A Splitting Result for Real Submanifolds of a Kähler Manifold

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Abstract

Let (Z, ω) be a connected Kähler manifold with an holomorphic action of the complex reductive Lie group $U^{\mathbb{C}}$, where U is a compact connected Lie group acting in a hamiltonian fashion. Let G be a closed compatible Lie group of $U^{\mathbb{C}}$ and let M be a G -invariant connected submanifold of Z . Let $x \in M$. If G is a real form of $U^{\mathbb{C}}$, we investigate conditions such that $G \cdot x$ compact implies $U^{\mathbb{C}} \cdot x$ is compact as well. The vice-versa is also investigated. We also characterize G -invariant real submanifolds such that the norm-square of the gradient map is constant. As an application, we prove a splitting result for real connected submanifolds of (Z, ω) generalizing a result proved in Gori and Podestà (Ann Global Anal Geom 26: 315–318, 2004), see also Bedulli and Gori (Results Math 47: 194–198, 2005), Biliotti (Bull Belg Math Soc Simon Stevin 16: 107–116 2009).

Keywords Gradient map · Real reductive Lie groups · Carton decomposition

Mathematics Subject Classification 22E45 · 53D20

1 Introduction

Let (Z, ω) be a Kähler manifold. Assume that $U^{\mathbb{C}}$ acts holomorphically on Z , that U preserves ω and that there is a momentum map for the U action on Z . This means there is a map $\mu : Z \rightarrow \mathfrak{u}^*$, where $\mathfrak{u}$ is the Lie algebra of U and $\mathfrak{u}^*$ is its dual, which is U -equivariant with respect to the given action of U on Z and the coadjoint action Ad^* of U on $\mathfrak{u}^*$ and satisfying the following condition. Let $\xi \in \mathfrak{u}$. We denote

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by ξ_Z the induced vector field on Z , i.e., $\xi_Z(p) = \frac{d}{dt}|_{t=0} \exp(t\xi)p$. Let μ^ξ be the function $\mu^\xi(z) := \mu(z)(\xi)$, i.e., the contraction of the momentum map along ξ . Then $d\mu^\xi = i_{\xi_Z}\omega$.

Let G be a closed connected subgroup of $U^\mathbb{C}$ compatible with respect to the Cartan decomposition of $U^\mathbb{C}$, i.e. $G = K \exp(\mathfrak{p})$, for $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ (Helgason 2001; Knapp 2002). The inclusion $i\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces by restriction a K -equivariant map $\mu_{i\mathfrak{p}} : Z \longrightarrow (i\mathfrak{p})^*$ (Heinzner et al. 2008; Heinzner and Stötzl 2006).

Let $\langle \cdot, \cdot \rangle$ be a U -invariant scalar product on $\mathfrak{u}$. Let $\langle \cdot, \cdot \rangle$ denote also the inner product on $i\mathfrak{u}$ such that multiplication by i be an isometry of $\mathfrak{u}$ into $i\mathfrak{u}$. Hence we may identify $\mathfrak{u}^*$ and $\mathfrak{u}$ by means of $\langle \cdot, \cdot \rangle$ and so we view μ as a map $\mu : Z \longrightarrow \mathfrak{u}$. Therefore, we may view $\mu_{i\mathfrak{p}}$ as a map $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$ as follows:

$$\langle \mu_{\mathfrak{p}}(x), \beta \rangle = -\langle \mu(x), i\beta \rangle, \forall \beta \in \mathfrak{p}.$$

We call $\mu_{\mathfrak{p}}$ the G -gradient map associated with μ . We also set $\mu_{\mathfrak{p}}^\beta := \langle \mu_{\mathfrak{p}}, \beta \rangle$. By definition, it follows that $\text{grad} \mu_{\mathfrak{p}}^\beta = \beta_Z$. If M is a G -stable locally closed real submanifold of Z , we may consider $\mu_{\mathfrak{p}}$ as a mapping $\mu_{\mathfrak{p}} : M \longrightarrow \mathfrak{p}$ such that $\text{grad} \mu_{\mathfrak{p}} = \beta_M$, where the gradient is computed with respect to the induced Riemannian metric on M . Since M is G -stable it follows $\beta_Z(p) = \beta_M(p)$ for any $p \in M$.

Assume that G is a real form of $U^\mathbb{C}$. If $U^\mathbb{C} \cdot x$ is compact, then it is well-known that G has a closed orbit contained in $U^\mathbb{C} \cdot x$ (Heinzner et al. 2008). On the other hand, if $G \cdot x$ is closed then it is not in general true that $U^\mathbb{C} \cdot x$ is closed as well (Guivarc'h et al. 1998). In Sect. 2, we investigate conditions such that $G \cdot x$ compact implies $U^\mathbb{C} \cdot x$ is compact. If $G \cdot x$ is compact then we give a necessary condition for $U^\mathbb{C} \cdot x$ to be compact. If M is Lagrangian and $x \in M$, then $U^\mathbb{C} \cdot x$ being compact implies $G \cdot x$ is a Lagrangian submanifold of $U^\mathbb{C} \cdot x$. Finally, we study the case when Z is $U^\mathbb{C}$ -semistable, M is G -semistable and is contained in the zero level set of the gradient map of $K^\mathbb{C}$. As an application we obtain a well-known result of Birkes (Birkes 1971).

A strategy for analyzing the G -action on M is to view the function $\nu_{\mathfrak{p}} : M \longrightarrow \mathbb{R}$,

$$\nu_{\mathfrak{p}}(x) = \|\mu_{\mathfrak{p}}(x)\|^2$$

as a Morse-like function. The function $\nu_{\mathfrak{p}}$ is called the norm-square of the gradient map. If M is compact or $\mu_{\mathfrak{p}}$ is proper, then associated to the critical points of $\nu_{\mathfrak{p}}$ we have G -stable submanifold of M that they are strata of a Morse type stratification of M (Heinzner et al. 2008; Kirwan 1984). In Sect. 3, we investigate under which condition $\nu_{\mathfrak{p}}$ is constant. The following result has some interest by itself.

Proposition 1 *Let M be a G -stable connected submanifold of Z and let $\mu_{\mathfrak{p}} : M \longrightarrow \mathfrak{p}$ be the restricted gradient map. Then the norm-square of the gradient map $\nu_{\mathfrak{p}} : M \longrightarrow \mathbb{R}$ is constant if and only if any G -orbit is compact.*

By the stratification Theorem (Heinzner et al. 2008), it follows that M coincides with a maximal pre-stratum and $\mu_{\mathfrak{p}}(M) = K \cdot \beta$. Moreover, $M = K \times_{K^\beta} \mu_{\mathfrak{p}}^{-1}(\beta)$, where $K^\beta = \{k \in K : \text{Ad}(k)(\beta) = \beta\}$. Let $x \in \mu_{\mathfrak{p}}^{-1}(\beta)$. By the K -equivariance of $\mu_{\mathfrak{p}}$,

it follows that the stabilizer $K_x \subseteq K^\beta$. Although $G \cdot x$ is compact, it is not true in general $K_x = K^\beta$. Indeed, let U be a connected, compact semisimple Lie group and let $\rho : U \rightarrow \mathrm{SL}(W)$ be a complex representation. Let G be a noncompact connected semisimple real form of $U^\mathbb{C}$. It is well known that $U^\mathbb{C}$ has a compact orbit in $\mathbb{P}(W)$, which is a complex U -orbit (Guillemin and Sternberg 1990). Let $\mathcal{O}$ denote a compact orbit of $U^\mathbb{C}$. If $x \in \mathcal{O}$ realizes the maximum of the norm-square of the G -gradient map restricted to $\mathcal{O}$, then $G \cdot x$ is closed and it is a K -orbit (Heinzner et al. 2008). Now, $K_x = K \cap U^{\mu(x)}$ and $U^{\mu(x)} = U_x$ since $U \cdot x$ is complex (Guillemin and Sternberg 1990). However, $\mu(x) \notin \mathfrak{p}$ and so K_x does not coincide in general with $K^{\mu_{\mathfrak{p}}(x)}$.

If M is a U -invariant compact connected complex submanifold of (Z, ω) , then v_{iu} constant is equivalent to U being semisimple and $M = U/U_\beta \times \mu^{-1}(\beta)$. The above splitting is Riemannian (Gori and Podestà 2004) (see also Bedulli and Gori 2005; Biliotti 2009 for the same result under the assumption that M is symplectic). In this paper we prove this splitting result without any assumption on M .

Theorem 2 *Let M be a $U^\mathbb{C}$ -stable connected submanifold of a Kähler manifold Z and let $\mu : M \rightarrow \mathfrak{u}$ be the restricted momentum map. Then the norm-square of the momentum map $\|\mu\|^2$ is constant if and only if U is semisimple and M is U -equivariantly isometric to the product of a flag manifold and an embedded, closed submanifold which is acted on trivially by U .*

Assume that G is a real form of U . The momentum map of U on Z induces a gradient map $\mu_{i\mathfrak{k}}$ of $K^\mathbb{C}$ in Z . We say that M is G -semistable if $M = \{p \in M : \overline{G \cdot p} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset\}$.

Theorem 3 *Assume that Z is compact and $U^\mathbb{C}$ -semistable and M is a G -semistable real connected submanifold of Z . Assume also M is contained in the zero fiber of $\mu_{i\mathfrak{k}}$. Then the norm-square of the G -gradient map $\|\mu_{\mathfrak{p}}\|^2$ is constant if and only if G is semisimple and M is K -equivariantly isometric to the product of a real flag and an embedded closed submanifold which is acted on trivially by K .*

2 Closed Orbits and Gradient Map

Let (Z, ω) be a Kähler manifold. Assume that $U^\mathbb{C}$ acts holomorphically on Z , that U preserves ω and that there is a momentum map for the U action on Z . Let $G \subset U^\mathbb{C}$ be a closed compatible subgroup and let M be a G -invariant submanifold of (Z, ω) and let $\mu_{\mathfrak{p}} : M \rightarrow \mathfrak{p}$ be the associated G -gradient map.

Lemma 4 *Let $x \in M$. Then:*

- if x realizes a local maximum of $v_{\mathfrak{p}}$, then $G \cdot x = K \cdot x$ and so it is compact;
- if $G \cdot x$ is compact, then $G \cdot x = K \cdot x$ and x is a critical point of $v_{\mathfrak{p}}$.

Proof If x realizes a local maximum for $v_{\mathfrak{p}}$, then $v_{\mathfrak{p}} : G \cdot x \rightarrow \mathbb{R}$ has a local maximum at x . By Corollary 6.12, p.21 in Heinzner et al. (2008), it follows $G \cdot x = K \cdot x$.

Assume $G \cdot x$ is compact. Then $v_{\mathfrak{p}} : G \cdot x \rightarrow \mathbb{R}$ has a local maximum. Applying, again, Corollary 6.12 p.21 in Heinzner et al. (2008), we get $G \cdot x = K \cdot x$. We compute

the differential of $\nu_{\mathfrak{p}}$ at x . It is easy to check

$$d\nu_{\mathfrak{p}}(v) = 2\langle (d\mu_{\mathfrak{p}})_x(v), \mu_{\mathfrak{p}}(x) \rangle.$$

Therefore, keeping in mind that $\text{Ker}(d\mu_{\mathfrak{p}})_x = (\mathfrak{p} \cdot x)^{\perp}$, where $\mathfrak{p} \cdot x = \{\xi_Z(x) : \xi \in \mathfrak{p}\}$ see Heinzner and Schwarz (2007), it follows $(d\nu_{\mathfrak{p}})_x = 0$ on $(\mathfrak{p} \cdot x)^{\perp}$. Since $G \cdot x = K \cdot x$, it follows $\mathfrak{p} \cdot x \subset \mathfrak{k} \cdot x$ and so, keeping in mind that $\nu_{\mathfrak{p}}$ is K -invariant, $(d\nu_{\mathfrak{p}})_x = 0$ on $\mathfrak{p} \cdot x$ as well, proving x is a critical point of $\nu_{\mathfrak{p}}$. $\square$

Lemma 5 *Let $x \in M$ be such that $G \cdot x$ is compact. Let $\beta = \mu_{\mathfrak{p}}(x)$. Then*

$$\mathfrak{k} \cdot x = \mathfrak{p} \cdot x \oplus^{\perp} \mathfrak{k}^{\beta} \cdot x.$$

Therefore $\mathfrak{k} \cdot x = \mathfrak{p} \cdot x$ if and only if $\dim K \cdot x = \dim K \cdot \beta$.

Proof Since $G \cdot x$ is compact, by the above Lemma $G \cdot x = K \cdot x$. By the K -equivariance of $\mu_{\mathfrak{p}}$, it follows that $\mu_{\mathfrak{p}} : K \cdot x \rightarrow K \cdot \beta$ is a smooth fibration. Therefore, keeping in mind that $\text{Ker}(d\mu_{\mathfrak{p}})_x = (\mathfrak{p} \cdot x)^{\perp}$, we have

$$(\mathfrak{p} \cdot x)^{\perp} \cap \mathfrak{k} \cdot x = \mathfrak{k}^{\beta} \cdot x.$$

Since $G \cdot x = K \cdot x$, we get

$$\mathfrak{k} \cdot x = \mathfrak{p} \cdot x \oplus^{\perp} ((\mathfrak{p} \cdot x)^{\perp} \cap \mathfrak{k} \cdot x) = \mathfrak{p} \cdot x \oplus^{\perp} \mathfrak{k}^{\beta} \cdot x.$$

This also implies $\mathfrak{k} \cdot x = \mathfrak{p} \cdot x$ if and only if $\dim K \cdot x = \dim K \cdot \beta$, concluding the proof. $\square$

Assume that G is a real form of $U^{\mathbb{C}}$. If $G \cdot x$ is compact then it is not in general true that $U^{\mathbb{C}} \cdot x$ is compact. Indeed, let V be a complex vector space and let $\tau : G \rightarrow \text{PGL}(V)$ be an irreducible faithful projective representation. Since the center of G acts trivially, we may assume that G is semisimple. The representation τ extends to an irreducible projective representation of $U^{\mathbb{C}}$. It is well-known that $U^{\mathbb{C}}$ has a unique closed orbit (Guillemin and Sternberg 1990). It is the orbit through a highest vector. On the other hand G could have more than one closed orbit in $\mathbb{P}(V)$ Guivarc'h et al. (1998, Proposition 4.28, p. 58). The following result tells us that there exists a unique compact G -orbit contained in the unique compact orbit of $U^{\mathbb{C}}$.

Proposition 6 *Let $M = U^{\mathbb{C}} \cdot x$ be a compact orbit. If G is a real form of U , then there exists exactly one closed G -orbit in M .*

Proof $U^{\mathbb{C}} \cdot x = U \cdot x$ and it is a flag manifold (Heinzner et al. 2008; Guillemin and Sternberg 1990). Applying a beautiful old Theorem of Wolf (1969), it follows that G has a unique closed orbit in M . The G orbit is given by the orbit throughout the maximum of the norm square of the gradient map (Heinzner et al. 2008). $\square$

The following result arises from Lemma 5.

Corollary 7 *Let $x \in M$ be such that $G \cdot x$ is compact. If $\dim K \cdot x = \dim K \cdot \mu_{\mathfrak{p}}(x)$, then $U^{\mathbb{C}} \cdot x$ is compact.*

Proof Since $u = \mathfrak{k} \oplus i\mathfrak{p}$, it follows $u \cdot x = \mathfrak{k} \cdot x + i\mathfrak{p} \cdot x$. By Lemma 5, $\mathfrak{k} \cdot x = \mathfrak{p} \cdot x$ and so $u^{\mathbb{C}} \cdot x = u \cdot x$. This implies $U \cdot x$ is open and closed in $U^{\mathbb{C}} \cdot x$. Therefore $U^{\mathbb{C}} \cdot x = U \cdot x$, concluding the proof. $\square$

The following result gives a necessary and sufficient condition so that $U^{\mathbb{C}} \cdot x$ is compact whenever $G \cdot x$ is.

Proposition 8 *Let $x \in M$ be such that $G \cdot x$ is compact. If G is a real form of $U^{\mathbb{C}}$, then $U^{\mathbb{C}} \cdot x$ is compact if and only if $i\mathfrak{k}^{\mu_{\mathfrak{p}}(x)} \cdot x \subseteq u \cdot x \cap i(\mathfrak{p} \cdot x)^{\perp}$. If M is Lagrangian, then $U^{\mathbb{C}} \cdot x$ is compact if and only if $\mu_{\mathfrak{p}} : K \cdot x \rightarrow K \cdot \mu_{\mathfrak{p}}(x)$ is a covering map. Moreover, $G \cdot x$ is a Lagrangian submanifold of $U^{\mathbb{C}} \cdot x$.*

Proof Set $\beta = \mu_{\mathfrak{p}}(x)$. By Lemma 5, $\mathfrak{k} \cdot x = \mathfrak{p} \cdot x \oplus^{\perp} \mathfrak{k}^{\beta} \cdot x$. Therefore, keeping in mind $u = \mathfrak{k} \oplus i\mathfrak{p}$, we have

$$u \cdot x = \mathfrak{p} \cdot x \oplus^{\perp} \mathfrak{k}^{\beta} \cdot x + i\mathfrak{p} \cdot x.$$

Since $i\mathfrak{k}^{\beta} \cdot x$ is orthogonal to $i\mathfrak{p} \cdot x$, it follows that $u \cdot x = u^{\mathbb{C}} \cdot x$, if and only if $i\mathfrak{k}^{\beta} \cdot x \subset u \cdot x \cap i(\mathfrak{p} \cdot x)^{\perp}$. If M is Lagrangian, then $T_x Z = T_x M \oplus^{\perp} J(T_x M)$. Therefore

$$u \cdot x = \mathfrak{p} \cdot x \oplus^{\perp} \mathfrak{k}^{\beta} \cdot x \oplus^{\perp} i\mathfrak{p} \cdot x.$$

This implies $u \cdot x = u^{\mathbb{C}} \cdot x$ if and only if $i\mathfrak{k}^{\beta} \cdot x \subseteq i\mathfrak{p} \cdot x$. By the first part of the proof we get $U^{\mathbb{C}} \cdot x$ is compact if and only if $\mathfrak{k}^{\beta} \cdot x = \{0\}$ and so if and only if $\dim K \cdot x = \dim K \cdot \beta$. In particular $\mathfrak{p} \cdot x = \mathfrak{k} \cdot x$. This implies $\dim_{\mathbb{R}} G \cdot x = \dim_{\mathbb{C}} U^{\mathbb{C}} \cdot x$ and so $G \cdot x$ is a compact Lagrangian submanifold of $U^{\mathbb{C}} \cdot x$. $\square$

Proposition 9 *Let M be a G -invariant Lagrangian submanifold of (Z, ω) . Let $x \in M$. Then $U^{\mathbb{C}} \cdot x$ is compact if and only if $\mathfrak{k} \cdot x = \mathfrak{p} \cdot x$. In particular $G \cdot x$ is compact and it is a Lagrangian submanifold of $U^{\mathbb{C}} \cdot x$.*

Proof Since M is Lagrangian, we have

$$u \cdot x = \mathfrak{k} \cdot x \oplus^{\perp} i\mathfrak{p} \cdot x.$$

Therefore $u \cdot x = u^{\mathbb{C}} \cdot x$ if and only if $i\mathfrak{k} \cdot x \subseteq i\mathfrak{p} \cdot x$ and $\mathfrak{p} \cdot x \subseteq \mathfrak{k} \cdot x$ hence if and only if $\mathfrak{k} \cdot x = \mathfrak{p} \cdot x$. This also implies $G \cdot x$ is compact, $\dim_{\mathbb{R}} G \cdot x = \dim_{\mathbb{C}} U^{\mathbb{C}} \cdot x$ and so $G \cdot x$ is a compact Lagrangian submanifold of $U^{\mathbb{C}} \cdot x$.

Proposition 10 *Let $x \in Z$. Assume that both $G \cdot x$ and $U^{\mathbb{C}} \cdot x$ are compact. Then $\dim_{\mathbb{R}} U^{\mathbb{C}} \cdot x \leq 2 \dim G \cdot x$. If the equality holds then $G \cdot x$ is totally real.*

Proof By Lemma 4 $U^{\mathbb{C}} \cdot x = U \cdot x$ and $G \cdot x = K \cdot x$. Since $u \cdot x = \mathfrak{k} \cdot x + i\mathfrak{p} \cdot x$ and $\mathfrak{p} \cdot x \subseteq \mathfrak{k} \cdot x$, it follows that

$$\dim_{\mathbb{R}} U^{\mathbb{C}} \cdot x \leq 2 \dim G \cdot x.$$

Note also that $\mathfrak{k}^{\mathbb{C}} \cdot x = \mathfrak{u}^{\mathbb{C}} \cdot x$. This implies $K^{\mathbb{C}} \cdot x$ is open in $U^{\mathbb{C}} \cdot x$. This remark is not new, see Heinzner et al. (2008), Heinzner and Stötzel (2006), and it arises from the Matsuki duality (Matsuki 1982). Finally, $2 \dim G \cdot x = \dim_{\mathbb{R}} U^{\mathbb{C}}$ if and only if $\mathfrak{k} \cdot x = \mathfrak{p} \cdot x$ and $\mathfrak{u} \cdot x = \mathfrak{k} \cdot x \oplus i\mathfrak{p} \cdot x$. In particular $G \cdot x$ is totally real in $U^{\mathbb{C}} \cdot x$. $\square$

The momentum map of U on Z induces a gradient map $\mu_{i\mathfrak{k}}$ of $K^{\mathbb{C}}$ in Z . Assume that M is contained in the zero fiber of $\mu_{i\mathfrak{k}}$.

Proposition 11 *Let $x \in M$. If $U^{\mathbb{C}} \cdot x$ is compact, then $G \cdot x$ is compact.*

Proof Let $y \in U^{\mathbb{C}} \cdot x$. Since $\mu = \mu_{i\mathfrak{k}} + \mu_{\mathfrak{p}}$, $M \subseteq \mu_{i\mathfrak{k}}^{-1}(0)$ and keeping in mind that $U^{\mathbb{C}} \cdot x = U \cdot x$, it follows that

$$\| \mu_{\mathfrak{p}}(y) \|^2 \leq \| \mu(y) \|^2 = \| \mu(x) \|^2 = \| \mu_{\mathfrak{p}}(x) \|^2.$$

Hence $\nu_{\mathfrak{p}} : U^{\mathbb{C}} \cdot x \rightarrow \mathbb{R}$ achieves its maximum in x . By Lemma 4, $G \cdot x$ is compact.

We say that M is G -semistable if $M = \{p \in M : \overline{U^{\mathbb{C}} \cdot p} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset\}$. In the papers (Heinzner and Schwarz 2007; Heinzner et al. 2008), the authors proved if M is G -semistable then $G \cdot x$ is closed if and only if $G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$. As an application we get the following result.

Proposition 12 *Assume that (Z, ω) is $U^{\mathbb{C}}$ -semistable and M is G -semistable, it is contained in the zero fiber of $\mu_{i\mathfrak{k}}$. Let $x \in M$. If $G \cdot x$ is closed, then $U^{\mathbb{C}} \cdot x$ is closed as well.*

Proof If $G \cdot x$ is closed then $G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$. Since $\mu_{\mathfrak{p}}^{-1}(0) \cap M = \mu^{-1}(0) \cap M$, it follows that $U^{\mathbb{C}} \cdot x \cap \mu^{-1}(0) \neq \emptyset$ as well and so $U^{\mathbb{C}} \cdot x$ is closed. $\square$

A corollary we prove a well-known result of Birkes (1971), see also Heinzner and Stötzel (1962).

Corollary 13 *Let G be a real form of U . Let V be complex vector space and W be real subspace of V such that $V = W^{\mathbb{C}}$. Assume that G acts on W . Let $w \in W$. Then $G \cdot w$ is closed if and only if $U^{\mathbb{C}} \cdot x$ is closed.*

Proof It is well-known that V , respectively W , is $U^{\mathbb{C}}$ -semistable, respectively G -semistable (Richardson and Slodowoy 1990), see also Biliotti (2021). Since W is a Lagrangian subspace of V , applying the above Proposition it follows that $G \cdot x$ closed implies $U^{\mathbb{C}} \cdot x$ is closed as well. Vice-versa, assume that $U^{\mathbb{C}} \cdot x$ is closed. The G -action on V is holomorphic and W is a real form of V . Since G is a real form of $U^{\mathbb{C}}$ the dimension of any G -orbit in $U^{\mathbb{C}} \cdot x \cap W$ coincides with the real dimension of $U^{\mathbb{C}} \cdot x$ (Borel and Harish-Chandra 1962, Proposition 2.3). Then all these orbits, and so $G \cdot x$, are closed. $\square$

3 Norm Square of the Gradient Map

We investigate splitting results for G -invariant real submanifolds of (Z, ω) .

Proposition 14 *Let M be a G -stable connected submanifold of Z and let $\mu_{\mathfrak{p}} : M \rightarrow \mathfrak{p}$ be the restricted gradient map. Then the norm-square of the gradient map $\nu_{\mathfrak{p}} : M \rightarrow \mathbb{R}$ is constant if and only if any G -orbit is compact.*

Proof Assume $\nu_{\mathfrak{p}}$ is constant. Let $x \in M$. Then $\nu_{\mathfrak{p}} : G \cdot x \rightarrow \mathbb{R}$ is constant and so $\nu_{\mathfrak{p}}$ has a maximum on x . By Lemma 4 $G \cdot x = K \cdot x$ and so it is compact. Vice-versa, assume that any G -orbit is compact. By Lemma 4 $(d\nu_{\mathfrak{p}})_x = 0$ for any $x \in M$. Since M is connected it follows $\nu_{\mathfrak{p}}$ is constant. $\square$

The following result is proved in Heinzner et al. (2008). For the sake of completeness we give a proof.

Proposition 15 *Let M be a G -stable connected submanifold of Z and let $\mu_{\mathfrak{p}} : M \rightarrow \mathfrak{p}$ be the restricted gradient map. If $\nu_{\mathfrak{p}}$ is constant, then $\mu_{\mathfrak{p}}(M) = K \cdot \beta$, $\mu_{\mathfrak{p}}^{-1}(\beta)$ is a submanifold and the following splitting*

$$M = K \times_{K^{\beta}} \mu_{\mathfrak{p}}^{-1}(\beta),$$

holds.

Proof Since $\nu_{\mathfrak{p}}$ is constant, it follows that $M = S_{\beta}$, where S_{β} is the maximal strata, and $\mu_{\mathfrak{p}}(S_{\beta}) = \mu_{\mathfrak{p}}(M) = K \cdot \beta$ (Heinzner et al. (2008, p. 21). In particular $M = K \mu_{\mathfrak{p}}^{-1}(\beta)$ and we may think $\mu_{\mathfrak{p}} : M \rightarrow K \cdot \beta$. Therefore β is a regular value and so $\mu_{\mathfrak{p}}^{-1}(\beta)$ is a K^{β} -invariant submanifold of M .

Let $x \in \mu_{\mathfrak{p}}^{-1}(\beta)$. By the K -equivariance of $\mu_{\mathfrak{p}}$, it is easy to check $K \cdot x \cap \mu_{\mathfrak{p}}^{-1}(\beta) = K^{\beta} \cdot x$. We claim that the same holds infinitesimally, i.e., $T_x \mu_{\mathfrak{p}}^{-1}(\beta) \cap \mathfrak{k} \cdot x = \mathfrak{k}^{\beta} \cdot x$. Indeed, let $v \in T_x \mu_{\mathfrak{p}}^{-1}(\beta) \cap \mathfrak{k} \cdot x$. Let $\xi \in \mathfrak{k}$ such that $v = \xi_M(x)$. Since $T_x \mu_{\mathfrak{p}}^{-1}(\beta) = \text{Ker}(d\mu_{\mathfrak{p}})_x$, we get

$$0 = \left. \frac{d}{dt} \right|_{t=0} \mu_{\mathfrak{p}}(\exp(t\xi)x) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}(\exp(t\xi))\beta,$$

and so $v \in \mathfrak{k}^{\beta} \cdot x$.

We define the map

$$\Psi : K \times_{K^{\beta}} \mu_{\mathfrak{p}}^{-1}(\beta) \rightarrow M \quad [k, x] \mapsto kx.$$

It is easy to check that Ψ is K -equivariant and smooth. Since $\mu_{\mathfrak{p}}(M) = K \cdot \beta$ it follows $M = K \cdot \mu_{\mathfrak{p}}^{-1}(\beta)$ and so Ψ is surjective. It is also injective since $kx = k'x$ if and only if $k'^{-1}k \in K^{\beta}$, proving it is bijective. Now, we proof that Ψ is a local diffeomorphism. This implies that Ψ is a diffeomorphism concluding the proof. Note that it is enough to prove $d\Psi_{[e, x]}$ is a diffeomorphism by the K -equivariance. Now,

$$T_x M = (\mathfrak{p} \cdot x) \oplus (\mathfrak{p} \cdot x)^{\perp} = (\mathfrak{p} \cdot x) \oplus T_x \mu_{\mathfrak{p}}^{-1}(\beta).$$

By Proposition 14 any G orbit is a K orbit. This implies $\mathfrak{p} \cdot x \subset \mathfrak{k} \cdot x$. Since $\mathfrak{k}^\beta \cdot x \subset (\mathfrak{p} \cdot x)^\perp$, it follows that the map

$$\mathfrak{p} \cdot x \hookrightarrow \mathfrak{k} \cdot x \longrightarrow \mathfrak{k} \cdot x / \mathfrak{k}^\beta \cdot x,$$

is injective. Therefore $d\Psi_{[e,x]}$ is surjective. Since Ψ is bijective it follows that $d\Psi_{[e,x]}$ must be bijective. $\square$

We are ready to prove the splitting results.

Proof of Theorem 2 Since ν is constant, applying Proposition 14 it follows that any $U^\mathbb{C}$ -orbit is compact and it is a complex U orbit. Then for any $x \in M$, we have $U_x = U_{\mu(x)}$ (Guillemin and Sternberg 1990). Since $U_{\mu(x)}$ is a centralizer of a torus, then the center of U does not act on M and so U is semisimple. By the above proposition $M = U/U^\beta \times \mu^{-1}(\beta)$ and for every $x \in \mu^{-1}(\beta)$, $U_x = U^\beta$ and so U_x acts trivially on $\mu^{-1}(\beta)$. If $x \in \mu^{-1}(\beta)$, then

$$T_x M = (iu \cdot x) \oplus T_x \mu^{-1}(\beta) = T_x U \cdot x \oplus T_x \mu^{-1}(\beta).$$

This implies that the U -action on M is polar with section $\mu^{-1}(\beta)$ (Dadok 1985) and so $\mu^{-1}(\beta)$ is totally geodesic. We claim that the above splitting is Riemannian.

Let $\xi \in \mathfrak{u}$ and let ξ_M the induced vector field. It is enough to prove that the function $g(\xi_M, \xi_M)$ is constant when restricted to $\mu^{-1}(\beta)$.

Let $x \in \mu^{-1}(p)$ and $v \in T_x \mu^{-1}(p)$. We may extend v to a vector field on a neighborhood of p , that we denote by X , such that $g(X, \xi_M) = 0$ for any $z \in W$ and for any $\xi \in \mathfrak{u}$. Indeed, let $\xi_1, \dots, \xi_k \in \mathfrak{u}$ such that $(\xi_1)_M(x), \dots, (\xi_k)_M(x)$ is a basis of $T_x U \cdot x$. Since the U action on M has only one type of orbit, it follows that there exists a neighborhood W of x such that $(\xi_1)_M(y), \dots, (\xi_k)_M(y)$ is a basis of $T_y U \cdot y$ for any $y \in W$. Applying a Gram–Schmidt process we get an orthonormal basis $\{Y_1, \dots, Y_k\}$ of $T_y U \cdot y$ for any $y \in W$. Let $\tilde{X}$ any local extension of v . Then

$$X = \tilde{X} - g(Y_1, \tilde{X})Y_1 - \dots - g(Y_k, \tilde{X})Y_k,$$

satisfies the above conditions. Moreover, for any $z \in \mu_p^{-1}(\beta) \cap W$, the vector field X lies in $T_z \mu_p^{-1}(\beta)$ due to the orthogonal splitting $T_z M = T_z U \cdot z \oplus T_z \mu_p^{-1}(\beta)$.

Let $\nu_M = -J(\xi_M)$. Then $J(\nu_M) = \xi_M$. Since $M = U/U^\beta \times \mu^{-1}(p)$, it follows $[X, \xi_M] = [X, \nu_M] = 0$ along $\mu_p^{-1}(\beta)$. By the closedness of ω , we have

$$d\omega(v, \nu_M(x), \xi_M(x)) = 0.$$

On the other hand, by the expression of the differential (Kobayashi and Nomizu 1996), we have

$$\begin{aligned} d\omega(v, \nu_M(x), \xi_M(x)) &= X\omega(\nu_M, \xi_M) + \nu_M\omega(\xi_M, X) + \xi_M\omega(X, \nu_M) \\ &\quad - \omega([X, \nu_M], \xi_M) - \omega([\nu_M, \xi_N], X) - \omega([\xi_M, X], Y). \end{aligned}$$

Now, $\omega([X, v_M], \xi_M) = \omega([\xi_M, X], Y) = 0$ due to the fact that $[X, v_M](x) = [\xi_M, X](x) = 0$, The term $\omega([v_M, \xi_N], X) = 0$, since

$$\omega([v_M, \xi_N], X) = g(J([v_M, \xi_M], X)) = 0$$

due to the facts that any U -orbit is complex and the splitting $T_x M = T_x \mu^{-1}(\beta) \stackrel{\perp}{\oplus} T_x U \cdot x$ holds. Finally, $v_M \omega(\xi_M, X) = 0$, respectively $\xi_M \omega(X, v_M) = 0$, due to the fact that

$$\omega(\xi_M, X) = g(J\xi_M, X) = 0,$$

respectively,

$$\omega(X, v_M) = g(JX, v_M) = -g(X, Jv_M) = 0,$$

along $U \cdot x$. Therefore

$$0 = d\omega(v, v_M(x), \xi_M(x)) = X\omega(v_M, \xi_M) = Xg(J(v_M), \xi_M) = Xg(\xi_M, \xi_M),$$

and so $g(\xi_M, \xi_M)$ is constant along $\mu_p^{-1}(\beta)$ and the result is proved. $\square$

Proof of Theorem 3 By Proposition 15 $M = K \times_{K^\beta} \mu_p^{-1}(\beta)$. By Proposition 14 it follows $G \cdot x$ is compact for any $x \in \mu_p^{-1}(\beta)$. Let $x \in \mu_p^{-1}(\beta)$. By Proposition 12, $U^\mathbb{C} \cdot x$ is compact as well. Since M is contained in the zero fiber of $\mu_{t\mathfrak{k}}$, it follows that $\mu_p(x) = \mu(x) = \beta$. This implies $K_x = K \cap U_x = K \cap U^\beta = K^\beta$ for any $x \in \mu_p^{-1}(\beta)$ and so $M = K/K^\beta \times \mu_p^{-1}(\beta)$. The Lie algebra of the center of G is contained in the Lie algebra of the center of $U^\mathbb{C}$. On the other hand, the Lie algebra of the center of $U^\mathbb{C}$ is the complexification of the Lie algebra of the center of U which acts trivially on M . This implies G is semisimple. Finally, keeping in mind that ω is closed and $U^\mathbb{C} \cdot x$ is compact for any $x \in \mu_p^{-1}(\beta)$, applying the same idea of the above proof we get the splitting $M = K/K^\beta \times \mu_p^{-1}(\beta)$ is Riemannian. $\square$

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