M_0,P_0}$ we have that $\nu_E(\omega) \leq \nu_{\mathcal{C}}(\omega)$. A useful consequence of this fact is the following one:

Lemma 5.3 *A basic 1-form $\omega \in \Omega^1_{M_0,P_0}$ is resonant if and only if $\nu_{\mathcal{C}}(\omega) > \nu_E(\omega)$.*

Proof Write $\omega = W + \tilde{\omega}$, where W is the initial form of ω. Denote $d = \nu_E(\omega) < nm$. Recall that $\nu_E(\tilde{\omega}) > \nu_E(W) = d$, we conclude that $\nu_{\mathcal{C}}(\omega) > d$ if and only if $\nu_{\mathcal{C}}(W) > d$. Since ω is a basic 1-form, we can write

$$W = x^a y^b \left\{ \mu \frac{dx}{x} + \zeta \frac{dy}{y} \right\}, \quad an + bm = d.$$

We have

$$\phi^* W = (t^n)^a (t^m + t^{m+1}\xi(t))^b \left(n\mu + m\zeta + t\psi(t)\right) \frac{dt}{t}.$$

The fact that $\nu_{\mathcal{C}}(W) > d$ is equivalent to say that $n\mu + m\zeta = 0$ and hence it is equivalent to say that ω is resonant. □

Corollary 5.4 *If $\nu_{\mathcal{C}}(\omega) \notin \Gamma$, then ω is a resonant basic 1-form.*

Proof Since $\nu_{\mathcal{C}}(\omega) \notin \Gamma$, this differential value is bounded by the conductor $c_\Gamma = (n-1)(m-1)$, hence we have that

$$\nu_E(\omega) \leq \nu_{\mathcal{C}}(\omega) < (n-1)(m-1) < nm.$$

Then ω is a basic 1-form. Moreover, since $\nu_E(\omega) \in \Gamma$ and $\nu_{\mathcal{C}}(\omega) \notin \Gamma$, we have that $\nu_E(\omega) < \nu_{\mathcal{C}}(\omega)$ and we conclude that ω is a resonant basic 1-form. □

5.2 Reachability Between Resonant Basic Forms

Let ω, ω' be two 1-forms $\omega, \omega' \in \Omega^1_{M_0,P_0}$. We say that ω' is *reachable* from ω if and only if there are nonnegative integer numbers a, b and a constant $\mu \in \mathbb{C}$ such that

$$\nu_E(\omega' - \mu x^a y^b \omega) > \nu_E(\omega').$$

Note that the constant μ and the pair (a, b) are necessarily unique.

We are interested in the case when ω and ω' are basic and resonant. In this situation, the initial parts are respectively given by

$$W = \mu x^c y^d \left\{ m\frac{dx}{dy} - n\frac{dy}{y} \right\}, \quad W' = \mu' x^{c'} y^{d'} \left\{ m\frac{dx}{x} - n\frac{dy}{y} \right\}.$$

Note that $a, a', b, b' \geq 1$ since ω, ω' are holomorphic 1-forms. We have that ω' is reachable from ω if and only if $c' \geq c$ and $d' \geq d$; in this case we have that

$$\nu_E \left(\omega' - \frac{\mu'}{\mu} x^{c'-c} y^{d'-d} \omega \right) > \nu_E(\omega') = c'n + d'm.$$

Note also that the minimum divisorial value of a basic and resonant 1-form is $n + m$ and its initial part is necessarily of the type

$$\mu xy \left\{ m\frac{dx}{x} - n\frac{dy}{y} \right\} = \mu(mydx - nxdy).$$

If ω is basic and resonant with $\nu_E(\omega) = n + m$, then any basic and resonant 1-form is reachable from ω.

6 Cuspidal Semimodules

In this section we develop certain features of semimodules over the semigroup Γ generated by the Puiseux pair (n, m). We consider, unless it is specified, only the singular case $n \geq 2$; in this case the conductor is $c_\Gamma = (n - 1)(m - 1)$ and we have the interesting property that any $p \in \Gamma$ with $p < nm$ is written as $p = an + bm$ in a unique way, with $a, b \geq 0$.

We proceed in a self contained way in order to help the reader, several results are true for more general semigroups, but we focus on the cuspidal semigroup Γ to shorten the arguments.

6.1 The Basis of a Semimodule

A nonempty subset $\Lambda \subset \mathbb{Z}_{\geq 0}$ is a Γ-*semimodule* if $\Lambda + \Gamma \subset \Lambda$. We say that Λ is *normalized* if $0 \in \Lambda$, this is equivalent to say that $\Gamma \subset \Lambda$. As for the case of semigroups, the *conductor* c_Λ is defined by

$$c_\Lambda = \min\{p \in \mathbb{Z}_{\geq 0};\ \{q \in \mathbb{Z}; q \geq p\} \subset \Lambda\}.$$

Note that if λ_{-1} is the minimum of Λ, then we have that $c_\Lambda \leq c_\Gamma + \lambda_{-1}$.

Definition 6.1 Let Λ be a Γ-semimodule. A nonempty finite increasing sequence of nonnegative integer numbers $\mathcal{B} = (\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$ is a *basis for* Λ if for any $0 \leq j \leq s$ we have that $\lambda_j \notin \Gamma(\mathcal{B}_{j-1})$, where $\Gamma(\mathcal{B}_{j-1}) = (\lambda_{-1}+\Gamma)\cup(\lambda_0+\Gamma)\cup\cdots\cup(\lambda_{j-1}+\Gamma)$.

If $\Lambda = \Gamma(\mathcal{B})$, we have a chain of semimodules

$$\lambda_{-1} + \Gamma = \Lambda_{-1} \subset \Lambda_0 \subset \cdots \subset \Lambda_s = \Lambda, \tag{8}$$

where $\Lambda_j = \Gamma(\mathcal{B}_j)$. We call *decomposition sequence of* Λ to this chain of semimodules. Let us note that

$$\lambda_j = \min(\Lambda \setminus \Lambda_{j-1}), \quad 0 \le j \le s. \tag{9}$$

This definitions are justified by next Proposition 6.2

Proposition 6.2 *Given a semimodule Λ, there is a unique basis $\mathcal{B}$ such that $\Lambda = \Gamma(\mathcal{B})$.*

Proof We start with $\lambda_{-1} = \min \Lambda$. Note that $\Gamma(\lambda_{-1}) \subset \Lambda$. If $\Gamma(\lambda_{-1}) = \Lambda$ we stop and we put $s = -1$. If $\Gamma(\lambda_{-1}) \ne \Lambda$, we put $\lambda_0 = \min(\Lambda \setminus \Gamma(\lambda_{-1}))$. Note that $\Gamma(\lambda_{-1}, \lambda_0) \subset \Lambda$. We continue in this way and, since $\lambda_j \not\equiv \lambda_k \mod n$ for $j \ne k$, after finitely many steps we obtain that $\Lambda = \Gamma(\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$. Let us show the uniqueness of $\mathcal{B} = (\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$. Assume that $\Lambda = \Gamma(\mathcal{B}')$, for another Γ-basis $\mathcal{B}' = (\lambda'_{-1}, \lambda'_0, \ldots, \lambda'_{s'})$. Note that $\lambda_{-1} = \min \Lambda = \lambda'_{-1}$. Assume that $\lambda_j = \lambda'_j$ for any $0 \le j \le k-1$. In view of Eq. (9) we have that $\lambda_k = \lambda'_k = \min(\Lambda \backslash \Gamma(\mathcal{B}_{k-1})) = \min\left(\Lambda \backslash \Gamma(\mathcal{B}'_{k-1})\right)$. This ends the proof. □

We say that $\mathcal{B} = (\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$ is *the basis of* $\Lambda = \Gamma(\mathcal{B})$ and that s is the *length* of Λ.

Consider a semimodule $\Lambda = \Gamma(\mathcal{B})$, an element $\lambda \in \mathbb{Z}_{\ge 0}$ is said to be Λ-*independent* if and only if $\lambda \notin \Lambda$ and $\lambda > \lambda_s$, where λ_s is the last element in the basis $\mathcal{B}$. In this case we obtain a basis $\mathcal{B}(\lambda)$, just by adding λ to $\mathcal{B}$ as being the last element. The new semimodule is denoted $\Lambda(\lambda)$, thus we have $\Lambda(\lambda) = \Lambda \cup (\lambda + \Gamma) = \Gamma(\mathcal{B}(\lambda))$.

Given a semimodule $\Lambda = \Gamma(\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$, we define the *axes* $u_i = u_i(\Lambda)$ by

$$u_0 = \lambda_{-1}; \quad u_i = \min\left(\Lambda_{i-2} \cap (\lambda_{i-1} + \Gamma)\right), \ 1 \le i \le s+1. \tag{10}$$

Note that $u_i(\Lambda_j) = u_i(\Lambda)$, for $0 \le i \le j+1 \le s+1$.

Definition 6.3 A semimodule $\Lambda = \Gamma(\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$ is *increasing* if and only if $\lambda_i > u_i$ for any $i = 0, 1, \ldots, s$.

Remark 6.4 If Λ is an increasing semimodule, each element Λ_i of the decomposition sequence is also an increasing semimodule. Moreover, if λ' is a Λ-independent value with $\lambda' > u_{s+1}$, then $\Lambda(\lambda')$ is also an increasing Γ-semimodule.

Given a semimodule $\Lambda = \Gamma(\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$, the semimodule $\widetilde{\Lambda} = \Lambda - \lambda_{-1}$ is called the *normalization of* Λ. Next features allow to deduce properties of Λ from properties of its normalization:

(1) The basis of $\widetilde{\Lambda}$ is $(0, \lambda_0 - \lambda_{-1}, \ldots, \lambda_s - \lambda_{-1})$.
(2) $\widetilde{\Lambda}_i = \Lambda_i - \lambda_{-1}$, for $i = -1, 0, \ldots, s$.
(3) $u_i(\widetilde{\Lambda}) = u_i(\Lambda) - \lambda_{-1}, \quad i = 0, 1, \ldots, s+1$.
(4) $c_{\widetilde{\Lambda}} = c_\Lambda - \lambda_{-1}$.
(5) Λ is increasing if and only if $\widetilde{\Lambda}$ is increasing.

6.2 Axes and Conductor

We precise the expressions of the axes and we bound them by the conductors.

Lemma 6.5 *Consider a semimodule* Λ *of length* s *and two indices* $0 \leq k < i \leq s$. *Then we have that* $u_{i+1} < c_{\Lambda_k} + n$.

Proof We can assume that $i = s$, $k = s - 1$. Note that $\lambda_s < c_{\Lambda_{s-1}}$, since $\lambda_s \notin \Lambda_{s-1}$. Then, there is a unique $\alpha \in \mathbb{Z}_{>0}$ such that $0 \leq \lambda_s - c_{\Lambda_{s-1}} + \alpha n < n$. We have that $\lambda_s + \alpha n \in \lambda_s + \Gamma$ and $\lambda_s + \alpha n \geq c_{\Lambda_{s-1}}$. We obtain

$$\lambda_s + \alpha n \in \Lambda_{s-1} \cap (\lambda_s + \Gamma).$$

We deduce that $u_{s+1} \leq \lambda_s + \alpha n < c_{\Lambda_{s-1}} + n$. □

Corollary 6.6 *Consider a semimodule* Λ *of the form* $\Lambda = \Gamma(n, m, \lambda_1, \ldots, \lambda_s)$. *Then* $u_{i+1} < nm$, *for any* $0 \leq i \leq s$.

Proof By Lemma 6.5, we have that $u_{i+1} \leq c_{\Lambda_0} + n$, but in this situation, we have that $\Lambda_0 \cup \{0\} = \Gamma$ and thus

$$c_{\Lambda_0} = c_\Gamma = (n-1)(m-1).$$

Hence $u_{i+1} \leq c_{\Lambda_0} + n = (n-1)(m-1) + n < nm$. □

Lemma 6.7 *Consider* $\Lambda = \Gamma(\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$. *There is a unique index* k *with* $-1 \leq k \leq s - 1$ *such that* $u_{s+1} \in \lambda_k + \Gamma$ *and there are unique expressions*

$$u_{s+1} = \lambda_s + na + mb, \quad a, b \in \mathbb{Z}_{\geq 0} \tag{11}$$

$$u_{s+1} = \lambda_k + nc + md, \quad c, d \in \mathbb{Z}_{\geq 0}. \tag{12}$$

In these expressions, we have $ac = bd = ab = cd = 0$ *and* $(a, b) \neq (0, 0) \neq (c, d)$.

Proof The existence of the expressions (11) and (12) is given by the definition of u_{s+1} as $u_{s+1} = \min(\Lambda_{s-1} \cap (\lambda_s + \Gamma))$. By the minimality of u_{s+1} and the fact that $u_{s+1} \neq \lambda_s$ and $u_{s+1} \neq \lambda_k$, we deduce the properties $ac = bd = 0$ and $(a, b) \neq (0, 0) \neq (c, d)$. Moreover, if $ab \neq 0$ we should have that $c = d = 0$ which is not possible; in the same way we see that $cd = 0$.

Let us show the uniqueness of the index k. Assume that there are two indices $-1 \leq k < k' \leq s - 1$ with $u_{s+1} = \lambda_k + cn + dm = \lambda_{k'} + c'n + d'm$. Take the case when $a \neq 0$, then we have that $c = c' = 0$ and we can write

$$\lambda_{k'} = \lambda_k + m(d - d') \in \lambda_k + \Gamma,$$

this is a contradiction. Same argument if $b \neq 0$.

If we normalize Λ, we have that $u_{s+1} - \lambda_s = \tilde{u}_{s+1} - \tilde{\lambda}_s$. By Lemma 6.5 we have

$$\tilde{u}_{s+1} - \tilde{\lambda}_s \leq \tilde{u}_{s+1} < c_{\tilde{\Lambda}_{s-1}} + n \leq c_\Gamma + n = (n-1)(m-1) + n < nm.$$

Hence, we have that $u_{s+1} - \lambda_s$ (and with the same argument $u_{s+1} - \lambda_k$) are strictly smaller than nm. Thus, the expression of these elements of Γ as a linear combination of n, m with non-negative coefficients is unique. □

6.3 The Limits

Consider a semimodule $\Lambda = \Gamma(\lambda_{-1}, \lambda_0, \ldots, \lambda_s)$ with $s \geq 0$. The *first and second limits* ℓ_1 *and* ℓ_2 *of* Λ are defined by

$$\ell_1 = \min\{p;\ np + \lambda_s \in \Lambda_{s-1}\}. \tag{13}$$

$$\ell_2 = \min\{q;\ mq + \lambda_s \in \Lambda_{s-1}\}. \tag{14}$$

Remark 6.8 We have that $\ell_1\ell_2 \neq 0$ and

$$u_{s+1} = \min\{\ell_1 n + \lambda_s,\ \ell_2 m + \lambda_s\}. \tag{15}$$

Indeed, by Lemma 6.7, we have either $u_{s+1} = an + \lambda_s$ or $u_{s+1} = bm + \lambda_s$; if $u_{s+1} = an+\lambda_s$, by minimality we have that $a = \ell_1$, in the same way, if $u_{s+1} = bm+\lambda_s$ we have that $b = \ell_2$. Moreover, there is a unique index k with $-1 \leq k < s$ such that

(1) If $u_{s+1} = \ell_1 n + \lambda_s$, then $u_{s+1} = \lambda_k + bm$.
(2) If $u_{s+1} = \ell_2 m + \lambda_s$, then $u_{s+1} = \lambda_k + an$.

Lemma 6.9 *If* $an + bm + \lambda_s \in \Lambda_{s-1}$*, then either* $a \geq \ell_1$ *or* $b \geq \ell_2$.

Proof Let us write $an + bm + \lambda_s = cn + dm + \lambda_j$ for a certain $j \leq s - 1$. If $ac \neq 0$, we find

$$(a-1)n + bm + \lambda_s = (c-1)n + dm + \lambda_j \in \Lambda_{s-1}.$$

Repeating the argument and working in a similar way with the coefficients b, d, we find an element

$$\tilde{a}n + \tilde{b}m + \lambda_s = \tilde{c}n + \tilde{d}m + \lambda_j$$

such that $a \geq \tilde{a}$ and $b \geq \tilde{b}$, with the property that $\tilde{a}\tilde{c} = 0$ and $\tilde{b}\tilde{d} = 0$. Moreover, we have that $(\tilde{c}, \tilde{d}) \neq (0, 0)$, since otherwise $\lambda_s \leq \lambda_{s-1}$. Suppose that $\tilde{c} \neq 0$, then $\tilde{a} = 0$ and

$$\tilde{b}m + \lambda_s = \tilde{c}n + \tilde{d}m + \lambda_j \in \Lambda_{s-1}.$$

By the minimality property of ℓ_2, we have that $\tilde{b} \geq \ell_2$ and then $b \geq \ell_2$. In a similar way, we show that if $\tilde{d} \neq 0$ we have that $a \geq \ell_1$. □

Let us note that the limits of the normalization $\tilde{\Lambda}$ are the same ones as for Λ.

Example 6.10 Consider the semigroup $\Gamma = \langle 5, 11\rangle$ and the Γ-semimodule $\Lambda = \Gamma(5, 11, 17, 23, 29)$. Let us compute the axes and the limits for this semimodule. Note that $s = 3$. We have $u_0 = \lambda_{-1} = n = 5$, and $\lambda_0 = m = 11$. In order to compute the limits of Λ_0 we have to find the minimal non-negative integers

- ℓ_1^1 such that $11 + 5\ell_1^1 = 5 + 11b$, hence $\ell_1^1 = b = 1$;
- ℓ_2^1 such that $11 + 11\ell_2^1 = 5 + 5a$ and we obtain $\ell_2^1 = 4$ and $a = 10$.

Hence, $\lambda_0 + n\ell_1^1 = 16$ and $\lambda_0 + m\ell_2^1 = 55$ and $u_1 = \min\{16, 55\} = 16$. Now, let us compute the limits of Λ_1 where $\lambda_1 = 17$. We search ℓ_1^2 and ℓ_2^2 minimal such that

- $17 + 5\ell_1^2 = 11 + 11b$, hence $\ell_1^2 = b = 1$ and $\lambda_1 + n\ell_1^2 = 22$;
- $17 + 11\ell_2^2 = 5 + 5a$ and we have that $\ell_2^2 = 3$ and $a = 9$, then $\lambda_1 + m\ell_2^2 = 50$.

We get that $u_2 = 22$. In a similar way, taking into account that $\lambda_2 = 23$ and $\lambda_3 = 29$, we get that $u_3 = 28$ and $u_4 = 34$. Since $u_i < \lambda_i$, for $i = 0, 1, 2, 3$, we obtain that the semimodule Λ is increasing.

7 Standard Bases

From now on, we fix a cusp $\mathcal{C}$ in $\mathrm{Cusps}(E)$ and we consider the semimodule Λ of differential values of $\mathcal{C}$:

$$\Lambda = \Lambda^{\mathcal{C}} = \{\nu_{\mathcal{C}}(\omega);\ \omega \in \Omega^1_{M_0,P_0}\} \setminus \{\infty\}. \tag{16}$$

We recall that $\Gamma \setminus \{0\} \subset \Lambda$.

Lemma 7.1 *If $(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$ is the basis of $\Lambda^{\mathcal{C}}$, then $\lambda_{-1} = n$ and $\lambda_0 = m$.*

Proof Let $(x, y) = (t^n, t^m\xi(t))$ be a Puiseux parametrization of $\mathcal{C}$, where $\xi(0) \neq 0$. Recall that $\nu_{\mathcal{C}}(adx + bdy)$ is the order in t of the expression

$$nt^n a(t^n, t^m\xi(t)) + t^m\xi(t)b(t^n, t^m\xi(t))\{m + t\xi'(t)/\xi(t)\}. \tag{17}$$

We see that this order is $\geq n$ and that $\nu_{\mathcal{C}}(dx) = n$. Hence $n = \lambda_{-1}$. Moreover, the terms in Eq. (17) of degree $< m$ come only from the first part $nt^n a(t^n, t^m\xi(t))$ of the sum, so, they are values in Γ. Since $m = \nu_{\mathcal{C}}(dy)$, we conclude that $\lambda_0 = m$. □

Definition 7.2 Write $\Lambda^{\mathcal{C}} = \Gamma(n, m, \lambda_1, \ldots, \lambda_s)$. A *standard basis for $\mathcal{C}$* is a list of 1-forms $\mathcal{G} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ such that $\nu_{\mathcal{C}}(\omega_i) = \lambda_i$, for $i = -1, 0, 1, \ldots, s$.

Remark 7.3 There is at least one standard basis, by definition of the semimodule of differential values. The standard bases are not in general unique. For instance, we have that

$$\nu_{\mathcal{C}}(hdx) = n, \quad \nu_{\mathcal{C}}(hdy) = m, \quad h(0) \neq 0.$$

On the other hand, for $i = 1, 2, \ldots, s$, we have that $\nu_{\mathcal{C}}(\omega_i) = \lambda_i \notin \Gamma$, then, in view of Corollary 5.4, the 1-form ω_i is basic resonant.

Remark 7.4 Let $\mathcal{G} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s)$ be a standard basis for $\mathcal{C}$. Then ω_{-1}, ω_0 have the form

$$\omega_{-1} = hdx + gdy,\ h(0) \neq 0; \quad \omega_0 = fdx + \psi dy,\ \psi(0) \neq 0,\ \nu_{\mathcal{C}}(fdx) > m.$$

Thus, we can write any differential 1-form ω in a unique way as $\omega = a\omega_{-1} + b\omega_0$. Anyway, we are mainly interested in the study of the 1-forms ω_i, for $1 \leq i \leq s$. From the standard basis $\mathcal{G}$ we can obtain a new one "adapted to the coordinates" given by

$$\mathcal{G} = (dx, dy, \omega_1, \ldots, \omega_s).$$

Just for simplifying the presentation of the computations, we will consider only this kind of standard bases.

7.1 The Zariski Invariant

In this subsection, we deal with properties of divisorial orders and differential values around the element λ_1, where

$$\Lambda^{\mathcal{C}} = \Gamma(n, m, \lambda_1, \ldots, \lambda_s).$$

This is the first step for a general result. Anyway, let us recall that $\lambda_1 - n$ is the classical Zariski invariant. Let us cite the work of Gómez-Martínez (2021) that essentially contains several of the results in this section.

Proposition 7.5 *We have the following properties:*

(1) *If* $s = 0$*, then* $\infty = \sup\{\nu_{\mathcal{C}}(\omega);\ \omega \in \Omega^1_{M_0,P_0},\ \nu_E(\omega) = n + m\}$.

(2) *If* $s \geq 1$*, then* $\lambda_1 = \sup\{\nu_{\mathcal{C}}(\omega);\ \omega \in \Omega^1_{M_0,P_0},\ \nu_E(\omega) = n + m\}$.

Proof Assume that $s = 0$ and hence $\Lambda^{\mathcal{C}} = \Gamma \setminus \{0\}$. Let us consider the 1-form $\eta = mydx - nxdy$. We have that $\nu_{\mathcal{C}}(\eta) > n + m = \nu_E(\eta)$. Moreover, since $s = 0$ we have that $\nu_{\mathcal{C}}(\eta) \in \Gamma$; then there is a monomial function f such that

$$\nu_E(df) = \nu_{\mathcal{C}}(df) = \nu_{\mathcal{C}}(\eta) > n + m.$$

In particular, there is a constant $\mu \neq 0$ such that $\nu_{\mathcal{C}}(\eta - \mu df) > \nu_{\mathcal{C}}(\eta)$. Write $\eta^1 = \eta - \mu df$; we have that $\nu_E(\eta^1) = \nu_E(\eta) = n + m$ and $\nu_{\mathcal{C}}(\eta^1) > \nu_{\mathcal{C}}(\eta)$. We repeat the argument with η^1 and in this way we obtain 1-forms η^k with $\nu_E(\eta^k) = n+m$ and $\nu_{\mathcal{C}}(\eta^k) \geq n + m + 1 + k$. This proves the first statement.

Assume now that $s \geq 1$. Let us first show that

$$\lambda_1 \leq \sup\{\nu_{\mathcal{C}}(\omega);\ \omega \in \Omega^1_{M_0,P_0},\ \nu_E(\omega) = n + m\}.$$

If $\nu_{\mathcal{C}}(\eta) \notin \Gamma$, we have that $\nu_{\mathcal{C}}(\eta) \geq \lambda_1$ since λ_1 is the minimum of the differential values not in Γ, then we are done. Assume that $\nu_{\mathcal{C}}(\eta) \in \Gamma$ and hence

$$\nu_{\mathcal{C}}(\eta) = an + bm > n + m.$$

Taking the function $f=x^ay^b$, up to multiply df by a constant c_1 we obtain that

$$\nu_{\mathcal{C}}(\eta_1)>\nu_{\mathcal{C}}(\eta)=an+bm,\quad \eta_1=\eta-c_1df.$$

Note that $\nu_E(\eta_1)=n+m$, since $\nu_E(df)=an+bm>n+m$. We restart with η_1 instead of η, noting that $\nu_{\mathcal{C}}(\eta)<\nu_{\mathcal{C}}(\eta_1)$. Repeating finitely many times this procedure, we obtain a new 1-form $\tilde{\eta}=\eta-d\tilde{f}$ such that $\nu_E(\tilde{\eta})=n+m$ and either $\nu_{\mathcal{C}}(\tilde{\eta})\geq c_\Gamma=(n-1)(m-1)$ or $\nu_{\mathcal{C}}(\tilde{\eta})\notin\Gamma$, in both cases we have that $\nu_{\mathcal{C}}(\tilde{\eta})\geq\lambda_1$ and we are done.

It remains to show that $\lambda_1\geq\sup\{\nu_{\mathcal{C}}(\omega);\ \omega\in\Omega^1_{M_0,P_0},\ \nu_E(\omega)=n+m\}$. Let us consider ω_1 such that $\nu_{\mathcal{C}}(\omega_1)=\lambda_1$ and let us show that it is not possible to have $\tilde{\omega}$ such that $\nu_E(\tilde{\omega})=n+m$ and $\nu_{\mathcal{C}}(\tilde{\omega})>\nu_{\mathcal{C}}(\omega_1)$. In this situation, both ω_1 and $\tilde{\omega}$ are basic resonant. We know that ω_1 is reachable from $\tilde{\omega}$ and thus there is a constant μ and $a,b\geq 0$ such that

$$\nu_E(\omega_1^1)>\nu_E(\omega_1),\quad \omega_1^1=\omega_1-\mu x^ay^b\tilde{\omega}.$$

We have that $\nu_{\mathcal{C}}(\omega_1^1)=\nu_{\mathcal{C}}(\omega_1)=\lambda_1$. We restart with the pair $\omega_1^1,\tilde{\omega}$; in this way, we obtain an infinite sequence of 1-forms $\omega_1,\omega_1^1,\omega_1^2,\dots$ with strictly increasing divisorial orders. Up to a finite number of steps, we find an index k such that $\nu_E(\omega_1^k)>\lambda_1=\nu_{\mathcal{C}}(\omega_1^k)$. This contradicts with the fact $\nu_{\mathcal{C}}(\omega_1^k)\geq\nu_E(\omega_1^k)$. □

Corollary 7.6 *Any 1-form $w\in\Omega^1_{M_0,P_0}$ such that $\nu_E(\omega)=n+m$ and $\nu_{\mathcal{C}}(\omega)\notin\Gamma$ satisfies that $\nu_{\mathcal{C}}(\omega)=\lambda_1$.*

Proof In view of the previous result, we have that $\nu_{\mathcal{C}}(\omega)\leq\lambda_1$. Since $\nu_{\mathcal{C}}(\omega)\notin\Gamma$, we also have that $\nu_{\mathcal{C}}(\omega)\geq\lambda_1$. □

Corollary 7.7 *Any 1-form $w\in\Omega^1_{M_0,P_0}$ such that $\nu_{\mathcal{C}}(\omega)=\lambda_1$ satisfies that $\nu_E(\omega)=n+m$.*

Proof Take ω_1 such that $\nu_{\mathcal{C}}(\omega_1)=\lambda_1$ and $\nu_E(\omega_1)=n+m$. Assume that

$$\nu_E(\omega)>n+m$$

in order to obtain a contradiction. Since $\lambda_1\notin\Gamma$, both ω and ω_1 are basic resonant and ω is reachable from ω_1. Then there is a function f with $\nu_{\mathcal{C}}(f)>0$ such that

$$\nu_E(\omega-f\omega_1)>\nu_E(\omega).$$

Put $\omega^1=\omega-f\omega_1$, since $\nu_{\mathcal{C}}(f\omega_1)>\lambda_1$, we have that $\nu_{\mathcal{C}}(\omega^1)=\lambda_1$. We restart with the pair ω^1,ω. After finitely many repetitions we find ω^k with $\nu_{\mathcal{C}}(\omega^k)=\lambda_1$ and $\nu_E(\omega^k)>\lambda_1$, contradiction. □

The following two lemmas are necessary steps in order to prove an inductive version of Proposition 7.5 valid for all indices $i=1,2,\dots,s$:

Lemma 7.8 *Assume that $s \geq 1$ and take ω_1 such that $\nu_{\mathcal{C}}(\omega_1) = \lambda_1$. Consider an integer number $k = na + mb + \lambda_1 \in \lambda_1 + \Gamma$. The following statements are equivalent:*

(1) $k \notin \Gamma$.
(2) $\nu_E(\omega) \leq \nu_E(x^a y^b \omega_1)$, *for any* $\omega \in \Omega^1_{M_0, P_0}$ *such that* $\nu_{\mathcal{C}}(\omega) = k$.

Proof Note that $k = \nu_{\mathcal{C}}(x^a y^b \omega_1) > (a+1)n + (b+1)m = \nu_E(x^a y^b \omega_1)$.

Assume that $k \in \Gamma$, then $k = na' + mb' > \nu_E(x^a y^b \omega_1)$. Taking $\omega = d(x^{a'} y^{b'})$, we have $\nu_{\mathcal{C}}(\omega) = \nu_E(\omega) = k > \nu_E(x^a y^b \omega_1)$.

Now assume that $k \notin \Gamma$. Let us reason by contradiction assuming that there is ω with $\nu_{\mathcal{C}}(\omega) = k$ with $\nu_E(\omega) > \nu_E(x^a y^b \omega_1)$. We have that ω is basic resonant, since $\nu_{\mathcal{C}}(\omega) \notin \Gamma$. Then ω is reachable from ω_1. Then there is $a', b' \geq 0$ and a constant μ such that $\nu_E(x^{a'} y^{b'} \omega_1) = \nu_E(\omega)$ and

$$\nu_E(\omega - c x^{a'} y^{b'} \omega_1) > \nu_E(\omega) > \nu_E(x^a y^b \omega_1).$$

Since $na' + mb' > na + mb$, we have that $\nu_{\mathcal{C}}(x^{a'} y^{b'} \omega_1) > k$ and hence $\nu_{\mathcal{C}}(\omega^1) = k$, where $\omega^1 = \omega - c x^{a'} y^{b'} \omega_1$. Repeating the procedure with the pair ω^1, ω_1, we obtain a sequence

$$\omega, \omega^1, \omega^2, \ldots$$

with strictly increasing divisorial order and such that $\nu_{\mathcal{C}}(\omega^j) = k$ for any j. This is a contradiction. □

Lemma 7.9 *Take ω_1 with $\nu_{\mathcal{C}}(\omega_1) = \lambda_1$. Let $\omega \in \Omega^1_{M_0, P_0}$ be a 1-form such that $\nu_{\mathcal{C}}(\omega) = \lambda \notin \Gamma$. There are unique $a, b \geq 0$ such that $\nu_E(\omega) = \nu_E(x^a y^b \omega_1)$. Moreover, we have that $\lambda \geq na + mb + \lambda_1$.*

Proof Note that ω is basic resonant and thus the existence and uniqueness of a, b is assured. Moreover, if $\lambda < na + mb + \lambda_1$, we can find a constant μ such that

$$\nu_E(\omega - \mu x^a y^b \omega_1) > \nu_E(x^a y^b \omega_1)$$

and $\nu_{\mathcal{C}}(\omega - \mu x^a y^b \omega_1) = \lambda$. Put $\omega^1 = \omega - \mu x^a y^b \omega_1$, we have that $\nu_{\mathcal{C}}(\omega^1) = \lambda \notin \Gamma$. As before, we have that

$$\nu_E(\omega^1) = \nu_E(x^{a_1} y^{b_1} \omega_1), \quad a_1 n + b_1 m > an + bm$$

and thus $\lambda < a_1 n + b_1 m + \lambda_1$. We repeat the process with the pair ω^1, ω_1, where $\omega^1 = \omega - \mu x^a y^b \omega_1$ in order to have a sequence $\omega, \omega^1, \omega^2, \ldots$ with strictly increasing divisorial orders and such that $\nu_{\mathcal{C}}(\omega^j) = \lambda$ for any j. This is a contradiction. □

7.2 Critical Divisorial Orders

Recall that we are considering a cusp $\mathcal{C}$ in $\mathrm{Cusps}(E)$, whose semimodule of differential values is

$$\Lambda^{\mathcal{C}} = \Gamma(n, m, \lambda_1, \ldots, \lambda_s).$$

The *critical divisorial orders* t_i, for $i = -1, 0, \ldots, s+1$ are defined as follows:

- We put $t_{-1} = n$ and $t_0 = m$.
- For $1 \leq i \leq s+1$, we put $t_i = t_{i-1} + u_i - \lambda_{i-1}$.

Let us note that $t_1 = m + (n+m) - m = n + m$.

Lemma 7.10 *Consider the semimodule* $\Lambda = \Gamma(n, m, \lambda_1, \ldots, \lambda_s)$ *and take an index* $1 \leq i \leq s$. *If* $\lambda_\ell > u_\ell$, *for any* $0 \leq \ell \leq i$, *we have that*

$$\lambda_j - \lambda_k > t_j - t_k, \quad -1 \leq k < j \leq i. \tag{18}$$

Proof We have that $\lambda_j - \lambda_{j-1} > t_j - t_{j-1}$ if and only if

$$t_j = t_{j-1} + u_j - \lambda_{j-1} > t_j + u_j - \lambda_j,$$

which is true, since $u_j - \lambda_j < 0$. Noting that

$$\lambda_j - \lambda_k = \sum_{\ell=k}^{j-1} (\lambda_{\ell+1} - \lambda_\ell) > \sum_{\ell=k}^{j-1} (t_{\ell+1} - t_\ell) = t_j - t_k,$$

The proof is ended. □

Lemma 7.11 *If the semimodule* $\Lambda = \Gamma(n, m, \lambda_1, \ldots, \lambda_s)$ *is increasing, we have that* $t_i < nm$, *for any* $i = -1, 0, 1, \ldots, s+1$.

Proof If $i \in \{-1, 0\}$ we have that $t_{-1} = n$, $t_0 = m$ and we are done. Assume that $1 \leq i \leq s$, we have that

$$t_i - n = t_i - t_{-1} = \sum_{\ell=0}^{i} (t_\ell - t_{\ell-1}) \leq \sum_{\ell=0}^{i} (\lambda_\ell - \lambda_{\ell-1}) = \lambda_i - \lambda_{-1} = \lambda_i - n.$$

Then $t_i \leq \lambda_i < c_\Gamma < nm$. Consider the case $i = s+1$. We have that

$$t_{s+1} = t_s + u_{s+1} - \lambda_s = u_{s+1} + (t_s - \lambda_s) \leq u_{s+1} < c_\Gamma + n < nm.$$

See Lemma 6.5. □

Remark 7.12 As a consequence of Lemma 7.11 we have that any 1-form ω such that $\nu_E(\omega) = t_i$ is a basic 1-form; moreover, if $t_i = \nu_E(\omega) < \nu_{\mathcal{C}}(\omega)$, then it is basic and resonant.

The critical divisorial orders are the divisorial orders of the elements of a standard basis, in view of the following

Theorem 7.13 *For each* $1 \leq i \leq s$ *we have the following statements*

(1) $\lambda_i = \sup\{\nu_{\mathcal{C}}(\omega) : \nu_E(\omega) = t_i\}$.

(2) *If* $\nu_{\mathcal{C}}(\omega) = \lambda_i$, *then* $\nu_E(\omega) = t_i$.
(3) *For each 1-form* ω *with* $\nu_{\mathcal{C}}(\omega) \notin \Lambda_{i-1}$, *there is a unique pair* $a, b \geq 0$ *such that* $\nu_E(\omega) = \nu_E(x^a y^b \omega_i)$. *Moreover, we have that* $\nu_{\mathcal{C}}(\omega) \geq \lambda_i + na + mb$.
(4) *We have that* $\lambda_i > u_i$.
(5) *Let* $k = \lambda_i + na + mb$, *then* $k \notin \Lambda_{i-1}$ *if and only if for all* ω *such that* $\nu_{\mathcal{C}}(\omega) = k$ *we have that* $\nu_E(\omega) \leq \nu_E(x^a y^b \omega_i)$.

In particular, the semimodules Λ_i *are increasing, for* $i = 1, 2, \ldots, s$.

A proof of this Theorem 7.13 is given in Appendix B.

Remark 7.14 Note that if $\mathcal{B} = (\omega_{-1} = dx, \omega_0 = dy, \omega_1, \omega_2, \ldots, \omega_s)$ is a standard basis, Theorem 7.13 says that $\nu_E(\omega_i) = t_i$ for any $i = -1, 0, 1, \ldots, s$ and that ω_{j+1} is reachable from ω_j, for any $1 \leq j \leq s-1$. That is, the initial parts of the 1-forms ω_i are given by

$$W_i = \mu_i x^{a_i} y^{b_i} \left\{ m\frac{dx}{x} - n\frac{dy}{y} \right\}, \quad \mu_i \neq 0,$$

where $(a_1, b_1) = (1, 1)$ and

$$1 = a_1 \leq a_2 \leq \cdots \leq a_s, \quad 1 = b_1 \leq b_2 \leq \cdots \leq b_s.$$

Moreover, we have that $na_i + mb_i = t_i$, for $i = 1, 2, \ldots, s$.

Remark 7.15 Note that $\mathcal{B} = (\omega_{-1} = dx, \omega_0 = dy, \omega_1, \omega_2, \ldots, \omega_s)$ is a standard basis if and only if $\nu_E(\omega_i) = t_i$ and $\nu_{\mathcal{C}}(\omega_i) \notin \Lambda_{i-1}$, for any $i = 1, 2, \ldots, s$. Moreover, Theorem 7.13 justifies an algorithm of construction of a standard basis as follows:

> *Assume we have obtained* ω_j, *for* $j = -1, 0, 1, \ldots, s'$. *We can produce the axis* $u_{s'+1}$ *and the critical divisorial order* $t_{s'+1} = t_{s'} + u_{s'+1} - \lambda_{s'}$. *There is an expression* $t_{s'+1} = an + bm$. *We consider the 1-form*
>
> $$\omega^0_{s'+1} = x^a y^b \left\{ m\frac{dx}{x} - n\frac{dy}{y} \right\}.$$
>
> *If* $\nu_{\mathcal{C}}(\omega^0_{s'+1}) \notin \Lambda_{s'}$, *we know that* $\lambda_{s'+1} = \nu_{\mathcal{C}}(\omega^0_{s'+1})$ *and* $s \geq s'+1$. *If* $\nu_{\mathcal{C}}(\omega^0_{s'+1}) \in \Lambda_{s'}$, *there is* $j \leq s'$ and $c, d \geq 0$ *such that*
>
> $$\nu_{\mathcal{C}}(x^c y^d \omega_j) = \nu_{\mathcal{C}}(\omega^0_{s'+1}).$$
>
> *We take a constant* μ *such that* $\nu_{\mathcal{C}}(\omega^0_{s'+1} - \mu x^c y^d \omega_j) > \nu_{\mathcal{C}}(\omega^0_{s'+1})$. *Put* $\omega^1_{s'+1} = \omega^0_{s'+1} - \mu x^c y^d \omega_j$. *We have that* $\nu_E(\omega^1_{s'+1}) = t_{s'+1}$. *We repeat the procedure with* $\omega^1_{s'+1}$. *After finitely many steps we get that either* $\nu_{\mathcal{C}}(\omega^k_{s'+1}) \notin \Lambda_{s'}$ *or* $\nu_{\mathcal{C}}(\omega^k_{s'+1}) \geq c_\Gamma$. *In the first case, we put* $\lambda_{s'+1} = \nu_{\mathcal{C}}(\omega^k_{s'+1})$, *in the second case we know that* $s = s'$.

Example 7.16 Consider the semigroup $\Gamma = \langle 5, 11\rangle$ and the Γ-semimodule $\Lambda = \Gamma(5, 11, 17, 23, 29)$ as in Example 6.10. The computation of the critical divisorial orders t_i, $i = -1, 0, 1, 2, 3$, gives

$$t_{-1} = 5, \quad t_0 = 11, \quad t_1 = 16, \quad t_2 = 21, \quad t_3 = 26, \quad t_4 = 31.$$

Since the semimodule Λ is increasing, by Alberich-Carramiñana et al. (2022) there exists a curve whose semimodule is Λ. In particular, the curve $\mathcal{C}$ given by the Puiseux parametrization $\phi(t) = (t^5, t^{11} + t^{12} + t^{13})$ has semigroup Γ and semimodule of differential values Λ. An extended standard basis is given by $\{\omega_{-1}, \omega_0, \omega_1, \omega_2, \omega_3, \omega_4\}$ where

$$\omega_{-1} = dx, \quad \omega_0 = dy, \quad \omega_1 = 5xdy - 11ydx,$$
$$\omega_2 = 11x\omega_1 - 5ydy, \quad \omega_3 = x\omega_2 + y\omega_1$$

and $\omega_4 = x\omega_3 - 33y\omega_2 - 1199x^6dx - 2035x^4ydx - 407x^4\omega_1 - 1595x^3\omega_2 + \cdots$. The reader can check that $\nu_E(\omega_i) = t_i$ as stated in Theorem 7.13.

8 Extended Standard Basis and Analytic Semiroots

As in previous sections, we consider a cusp $\mathcal{C}$ in Cusps(E), whose semimodule of differential values is

$$\Lambda = \Gamma(n, m, \lambda_1, \ldots, \lambda_s).$$

Let us recall that the axis $u_{s+1} = \min(\Lambda_{s-1} \cap (\lambda_s + \Gamma))$ is well defined and we have also a well defined critical divisorial order

$$t_{s+1} = t_s + u_{s+1} - \lambda_s.$$

Let us also remark that $t_\ell < nm$, for $0 \leq \ell \leq s+1$, in view of Lemma 7.11.

Definition 8.1 We say that a differential 1-form ω is *dicritically adjusted* to $\mathcal{C}$ if and only if $\nu_E(\omega) = t_{s+1}$ and $\nu_{\mathcal{C}}(\omega) = \infty$. An *extended standard basis* for $\mathcal{C}$ is a list

$$\omega_{-1} = dx, \omega_0 = dy, \omega_1, \ldots, \omega_s;\ \omega_{s+1}$$

where $\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s$ is a standard basis and ω_{s+1} is dicritically adjusted to $\mathcal{C}$.

Lemma 8.2 *Assume that $\nu_E(\eta) = t_{s+1}$ and $\nu_{\mathcal{C}}(\eta) > u_{s+1}$. Then, there is a 1-form $\tilde{\eta}$ such that $\nu_E(\tilde{\eta}) = t_{s+1}$ and $\nu_{\mathcal{C}}(\tilde{\eta}) > \nu_{\mathcal{C}}(\eta)$.*

Proof Take a standard basis $dx, dy, \omega_1, \ldots, \omega_s$ for $\mathcal{C}$. There is an index k such that $\nu_{\mathcal{C}}(\eta) = an + bm + \lambda_k$. Consider the 1-form $x^a y^b \omega_k$. Note that $\nu_{\mathcal{C}}(x^a y^b \omega_k) = \nu_{\mathcal{C}}(\eta)$.

If we show that $an + bm + t_k > t_{s+1}$, we are done, by taking $\tilde{\eta} = \eta - \mu x^a y^b \omega_k$ for a convenient constant μ. We have

$$an + bm + \lambda_k > u_{s+1} \Rightarrow an + bm + t_k > u_{s+1} - \lambda_k + t_k. \tag{19}$$

It remains to show that $u_{s+1} - \lambda_k + t_k \geq t_{s+1}$. We have

$$u_{s+1} - \lambda_k + t_k \geq t_{s+1} \Leftrightarrow u_{s+1} - \lambda_k + t_k \geq t_s + u_{s+1} - \lambda_s \Leftrightarrow \lambda_s - \lambda_k \geq t_s - t_k.$$

We are done by Lemma 7.10. □

Proposition 8.3 *There is at least one 1-form ω dicritically adjusted to $\mathcal{C}$.*

Proof Take a standard basis $dx, dy, \omega_1, \ldots, \omega_s$ for $\mathcal{C}$. There is an index $k < s$ such that $u_{s+1} = an + bm + \lambda_s = cn + dm + \lambda_k$. Note that

$$an + bm + t_s < cn + dm + t_k,$$

since $t_s - t_k < \lambda_s - \lambda_k$. In this way, we have

(1) $t_{s+1} = \nu_E(x^a y^b \omega_s) < \nu_E(x^c y^d \omega_k)$.
(2) $u_{s+1} = \nu_{\mathcal{C}}(x^a y^b \omega_s) = \nu_{\mathcal{C}}(x^c y^d \omega_k)$.

Taking $\eta = x^a y^b \omega_s - \mu x^c y^d \omega_k$, for a convenient constant μ, we have that

$$\nu_E(\eta) = t_{s+1}, \quad \nu_{\mathcal{C}}(\eta) > u_{s+1}.$$

By a repeated application of Lemma 8.2, we find a 1-form $\tilde{\eta}$ such that

$$\nu_E(\tilde{\eta}) = t_{s+1} < \nu_{\mathcal{C}}(\tilde{\eta}), \quad \nu_{\mathcal{C}}(\tilde{\eta}) > c_\Gamma + 1.$$

Now, we can integrate $\tilde{\eta}$ as follows. Let $\phi : t \mapsto \phi(t) = (\phi_1(t), \phi_2(t))$ be a reduced parametrization of the curve $\mathcal{C}$. Then ϕ induces a morphism

$$\phi^\# : \mathbb{C}\{x, y\} \to \mathbb{C}\{t\}, \quad h \mapsto h \circ \phi,$$

whose kernel is generated by a local equation of $\mathcal{C}$ and moreover, the conductor ideal $t^{c_\Gamma}\mathbb{C}\{t\}$ is contained in the image of $\phi^\#$. Let us write

$$\phi^* \tilde{\eta} = \xi(t)dt, \quad \xi(t) \in t^{c_\Gamma}\mathbb{C}\{t\}.$$

By integration, there is a series $\psi(t)$ such that $\psi'(t) = \xi(t)$, with $\psi(t) \in t^{c_\Gamma + 1}\mathbb{C}\{t\}$. In view of the properties of the conductor ideal, there is a function $h \in \mathbb{C}\{x, y\}$ such that $h \circ \phi(t) = \psi(t)$. If we consider $\omega = \tilde{\eta} - dh$, we have that $\nu_E(\omega) = \nu_E(\tilde{\eta}) = t_{s+1}$ and $\nu_{\mathcal{C}}(\omega) = \infty$. □

Proposition 8.4 *If ω is dicritically adjusted to $\mathcal{C}$, we have that ω is basic and resonant (hence it is E-totally dicritical) and $\mathcal{C}$ is an ω-cusp.*

Proof Recall that $t_{s+1} < nm$ and $\nu_{\mathcal{C}}(\omega) = \infty > t_{s+1}$. □

8.1 Delorme's Decompositions

Delorme's decompositions are described in next Theorem 8.5. This result is the main tool we need to use in our statements on the analytic semiroots. We provide a proof of it, using a different approach to the one of Delorme, in Appendix C.

Theorem 8.5 (Delorme's decomposition) *Let $\mathcal{C}$ be a cusp in* $\mathrm{Cusps}(E)$*, consider an extended standard basis $\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s; \omega_{s+1}$ of $\mathcal{C}$ and denote by*

$$\Lambda = \Gamma(n, m, \lambda_1, \ldots, \lambda_s)$$

the semimodule of differential values of $\mathcal{C}$. For any indices $0 \le j \le i \le s$, there is a decomposition

$$\omega_{i+1} = \sum_{\ell=-1}^{j} f_\ell^{ij} \omega_\ell$$

such that, for any $-1 \le \ell \le j$ we have $\nu_{\mathcal{C}}(f_\ell^{ij}\omega_\ell) \ge v_{ij}$, where $v_{ij} = t_{i+1} - t_j + \lambda_j$ and there is exactly one index $-1 \le k \le j-1$ such that $\nu_{\mathcal{C}}(f_k^{ij}\omega_k) = \nu_{\mathcal{C}}(f_j^{ij}\omega_j) = v_{ij}$.

Remark 8.6 Note that $v_{ii} = t_{i+1} - t_i + \lambda_i = u_{i+1}$. We also have that

$$v_{ij} = u_{i+1} - (\lambda_i - u_i) - \cdots - (\lambda_{j+1} - u_{j+1}).$$

In particular we have that $v_{ij} \le u_{i+1} < c_\Gamma + n < nm$. See Lemma 6.5.

8.2 Analytic Semiroots

Let $\Lambda = \Gamma(n, m, \lambda_1, \ldots, \lambda_s)$ be the semimodule of differential values of the E-cusp $\mathcal{C}$ and let us consider an extended standard basis

$$\mathcal{E} = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s; \omega_{s+1})$$

We recall that $\omega_1, \omega_2, \ldots, \omega_{s+1}$ are basic and resonant. Fix a free point $P \in E$. For each $i = 1, 2, \ldots, s+1$, there is an E-cusp $\mathcal{C}_P^i$ passing through P and invariant by ω_i. Let us note that if P is the infinitely near point of $\mathcal{C}$ in E, we have that $\mathcal{C}_P^{s+1} = \mathcal{C}$.

Definition 8.7 The cusps $\mathcal{C}_P^i$, for $i = 1, 2, \ldots, s+1$ are called the *analytic semiroots* of $\mathcal{C}$ through P with respect to the extended standard basis $\mathcal{E}$.

Let us denote $\mathcal{E}_i = (\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_{i-1}; \omega_i)$ for any $1 \le i \le s+1$. The main objective of this paper is to show the following Theorem

Theorem 8.8 *For any $1 \le i \le s+1$ and any free point $P \in E$, we have*

(a) *Λ_{i-1} is the semimodule of differential values of $\mathcal{C}_P^i$.*

(b) *$\mathcal{E}_i$ is an extended standard basis for $\mathcal{C}_P^i$.*

Let us consider an index $1 \leq i \leq s+1$ and an analytic semiroot $\mathcal{C}_P^i$ in order to prove Theorem 8.8.

Lemma 8.9 *We have that $\nu_{\mathcal{C}_P^i}(\omega_j) = \lambda_j$, for any $j = -1, 0, 1, \ldots, i-1$.*

Proof For any basic non resonant 1-form ω, we have that

$$\nu_{\mathcal{C}}(\omega) = \nu_E(\omega) = \nu_{\mathcal{C}_P^i}(\omega).$$

This is particularly true for the case of 1-forms of the type $\omega = df$, where we know that $\nu_E(\omega) = \nu_E(f)$. Hence, for any germ of function $f \in \mathbb{C}\{x, y\}$ we have that

$$\min\{\nu_{\mathcal{C}}(df), nm\} = \min\{\nu_{\mathcal{C}_P^i}(df), nm\}. \tag{20}$$

The statement of the Lemma is true for $\ell = -1, 0$, since

$$\nu_{\mathcal{C}}(dx) = \nu_{\mathcal{C}_P^i}(dx) = n, \quad \nu_{\mathcal{C}}(dy) = \nu_{\mathcal{C}_P^i}(dy) = m.$$

Let us assume that it is true for any $\ell = -1, 0, 1, \ldots, j$, with $0 \leq j \leq i-2$. We apply Theorem 8.5 to obtain a decomposition

$$\omega_i = \sum_{\ell=-1}^{j+1} f_\ell \omega_\ell$$

such that $\nu_{\mathcal{C}}(f_\ell \omega_\ell) \geq v_{i-1,j+1}$, where $v_{i-1,j+1} < nm$ (see Remark 8.6) and there is a single $k \leq j$ such that $\nu_{\mathcal{C}}(f_{j+1}\omega_{j+1}) = \nu_{\mathcal{C}}(f_k \omega_k) = v_{i-1,j+1}$. We deduce that

$$\nu_{\mathcal{C}}\left(\sum_{\ell=-1}^{j} f_\ell \omega_\ell\right) = \nu_{\mathcal{C}}(f_k \omega_k) = v_{i-1,j+1}.$$

On the other hand, by induction hypothesis and noting that $v_{i-1,j+1} < nm$, we have that

$$\min\{\nu_{\mathcal{C}}(f_\ell \omega_\ell), nm\} = \min\{\nu_{\mathcal{C}_P^i}(f_\ell \omega_\ell), nm\}, \quad \ell = -1, 0, 1, \ldots, j.$$

In particular, we have that

$$\nu_{\mathcal{C}_P^i}(f_k \omega_k) = v_{i-1,j+1}, \quad \nu_{\mathcal{C}_P^i}\left(\sum_{\ell=-1}^{j} f_\ell \omega_\ell\right) = v_{i-1,j+1}.$$

Since $\nu_{\mathcal{C}_P^i}(\omega_i)=\infty$, we have that $\nu_{\mathcal{C}_P^i}(f_{j+1}\omega_{j+1})=v_{i-1,j+1}$. Hence we have

$$v_{i-1,j+1}=\nu_{\mathcal{C}}(f_{j+1}\omega_{j+1})=\nu_{\mathcal{C}_P^i}(f_{j+1}\omega_{j+1}).$$

Noting that $\nu_{\mathcal{C}}(f_{j+1})=\nu_{\mathcal{C}_P^i}(f_{j+1})$, we conclude that $\nu_{\mathcal{C}}(\omega_{j+1})=\nu_{\mathcal{C}_P^i}(\omega_{j+1})$. The proof is ended. □

Corollary 8.10 $\Lambda^{\mathcal{C}_P^i}\supset\Lambda_{i-1}$.

Proof It is enough to remark that $\lambda_j\in\Lambda^{\mathcal{C}_P^i}$ for any $j=-1,0,1,\ldots,i-1$. □

Proposition 8.11 $\Lambda^{\mathcal{C}_P^i}=\Lambda_{i-1}$.

Proof We already know that $\Lambda^{\mathcal{C}_P^i}\supset\Lambda_{i-1}$. Assume that $\Lambda^{\mathcal{C}_P^i}\neq\Lambda_{i-1}$ and take the number

$$\lambda=\min\left(\Lambda^{\mathcal{C}_P^i}\setminus\Lambda_{i-1}\right).$$

Note that $\lambda>m$ and hence there is a maximum index $0\le j\le i-1$ such that $\lambda_j<\lambda$. We have that

$$\nu_E(\omega_j)=t_j=t_j(\mathcal{C}_P^i),\quad t_j=t_j(\mathcal{C}).$$

where $t_j(*)$ denotes the critical divisorial order t_j with respect to the curve $*$. Assume first that $j\le i-2$.

Let $\tilde{\omega}$ be a 1-form in a standard basis for $\mathcal{C}_P^i$ that corresponds to the differential value λ. Note that λ is the differential value in the basis of $\Lambda^{\mathcal{C}_P^i}$ that immediately follows λ_j and the previous ones correspond to the values in the basis of Λ. We have that

$$\nu_E(\omega_{j+1})=t_{j+1}=t_j+u_{j+1}-\lambda_j=t_{j+1}(\mathcal{C}_P^i)=\nu_E(\tilde{\omega}).$$

In view of the property (1) in Theorem 7.13, we have that

$$\lambda=\max\{\nu_{\mathcal{C}_P^i}(\omega);\ \nu_E(\omega)=t_{j+1}(\mathcal{C}_P^i)=t_{j+1}\}$$

In view of Lemma 8.9, we know that $\nu_{\mathcal{C}_P^i}(\omega_{j+1})=\lambda_{j+1}$ and $\nu_E(\omega_{j+1})=t_{j+1}$. This implies that $\lambda_{j+1}<\lambda$, contradiction.

Let us consider now the case when $j=i-1$. We shall prove that it is not possible to have $s(\mathcal{C}_P^i)>i-1$ where $s(*)$ refers to "concept s" with respect to the curve $*$ (that is, $s+2$ is the number of elements of the basis of the semimodule of the curve). If $s(\mathcal{C}_P^i)>i-1$, we have that

$$\lambda=\max\{\nu_{\mathcal{C}_P^i}(\omega);\ \nu_E(\omega)=t_i(\mathcal{C}_P^i)=t_i\}.$$

But we know that $\nu_{\mathcal{C}_P^i}(\omega_i)=\infty$ and $\nu_E(\omega_i)=t_i$, this is the desired contradiction. □

Example 8.12 Let us compute the analytic semiroots of the curve $\mathcal{C}$ given in Example 7.16. The Puiseux parametrizations for the E-cusps $\mathcal{C}_a^i$ of ω_i, $i = 1, 2, 3$, are given by

$$\phi_1^a(t) = (t^5, at^{11})$$
$$\phi_2^a(t) = (t^5, at^{11} + a^2t^{12} + \tfrac{23}{22}a^3t^{13} + \tfrac{136}{121}a^4t^{14} + \cdots)$$
$$\phi_3^a(t) = \left(t^5, \sum_{i\geq 11} a^{i-10}t^i\right)$$

with $a \in \mathbb{C}^*$. Hence, the analytic semiroots of $\mathcal{C}$ are the curves $\mathcal{C}_1^1$, $\mathcal{C}_1^2$ and $\mathcal{C}_1^3$ given by the above parametrizations $\phi_i^1(t)$, $i = 1, 2, 3$, with $a = 1$.

Note that in this example, for any $i = 1, 2, 3$, all the E-cusps of the family $\{\mathcal{C}_a^i\}_a$ are analytically equivalent. To see this it is enough to consider the new parameter $t = a^{-1}u$ and the change of variables $x_1 = a^5x$, $y_1 = a^{10}y$.

Example 8.13 We would like to remark that, in general, the E-cusps of an element ω_i, $i \geq 2$, of a standard basis are not analytically equivalent as the following example shows. Consider the curve $\mathcal{C}$ given by the Puiseux parametrization $\phi(t) = (t^7, t^{17} + t^{30} + t^{33} + t^{36})$ with $\Gamma = \langle 7, 17\rangle$ and $\Lambda = \Gamma(7, 17, 37, 57)$. A standard basis is given by $\omega_{-1} = dx, \quad \omega_0 = dy, \quad \omega_1 = 7xdy - 17ydx$ and

$$\omega_2 = 3757x^2ydx - 1547x^3dy - 4624y^2dx + 1904xydy + 1183y^2dy.$$

The E-cusps of ω_2 are the curves given by the Puiseux parametrization

$$\varphi_a(t) = (t^7, at^{17} + a^3t^{30} + a^4t^{33} + \cdots)$$

with $a \in \mathbb{C}^*$. If we consider a new parameter $t = a^{-2/13}u$ and the we make the change of variables $x_1 = a^{14/13}x$, $y_2 = a^{21/13}y$, we obtain that the family of E-cusps of ω_2 are the curves $\mathcal{C}_a^2$ given by the parametrizations

$$\phi_a(t) = (t^7, t^{17} + t^{30} + a^{7/13}t^{33} + \cdots)$$

From the results above, we have that $\Lambda^{\mathcal{C}_a^2} = \Lambda_1 = \Gamma(7, 17, 37)$ for all $a \in \mathbb{C}^*$. Since $33 \notin \Lambda_1 - 7$, by Theorem 2.1 in Hefez and Hernandes (2011), two curves $\mathcal{C}_{a_1}^2$ and $\mathcal{C}_{a_2}^2$ are not, in general, analytically equivalent for $a_1, a_2 \in \mathbb{C}^*$.

Example 8.14 Let us consider the 1-form ω of example 4.7 in Gómez-Martínez (2021) given by

$$\omega = (7y^5 + 2x^9y - 2x^9y^2 - 9x^2y^4)dx + (4y^3x^3 - x^{10} + 2x^{10}y - 3xy^4 - x^8y^2)dy.$$

This 1-form is pre-basic and resonant for the pair (4, 9) since $\nu_E(\omega) = 48$, the co-pair of (4, 9) is (3, 7) and $\mathrm{Cl}(\omega; x, y) \subset R^{4,9}(3, 4)$ where

$$R^{4,9}(3,4) = \{(\alpha, \beta) \in \mathbb{R}^2 \;\; \alpha + 2\beta \geq 11\} \cap \{(\alpha, \beta) \in \mathbb{R}^2 \;\; 3\alpha + 7\beta \geq 37\}.$$

Moreover, the weighted initial part of ω is given by $\mathrm{In}^{48}_{4,9;x,y} = x^2 y^3(-9ydx + 4xdy)$. Consequently ω is totally E-dicritical for the last divisor E associated to the cuspidal sequence $\mathcal{S}^{4,9}_{y=0}$. Note that ω is not a basic 1-form since $\nu_E(\omega) > nm = 36$.

The invariant curves of ω which are transversal to the dicritical component E are the curves $\mathcal{C}_a$, $a \in \mathbb{C}^*$, given by

$$y^4 - ax^9 + (a-1)x^7 y + x^7 y^2 = 0.$$

Note that these curves do not have the same semimodule of differential values since the curves $\mathcal{C}_a$, with $a \neq 1$, have Zariski invariant equal to 10 whereas the curve $\mathcal{C}_1$ has Zariski invariant equal to 19.

Acknowledgements We are grateful to Marcelo E. Hernandes for all the conversations and suggestions on the subject. The work of Oziel Gómez has influenced our work in this paper, as well as the guidance of Patricio Almirón on the study of semimodules.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Appendix A: Bounds for the Conductor

In this Appendix we present some bounds for the conductor of the semimodules Λ_i in the decomposition series of

$$\Lambda = \Gamma(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_s)$$

that will be enough to prove the results we need relative to the structure of the semimodule of differential values. We recall that the semigroup Γ is generated by a Puiseux pair (n, m), with $2 \leq n$.

Given two integer numbers $r \leq s$, we denote $[r, s]$ the set of the integer numbers ℓ such that $r \leq \ell \leq s$. For any $q \geq 0$ we denote by I_q the interval $I_q = [nq, n(q+1)-1]$, in particular, we have that $I_0 = \{0, 1, 2, \ldots, n-1\}$. For any $r, s \in I_0$, we define the *circular interval* $\langle r, s \rangle$ by

$$\langle r, s \rangle = [r, s], \quad \text{if } r \leq s; \quad \langle r, s \rangle = [r, n-1] \cup [0, s], \quad \text{if } r > s.$$

We denote by $\rho : \mathbb{Z} \to \mathbb{Z}/(n)$ the canonical map and we also use the notation $\rho(p) = \bar{p}$. Since $\gcd(n, m) = 1$, there is a bijection

$$\xi : \mathbb{Z}/(n) \to I_0 = \{0, 1, 2, \ldots, n-1\}, \quad \xi^{-1}(k) = \rho(km).$$

For any $q \geq 0$ and any subset $S \subset \mathbb{Z}_{\geq 0}$, we define the *q-level set* $R_q(S) \subset I_0$ by $R_q(S) = \xi\left(\rho(S \cap I_q)\right)$.

Remark A.1 We have that $R_q(\Lambda) \subset R_{q'}(\Lambda)$ for all $q' \geq q$.

Remark A.2 For any $\mu \in \mathbb{Z}_{\geq 0}$ and $q \geq 1$, we have

$$\# R_q(\mu + \Gamma) \leq \# R_{q-1}(\mu + \Gamma) + 1.$$

Indeed, this is equivalent to show that $\#\rho((\mu + \Gamma) \cap I_q) \leq \#\rho((\mu + \Gamma) \cap I_{q-1}) + 1$. Assume that $\bar{p}_1, \bar{p}_2 \in \rho((\mu+\Gamma)\cap I_q)\setminus\rho((\mu+\Gamma)\cap I_{q-1})$. We can take representatives $p_1, p_2 \in (\mu + \Gamma) \cap I_q$ of $\bar{p}_1$ and $\bar{p}_2$ of the form $p_1 = \mu + b_1 m$, $p_2 = \mu + b_2 m$. If $p_1 \neq p_2$, we have that $|p_1 - p_2| \geq m > n$ and this is not possible.

Lemma A.3 *Consider $\mu \in I_v$, denote $r = \xi(\bar{\mu})$ and let q be such that $q \geq v$. For any $p \in R_q(\mu + \Gamma)$ we have that $\langle r, p\rangle \subset R_q(\mu + \Gamma)$. In particular, the set $R_q(\mu + \Gamma)$ is a circular interval.*

Proof The second statement is straightforward, since the union of circular intervals with a common point is a circular interval. To prove the first statement, we proceed by induction on the number ℓ of elements in $\langle r, p\rangle$. If $\ell \leq 2$, we are done, since $\langle r, p\rangle \subset \{r, p\} \subset R_q(\mu + \Lambda)$. Assume that $\ell > 2$; in particular we have that $r \neq p$. Consider the point $\tilde{p} \in I_0$ given by $\tilde{p} = p - 1$, if $p \geq 1$ and $\tilde{p} = n - 1$, if $p = 0$. We have that $\langle r, p\rangle = \langle r, \tilde{p}\rangle \cup \{p\}$ and the length of $\langle r, \tilde{p}\rangle$ is $\ell - 1$. Then, it is enough to show that $\tilde{p} \in R_q(\mu + \Gamma)$. Take an element $\mu + an + bm \in I_q \cap (\mu + \Gamma)$ such that $\rho(\mu + an + bm) = \rho(pm)$. Noting that $r \neq p$, we have that $b \geq 1$. There is $q' \leq q$ such that $\mu + an + (b-1)m \in I_{q'}$ and hence

$$\mu + (a + q - q')n + (b-1)m \in I_q \cap (\mu + \Gamma).$$

We have that $\rho(\mu + (a + q - q')n + (b-1)m) = \rho(\tilde{p}m)$ and thus $\tilde{p} \in R_q(\mu + \Gamma)$.

□

Definition A.4 We define the *tops q_1 and q_2 of* Λ by the property that

$$\lambda_s + n\ell_1 \in I_{q_1}, \quad \lambda_s + m\ell_2 \in I_{q_2}$$

where ℓ_1 and ℓ_2 are the limits of Λ. The *main top* Q_Λ is the maximum of q_1, q_2.

Proposition A.5 *Consider a normalized semimodule $\Lambda = \Gamma(0, \lambda_0, \lambda_1, \ldots, \lambda_s)$. Let v be such that $\lambda_s \in I_v$ and assume that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq v$. Denote by q_1, q_2 the tops of Λ and put $r = \xi(\bar{\lambda}_s)$. Then we have that*

(1) $[0, r-1] \subset R_q(\Lambda_{s-1})$, *for all* $q \geq q_1 - 1$.
(2) $[r, n-1] \subset R_q(\Lambda)$, *for all* $q \geq q_2 - 1$.

In particular $c_\Lambda \leq n(Q_\Lambda - 1)$.

Proof Note that Statements (1) and (2) imply that

$$I_0 = [0, n-1] = [0, r] \cup [r, n-1] \subset R_q(\Lambda), \quad q \geq Q_\Lambda - 1$$

and thus, we have that $c_\Lambda \leq n(Q_\Lambda - 1)$.
Proof of Statement (1) By Remark A.1 it is enough to show that we have $[0, r-1] \subset R_{q_1-1}(\Lambda_{s-1})$. Since $\lambda_s + n\ell_1 \in \Lambda_{s-1}$, there is an index $k \leq s-1$ such that $\lambda_s + n\ell_1 = \lambda_k + an + bm$. By the minimality of ℓ_1, we have that $a = 0$ and hence $\lambda_s + n\ell_1 = \lambda_k + bm$. Denote $r_k = \xi(\bar{\lambda}_k)$. Note that $r_k \neq r$, since $r \notin R_v(\Lambda_{s-1})$.

Assume that the next statements are true:

(a) If $r_k > r$, then $[0, r] \subset R_{q_1}(\lambda_k + \Gamma)$.
(b) If $r_k < r$, then $[r_k, r] \subset R_{q_1}(\lambda_k + \Gamma)$ and $[0, r_k] \subset R_{q_1-1}(\Lambda_{s-1})$.

If $r_k > r$, by the minimality of ℓ_1, we have that $r \notin R_{q_1-1}(\lambda_k + \Gamma)$; now, in view of Remark A.2 and noting that $[0, r] = [0, r-1] \cup \{r\}$, we obtain that

$$[0, r-1] \subset R_{q_1-1}(\lambda_k + \Gamma) \subset R_{q_1-1}(\Lambda_{s-1}).$$

If $r_k < r$, we obtain as above that $[r_k, r-1] \subset R_{q_1-1}(\lambda_k + \Gamma)$, then

$$[0, r-1] = [0, r_k] \cup [r_k, r-1] \subset R_{q_1-1}(\Lambda_{s-1}).$$

If remains to prove (a) and (b).

Proof of (a): We can apply Lemma A.3 to have that $\langle r_k, r\rangle \subset R_{q_1}(\lambda_k + \Gamma)$. We end by noting that $[0, r] \subset \langle r_k, r\rangle$.

Proof of (b): We apply Lemma A.3 to have that $\langle r_k, r\rangle = [r_k, r] \subset R_{q_1}(\lambda_k + \Gamma)$. On the other hand, we know that $R_{q_1-1}(\Lambda_{s-1})$ is a circular interval since $q_1 - 1 \geq v$ and it contains 0 and r_k. Moreover $r \notin R_{q_1-1}(\Lambda_{s-1})$ and $r > r_k$, then the circular interval $R_{q_1-1}(\Lambda_{s-1})$ contains $[0, r_k]$.

Proof of Statement (2): It is enough to show that $[r, n-1] \subset R_{q_2-1}(\Lambda)$. By an argument as before, there is an index $k \leq s-1$ such that $\lambda_s + m\ell_2 = \lambda_k + na$. Take $r_k \neq r$ as above. By Lemma A.3, we have that $\langle r, r_k\rangle \subset R_{q_2}(\lambda_s + \Gamma)$. Let us see that $r_k \notin R_{q_2-1}(\lambda_s + \Gamma)$. For this, let us show that the property

$$r_k \in R_{q_2-1}(\lambda_s + \Gamma)$$

leads to a contradiction. This property should imply that $\lambda_k + n(a-1) \in \lambda_s + \Gamma$ and hence there are nonnegative integer numbers α, β such that

$$\lambda_s + n\alpha + m\beta = \lambda_k + n(a-1).$$

If $a-1 \leq \alpha$, we have that $\lambda_k = \lambda_s + (\alpha - a + 1)n + \beta m$ and this contradicts the fact that $\lambda_k < \lambda_s$; hence $a-1 > \alpha$ and we have

$$\lambda_s + m\beta = \lambda_k + n(a - 1 - \alpha).$$

Since $a - 1 - \alpha < a$, we have that $\beta < \ell_2$. This contradicts the minimality of ℓ_2.

Since $r_k \notin R_{q_2-1}(\lambda_s + \Gamma)$, we can apply Remark A.2 that tells us that

$$\langle r, r_k \rangle \setminus \{r_k\} \subset R_{q_2-1}(\lambda_s + \Gamma) \subset R_{q_2-1}(\Lambda).$$

Note also that $r_k \in R_{q_2-1}(\Lambda)$. Then we have that $\langle r, r_k \rangle \subset R_{q_2-1}(\Lambda)$.

If $r > r_k$, then $[r, n-1] \subset \langle r, r_k \rangle \subset R_{q_2-1}(\Lambda)$. Assume now that $r < r_k$. Recall that $r \notin R_v(\Lambda_{s-1})$; since $R_v(\Lambda_{s-1})$ is a circular interval containing r_k and 0, but not containing r, we have that

$$[r_k, n-1] \subset R_v(\Lambda_{s-1}) \subset R_{q_2-1}(\Lambda_{s-1}) \subset R_{q_2-1}(\Lambda).$$

We conclude that $[r, n-1] = \langle r, r_k \rangle \cup [r_k, n-1] \subset R_{q_2-1}(\Lambda)$. □

Proposition A.6 *Let Λ be a normalized increasing semimodule of length s and let v be such that $u_{s+1} \in I_v$. Then $R_q(\Lambda)$ is a circular interval for any $q \geq v$.*

Proof Let us proceed by induction on the length s of Λ. If $s = -1$, we have $\Lambda = \Lambda_{-1} = \Gamma$. By Lemma A.3 applied to $\mu = 0$, we are done. Let us suppose that $s \geq 0$ and assume by induction that the result is true for Λ_{s-1}. We have that $\Lambda = \Lambda_{s-1} \cup (\lambda_s + \Gamma)$. This implies that

$$R_q(\Lambda) = R_q(\Lambda_{s-1}) \cup R_q(\lambda_s + \Gamma), \quad q \geq 0.$$

Let v' be such that $u_s \in I_{v'}$. By induction hypothesis, we know that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq v'$. Moreover, by the increasing property, we have that

$$u_{s+1} > \lambda_s > u_s \geq \lambda_{s-1}.$$

In particular, we have that $v \geq v'$ and $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq v$. On the other hand, take v'' such that $\lambda_s \in I_{v''}$. By Lemma A.3, we know that $R_q(\lambda_s + \Gamma)$ is a circular interval for any $q \geq v''$. Since $v \geq v''$, we have that $R_q(\lambda_s + \Gamma)$ is a circular interval for any $q \geq v$. Thus, both

$$R_q(\Lambda_{s-1}) \quad \text{and} \quad R_q(\lambda_s + \Gamma)$$

are circular intervals for $q \geq v$. We need to show that their union is also a circular interval. Since $v \geq v''$, we have that $r \in R_q(\lambda_s + \Gamma)$ for $r = \xi(\overline{\lambda}_s)$. Noting that $0 \in R_q(\Lambda_{s-1})$ and $r \in R_q(\lambda_s + \Gamma)$, in order to prove that $R_q(\Lambda)$ is a circular interval, it is enough to show that one of the following properties holds

$$(a): \quad [0, r] \subset R_q(\Lambda); \qquad (b): \quad [r, n-1] \subset R_q(\Lambda).$$

We can apply Proposition A.5 to $\Lambda = \Lambda_{s-1}(\lambda_s)$. Indeed, by induction hypothesis we know that $R_p(\Lambda_{s-1})$ is a circular interval for any $p \geq v'$; since $v' \leq v''$, we have that $R_p(\Lambda_{s-1})$ is a circular interval for any $p \geq v''$ and thus we are in the hypothesis of Proposition A.5. Now, by Eq. (15) in Remark 6.8, we have either $u_{s+1} = \lambda_s + nl_1$ or $u_{s+1} = \lambda_s + ml_2$.

(a) If we have $u_{s+1} = \lambda_s + nl_1$, noting that $q_1 = v$, we apply Proposition A.5 and we obtain that $[0, r] \subset R_q(\Lambda_{s-1}) \subset R_q(\Lambda)$.
(b) If we have $u_{s+1} = \lambda_s + ml_2$, noting that $q_2 = v$, we apply Proposition A.5 and we obtain that $[r, n-1] \subset R_q(\Lambda)$.

□

Corollary A.7 *Let Λ be an increasing semimodule such that its minimum element λ_{-1} is a multiple of n. Let Q_Λ be the main top of Λ. Then $c_\Lambda \leq n(Q_\Lambda - 1)$.*

Proof Assume first that Λ is normalized. Let v' be such that $u_s \in I_{v'}$ and v'' such that $\lambda_s \in I_{v''}$. We know that $v'' \geq v'$. By Proposition A.6, we know that $R_q(\Lambda_{s-1})$ is a circular interval for any $q \geq v'$ and hence for any $q \geq v''$. Then we are in the hypothesis of Proposition A.5 and we conclude.

Assume now that $\lambda_{-1} = kn$ and consider the normalization $\widetilde{\Lambda} = \Lambda - kn$. Let us note that the tops are related by the property $\tilde{q}_j = q_j - k$, for $j = 1, 2$ and hence $Q_{\widetilde{\Lambda}} = Q_\Lambda - k$. On the other hand $c_{\widetilde{\Lambda}} = c_\Lambda - nk$. We conclude that

$$c_\Lambda = c_{\widetilde{\Lambda}} + nk \leq n(Q_{\widetilde{\Lambda}} - 1) + nk = n(Q_\Lambda - 1).$$

□

Appendix B: Structure of the Semimodule

In this Appendix we present a proof, using a different approach to the one of Delorme, of the main results on the structure of the semimodule of differential values for an E-cusp $\mathcal{C}$. As before, we denote $\Lambda = \Gamma(n, m, \lambda_1, \ldots, \lambda_s)$, $n \geq 2$, the semimodule of differential values of $\mathcal{C}$ and we select a standard basis $\omega_{-1} = dx$, $\omega_0 = dy$, $\omega_1, \ldots, \omega_s$ of the cusp $\mathcal{C}$.

Proposition B.1 *For each $1 \leq i \leq s$ we have the following statements*

(1) $\lambda_i = \sup\{\nu_{\mathcal{C}}(\omega) : \nu_E(\omega) = t_i\}$.
(2) *If $\nu_{\mathcal{C}}(\omega) = \lambda_i$, then $\nu_E(\omega) = t_i$.*
(3) *For each 1-form ω with $\nu_{\mathcal{C}}(\omega) \notin \Lambda_{i-1}$, there is a unique pair $a, b \geq 0$ such that $\nu_E(\omega) = \nu_E(x^a y^b \omega_i)$. Moreover, we have that $\nu_{\mathcal{C}}(\omega) \geq \lambda_i + na + mb$.*
(4) *We have that $\lambda_i > u_i$.*
(5) *Let $k = \lambda_i + na + mb$, then $k \notin \Lambda_{i-1}$ if and only if for all ω such that $\nu_{\mathcal{C}}(\omega) = k$ we have that $\nu_E(\omega) \leq \nu_E(x^a y^b \omega_i)$.*

In particular, the semimodules Λ_i are increasing, for $i = 1, 2, \ldots, s$.

Proof Assume that $i = 1$ and then $t_1 = n + m = u_1$. We have

- Statement (1) is proven in Proposition 7.5.
- Statement (2) is proven in Corollary 7.7.
- Statement (3) is proven in Lemma 7.9.
- Statement (4) follows from the fact that $\lambda_1 > n + m = \nu_E(\omega_1) = u_1$.
- Statement (5) is proven in Lemma 7.8.

Now, let us assume that $i \geq 2$ and take the induction hypothesis that the statements (1)-(5) are true for indices ℓ with $1 \leq \ell \leq i - 1$.

Denote by ℓ_1 and ℓ_2 the Λ_{i-1}-limits. By Eq. (15) in Remark 6.8, we have two possibilities: either $u_i = \lambda_{i-1} + n\ell_1$ or $u_i = \lambda_{i-1} + m\ell_2$. We assume that $u_i = \lambda_{i-1} + n\ell_1$, the computations in the case $u_i = \lambda_{i-1} + m\ell_2$ are similar ones.

The proof is founded in three claims as follows:

- *Claim 1: There is a 1-form η with $\nu_E(\eta) = t_i$, whose initial part is proportional to the initial part of $x^{\ell_1}\omega_{i-1}$ and such that either $\nu_{\mathcal{C}}(\eta) \geq c_\Gamma$ or $\nu_{\mathcal{C}}(\eta) \notin \Lambda_{i-1}$.*
- *Claim 2: Any 1-form ω with $\nu_{\mathcal{C}}(\omega) \notin \Lambda_{i-1}$ is reachable from $x^{\ell_1}\omega_{i-1}$.*
- *Claim 3: Let η be a 1-form such that $\nu_E(\eta) = t_i$ whose initial part is proportional to the initial* part of *$x^{\ell_1}\omega_{i-1}$ and such that either $\nu_{\mathcal{C}}(\eta) \geq c_\Gamma$ or $\nu_{\mathcal{C}}(\eta) \notin \Lambda_{i-1}$. Then $\nu_{\mathcal{C}}(\eta) = \lambda_i$.*

We recall to the reader that the notion “initial part” refers to the concept of weighted initial part defined in Sect. 3.2.

Proof of Claim 1 Recall that $t_i = \nu_E(\omega_{i-1}) + u_i - \lambda_{i-1} = \nu_E(\omega_{i-1}) + n\ell_1$. Let us start with $\eta_1 = x^{\ell_1}\omega_{i-1}$. We have that

$$\nu_E(\eta_1) = n\ell_1 + \nu_E(\omega_{i-1}) = t_i, \quad \nu_{\mathcal{C}}(\eta_1) = n\ell_1 + \lambda_{i-1} = u_i \in \Lambda_{i-2}.$$

By Statement (5) applied to $\nu_{\mathcal{C}}(\eta_1) \in \Lambda_{i-2}$, there is η_1' with $\nu_{\mathcal{C}}(\eta_1') = \nu_{\mathcal{C}}(\eta_1)$ and $\nu_E(\eta_1') > \nu_E(\eta_1)$. Since $\nu_{\mathcal{C}}(\eta_1') = \nu_{\mathcal{C}}(\eta_1)$, there is a non-null constant μ such that

$$\nu_{\mathcal{C}}(\tilde{\eta}) > \nu_{\mathcal{C}}(\eta_1), \quad \text{where } \tilde{\eta} = \eta_1 - \mu\eta_1'.$$

Since $\nu_E(\eta_1') > \nu_E(\eta_1)$, we have that $\nu_E(\tilde{\eta}) = \nu_E(\eta_1) = t_i$ and the initial part of $\tilde{\eta}$ is the same one as the initial part of $\eta_1 = x^{\ell_1}\omega_{i-1}$. If $\nu_{\mathcal{C}}(\tilde{\eta}) \geq c_\Gamma$ or $\nu_{\mathcal{C}}(\tilde{\eta}) \notin \Lambda_{i-1}$, we put $\eta = \tilde{\eta}$ and we are done. Assume that $\nu_{\mathcal{C}}(\tilde{\eta}) \in \Lambda_{i-1}$. Let us write

$$\nu_{\mathcal{C}}(\tilde{\eta}) = an + bm + \lambda_\ell, \quad \ell \leq i - 1.$$

Let us see that $\nu_E(\tilde{\eta}) < \nu_E(x^a y^b \omega_\ell)$; this is equivalent to verify that $t_i - t_\ell < na + mb$. Since $\nu_{\mathcal{C}}(\tilde{\eta}) > u_i$, in view of Lemma 7.10 we have

$$na + mb > u_i - \lambda_\ell = n\ell_1 + \lambda_{i-1} - \lambda_\ell \geq n\ell_1 + t_{i-1} - t_\ell = t_i - t_\ell.$$

On the other hand, we have that $\nu_{\mathcal{C}}(\tilde{\eta}) = \nu_{\mathcal{C}}(x^a y^b \omega_\ell)$. Thus, there is a constant μ_1, such that $\nu_{\mathcal{C}}(\tilde{\eta}_1) > \nu_{\mathcal{C}}(\tilde{\eta})$, $\nu_E(\tilde{\eta}_1) = \nu_E(\tilde{\eta})$, where $\tilde{\eta}_1 = \tilde{\eta} - \mu_1 x^a y^b \omega_\ell$, and the initial part of $\tilde{\eta}_1$ is the same one as the initial part of $x^{\ell_1}\omega_{i-1}$.

If $\nu_{\mathcal{C}}(\tilde{\eta}_1) \in \Lambda_{i-1}$, we repeat the procedure starting with $\tilde{\eta}_1$, to obtain $\tilde{\eta}_2$ such that $\nu_{\mathcal{C}}(\tilde{\eta}_2) > \nu_{\mathcal{C}}(\tilde{\eta}_1)$ and $\nu_E(\tilde{\eta}_2) = t_i$. After finitely many repetitions, we get a 1-form η such that $\nu_E(\eta) = t_i$, whose initial part is the same one as $x^{\ell_1}\omega_{i-1}$ and either $\nu_{\mathcal{C}}(\eta) \geq c_\Gamma$ or $\nu_{\mathcal{C}}(\eta) \notin \Lambda_{i-1}$. This proves Claim 1.

Proof of Claim 2 Take ω such that $\lambda = \nu_{\mathcal{C}}(\omega) \notin \Lambda_{i-1}$. Note that $\lambda \notin \Lambda_{i-2}$. By Statement (3), we have that ω is reachable from ω_{i-1}. Thus, there are $a, b \geq 0$ and a constant μ such that

$$\nu_E(\omega - \mu x^a y^b \omega_{i-1}) > \nu_E(\omega) = \nu_E(x^a y^b \omega_{i-1}) = an + bm + t_{i-1}.$$

and moreover, we have that $\lambda = \nu_{\mathcal{C}}(\omega) > an + bm + \lambda_{i-1} = k$ (note that $\lambda \neq k$, since $\lambda \notin \Lambda_{i-1}$).

Consider the 1-form $\omega' = \omega - \mu x^a y^b \omega_{i-1}$. We know that

$$\nu_{\mathcal{C}}(\omega') = k, \quad \nu_E(\omega') > \nu_E(x^a y^b \omega_{i-1}).$$

By Statement (5), we conclude that $k \in \Lambda_{i-2}$. Hence $k \in \Lambda_{i-2} \cap (\lambda_{i-1} + \Gamma)$. Let us show that we necessarily have that $a \geq \ell_1$. Write

$$k = an + bm + \lambda_{i-1} = \tilde{a}n + \tilde{b}m + \lambda_j, \quad j \leq i-2.$$

Since $\lambda_{i-1} > \lambda_j$, we have that $an + bm < \tilde{a}n + \tilde{b}m$. Thus, we have either $a < \tilde{a}$ or $b < \tilde{b}$. If $b < \tilde{b}$, we have that $an + \lambda_{i-1} = \tilde{a}n + (\tilde{b} - b)m + \lambda_j \in \Lambda_{i-2} \cap (\lambda_{i-1} + \Gamma)$. In view of the minimality of ℓ_1 we should have that $\ell_1 \leq a$ and then ω is reachable from $x^{\ell_1}\omega_{i-1}$. Assume that $a < \tilde{a}$ and let us obtain a contradiction. We have

$$bm + \lambda_{i-1} = (\tilde{a} - a)n + \tilde{b}m + \lambda_j \in \Lambda_{i-2} \cap (\lambda_{i-1} + \Gamma).$$

We deduce that $b \geq \ell_2$. By Statement (4), we know that Λ_{i-1} is an increasing semimodule, starting at $\lambda_{-1} = n$. By Corollary A.7, we know that $c_{\Lambda_{i-1}} \leq n(Q_{\Lambda_{i-1}} - 1)$, where $Q_{\Lambda_{i-1}} = \max\{q_1, q_2\}$ and q_1, q_2 are the tops of Λ_{i-1}. Suppose that $\lambda \in I_d$, we have

(1) $\lambda > k = an + bm + \lambda_{i-1} \geq u_i = \ell_1 n + \lambda_{i-1}$ and hence $d \geq q_1$.
(2) $\lambda > k = an + bm + \lambda_{i-1} \geq \ell_2 m + \lambda_{i-1}$ and hence $d \geq q_2$.

We conclude that $\lambda \in \Lambda_{i-1}$, contradiction. This ends the proof of Claim 2.

Proof of Claim 3 Note that $\nu_{\mathcal{C}}(\eta) \geq \lambda_i$. Assume that $\lambda = \nu_{\mathcal{C}}(\eta) > \lambda_i$. Recalling that $\nu_{\mathcal{C}}(\omega_i) = \lambda_i \notin \Lambda_{i-1}$ and that the initial part of η is proportional to the initial part of $x^{\ell_1}\omega_{i-1}$, we can apply Claim 2 and we get that ω_i is reachable from η. Then there are $a, b \geq 0$ and a constant μ such that $\nu_E(\omega_i - \mu x^a y^b \eta) > \nu_E(\omega_i)$. Put $\omega_i^1 = \omega_i - \mu x^a y^b \eta$. We have that $\nu_{\mathcal{C}}(\omega_i^1) = \lambda_i$, since $\nu_{\mathcal{C}}(\mu x^a y^b \eta) \geq \lambda > \lambda_i$. In this way we produce an infinite list of strictly increasing divisorial order 1-forms

$$\omega_i = \omega_i^0, \omega_i^1, \omega_i^2, \ldots$$

such that $\nu_{\mathcal{C}}(\omega_i^j) = \lambda_i$, for any $i \geq 0$. For an index j we have that $\nu_E(\omega_i^j) \geq c_\Gamma$ and then $\lambda_i \geq \nu_E(\omega_i^j) \geq c_\Gamma$ and this is a contradiction. So we necessarily have that $\nu_{\mathcal{C}}(\eta) = \lambda_i$. This ends the proof of Claim 3.

Proof of Statements (1) and (2): In view of Claim 1 and Claim 3, there is a 1-form η with $\nu_E(\eta) = t_i$ such that $\nu_{\mathcal{C}}(\eta) = \lambda_i$, whose initial part is proportional to the initial part of $x^{\ell_1}\omega_{i-1}$. In order to prove Statement (1), it remains to prove that if $\nu_E(\omega) = t_i$ then $\nu_{\mathcal{C}}(\omega) \leq \lambda_i$. Assume that $\lambda = \nu_{\mathcal{C}}(\omega) > \lambda_i = \nu_{\mathcal{C}}(\eta)$. The 1-form ω is basic and resonant and it has the same divisorial order as η. Hence there is a constant $\mu \neq 0$ such that

$$\nu_E(\eta^1) > t_i = \nu_E(\eta) = \nu_E(\omega), \quad \eta^1 = \eta - \mu\omega.$$

The 1-form η^1 satisfies that $\nu_{\mathcal{C}}(\eta^1) = \lambda_i \notin \Lambda_{i-1}$; by Claim 2, there are $a, b \geq 0$ and a constant μ' such that

$$\nu_E(\eta^2) > \nu_E(\eta^1), \quad \eta^2 = \eta^1 - \mu' x^a y^b \eta.$$

We have that $\nu_{\mathcal{C}}(\eta^2) = \lambda_i$ and $\nu_E(\eta^2) > \nu_E(\eta^1)$. Repeating this procedure, we have a list of 1-forms $\eta^1, \eta^2, \ldots$ with strictly increasing divisorial order such that $\nu_{\mathcal{C}}(\eta^j) = \lambda_i$ for any j. We find a contradiction just by considering one of such η^j with $\nu_E(\eta^j) \geq c_\Gamma$. This ends the proof of Statement (1).

Let us prove Statement (2). Choose ω with $\nu_{\mathcal{C}}(\omega) = \lambda_i$. By Claim 2, we have that ω is reachable from η and hence $\nu_E(\omega) \geq t_i$. Assume by contradiction that $\nu_E(\omega) > t_i$. There is a constant μ and $a, b \geq 0$ with $a + b \geq 1$ such that

$$\nu_E(\omega^1) > \nu_E(\omega), \quad \omega^1 = \omega - \mu x^a y^b \eta.$$

Since $\nu_{\mathcal{C}}(\mu x^a y^b \eta) = an + bm + \lambda_i > \lambda_i$, we have that $\nu_{\mathcal{C}}(\omega^1) = \lambda_i$. Repeating the argument, we get a sequence of 1-forms $\omega^0 = \omega, \omega^1, \ldots$ with strictly increasing divisorial order such that $\nu_{\mathcal{C}}(\omega^j) = \lambda_i$ for any j. This is a contradiction.

Proof of Statement (3): By Claim 2, we have that ω_i is reachable from $x^{\ell_1}\omega_{i-1}$. By Statement (2) (already proved) we have that $\nu_E(\omega_i) = t_i$. Hence the initial part of ω_i is proportional to the initial part of $x^{\ell_1}\omega_{i-1}$. Consider a 1-form ω with $\nu_{\mathcal{C}}(\omega) \notin \Lambda_{i-1}$. By Claim 2 the 1-form ω is reachable from $x^{\ell_1}\omega_{i-1}$ and hence it is reachable from ω_i. Then, there are $a, b \geq 0$ such that

$$\nu_E(x^a y^b \omega_i) = an + bm + t_i = \nu_E(\omega).$$

Since $\nu_{\mathcal{C}}(\omega) \notin \Lambda_{i-1}$, we have that $nm > \nu_{\mathcal{C}}(\omega) > \nu_E(\omega) > an + bm$, this implies the uniqueness of a, b. Let us show that $\nu_{\mathcal{C}}(\omega) \geq an + bm + \lambda_i$. Assume by contradiction that $\nu_{\mathcal{C}}(\omega) < an + bm + \lambda_i$. Consider $\omega^1 = \omega - \mu x^a y^b \omega_i$ such that $\nu_E(\omega^1) > \nu_E(\omega)$. In view of the contradiction hypothesis, we have that $\nu_{\mathcal{C}}(\omega^1) = \nu_{\mathcal{C}}(\omega)$. Moreover, if $\nu_E(\omega^1) = \nu_E(x^{a_1} y^{b_1} \omega_i)$ we also have that $\nu_{\mathcal{C}}(\omega) < a_1 n + b_1 m + \lambda_i$. The situation repeats and we obtain an infinite sequence of 1-forms $\omega^0 = \omega, \omega^1, \omega^2, \ldots$ with strictly increasing divisorial orders, such that $\nu_{\mathcal{C}}(\omega^j) = \nu_{\mathcal{C}}(\omega)$ for any $j \geq 0$. This is a contradiction.

Proof of Statement (4): Noting that $\nu_E(x^{\ell_1}\omega_{i-1}) = t_i$, by Statement (1) we have $\lambda_i \geq \nu_{\mathcal{C}}(x^{\ell_1}\omega_{i-1}) = n\ell_1 + \lambda_{i-1} = u_i$. On the other hand, since $\lambda_i \notin \Lambda_{i-1}$, we have that $\lambda_i \neq u_i$ and hence $\lambda_i > u_i$.

Proof of Statement (5): Consider $k = \lambda_i + na + mb$. Assume first that $k \notin \Lambda_{i-1}$. Let ω be such that $\nu_{\mathcal{C}}(\omega) = k$. We have to prove that

$$\nu_E(\omega) \leq \nu_E(x^a y^b \omega_i) = an + bm + t_i.$$

In view of Statement (3), we know that ω is reachable from ω_i. Hence there are $a', b' \geq 0$ and a constant μ such that $\nu_E(\omega - \mu x^{a'} y^{b'} \omega_i) > \nu_E(\omega)$. Hence

$$\nu_E(\omega) = \nu_E(x^{a'} y^{b'} \omega_i) = a'n + b'm + t_i.$$

Assume by contradiction that $\nu_E(\omega) > \nu_E(x^a y^b \omega_i) = an + bm + t_i$. This implies that $a'n + b'm > an + bm$ and thus

$$\nu_{\mathcal{C}}(x^{a'} y^{b'} \omega_i) = a'n + b'm + \lambda_i > k = an + bm + \lambda_i = \nu_{\mathcal{C}}(x^a y^b \omega_i) = \nu_{\mathcal{C}}(\omega).$$

Put $\omega^1 = \omega - \mu x^{a'} y^{b'} \omega_i$. We have that $\nu_{\mathcal{C}}(\omega^1) = k$. Repeating the argument with ω^1, we obtain an infinite list of increasing divisorial orders 1-forms $\omega^0 = \omega, \omega^1, \omega^2, \ldots$ such that $\nu_{\mathcal{C}}(\omega^j) = k \notin \Lambda_{i-1}$. This is a contradiction.

Assume now that $k \in \Lambda_{i-1}$. There is an index $\ell \leq i - 1$ such that

$$k = an + bm + \lambda_i = a'n + b'm + \lambda_\ell.$$

By Lemma 7.10, we have that $\lambda_i - \lambda_\ell > t_i - t_\ell$ and hence $an + bm + t_i < a'n + b'm + t_\ell$. The 1-form $x^{a'} y^{b'} \omega_\ell$ satisfies that $k = \nu_{\mathcal{C}}(x^{a'} y^{b'} \omega_\ell)$ and

$$\nu_E(x^{a'} y^{b'} \omega_\ell) = a'n + b'm + t_\ell > an + bm + t_i = \nu_E(x^a y^b \omega_i).$$

This ends the proof. □

Appendix C: Delorme's Decompositions

In this Appendix, we provide a proof, using another approach, of Delorme's decompositions stated in Theorem 8.5. That is, we consider a cusp $\mathcal{C} \in \text{Cusps}(E)$, an extended standard basis $\omega_{-1}, \omega_0, \omega_1, \ldots, \omega_s; \omega_{s+1}$ of $\mathcal{C}$, where $\Lambda = \Gamma(n, m, \lambda_1, \ldots, \lambda_s)$ is the semimodule of differential values of $\mathcal{C}$. We have to prove that for any indices $0 \leq j \leq i \leq s$, there is a decomposition

$$\omega_{i+1} = \sum_{\ell=-1}^{j} f_\ell^{ij} \omega_\ell$$

such that, for any $-1 \leq \ell \leq j$ we have $\nu_{\mathcal{C}}(f_\ell^{ij}\omega_\ell) \geq v_{ij}$, where $v_{ij} = t_{i+1} - t_j + \lambda_j$ and there is exactly one index $-1 \leq k \leq j-1$ such that $\nu_{\mathcal{C}}(f_k^{ij}\omega_k) = \nu_{\mathcal{C}}(f_j^{ij}\omega_j) = v_{ij}$.

Note that the case $i = 0$ is straightforward. Indeed, we have $v_{00} = n + m$ and if we write $\omega_1 = adx + bdy$, we necessarily have that $\nu_{\mathcal{C}}(adx) = \nu_{\mathcal{C}}(bdy) = n + m$ in view of the fact that the initial part of ω_1 is proportional to $mydx - nxdy$.

Thus, we assume that $i \geq 1$.

Lemma C.1 *Given a 1-form η with $\nu_{\mathcal{C}}(\eta) > u_{i+1}$ and $\nu_E(\eta) > t_{i+1}$, we have*

(a) *If $\nu_E(\eta) < nm$, there is a 1-form α such that $\nu_E(\eta - \alpha) > \nu_E(\eta)$ that can be decomposed as $\alpha = \sum_{\ell=-1}^{i} g_\ell\omega_\ell$, where $\nu_{\mathcal{C}}(g_\ell\omega_\ell) > u_{i+1}$ for $-1 \leq \ell \leq i$.*

(b) *If $\nu_E(\eta) \geq nm$, there is a decomposition $\eta = \sum_{\ell=-1}^{i} h_\ell\omega_\ell$ where each summand $h_\ell\omega_\ell$ satisfies that $\nu_{\mathcal{C}}(h_\ell\omega_\ell) > u_{i+1}$.*

Proof (b) Assume that $\nu_E(\eta) \geq nm$, we have $\eta = fdx + gdy = f\omega_{-1} + g\omega_0$, where $\nu_E(fdx) \geq nm$ and $\nu_E(gdy) \geq nm$. In view of Lemma 6.5, we have that $u_{i+1} < c_\Gamma + n < nm$. We are done by taking the decomposition $\eta = f\omega_{-1} + g\omega_0$.

(a) Assume that $\nu_E(\eta) < nm$. Note that η is a basic 1-form. There are two possible cases: η is resonant or not. Assume first that η is not resonant. Then

$$\nu_{\mathcal{C}}(\eta) = \nu_E(\eta) = \nu_E(\alpha) > u_{i+1},$$

where α is the initial part of η. Note that $\nu_E(\eta - \alpha) > \nu_E(\eta)$. We can write $\alpha = g_{-1}dx + g_0dy = g_{-1}\omega_{-1} + g_0\omega_0$, where

$$\nu_{\mathcal{C}}(g_\ell\omega_\ell) \geq \nu_E(g_\ell\omega_\ell) \geq \nu_E(\alpha) = \nu_E(\eta) = \nu_{\mathcal{C}}(\eta) > u_{i+1}, \quad \ell = -1, 0.$$

This is the desired decomposition.

Assume that η is resonant. Define $k = \max\{\ell \leq i;\ \eta \text{ is reachable from } \omega_\ell\}$. The fact that η is resonant implies that $k \geq 1$ (recall that $i \geq 1$). Consider $a, b \geq 0$ and a constant φ such that

$$\nu_E(\tilde{\eta}) > \nu_E(\eta), \quad \tilde{\eta} = \eta - \varphi x^a y^b \omega_k. \tag{21}$$

If we show that $\nu_{\mathcal{C}}(x^a y^b \omega_k) > u_{i+1}$, we are done. Let us do it. Assume first that $k = i$. We know that

$$\nu_E(\eta) = \nu_E(x^a y^b \omega_i) = an + bm + t_i > t_{i+1} = t_i + u_{i+1} - \lambda_i.$$

This implies that $an + bm > u_{i+1} - \lambda_i$ and then $\nu_{\mathcal{C}}(x^a y^b \omega_i) = an + bm + \lambda_i > u_{i+1}$.

Assume now that $1 \leq k \leq i - 1$. Let us reason by contradiction assuming that $\nu_{\mathcal{C}}(x^a y^b \omega_k) \leq u_{i+1}$. Denote by $\tilde{\eta} = \eta - \varphi x^a y^b \omega_k$. By Eq. (21), we know that

$$\nu_E(\tilde{\eta}) > \nu_E(x^a y^b \omega_k) = an + bm + t_k. \tag{22}$$

Since $\nu_{\mathcal{C}}(x^a y^b \omega_k) \leq u_{i+1} < \nu_{\mathcal{C}}(\eta)$, we have that

$$\nu_{\mathcal{C}}(\tilde{\eta}) = \nu_{\mathcal{C}}(x^a y^b \omega_k) = an + bm + \lambda_k. \tag{23}$$

In view of Eqs. (23) and (22), we can apply Statement (5) in Proposition B.1 to conclude that

$$an + bm + \lambda_k \in \Lambda_{k-1}. \tag{24}$$

Let ℓ_1 and ℓ_2 be the Λ_k-limits. Since $an + bm + \lambda_k \in \Lambda_{k-1}$, we have that $a \geq \ell_1$ or $b \geq \ell_2$, by Lemma 6.9. We have four cases to be considered:

$$u_{k+1} = n\ell_1 + \lambda_k \text{ and } a \geq \ell_1; \quad u_{k+1} = n\ell_1 + \lambda_k \text{ and } b \geq \ell_2;$$
$$u_{k+1} = m\ell_2 + \lambda_k \text{ and } a \geq \ell_1; \quad u_{k+1} = m\ell_2 + \lambda_k \text{ and } b \geq \ell_2.$$

Assume that $u_{k+1} = n\ell_1 + \lambda_k$ and $a \geq \ell_1$. This implies that $x^a y^b \omega_k$, and hence η, is reachable from $x^{\ell_1}\omega_k$ and hence from ω_{k+1}. This contradicts the maximality of the index k.

Assume that $u_{k+1} = n\ell_1 + \lambda_k$ and $b \geq \ell_2$ and $a < \ell_1$. We have that

$$\nu_{\mathcal{C}}(x^a y^b \omega_k) \geq \nu_{\mathcal{C}}(y^{\ell_2}\omega_k) = m\ell_2 + \lambda_k > u_{k+1} = n\ell_1 + \lambda_k.$$

Let q_1 and q_2 be the tops of Λ_k and Q_{Λ_k} the main top, we have that

$$\nu_{\mathcal{C}}(x^a y^b \omega_k) \geq n Q_{\Lambda_k} \geq c_{\Lambda_k} + n.$$

Recall that $c_{\Lambda_k} \leq n(Q_{\Lambda_k} - 1)$ in view of Proposition A.5. On the other hand, we know by Lemma 6.5 that $u_{i+1} < c_{\Lambda_i} + n \leq c_{\Lambda_k} + n$. We have the contradiction $u_{i+1} < c_{\Lambda_k} + n \leq \nu_{\mathcal{C}}(x^a y^b \omega_k) \leq u_{i+1}$.

The two remaining cases with $u_{k+1} = m\ell_2 + \lambda_k$ may be considered in a similar way to the previous ones. □

Proposition C.2 *We can write $\omega_{i+1} = \sum_{\ell=-1}^{i} f_\ell \omega_\ell$ where $\nu_{\mathcal{C}}(f_\ell\omega_\ell) \geq u_{i+1}$ for $-1 \leq \ell \leq i$ and such that $\nu_{\mathcal{C}}(f_i\omega_i) = u_{i+1}$ and there is exactly one index $k \in \{-1, 0, 1, \ldots, i-1\}$ satisfying that $\nu_{\mathcal{C}}(f_k\omega_k) = u_{i+1}$.*

Proof Let us consider first the case $i = 0$. We know that $u_1 = t_1 = n + m$ and that ω_1 is basic resonant, with $\nu_E(\omega_1) = n + m$. Then, there is a constant μ such that $\nu_E(\eta) > n + m$, where $\eta = \omega_1 - \mu(mydx - nxdy)$. We can write $\eta = g_{-1}dx + g_0 dy$, where $\nu_E(g_{-1}) > m$ and $\nu_E(g_0) > n$. Let us put $f_{-1} = \mu my + g_{-1}$ and $f_0 = -\mu nx + g_0$. We have that $\nu_{\mathcal{C}}(f_{-1}) = m$ and $\nu_{\mathcal{C}}(f_0) = n$; hence $\omega_1 = f_{-1}dx + f_0 dy = f_{-1}\omega_{-1} + f_0\omega_0$ is the desired decomposition.

Assume now that $1 \leq i \leq s$. Let ℓ_1, ℓ_2 be the limits of Λ_i. By Remark 6.8, there is exactly one index k with $-1 \leq k \leq i - 1$ such that

(1) If $u_{i+1} = \ell_1 n + \lambda_i$, then $u_{i+1} = \lambda_k + bm$ (note that $b \geq 1$).
(2) If $u_{i+1} = \ell_2 m + \lambda_i$, then $u_{i+1} = \lambda_k + an$ (note that $a \geq 1$).

Assume that $u_{i+1} = \ell_1 n + \lambda_i$, the case $u_{i+1} = \ell_2 m + \lambda_i$ is symmetric to this one. We have that

$$\nu_{\mathcal{C}}(x^{\ell_1}\omega_i) = u_{i+1} = \nu_{\mathcal{C}}(y^b \omega_k). \tag{25}$$

On the other hand, we have that

$$\nu_E(x^{\ell_1}\omega_i) = t_i + \ell_1 n = t_i + (u_{i+1} - \lambda_i) = t_{i+1};$$
$$\nu_E(y^b\omega_k) = t_k + bm = t_k + u_{i+1} - \lambda_k.$$

By Lemma 7.10, we have that

$$t_k + u_{i+1} - \lambda_k - t_{i+1} = (\lambda_i - \lambda_k) - (t_i - t_k) > 0,$$

and hence $t_{i+1} = \nu_E(x^{\ell_1}\omega_i) < \nu_E(y^b\omega_k)$. Since both ω_{i+1} and $x^{\ell_1}\omega_i$ are basic resonant with the same divisorial order t_{i+1}, there is a constant φ such that

$$\nu_E(\omega_{i+1} - \varphi x^{\ell_1}\omega_i) > \nu_E(\omega_{i+1}) = t_{i+1}.$$

By Eq. (25), there is a constant μ such that

$$\nu_{\mathcal{C}}(\omega^0_{i+1}) > u_{i+1}, \text{ where } \omega^0_{i+1} = \varphi x^{\ell_1}\omega_i - \mu y^b\omega_k.$$

Put $\eta^0 = \omega_{i+1} - \omega^0_{i+1} = \omega_{i+1} - \varphi x^{\ell_1}\omega_i + \mu y^b\omega_k$. We have that $\nu_E(\eta^0) > t_{i+1}$ and $\nu_{\mathcal{C}}(\eta^0) > u_{i+1}$ in view of the following facts:

(1) $\nu_E(\eta^0) \geq \min\{\nu_E(\omega_{i+1} - \varphi x^{\ell_1}\omega_i), \nu_E(\mu y^b\omega_k)\} > t_{i+1}$.
(2) $\nu_{\mathcal{C}}(\eta^0) \geq \min\{\nu_{\mathcal{C}}(\omega_{i+1}), \nu_{\mathcal{C}}(\omega^0_{i+1})\} = \min\{\lambda_{i+1}, \nu_{\mathcal{C}}(\omega^0_i)\} > u_{i+1}$. Recall that Λ is an increasing semimodule; (here we put $\lambda_{s+1} = \infty$).

The proof is now a consequence of Lemma C.1 as follows. We start with η^0 as before. If $\nu_E(\eta^0) \geq nm$, we apply Lemma C.1 (b). We are done by taking the decomposition

$$\omega_{i+1} = \omega^0_{i+1} + \sum_{\ell=-1}^{i} h_\ell\omega_\ell = \varphi x^{\ell_1}\omega_i - \mu y^b\omega_k + \sum_{\ell=-1}^{i} h_\ell\omega_\ell.$$

If $\nu_E(\eta^0) < nm$, we apply Lemma C.1 (a) and we obtain $\eta^1 = \eta^0 - \sum_{\ell=-1}^{i} g_\ell\omega_\ell$ such that $\nu_E(\eta^1) > \nu_E(\eta^0) > t_{i+1}$ and $\nu_{\mathcal{C}}(\eta^1) > u_{i+1}$. If $\nu_E(\eta^1) < nm$, we re-apply Lemma C.1 (a) to η^1. After finitely many steps, we obtain

$$\tilde{\eta} = \eta^0 - \sum_{\ell=-1}^{i} \tilde{g}_\ell\omega_\ell,$$

such that $\nu_E(\tilde{\eta}) \geq nm$, $\nu_{\mathcal{C}}(\tilde{\eta}) > u_{i+1}$ and $\nu_E(\tilde{\eta}) > t_{i+1}$, where $\nu_{\mathcal{C}}(\tilde{g}_\ell\omega_\ell) > u_{i+1}$ for any $-1 \leq \ell \leq i$. We apply Lemma C.1 (b) to $\tilde{\eta}$ to obtain that $\tilde{\eta} = \sum_{\ell=-1}^{i} \tilde{h}_\ell\omega_\ell$ with $\nu_{\mathcal{C}}(\tilde{h}_\ell\omega_\ell) > u_{i+1}$. The desired decomposition is given by

$$\omega_{i+1} = \omega^0_{i+1} + \sum_{\ell=-1}^{i} (\tilde{g}_\ell + \tilde{h}_\ell)\omega_\ell = \varphi x^{\ell_1}\omega_i - \mu y^b\omega_k + \sum_{\ell=-1}^{i} (\tilde{g}_\ell + \tilde{h}_\ell)\omega_\ell.$$

This ends the proof. □

Let us end the proof of Theorem 8.5. We already know that it is true when $j = i$, in view of Proposition C.2. We assume that the result is true for $j + 1 \leq i$ and let us show that it is true for $0 \leq j \leq i$. In order to simplify notations, let us write

$$v_j = t_{i+1} - t_j + \lambda_j, \quad v_{j+1} = t_{i+1} - t_{j+1} + \lambda_{j+1}.$$

Recall that $v_{j+1} = v_j + \lambda_{j+1} - u_{j+1}$. By induction hypothesis, we have that

$$\omega_{i+1} = \sum_{\ell=-1}^{j+1} h_\ell \omega_\ell,$$

where $\nu_{\mathcal{C}}(h_\ell \omega_\ell) \geq v_{j+1}$ for any $-1 \leq \ell \leq j+1$ and $\nu_{\mathcal{C}}(h_{j+1}\omega_{j+1}) = v_{j+1}$. We apply Proposition C.2 to write $\omega_{j+1} = \sum_{\ell=-1}^{j} g_\ell \omega_\ell$, where $\nu_{\mathcal{C}}(g_\ell \omega_\ell) \geq u_{j+1}$ for any $-1 \leq \ell \leq j$ and there is exactly one index k such that $\nu_{\mathcal{C}}(g_j \omega_j) = \nu_{\mathcal{C}}(g_k \omega_k) = u_{j+1}$. Now, we have an expression

$$\omega_{i+1} = \sum_{\ell=-1}^{j} f_\ell \omega_\ell, \quad f_\ell = h_\ell + h_{j+1} g_\ell.$$

We have the following properties:

(1) $\nu_{\mathcal{C}}(h_\ell \omega_\ell) > v_j$, for any $-1 \leq \ell \leq j$. Indeed, we know that

$$v_{j+1} = v_j + (\lambda_{j+1} - u_{j+1}) > v_j,$$

recall that the semimodule is increasing and then $\lambda_{j+1} > u_{j+1}$.

(2) $\nu_{\mathcal{C}}(h_{j+1} g_\ell \omega_\ell) \geq v_j$ and k, j are the unique indices such that

$$\nu_{\mathcal{C}}(h_{j+1} g_j \omega_j) = \nu_{\mathcal{C}}(h_{j+1} g_k \omega_k) = v_j.$$

In order to prove this, it is enough to note that

$$\nu_{\mathcal{C}}(h_{j+1} g_\ell \omega_\ell) = (v_{j+1} - \lambda_{j+1}) + \nu_{\mathcal{C}}(g_\ell \omega_\ell) \geq (v_{j+1} - \lambda_{j+1}) + u_{j+1} = v_j$$

and the equality holds exactly for the indices $\ell = j, k$.

The desired result comes from the above properties (1) and (2), noting that

$$\nu_{\mathcal{C}}(f_\ell \omega_\ell) \geq \min\{\nu_{\mathcal{C}}(h_\ell \omega_\ell), \nu_{\mathcal{C}}(h_{j+1} g_\ell \omega_\ell)\}$$

and the equality holds when the two values are different.

References

Abhyankar, S.S., Moh, T.T.: Newton–Puiseux expansions and generalized Tschirnhausen transformation. I. J. Reine Angew. Math. **260**, 47–83 (1973a)

Abhyankar, S.S., Moh, T.T.: Newton–Puiseux expansions and generalized Tschirnhausen transformation. II. J. Reine Angew. Math. **261**, 29–54 (1973b)

Alberich-Carramiñana, M., Almirón, P., Moyano-Fernández, J.-J.: Curve singularities with one Puiseux pair and value sets of modules over their local rings (2022). arXiv:2105.07943

Almirón, P., Moyano-Fernández, J.J.: A formula for the conductor of a semimodule of a numerical semigroup with two generators. Semigroup Forum **103**, 278–285 (2021)

Aroca, J.M., Hironaka, H., Vicente, J.L.: Complex Analytic Desingularization. With a Foreword by Bernard Teissier. Springer, Tokyo (2018)

Corral, N., Hernandes, M.E., Hernandes, M.E.R.: Dicritical foliations and semiroots of plane branches (2023). arXiv:2304.01047

Delorme, C.: Sur les modules des singularités de courbes planes. Bull. Soc. Math. France **106**, 417–446 (1978)

Gómez-Martínez, O.: Zariski invariant for non-isolated separatrices through Jacobian curves of pseudo-cuspidal dicritical foliations. J. Singul. **23**, 236–270 (2021)

Hefez, A., Hernandes, M.E.: Computational methods in the local theory of curves. Publicações Matemáticas do IMPA [IMPA Mathematical Publications] 23 Colóquio Brasileiro de Matemática. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro (2001)

Hefez, A., Hernandes, M.E.: Standard basis for local rings of branches and the modules of differentials. J. Symb. Comput. **42**, 178–191 (2007)

Hefez, A., Hernandes, M.E.: The analytic classification of plane branches. Bull. Lond. Math. Soc. **43**(2), 289–298 (2011)

Seidenberg, A.: Reduction of singularities of the differential equation Ady = Bdx. Am. J. Math. **90**(1), 248–269 (1968)

Wall, C.T.C.: Singular Points of Plane Curves, vol. 63. Cambridge University Press, Cambridge (2004)

Zariski, O.: Characterization of plane algebroid curves whose module of differentials has maximum torsion. Proc. Natl. Acad. Sci. U.S.A. **56**, 781–786 (1966)

Zariski, O.: The moduli problem for plane branches. With an appendix by Bernard Teissier. Translated from the 1973 French original by Ben Lichtin. University Lecture Series, vol. 39. American Mathematical Society, Providence (2006)