Polytope Duality for Families of $K3$ Surfaces and Coupling

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Received: 12 November 2019 / Accepted: 30 May 2020 / Published online: 24 June 2020
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Abstract

We study a relation between coupling introduced by Ebeling (Kodai Math J 29:319–336, 2006) and the polytope duality among families of $K3$ surfaces.

Keywords Families of $K3$ surfaces · Coupling · Polytope duality

Mathematics Subject Classification 14J28 · 52B20

1 Introduction

A notion of coupling is introduced by Ebeling (2006) as a tone-down of the duality of weight systems by Kobayashi (2008). It is proved that the duality is also “polar dual”, in the sense that certain rational polytopes associated to weight systems are dual. In Ebeling (2006), there is given a list of coupling pairs for 95 weight systems of simple $K3$ hypersurface singularities classified by Yonemura (1990), and it is proved that the duality induces Saito’s duality, which is a relation between the zeta functions of the Milnor fibre of the singularities. It is interesting to note that these dualities translate a famous mirror symmetry.

Instead of rational polytopes, we are interested in Batyrev’s toric mirror symmetry (Batyrev 1994) for integral polytopes in this article. As a generalisation of Arnold’s strange duality for unimodal singularities, Ebeling and Takahashi (2011) defined a notion of strange duality for invertible polynomials. It is studied by Mase and Ueda (2015) that the strange duality for bimodal singularities defined by invertible polynomials extends to the polytope duality among families of $K3$ surfaces.

The polytope duality is focusing on more details in geometry of $K3$ surfaces such as resolution of singularities, as a compactification of some singularities in three dimensional space which should affect the geometry of the surfaces while the polar duality in Ebeling (2006), Kobayashi (2008) is determined only by the weight systems

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and thus, it is quite global. There are some profiles in coupling as is explained in Ebeling (2006) from the viewpoint of mirror symmetry, in particular, in terms of the Milnor fibres. It is expected that a study of the polar duality associated to coupling gives another explanation to coupling by Batyrev's mirror symmetry. In turn, we expect some relation can be extracted between the Milnor fibres of singularities and the geometry of associated $K3$ surfaces.

Motivated by this, and focusing on coupling, we consider the following problem.

PROBLEM. Let a and b be weight systems that are coupling pair and their weighted magic square is given by polynomials f and f' , respectively. Determine whether or not there exist reflexive polytopes Δ and Δ' , and projectivisations F and F' of f and f' in the weighted projective spaces $\mathbb{P}(a)$ and $\mathbb{P}(b)$, respectively, such that they are *polytope dual* in the sense that they satisfy the following conditions:

$$\Delta^* \simeq \Delta', \quad \Delta_F \subset \Delta \subset \Delta_a, \quad \Delta_{F'} \subset \Delta' \subset \Delta_b.$$

Here Δ_F and $\Delta_{F'}$ are Newton polytopes of F and F' , respectively, and Δ_a and Δ_b are polytopes that define the weighted projective spaces $\mathbb{P}(a)$ and $\mathbb{P}(b)$. Note that the isomorphism between Δ^* and Δ' is that of as integral (or lattice) polytopes.

The main theorem of the article, which is proved in Sect. 5 is stated:

Theorem 5.1 Any coupling pairs in Yonemura's list extend to the polytope dual except the following three pairs of weight systems : (1, 3, 4, 7; 15) (self-coupling), (1, 3, 4, 4; 12) (self-coupling), and (1, 1, 3, 5; 10) and (3, 5, 11, 19; 38). The complete choice of reflexive polytopes is given in Table 1.

In Sect. 2, we recall the definitions concerning the weighted projective spaces and the strange duality. In Sect. 3, we recall the definition of coupling. In Sect. 4, we explain the polytope duality after recalling necessary notions of toric geometry.

2 Preliminary

A $K3$ surface is a compact complex 2-dimensional non-singular algebraic variety with trivial canonical bundle and irregularity zero.

Let $(a_0, \dots, a_n)$ be a well-posed $(n+1)$ -tuple of positive integers, that is, $a_0 \leq \dots \leq a_n$, and any n -tuples out of them are coprime. Recall that the *weighted projective space* $\mathbb{P}(a) = \mathbb{P}(a_0, \dots, a_n)$ with *weight* $a = (a_0, \dots, a_n)$ is defined by

$$\mathbb{P}(a) = \mathbb{P}(a_0, \dots, a_n) := \mathbb{C}^{n+2} \setminus \{0\} / \sim,$$

where $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ if there exists a non-zero complex number λ such that

$$(y_0, \dots, y_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$$

holds. We call a_i the *weight* of the variable x_i , and denote it by $\text{wt } x_i$.

In case $n = 3$, we fix a system of variables W, X, Y, Z of the weighted projective space $\mathbb{P}(a_0, a_1, a_2, a_3)$ with weights

$$\text{wt } W = a_0, \quad \text{wt } X = a_1, \quad \text{wt } Y = a_2, \quad \text{wt } Z = a_3.$$

We say a polynomial F in $\mathbb{P}(a_0, a_1, a_2, a_3)$ is an *anticanonical section* if F is of degree $d := a_0 + a_1 + a_2 + a_3$. The tuple $(a_0, a_1, a_2, a_3; d)$ is called a *weight system*.

By Dolgachev (1982), the anticanonical sheaf of $\mathbb{P}(a)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(a)}(-d)$. All weight systems that give simple $K3$ hypersurface singularities are classified by Yonemura (1990). Namely, if a weight a is in Yonemura's list, general anticanonical sections of $\mathbb{P}(a)$ are birational to $K3$ surfaces. Thus, one can consider families of $K3$ surfaces.

For a polynomial f in three variables, a polynomial F in the weighted projective space $\mathbb{P}(a)$ is called a *projectivisation of f* if there exists a linear form l in $\mathbb{P}(a)$ such that

$$f = F|_{l=0}$$

holds. In this case, the form l is called a *section of f* for F .

3 Coupling

Recall the definition of *coupling* for weight systems with three entries. Let $w = (w_1, w_2, w_3; d)$ and $w' = (w'_1, w'_2, w'_3; h)$ be weight systems, with weights being well-posed.

A *weighted magic square C* for the weight systems w and w' is a square matrix of size 3 that satisfies relations

$$C^t(w_1 \ w_2 \ w_3) = {}^t(d \ d \ d) \quad \text{and} \quad (w'_1 \ w'_2 \ w'_3) C = (h \ h \ h).$$

The pair of weight systems (w, w') is called *coupled* if C is *almost primitive*, that is, if $|\det C| = \left(d - \sum_{i=1}^3 w_i\right)h = \left(h - \sum_{i=1}^3 w'_i\right)d$ hold. The pair of weight systems (w, w') is *strongly coupled* if it is coupled and the weighted magic square C has entries zero in every column and row.

Thus, one can assign polynomials f and f' to $C = (c_{ij})$ in such a way that

$$f = \sum_{i=1}^3 x^{c_{i1}} y^{c_{i2}} z^{c_{i3}}, \quad f' = \sum_{i=1}^3 x^{c_{1i}} y^{c_{2i}} z^{c_{3i}}.$$

In other words, there exist weight systems $(a_0, a_1, a_2, a_3; d)$ and $(b_0, b_1, b_2, b_3; h)$ such that there exist $i, j \in \{0, 1, 2, 3\}$ with properties

$$|\det C| = ha_i = kb_j, \quad \text{and} \quad {}^t A_{f'} = A_f.$$

Define an anticanonical section F of weight system $(a_0, a_1, a_2, a_3; d)$ so that l is the section of f for F , where l is a linear form defined by

$$l = \begin{cases} w^{h/a_0} & \text{if } |\det C| = ha_0, \\ x^{h/a_1} & \text{if } |\det C| = ha_1, \\ y^{h/a_2} & \text{if } |\det C| = ha_2, \\ z^{h/a_3} & \text{if } |\det C| = ha_3. \end{cases}$$

Note that the choice of variables is different from the original Ebeling's paper (Ebeling 2006). And then define a polynomial F' so that

$${}^tA_{F'} = A_F$$

holds. Note that F' is a projectivisation of f' , and an anticanonical section in the weighted projective space of weight system $(b_0, b_1, b_2, b_3; h)$.

Ebeling (2006, Tables 2 and 3) gives (strongly) coupling pairs among weighted systems in Yonemura's list.

4 Duality of Polytopes

Let M be a lattice of rank 3, and N be its dual lattice $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ that is again of rank 3. A *polytope* is a convex hull of finite number of points in $M \otimes \mathbb{R}$. If vertices of a polytope Δ are $v_1, \dots, v_r$, we denote it by

$$\Delta = \text{Conv}\{v_1, \dots, v_r\}.$$

We call a polytope *integral* if all the vertices of the polytope are in M . For a polytope Δ , define the *polar dual polytope* Δ^* by

$$\Delta^* := \{y \in N \otimes \mathbb{R} \mid \langle y, x \rangle_{\mathbb{R}} \geq -1 \text{ for all } x \in \Delta\},$$

where $\langle \cdot, \cdot \rangle$ is a natural pairing $N \times M \rightarrow \mathbb{Z}$, and $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is the extension to $\mathbb{R}$ -coefficients. Let Δ be an integral polytope that contains the origin in its interior as the only lattice point. The polytope Δ is *reflexive* if the polar dual Δ^* is also an integral polytope.

Recall an interesting property of reflexive polytopes related to $K3$ surfaces due to Batyrev (1994):

Theorem 4.1 (Batyrev 1994) *Denote by $\mathbb{P}_{\Delta}$ the toric 3-fold associated to an integral polytope Δ . The following conditions are equivalent.*

- (1) *The polytope Δ is reflexive.*
- (2) *General anticanonical sections of $\mathbb{P}_{\Delta}$ are birational to $K3$ surfaces.* □

In particular, the weighted projective space $\mathbb{P}(a)$ with weight system $a = (a_0, a_1, a_2, a_3)$ in Yonemura's list is a toric Fano 3-fold determined by a reflexive polytope $\Delta^{(n)}$ in the $\mathbb{R}$ -extension of the lattice

$$M_n := \left\{ (i, j, k, l) \in \mathbb{Z}^4 \mid a_0 i + a_1 j + a_2 k + a_3 l = 0 \right\},$$

where the weight system a is assigned No. n in Yonemura's list. The anticanonical sections are weighted homogeneous polynomial of degree $d := a_0 + a_1 + a_2 + a_3$, thus, there is a one-to-one correspondence between a lattice point (i, j, k, l) in M_n and a rational monomial $W^{i+1} X^{j+1} Y^{k+1} Z^{l+1}$. In this way, once a $\mathbb{Z}$ -basis is taken for M_n , we identify lattice points in $\Delta^{(n)}$ and monomials of weighted degree d .

5 Main Result

In this section, we prove the main theorem.

Theorem 5.1 *Any coupling pairs in Yonemura's list extend to the polytope dual except the following three pairs of weight systems : $(1, 3, 4, 7; 15)$ (self-coupling), $(1, 3, 4, 4; 12)$ (self-coupling), and $(1, 1, 3, 5; 10)$ and $(3, 5, 11, 19; 38)$. The complete choice of reflexive polytopes is given in Table 1.*

Remark 1 In Table 1, the reflexive polytopes Δ and Δ' are given as a set of monomials that are vertices of them. If there are more than one pairs, they are separated by a dotted line and polytopes in the same row give the polytope duality. Although they are not originally numbered, we label a number for each case treated accordingly from the top to the bottom in Tables 2 and 3 in Ebeling (2006).

Proof Take polynomials F and F' that are respectively anticanonical sections of the weighted projective spaces $\mathbb{P}(a)$ and $\mathbb{P}(b)$ as in the Tables 2 and 3 in Ebeling (2006).

Recall that a pair of reflexive polytopes Δ and Δ' is *polytope dual* if relations $\Delta_F \subset \Delta \subset \Delta_a$, $\Delta_{F'} \subset \Delta' \subset \Delta_b$, and $\Delta^* \simeq \Delta'$ hold.

The strategy of the proof is that in each case, after taking a basis of the lattice M_n , we observe if the Newton polytope Δ_F of the polynomial F is reflexive by a direct computation. If the polytope is not reflexive, then, we search a reflexive polytope Δ satisfying inclusions $\Delta_F \subset \Delta \subset \Delta^{(n)}$ of polytopes. The analogous observations should be made for F' . Once one gets a candidate reflexive polytope Δ and Δ' , we then study whether they satisfy a relation $\Delta^* \simeq \Delta'$.

The assertion is proved case by case.

5.1 No. 1–No. 10

We claim that the unique pair $(\Delta^{(14)}, \Delta^{(14)*})$ is polytope dual commonly for Nos. 1 to 10. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 14, 28, 45$, and 51 by

$$\begin{aligned} e_1^{(14)} &= (-6, 1, 0, 0), & e_2^{(14)} &= (-14, 0, 1, 0), & e_3^{(14)} &= (-21, 0, 0, 1), \\ e_1^{(28)} &= (-3, 1, 0, 0), & e_2^{(28)} &= (-7, 0, 1, 0), & e_3^{(28)} &= (-10, 0, 0, 1), \\ e_1^{(45)} &= (-4, 1, 0, 0), & e_2^{(45)} &= (-9, 0, 1, 0), & e_3^{(45)} &= (-14, 0, 0, 1), \end{aligned}$$

Table 1 Polytope duality associated to coupling pairs

No.	Δ'	$b; h$	$a; d$	Δ
1.	Z^2, W^{42}, X^7, Y^3	1, 6, 14, 21; 42	1, 6, 14, 21; 42	Z^2, W^{42}, X^7, Y^3
2.	WZ^2, W^{21}, X^7, Y^3	1, 3, 7, 10; 21	1, 6, 14, 21; 42	Z^2, W^{42}, X^7, Y^3
3.	Z^2, W^{28}, X^7, WY^3	1, 4, 9, 14; 28	1, 6, 14, 21; 42	Z^2, W^{42}, X^7, Y^3
4.	Z^2, W^{36}, WX^7, Y^3	1, 5, 12, 18; 36	1, 6, 14, 21; 42	Z^2, W^{42}, X^7, Y^3
5.	WZ^2, W^{21}, X^7, Y^3	1, 3, 7, 10; 21	1, 3, 7, 10; 21	WZ^2, W^{21}, X^7, Y^3
6.	Z^2, W^{28}, X^7, WY^3	1, 4, 9, 14; 28	1, 3, 7, 10; 21	WZ^2, W^{21}, X^7, Y^3
7.	Z^2, W^{36}, WX^7, Y^3	1, 5, 12, 18; 36	1, 3, 7, 10; 21	WZ^2, W^{21}, X^7, Y^3
8.	Z^2, W^{28}, X^7, WY^3	1, 4, 9, 14; 28	1, 4, 9, 14; 28	Z^2, W^{28}, X^7, WY^3
9.	Z^2, W^{36}, WX^7, Y^3	1, 5, 12, 18; 36	1, 4, 9, 14; 28	Z^2, W^{28}, X^7, WY^3
10.	Z^2, W^{36}, WX^7, Y^3	1, 5, 12, 18; 36	1, 5, 12, 18; 36	Z^2, W^{36}, WX^7, Y^3
11.	$Z^2, W^{30}, W^6X^6, X^5Y, Y^3$ $Z^2, W^{30}, W^2X^7, X^5Y, Y^3$	1, 4, 10, 15; 30	1, 6, 8, 15; 30	$Z^2, W^{30}, X^5, XY^3, W^6Y^3$ $Z^2, W^{30}, X^5, XY^3, W^{14}Y^2$
12.	$Z^2, W^{30}, W^6X^6, X^5Y, Y^3$ $Z^2, W^{30}, W^2X^7, X^5Y, Y^3$	1, 4, 10, 15; 30	1, 5, 7, 13; 26	$Z^2, W^{26}, WX^5, XY^3, W^5Y^3$ $Z^2, W^{26}, WX^5, XY^3, W^{12}Y^2$
13.	$Z^2, W^{30}, X^5, XY^3, W^6Y^3$ $Z^2, W^{30}, X^5, XY^3, W^{14}Y^2$	1, 6, 8, 15; 30	1, 3, 7, 11; 22	$Z^2, W^{22}, W^4X^6, X^5Y, WY^3$ $Z^2, W^{22}, WX^7, X^5Y, WY^3$
14.	$Z^2, W^{26}, WX^5, XY^3, W^5Y^3$ $Z^2, W^{26}, WX^5, XY^3, W^{12}Y^2$	1, 5, 7, 13; 26	1, 3, 7, 11; 22	$Z^2, W^{22}, W^6X^6, X^5Y, WY^3$ $Z^2, W^{22}, X^7, X^5Y, WY^3$
15.	$Z^2, W^{24}, W^6X^6, X^4Z, Y^3$ Z^2, W^{24}, X^8, Y^3	1, 3, 8, 12; 24	1, 6, 8, 9; 24	$W^6Z^2, W^{24}, X^4, XZ^2, Y^3$ XZ^2, W^{24}, X^4, Y^3

Table 1 continued

No.	Δ'	$b; h$	$a; d$	Δ
16.	$Z^2, W^{24}, W^6 X^6, X^4 Z, Y^3$ Z^2, W^{24}, X^8, Y^3	1,3,8,12;24	1,5,7,8;21	$W^5 Z^2, W^{21}, W X^4, X Z^2, Y^3$ $X Z^2, W^{21}, W X^4, Y^3$
17.	$W^6 Z^2, W^{24}, X^4, X Z^2, Y^3$	1,6,8,9;24	1,2,5,7;15	$W Z^2, W^{15}, W^3 X^6, X^4 Z, Y^3$
18.	$W^5 Z^2, W^{21}, W X^4, X Z^2, Y^3$ $Z^2, W^{22}, W^2 X^5, X^4 Y, X Y^3, W^{10} Y^2$ $Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{22}, W^2 X^5, X^4 Y, X Y^3, W^{10} Y^2$ $Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{22}, W^2 X^5, X^4 Y, X Y^3, W^4 Y^3$ $X Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$	1,5,7,8;21	1,2,5,7;15	$W Z^2, W^{15}, W^3 X^6, X^4 Z, Y^3$ $Z^2, W^{22}, W^2 X^5, X^4 Y, X Y^3, W^{10} Y^2$ $Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{22}, W^2 X^5, X^4 Y, X Y^3, W^4 Y^3$ $X Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$
19.	$Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^{10} Y^2$ $Z^2, W^{18}, X^6, X Y^3, W^8 Y^2$ $Z^2, W^{18}, X^6, X Y^3, W^2 Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $W^3 Z^2, W^{15}, X^5, X Z^2, Y^3$	1,4,6,11;22	1,4,6,11;22	$Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{22}, W^2 X^5, X^4 Y, X Y^3, W^4 Y^3$ $X Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$ $X Z^2, W^{18}, W^6 X^3, X^3 Y, Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $X Z^2, W^{15}, W^6 X^3, X^3 Z, Y^3$
20.	$Z^2, W^{18}, X^6, X Y^3, W^2 Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $W^3 Z^2, W^{15}, X^5, X Z^2, Y^3$	1,3,5,9;18	1,4,6,7;18	$Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$ $X Z^2, W^{18}, W^6 X^3, X^3 Y, Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $X Z^2, W^{15}, W^6 X^3, X^3 Z, Y^3$
21.	$Z^2, W^{20}, X^5, Y^2 Z$	1,3,5,6;15	1,3,5,6;15	$Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$ $X Z^2, W^{18}, W^6 X^3, X^3 Y, Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $X Z^2, W^{15}, W^6 X^3, X^3 Z, Y^3$
22.	$Z^2, W^{20}, X^5, Y^2 Z$	1,4,5,10;20	1,4,5,10;20	$Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$ $X Z^2, W^{18}, W^6 X^3, X^3 Y, Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $X Z^2, W^{15}, W^6 X^3, X^3 Z, Y^3$
23.	Z^2, W^{20}, X^5, Y^4	1,4,5,10;20	1,3,4,7;15	$Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$ $X Z^2, W^{18}, W^6 X^3, X^3 Y, Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $X Z^2, W^{15}, W^6 X^3, X^3 Z, Y^3$
24.	Not exist	1,3,4,7;15	1,3,4,7;15	$Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$ $X Z^2, W^{18}, W^6 X^3, X^3 Y, Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $X Z^2, W^{15}, W^6 X^3, X^3 Z, Y^3$
25.	$Z^2, W^{16}, W X^5, X^4 Y, Y^4$ $Z^2, W^{16}, W^4 X^4, X^4 Y, Y^4$	1,3,4,8;16	1,4,5,6;16	$Z^2, W^{22}, W^6 X^4, X^4 Y, X Y^3, W^4 Y^3$ $Z^2, W^{18}, W^2 X^4, X^3 Y, Y^3$ $X Z^2, W^{18}, W^6 X^3, X^3 Y, Y^3$ $X Z^2, W^{15}, X^5, Y^3$ $X Z^2, W^{15}, W^6 X^3, X^3 Z, Y^3$

Table 1 continued

No.	Δ'	$b; h$	$a; d$	Δ
26.	$W^3Z^2, Y^2Z, XZ^2, W^{13}, W^4X^3, X^3Y, WY^3$			$W^3Z^2, Y^2Z, XZ^2, W^{13}, W^4X^3, X^3Y, W^9Y$
	$W^8Z, Y^2Z, XZ^2, W^{13}, WX^4, X^3Y, WY^3$			$W^8Z, Y^2Z, XZ^2, W^{13}, WX^4, X^3Y, W^9Y$
	$W^3Z^2, Y^2Z, XZ^2, W^{13}, WX^4, X^3Y, W^9Y$	1,3,4,5;13	1,3,4,5;13	$W^8Z, Y^2Z, XZ^2, W^{13}, WX^4, X^3Y, WY^3$
	$W^3Z^2, Y^2Z, XZ^2, W^{13}, WX^4, X^3Y, WY^3$			$W^8Z, Y^2Z, XZ^2, W^{13}, WX^4, X^3Y, WY^3$
	$W^3Z^2, Y^2Z, XZ^2, W^{13}, WX^4, X^3Y, WY^3$			$W^8Z, Y^2Z, XZ^2, W^{13}, WX^4, X^3Y, WY^3$
27.	Not exist	1,3,4,4;12	1,3,4,4;12	Not exist
28.	Z^2, W^{18}, X^9, Y^3			WZ^2, W^9X^2, X^8, Y^3
	Z^2, W^{18}, W^2X^8, Y^3	1,2,6,9;18	2,3,8,11;24	WZ^2, W^{12}, X^8, Y^3
	Z^2, W^{18}, X^9, Y^3			$Z^2, W^{16}X^2, WX^8, Y^3$
29.	Z^2, W^{18}, W^2X^8, Y^3	1,2,6,9;18	2,5,14,21;42	Z^2, W^{21}, WX^8, Y^3
30.	$Z^2, W^{14}, X^7, XY^3, W^2Y^3$			$Z^2, W^{10}X^2, W^4X^6, X^6Y, WY^3$
	$Z^2, W^{14}, X^7, XY^3, W^6Y^2$	1,2,4,7;14	2,3,8,13;26	$Z^2, W^{10}X^2, WX^8, X^6Y, WY^3$
	$XZ^2, W^{12}, W^2X^5, Y^3$			Z^2, W^{15}, X^{10}, Y^3
31.	XZ^2, W^{12}, X^6, Y^3			$Z^2, W^{12}X^2, X^{10}, Y^3$
	$Y^3, XZ^2, W^{12}, X^6, XZ^2$	1,2,4,5;12	2,3,10,15;30	$Y^3, Z^2, W^{12}X^2, W^6X^6, X^5Z$
	$Y^3, XZ^2, X^4Y, W^2X^5, W^{12}, W^2Z^2$			$Y^3, Z^2, W^5Y^2, W^{12}X^2, W^6X^6, X^5Z$
	$W^2Z^2, W^{12}, W^2X^5, X^2Y^2, Y^3, XZ^2$			$Z^2, Y^3, W^6X^6, X^5Z, W^{12}X^2, W^{10}Y$
	$W^2Z^2, W^{12}, W^2X^5, X^2Y^2, Y^3, XZ^2$			$Z^2, Y^3, W^6X^6, X^5Z, W^{12}X^2, W^{10}Y$

Table 1 continued

No.	Δ'	b, h	a, d	Δ
32.	Z^2, W^{10}, X^5, Y^5 $Z^2, W^{10}, X^5, W^2Y^4, X^{a+1}Y^{-a+4}$ $a = 0, 1, 2, 3.$ Z^2, W^{10}, X^5, Y^5	1,2,2,5;10	2,4,5,9;20	WZ^2, W^5Y^2, X^5, Y^4 $WZ^2, W^5Y^2, X^5, Y^4, W^{2(a+1)}X^{-a+4}$ $a = 0, 1, 2, 3.$ Z^2, W^5Y^2, X^5, WY^4
33.	$Z^2, W^{10}, X^5, W^2Y^4, X^{a+1}Y^{-a+4}$ $a = 0, 1, 2, 3.$ Z^2, W^{10}, X^5, Y^5	1,2,2,5;10	2,6,7,15;30	$Z^2, W^8Y^2, X^5, WY^4, W^{3(a+1)}X^{-a+4}$ $a = 0, 1, 2, 3.$ Z^2, W^8X^2, X^4Y, WY^4
34.	$Z^2, W^{10}, X^5, W^2Y^4, X^{a+1}Y^{-a+4}$ $a = 0, 1, 2, 3.$ Z^2, W^{10}, X^5, Y^5	1,2,2,5;10	2,5,6,13;26	$Z^2, W^8X^2, X^4Y, WY^4, W^{3a+4}Y^{-a+3}$ $a = 0, 1, 2, 3.$ XZ^2, W^{11}, WX^6, Y^3
35.	Z^2, W^{12}, X^{12}, Y^3	1,1,4,6;12	3,5,11,14;33	Z^2, W^{11}, WX^8, XY^3
36.	Z^2, W^{12}, X^{12}, Y^3	1,1,4,6;12	4,5,13,22;44	$Z^2, W^{12}X, X^{11}, Y^3$
37.	Z^2, W^{12}, X^{12}, Y^3	1,1,4,6;12	5,6,22,33;66	$XZ^2, W^{10}, W^6X^3, X^5Y, Y^3$
38.	$Z^2, W^{10}, X^{10}, XY^3, WY^3$ $Z^2, W^{10}, X^{10}, XY^3, W^4Y^2$	1,1,3,5;10	3,4,10,13;30	$XZ^2, W^{10}, W^2X^6, X^5Y, Y^3$ $XZ^2, W^{10}, W^2X^6, X^5Y, Y^3$ $XZ^2, W^{10}, W^2X^6, X^5Y, Y^3$
39.	Not exist	1,1,3,5;10	3,5,11,19;38	Not exist
40.	$Z^2, W^{10}, X^{10}, XY^3, WY^3$ $Z^2, W^{10}, X^{10}, X^4Y^2, WY^3$	1,1,3,5;10	4,5,18,27;54	$Z^2, W^9Y, W^6X^6, WX^{10}, Y^3$ $Z^2, W^{11}X^2, WX^{10}, Y^3, W^9Y$
41.	WZ^2, W^9, X^9, Y^3 $Y^3, WZ^2, W^9, X^9, XZ^2$	1,1,3,4;9	3,4,11,18;36	Z^2, W^{12}, X^9, WY^3 $WY^3, Z^2, W^6Z, W^4X^6, X^9$

Table 1 continued

No.	Δ'	$b; h$	$a; d$	Δ
42.	$Y^3, WZ^2, W^9, X^9, XZ^2$ WZ^2, W^9, X^9, Y^3	1,1,3,4;9	2,5,9,11;27	$Y^3, XZ^2, W^8Z, W^6X^3, WX^5$ Z^2, W^{16}, WX^9, Y^3
43.	$Y^3, WZ^2, W^9, X^9, XZ^2$ Z^2, W^8, X^8, Y^4	1,1,3,4;9	3,5,16,24;48	$Y^3, Z^2, W^8Z, W^6X^6, WX^9$ XZ^2, W^8, X^6, Y^2Z
44.	$Y^2Z, XZ^2, WZ^2, W^7, X^7$	1,1,2,4;8	3,4,7,10;24	$Z^2, W^7Y, W^4X^4, X^7, Y^4$
45.	$Y^3Z, WZ^2, X^3Z, W^5, X^5, Y^5$	1,1,2,3;7	3,4,7,14;28	$YZ^2, W^4Z, W^5X, WY^3, X^5$
46.	WZ^2, W^5, X^5, Y^5	1,1,1,2;5	4,5,7,9;25	Z^2, W^8, WX^5, Y^5
47.	Z^2, W^6, X^6, Y^6	1,1,1,2;5	5,7,8,20;40	YZ^2, W^6, X^5, XY^3
48.	Z^2, W^6, X^6, Y^6	1,1,1,3;6	5,6,8,11;30	Z^2, W^6X, X^5Y, Y^5
49.	Z^4, W^4, X^4, Y^4	1,1,1,3;6	7,8,10,25;50	Z^3, W^4X, X^3Z, Y^4
50.	$Z^2, W^7, W^aX^{7-a}, X^4Y^2, XY^4, WY^4$	1,1,1,1;4	7,8,9,12;36	$Z^2, X^4Y^2, WY^4, W^4Y^2, W^aX^{7-a}, X^7$
51.	$Z^2, W^7, W^aX^{7-a}, X^4Y^2, XY^4, W^4Y^2$	2,2,3,7;14	2,2,3,7;14	$Z^2, XY^4, WY^4, W^4Y^2, W^aX^{7-a}, X^7$ $a = 0, 1, \dots, 7$

$$e_1^{(51)} = (-5, 1, 0, 0), \quad e_2^{(51)} = (-12, 0, 1, 0), \quad e_3^{(51)} = (-18, 0, 0, 1).$$

In Kobayashi and Mase (2012), it is proved that the polytopes $\Delta^{(n)}$ for $n = 14, 28, 45$, and 51 are isomorphic to the polytope $\Delta^{(14)}$ that is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(6, -1, -1)$, and $(-1, 2, -1)$ under the above choice of basis. Since the polar dual $\Delta^{(14)*}$ is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-6, -14, -21)$, the linear map of $\mathbb{R}^3$ defined by a matrix $\begin{pmatrix} 6 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ gives an isomorphism from $\Delta^{(14)}$ to $\Delta^{(14)*}$ as lattice polytopes. Therefore, a relation $\Delta^{(14)*} \simeq \Delta^{(14)}$ holds.

Define 2-dimensional sections Γ_1 , Γ_2 , and Γ_3 of $\Delta^{(14)}$ by

$$\Gamma_1 = \text{Conv}\{(-1, -1, 0), (6, -1, -1), (-1, 2, -1)\},$$

$$\Gamma_2 = \text{Conv}\{(-1, 0, -1), (6, -1, -1), (-1, -1, 1)\},$$

$$\Gamma_3 = \text{Conv}\{(0, -1, -1), (-1, -1, 1), (-1, 2, -1)\}.$$

No. 1. The Newton polytope of $F = F'$ coincides with $\Delta^{(14)}$.

No. 2. Since the Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, coincides with $\Delta^{(28)}$, which is isomorphic to $\Delta^{(14)}$. Thus, we take Δ' to be $\Delta^{(14)}$. Since the polar dual polytope of $\Delta^{(14)}$ is $\Delta^{(14)}$ itself, and the polynomial F is an anticanonical member of the weight system $(1, 6, 14, 21)$, the Newton polytope Δ_F of F is a subpolytope of Δ , we can take Δ to be again $\Delta^{(14)}$.

No. 3. Since the Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, coincides with $\Delta^{(45)}$, which is isomorphic to $\Delta^{(14)}$. Thus, we take Δ' to be $\Delta^{(14)}$. As in No.2, we can take Δ to be $\Delta^{(14)}$.

No. 4. Since the Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, coincides with $\Delta^{(51)}$, which is isomorphic to $\Delta^{(14)}$. Thus, we take Δ' to be $\Delta^{(14)}$. As in No.2, we can take Δ to be $\Delta^{(14)}$.

Nos. 5, 6, and 7. The Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 0)$, $(-1, -1, 1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, contains a face Γ_1 . Since the polar dual of Γ_1 is a non-integral vertex $(3/10, 7/10, 21/10)$, the Newton polytope $\Delta_{F'}$ is not reflexive. In order to construct a reflexive polytope Δ' such that $\Delta_{F'} \subset \Delta'$, one may replace the face Γ_1 of $\Delta_{F'}$ by some sections. In other words, fixing vertices $(-1, -1, 1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, we may add one or two other vertices to the polytope $\Delta_{F'}$. This is done by a try and error construction with the following three cases.

First try-out The face Γ_1 is replaced by a section spanned by vertices $(-1, -1, 0)$, $(6, -1, -1)$, and $(-1, a, -1)$ with $a = 1, 0, -1$. The polar dual of this section is a vertex $v_1 = 1/(a+8)(a+1, 7, 7(a+1))$. The vertex is a lattice point if and only if $a = -1$. This means that we obtain the polytope $\Delta^{(14)}$ after the construction. Thus we may take $\Delta' = \Delta = \Delta^{(14)}$.

Second try-out The face Γ_1 is replaced by a section spanned by vertices $(-1, -1, 0)$, $(-1, 2, -1)$, and $(b, -1, -1)$ with $b = 5, 4, 3, 2, 1, 0, -1$. The polar dual of this section is a vertex $v_2 = 1/(b+4)(3, b+1, 3(b+1))$. The vertex is a lattice point if and only if $b = -1$. This means that we obtain the polytope $\Delta^{(14)}$ after the construction. Thus we may take $\Delta' = \Delta = \Delta^{(14)}$.

Third try-out The face Γ_1 is replaced by a section spanned by vertices $(-1, -1, 0)$, $(-1, a, -1)$, and $(b, -1, -1)$ with $a = 1, 0, -1$, and $b = 5, 4, 3, 2, 1, 0, -1$. Note that $a = -1$ if and only if $b = -1$ since this try-out means that we are constructing a polytope that contains a section which is a subset of the triangle spanned by $(-1, 2, -1)$, $(6, -1, -1)$, and $(-1, -1, -1)$ of $\Delta^{(14)}$. If $a = b = -1$, then, the resulting polytope coincides with $\Delta^{(14)}$, which is reflexive. Otherwise, the polar dual of this section is a vertex $v_3 = 1/(a+b+2)(a+1, b+1, (a+1)(b+1))$, which is never a lattice point. Therefore, we obtain the polytope $\Delta^{(14)}$ after the construction. Thus we may take $\Delta' = \Delta = \Delta^{(14)}$.

Nos. 8 and 9. The Newton polytope of F' , which is the convex hull of vertices $(-1, 0, -1)$, $(-1, -1, 1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, contains a face Γ_2 . Since the polar dual of Γ_2 is a non-integral vertex $(2/9, 14/9, 7/9)$, the Newton polytope $\Delta_{F'}$ is not reflexive. In order to construct a reflexive polytope Δ' such that $\Delta_{F'} \subset \Delta'$, one may replace the face Γ_2 of $\Delta_{F'}$ by some sections. In other words, fixing vertices $(-1, -1, 1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, we may add one or two other vertices to the polytope $\Delta_{F'}$. This is done by a try and error construction.

The idea to add a section two of which vertices are $(-1, -1, 1)$ and $(-1, -1, 0)$ is excluded by the arguments made in Nos. 5, 6, and 7. Thus we consider that we add a section spanned by vertices $(-1, -1, 1)$, $(-1, 0, -1)$, and $(a, -1, -1)$ with $a = 5, 4, 3, 2, 1, 0, -1$. This section has the polar dual spanned by a vertex $v = 1/(a+3)(2, 2(a+1), a+1)$. The vertex v is integral if and only if $a = -1$. Therefore, we may take Δ' to be $\Delta^{(14)}$, and Δ to be $\Delta^{(14)}$.

No. 10. The Newton polytope of F' , which is the convex hull of vertices $(0, -1, -1)$, $(-1, -1, 1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, contains the face Γ_3 . Since the polar dual of Γ_3 is a non-integral vertex $(6/5, 2/5, 3/5)$, the Newton polytope $\Delta_{F'}$ is not reflexive. In order to construct a reflexive polytope Δ' such that $\Delta_{F'} \subset \Delta'$, one may replace the face Γ_3 of $\Delta_{F'}$ by some sections. In other words, fixing vertices $(-1, -1, 1)$, $(6, -1, -1)$, and $(-1, 2, -1)$, we may add one or two other vertices to the polytope $\Delta_{F'}$. This is done by a try and error construction.

The idea to add a section two of which vertices are $(-1, -1, 1)$ and $(-1, -1, 0)$ is excluded by the arguments made in Nos. 5, 6, and 7. Thus we consider that we add a section spanned by vertices $(-1, -1, 1)$, $(0, -1, -1)$, and $(-1, a, -1)$ with $a = 1, 0, -1$. This section has the polar dual spanned by a vertex $v = 1/(a+3)(2(a+1), 2, a+1)$. The vertex v is integral if and only if $a = -1$. Therefore, we may take Δ' to be $\Delta^{(14)}$, and Δ to be $\Delta^{(14)}$.

5.2 No. 11–No. 14

We claim that there exist two polytope-dual pairs for Nos. 11 to 14. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 38, 50, 77$, and 82 by

$$\begin{aligned} e_1^{(38)} &= (-6, 1, 0, 0), & e_2^{(38)} &= (-8, 0, 1, 0), & e_3^{(38)} &= (-15, 0, 0, 1), \\ e_1^{(50)} &= (-4, 1, 0, 0), & e_2^{(50)} &= (-10, 0, 1, 0), & e_3^{(50)} &= (-15, 0, 0, 1), \\ e_1^{(77)} &= (-5, 1, 0, 0), & e_2^{(77)} &= (-7, 0, 1, 0), & e_3^{(77)} &= (-13, 0, 0, 1), \\ e_1^{(82)} &= (-3, 1, 0, 0), & e_2^{(82)} &= (-7, 0, 1, 0), & e_3^{(82)} &= (-11, 0, 0, 1). \end{aligned}$$

In Kobayashi and Mase (2012), it is proved that the polytopes $\Delta^{(38)}$ and $\Delta^{(77)}$, and $\Delta^{(50)}$, and $\Delta^{(82)}$ are respectively isomorphic to the polytopes

$$\begin{aligned} \Delta^{(38,77)} &:= \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (4, -1, -1), (0, 2, -1), (-1, 2, -1) \right\}, \\ \Delta^{(50,82)} &:= \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (6, -1, -1), (4, 0, -1), (-1, 2, -1) \right\}. \end{aligned}$$

Define polytopes Δ_1 , Δ'_1 , and Δ_2 , Δ'_2 by

$$\begin{aligned} \Delta_1 &:= \Delta^{(38,77)}, \\ \Delta'_1 &:= \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (5, -1, -1), (4, 0, -1), (-1, 2, -1) \right\}, \\ \Delta_2 &:= \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (6, -1, -1), (4, 0, -1), (-1, 2, -1) \right\}, \\ \Delta'_2 &:= \Delta^{(50,82)}. \end{aligned}$$

Since the polar dual polytopes Δ_1^* and Δ_2^* of Δ_1 and Δ_2 are the convex hulls of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, -2, -3)$, and $(-6, -8, -15)$, respectively, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-2, -4, -7)$, and $(-4, -10, -15)$, and the linear maps of $\mathbb{R}^3$ determined by matrices $A_1 := \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & 4 & 8 \end{pmatrix}$, and $A_2 := {}^t A_1$ respectively give isomorphisms from Δ'_1 to Δ_1^* and from Δ'_2 to Δ_2^* , the relations $\Delta_1^* \simeq \Delta'_1$ and $\Delta_2^* \simeq \Delta'_2$ hold.

No. 11. The Newton polytope of F , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(4, -1, -1)$, and $(0, 2, -1)$, is not reflexive. Indeed, Δ_F contains a face Γ spanned by $(-1, -1, 1)$, $(-1, -1, -1)$, and $(0, 2, -1)$ of which the polar dual is a non-integral vertex $1/2(3, -1, 0)$. In order to construct a reflexive polytope Δ such that $\Delta_F \subset \Delta$, we have to replace the face Γ by a union of some sections, which is done by add a vertex $(-1, a, -1)$ with $a = 1, 2$ so that there is a new face spanned by $(-1, -1, 1)$, $(0, 2, -1)$, and $(-1, a, -1)$. The polar dual of this new face is a vertex $1/(a-3)(2(a-2), 2, a+1)$. Thus, this vertex is an integral point if and only if $a = 1, 2$.

The Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(4, 0, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$

contains a face Γ' spanned by $(-1, -1, 1)$, $(-1, -1, -1)$, and $(4, 0, -1)$ of which the polar dual is a non-integral vertex $1/4(-1, 5, 0)$. In order to construct a reflexive polytope Δ' such that $\Delta_{F'} \subset \Delta'$, we have to replace the face Γ' by a union of some sections, which is done by add a vertex $(b, -1, -1)$ with $b = 6, 5, 4, 3, 2, 1, 0, -1$ so that there is a new face spanned by $(-1, -1, 1)$, $(4, 0, -1)$, and $(b, -1, -1)$. The polar dual of this new face is a vertex $1/(b-7)(2, 2(b-4), b+1)$. Thus, this vertex is an integral point if and only if $b = 6, 5$.

By the above observation, there are two polytope-dual pairs, that is, if $(a, b) = (2, 5)$, then, (Δ_1, Δ'_1) , and if $(a, b) = (1, 6)$, then, (Δ'_2, Δ_2) .

No. 12. The Newton polytope of F , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(4, -1, -1)$, and $(0, 2, -1)$, is isomorphic to the Newton polytope of F that we have already studied in No. 11. Thus, we may take Δ_1 or Δ_2 as a reflexive polytope Δ .

The Newton polytope of F' , which is the convex hull of vertices $(4, 0, -1)$, $(-1, -1, 1)$, $(0, -1, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains faces Γ_1 , which is spanned by $(-1, -1, 1)$, $(0, -1, -1)$, and $(4, 0, -1)$, and Γ_2 , which is spanned by $(-1, -1, 1)$, $(0, -1, -1)$, and $(-1, 2, -1)$ of which the polar duals are vertices $v_1 = 1/7(-2, 8, -1)$, and $v_2 = 1/5(6, 2, 3)$, respectively.

We have to replace the face Γ_1 by a union of two sections, in particular, we have to use a new face spanned by vertices $(-1, -1, 1)$, $(4, 0, -1)$, and $(a, -1, -1)$ with $a = 6, 5, 4, 3, 2, 1, 0, -1$. By the same argument made in No. 11 for polytope Δ' , we have to take $a = 6, 5$.

We have to replace Γ_2 by a union of two sections, in particular, we have to use a new face Γ spanned by vertices $(-1, -1, 1)$, $(0, -1, -1)$, and $(-1, b, -1)$ with $b = -1, 0, 1$. The polar dual of Γ is a vertex $1/(b+3)(2(b+1), 2, b+1)$. Thus, this vertex is an integral point if and only if $b = -1$.

Therefore, there are two polytope-dual pairs, that is, if $(a, b) = (5, -1)$, then, (Δ_1, Δ'_1) , and if $(a, b) = (6, -1)$, then, (Δ'_2, Δ_2) .

No. 13. The Newton polytope of F , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(4, 0, -1)$, and $(-1, 2, -1)$, is isomorphic to the Newton polytope of F' in No. 11. Thus, we may take Δ_1 or Δ_2 as a reflexive polytope Δ .

The Newton polytope of F' , which is the convex hull of vertices $(0, 2, -1)$, $(-1, -1, 1)$, $(-1, 0, -1)$, and $(4, -1, -1)$ is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains faces Γ_1 , which is spanned by $(-1, -1, 1)$, $(-1, 0, -1)$, and $(4, -1, -1)$, and Γ_2 , which is spanned by $(-1, -1, 1)$, $(-1, 0, -1)$, and $(0, 2, -1)$ of which the polar duals are vertices $v_1 = 1/7(2, 10, 5)$, and $v_2 = 1/3(4, -2, -1)$, respectively.

We have to replace the face Γ_1 by a union of two sections, in particular, we have to use a new face Γ'_1 spanned by vertices $(-1, -1, 1)$, $(-1, 0, -1)$, and $(a, -1, -1)$ with $a = 4, 3, 2, 1, 0, -1$. The polar dual of Γ'_1 is a vertex $1/(a+3)(2, 2(a+1), a+1)$. Thus, this vertex is an integral point if and only if $a = -1$.

We have to replace Γ_2 by a union of two sections, in particular, we have to use a new face Γ'_2 spanned by vertices $(-1, -1, 1)$, $(0, 2, -1)$, and $(-1, b, -1)$ with $b = 1, 2$. The polar dual of Γ'_2 is a vertex $1/(b-2)(2(b-2), 2, b+1)$. Thus, this vertex is an integral point if and only if $b = 1, 2$.

Therefore, there are two polytope-dual pairs, that is, if $(a, b) = (-1, 2)$, then, (Δ'_1, Δ_1) , and if $(a, b) = (-1, 1)$, then, (Δ_2, Δ'_2) .

No. 14. The Newton polytope of F coincides with that of F' in No. 12. Thus, one may take Δ_1 or Δ_2 to be Δ . Besides, the Newton polytope of F' coincides with that of F' in No. 13. Thus, one may take Δ'_1 or Δ'_2 to be Δ' . Therefore, there are two polytope-dual pairs, that is, (Δ'_1, Δ_1) and (Δ_2, Δ'_2) .

5.3 No. 15–No. 18

We claim that there exists a unique polytope-dual pair for Nos. 17, 18, and that two pairs for Nos. 15, 16. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 13, 20, 59$, and 72 by

$$\begin{aligned} e_1^{(13)} &= (-3, 1, 0, 0), & e_2^{(13)} &= (-8, 0, 1, 0), & e_3^{(13)} &= (-12, 0, 0, 1), \\ e_1^{(20)} &= (-6, 1, 0, 0), & e_2^{(20)} &= (-8, 0, 1, 0), & e_3^{(20)} &= (-9, 0, 0, 1), \\ e_1^{(59)} &= (-5, 1, 0, 0), & e_2^{(59)} &= (-7, 0, 1, 0), & e_3^{(59)} &= (-8, 0, 0, 1), \\ e_1^{(72)} &= (-2, 1, 0, 0), & e_2^{(72)} &= (-5, 0, 1, 0), & e_3^{(72)} &= (-7, 0, 0, 1). \end{aligned}$$

In Kobayashi and Mase (2012), it is proved that the polytopes $\Delta^{(20)}$ and $\Delta^{(59)}$ are isomorphic to the convex hull $\Delta^{(20,59)}$ of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(3, -1, -1)$, $(0, -1, 1)$, and $(-1, 2, -1)$.

Define polytopes Δ_1 , Δ'_1 , and Δ_2 , Δ'_2 by

$$\begin{aligned} \Delta_1 &:= \Delta^{(20,59)}, \\ \Delta'_1 &:= \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (5, -1, -1), (3, -1, 0), (-1, 2, -1) \right\}, \\ \Delta_2 &:= \text{Conv}\{(0, -1, 1), (-1, -1, -1), (3, -1, -1), (-1, 2, -1)\}, \\ \Delta'_2 &:= \Delta^{(13)} = \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (7, -1, -1), (-1, 2, -1) \right\}. \end{aligned}$$

Since the polar dual polytopes Δ_1^* and Δ_2^* of Δ_1 and Δ_2 are the convex hulls of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(0, -2, -3)$, and $(-6, -8, -9)$, respectively, $(0, 1, 0)$, $(0, 0, 1)$, $(2, 0, -1)$, and $(-6, -8, -9)$, and the linear maps of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 4 & 5 \end{pmatrix}$ gives isomorphisms from Δ'_1 to Δ_1^* and from Δ'_2 to Δ_2^* , the relations $\Delta_1^* \simeq \Delta'_1$ and $\Delta_2^* \simeq \Delta'_2$ hold.

No. 15. The Newton polytope of F , which is the convex hull of vertices $(0, -1, 1)$, $(-1, -1, -1)$, $(3, -1, -1)$, and $(-1, 2, -1)$, coincides with Δ_2 .

The Newton polytope of F' , which is the convex hull of vertices $(3, -1, 0)$, $(-1, -1, 1)$, $(-1, -1, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the Newton polytope $\Delta_{F'}$ of F' contains a face Γ spanned by vertices $(3, -1, 0)$, $(-1, -1, -1)$, and $(-1, 2, -1)$, and the polar dual of Γ is a non-integral vertex $1/3(-1, 0, 4)$. We have to replace the face Γ by a union of two sections, in particular, by using vertices $(3, -1, 0)$ and $(-1, 2, -1)$, we may choose an appropriate vertex of form $(a, -1, -1)$ with $a = 7, 6, 5, 4, 3, 2, 1, 0$, so that the resulting face is reflexive. The face Γ' spanned by $(3, -1, 0)$, $(-1, 2, -1)$ and $(a, -1, -1)$ with $a = 7, 6, 5, 4, 3, 2, 1, 0$ has polar

dual $v = 1/(a - 8)(3, a + 1, 3(a - 3))$. Thus, the vertex v is a lattice point if and only if $a = 7, 5$.

Therefore, there are two polytope-dual pairs, that is, if $a = 5$, then, $(\Delta^{(20)} = \Delta_1, \Delta'_1)$, and if $a = 7$, then, $(\Delta_2, \Delta'^{(13)} = \Delta'_2)$.

No. 16. The Newton polytope of F coincides with that of F in No. 15, namely, with Δ_2 .

The Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(0, -1, -1)$, $(3, -1, 0)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the Newton polytope $\Delta_{F'}$ contains two faces Γ_1 spanned by $(-1, -1, 1)$, $(0, -1, -1)$, and $(-1, 2, -1)$, and Γ_2 spanned by $(3, -1, 0)$, $(0, -1, -1)$, and $(-1, 2, -1)$, and the polar dual of these faces are respectively, $\Gamma_1^* = 1/5(6, 2, 3)$, and $\Gamma_2^* = 1/8(-3, -1, 9)$. We have to replace Γ_1 and Γ_2 by sums of other sections.

For Γ_1 , we have to use a section Γ'_1 spanned by $(-1, -1, 1)$, $(0, -1, -1)$, and $(-1, a, -1)$ with $a = 1, 0, -1$. The polar dual of Γ'_1 is a vertex $v_1 = 1/(a + 3)(2(a + 1), 2, a + 1)$. Thus, the vertex v_1 is a lattice point if and only if $a = -1$.

For Γ_2 , we have to use a section Γ'_2 spanned by $(3, -1, 0)$, $(-1, 2, -1)$, and $(b, -1, -1)$ with $b = 7, 6, 5, 4, 3, 2, 1$. The polar dual of Γ'_2 is a vertex $v_2 = 1/(b - 8)(3, b + 1, 3(b - 3))$. Thus, the vertex v_2 is a lattice point if and only if $b = 5, 7$.

Therefore, there are two polytope-dual pairs, that is, if $b = 5$, then, (Δ_1, Δ'_1) , and if $b = 7$, then, (Δ_2, Δ'_2) .

No. 17. The Newton polytope of F , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(3, -1, 0)$, and $(-1, 2, -1)$, coincides with that of F' in No. 15. Thus the same argument can be applied to Δ_F , so, one may take $\Delta^{(72)}$ or a polytope with a vertex $(5, -1, -1)$ instead of a face spanned by vertices $(3, -1, 0)$, $(-1, -1, -1)$, and $(-1, 2, -1)$ as Δ .

The Newton polytope of F' , which is the convex hull of vertices $(0, -1, 1)$, $(-1, -1, 0)$, $(3, -1, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains faces Γ_1 spanned by $(0, -1, 1)$, $(-1, -1, 0)$, and $(-1, 2, -1)$, and Γ_2 spanned by $(-1, -1, 0)$, $(3, -1, -1)$, and $(-1, 2, -1)$ of which the polar duals are $\Gamma_1^* = 1/2(3, -1, -3)$, and $\Gamma_2^* = 1/7(3, 4, 12)$. We must replace these faces by sums of sections by adding other vertices to $\Delta_{F'}$. For Γ_1 , the only possible way to add a vertex is to adding the vertex $(-1, -1, 1)$. For Γ_2 , there are three try-outs.

First try-out Consider a section Γ'_1 spanned by $(-1, -1, 0)$, $(-1, 2, -1)$, and $(a, -1, -1)$ with $a = 2, 1, 0, -1$. The polar dual of Γ'_1 is a vertex $v_1 = 1/(a + 4)(3, a + 1, 3(a + 1))$. The vertex v_1 is a lattice point if and only if $a = -1$.

Second try-out Consider a section Γ'_2 spanned by $(-1, -1, 0)$, $(3, -1, -1)$, and $(-1, b, -1)$ with $b = 1, 0, -1$. The polar dual of Γ'_2 is a vertex $v_2 = 1/(b + 5)(b + 1, 4, 4(b + 1))$. The vertex v_2 is a lattice point if and only if $b = -1$.

Third try-out Consider a section Γ'_3 spanned by $(-1, -1, 0)$, $(a, -1, -1)$, and $(-1, b, -1)$ with $a = 2, 1, 0, -1$ and $b = 1, 0, -1$. Note that $a = -1$ and $b = -1$ simultaneously occur. If $a = b = -1$, then, it is clear that Γ'_3 is reflexive. Otherwise, the polar dual of Γ'_3 is a vertex $v_3 = 1/(a + b + 2)(b + 1, a + 1, (a + 1)(b + 1))$, which is never a lattice point. Therefore Γ'_3 is reflexive if and only if $a = b = -1$.

Therefore, the pair (Δ'_1, Δ_1) is polytope-dual.

No. 18. The Newton polytope of F , which is the convex hull of vertices $(-1, -1, 1)$, $(0, -1, -1)$, $(3, -1, 0)$, and $(-1, 2, -1)$, coincides with that of F' in No. 16. Thus the same argument in No. 16 can be applied here.

The Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 0)$, $(0, -1, 1)$, $(3, -1, -1)$, and $(-1, 2, -1)$, coincides with that of F' in No. 17. Thus the same argument in No. 17 can be applied in this case.

Therefore, the pair (Δ'_1, Δ_1) is polytope-dual.

5.4 No. 19

We claim that there exist three polytope-dual pairs. Take a basis $\{e_1^{(78)}, e_2^{(78)}, e_3^{(78)}\}$ for a lattice M_{78} by

$$e_1^{(78)} = (-4, 1, 0, 0), \quad e_2^{(78)} = (-6, 0, 1, 0), \quad e_3^{(78)} = (-11, 0, 0, 1).$$

Define polytopes $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ by

$$\begin{aligned} \Delta_1 &:= \Delta^{(78)} = \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), (4, -1, -1), \\ (3, 0, -1), (0, 2, -1), (-1, 2, -1) \end{array} \right\}, \\ \Delta_2 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), (4, -1, -1), \\ (3, 0, -1), (0, 2, -1), (-1, 1, -1) \end{array} \right\}, \\ \Delta_3 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), (3, -1, -1), \\ (3, 0, -1), (0, 2, -1), (-1, 2, -1) \end{array} \right\}, \\ \Delta_4 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), (3, -1, -1), \\ (3, 0, -1), (0, 2, -1), (-1, 1, -1) \end{array} \right\}. \end{aligned}$$

Since the polar dual polytopes Δ_1^* , Δ_2^* , and Δ_3^* of Δ_1 , Δ_2 , and Δ_3 are the convex hulls of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-2, -2, -5)$, $(-4, -6, -11)$, $(0, -2, -3)$, resp., $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-2, -2, -5)$, $(-4, -6, -11)$, $(1, -1, -1)$, resp., $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-1, 0, -2)$, $(-4, -6, -11)$, $(0, -2, -3)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 6 \end{pmatrix}$ gives isomorphisms from Δ_4 to Δ_1^* , from Δ_2 to Δ_2^* , and from Δ_3 to Δ_3^* , the relations $\Delta_1^* \simeq \Delta_4$, $\Delta_2^* \simeq \Delta_2$, and $\Delta_3^* \simeq \Delta_3$ hold.

The Newton polytope of $F = F'$, which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(3, 0, -1)$, and $(0, 2, -1)$, is not reflexive. Indeed, the polytope Δ_F contains faces Γ_1 spanned by $(-1, -1, 1)$, $(-1, -1, -1)$, and $(0, 2, -1)$, and Γ_2 spanned by $(-1, -1, 1)$, $(-1, -1, -1)$, and $(3, 0, -1)$, of which the polar duals are non-integral vertices $v_1 = 1/2(3, -1, 0)$, and $v_2 = 1/3(-1, 4, 0)$, respectively. For these faces, we have to replace them by sums of other sections.

For Γ_1 , we have to use a section Γ'_1 spanned by $(-1, -1, 1)$, $(0, 2, -1)$, and $(-1, a, -1)$ with $a = 2, 1, 0$. The polar dual of Γ'_1 is a vertex $v_1 = 1/(a-3)(2(a-2), 2, a+1)$. The vertex v_1 is a lattice point if and only if $a = 2, 1$.

For Γ_2 , we have to use a section Γ'_2 spanned by $(-1, -1, 1)$, $(3, 0, -1)$, and $(b, -1, -1)$ with $b = 4, 3, 2, 1, 0$. The polar dual of Γ'_2 is a vertex $v_2 = 1/(b-5)(2, 2(b-3), b+1)$. The vertex v_2 is a lattice point if and only if $b = 4, 3$.

Therefore, there are there polytope-dual pairs, that is, if $(a, b) = (2, 4)$ for one side and $(a, b) = (1, 3)$ for the other, then, (Δ_1, Δ_4) , if $(a, b) = (1, 4)$ for both sides, then, (Δ_2, Δ_2) , if $(a, b) = (2, 3)$ for both sides, then, (Δ_3, Δ_3) .

5.5 No. 20

We claim that there exist two polytope-dual pairs. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of M_n for $n = 39, 60$ by

$$\begin{aligned} e_1^{(39)} &= (-3, 1, 0, 0), & e_2^{(39)} &= (-5, 0, 1, 0), & e_3^{(39)} &= (-9, 0, 0, 1), \\ e_1^{(60)} &= (-4, 1, 0, 0), & e_2^{(60)} &= (-6, 0, 1, 0), & e_3^{(60)} &= (-7, 0, 0, 1). \end{aligned}$$

Define polytopes $\Delta_1, \Delta'_1, \Delta_2$ and Δ'_2 by

$$\begin{aligned} \Delta_1 &:= \text{Conv} \left\{ (0, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (3, -1, -1), (2, 0, -1), (-1, 2, -1) \right\}, \\ \Delta'_1 &:= \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (5, -1, -1), (0, 2, -1), (-1, 1, -1) \right\}, \\ \Delta_2 &:= \text{Conv} \left\{ (0, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (2, -1, -1), (2, 0, -1), (-1, 2, -1) \right\}, \\ \Delta'_2 &:= \Delta^{(39)} = \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (5, -1, -1), (0, 2, -1), (-1, 2, -1) \right\}. \end{aligned}$$

Since the polar dual polytopes Δ_1^* and Δ_2^* of Δ_1 and Δ_2 are the convex hulls of vertices $(0, 1, 0)$, $(0, 0, 1)$, $(-2, -2, -3)$, $(-4, -6, -7)$, and $(2, 0, -1)$, respectively, $(0, 1, 0)$, $(0, 0, 1)$, $(-1, 0, -1)$, $(-4, -6, -7)$, and $(2, 0, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{pmatrix}$ gives isomorphisms from Δ'_1 to Δ_1^* , and from Δ_2 to Δ_2^* , the relations $\Delta_1^* \simeq \Delta'_1$ and $\Delta_2^* \simeq \Delta'_2$ hold.

The Newton polytope of F , which is the convex hull of vertices $(0, -1, 1)$, $(-1, -1, -1)$, $(2, 0, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope Δ_F contains a face Γ spanned by $(0, -1, 1)$, $(-1, -1, -1)$, and $(2, 0, -1)$, of which the polar dual is a non-integral vertex $1/5(-2, 6, 1)$. The face Γ has to be replaced by a sum of other sections: consider a section Γ' spanned by $(0, -1, 1)$, $(2, 0, -1)$, and $(a, -1, -1)$ with $a = 3, 2, 1, 0$. The polar dual of Γ' is a vertex $v' = 1/(a-4)(2, 2(a-2), a)$. Thus the vertex is a lattice point if and only if $a = 2, 3$.

The Newton polytope of F' , which is the convex hull of vertices $(0, 2, -1)$, $(-1, -1, 1)$, $(-1, -1, -1)$, and $(2, -1, 0)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains faces Γ_1 spanned by $(-1, -1, 1)$, $(-1, -1, -1)$, and $(0, 2, -1)$, and Γ_2 spanned by $(2, -1, 0)$, $(-1, -1, -1)$, and $(0, 2, -1)$, of which the polar duals are

respectively non-lattice vertices $1/2(3, -1, 0)$, and $1/7(-3, 1, 9)$. We have to replace these faces by sums of other sections.

For Γ_1 , consider a section Γ'_1 spanned by $(-1, -1, 1)$, $(0, 2, -1)$, and $(-1, b, -1)$ with $b = 2, 1, 0$. The polar dual of Γ'_1 is a vertex $v_1 = 1/(b-3)(2(b-2), 2, b+1)$. The vertex v_1 is a lattice point if and only if $b = 1, 2$.

For Γ_2 , consider a section Γ'_2 spanned by $(2, -1, 0)$, $(0, 2, -1)$, and $(b', -1, -1)$ with $b' = 5, 4, 3, 2, 1, 0$. The polar dual of Γ'_2 is a vertex $v_2 = 1/(b'-6)(3, b', 3(b'-2))$. The vertex v_2 is a lattice point if and only if $b' = 3, 5$.

Therefore, there are two polytope-dual pairs, that is, if $(a, b, b') = (3, 1, 5)$, then, (Δ_1, Δ'_1) , and if $(a, b, b') = (2, 2, 5)$, then, (Δ_2, Δ'_2) .

5.6 No. 21

We claim that there exist two polytope-dual pairs. Take a basis $\{e_1^{(22)}, e_2^{(22)}, e_3^{(22)}\}$ of a lattice M_{22} with by

$$e_1^{(22)} = (-3, 1, 0, 0), \quad e_2^{(22)} = (-5, 0, 1, 0), \quad e_3^{(22)} = (-6, 0, 0, 1).$$

Define polytopes Δ_1 , Δ_2 and Δ'_2 by

$$\Delta_1 := \text{Conv}\{(0, -1, 1), (-1, -1, -1), (4, -1, -1), (-1, 2, -1)\},$$

$$\Delta_2 := \text{Conv} \left\{ \begin{array}{l} (0, -1, 1), (-1, -1, -1), \\ (2, -1, -1), (2, -1, 0), (-1, 2, -1) \end{array} \right\},$$

$$\Delta'_2 := \Delta^{(22)} = \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), \\ (4, -1, -1), (0, -1, 1), (-1, 2, -1) \end{array} \right\}.$$

Since the polar dual polytopes Δ_1^* and Δ_2^* of Δ_1 and Δ_2 are the convex hulls of vertices $(0, 1, 0)$, $(0, 0, 1)$, $(-3, -5, -6)$, and $(2, 0, -1)$, respectively, $(0, 1, 0)$, $(0, 0, 1)$, $(-1, -1, 0)$, $(-3, -5, -6)$, and $(2, 0, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ gives isomorphisms from Δ_1 to Δ_1^* , and from Δ'_2 to Δ_2^* , the relations $\Delta_1^* \simeq \Delta_1$ and $\Delta_2^* \simeq \Delta'_2$ hold.

The Newton polytope of $F = F'$, which is the convex hull of vertices $(0, -1, 1)$, $(-1, -1, -1)$, $(2, -1, 0)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope Δ_F contains a face Γ spanned by $(2, -1, 0)$, $(-1, -1, -1)$, and $(-1, 2, -1)$ of which the polar dual is a non-integral vertex $1/2(-1, 0, 3)$. We may replace the face Γ by a union of other sections. Consider a section Γ' spanned by $(2, -1, 0)$, $(-1, 2, -1)$, and $(a, -1, -1)$ with $a = 4, 3, 2, 1, 0$. The polar dual of Γ' is a vertex $v = 1/(a-5)(3, a+1, 3(a-2))$. The vertex v is a lattice point if and only if $a = 2, 4$.

Therefore, there are two polytope-dual pairs, that is, if $a = 4$, then, (Δ_1, Δ_1) , and if $a = 2$ for Δ , and $a = 4$ for Δ' , then, (Δ_2, Δ'_2) .

5.7 No. 22–No. 24

We claim that there exist a unique polytope-dual pair for Nos. 22 and 23, and that none for No. 24. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 9$ and 71 by

$$\begin{aligned} e_1^{(9)} &= (-4, 1, 0, 0), & e_2^{(9)} &= (-5, 0, 1, 0), & e_3^{(9)} &= (-10, 0, 0, 1), \\ e_1^{(71)} &= (-3, 1, 0, 0), & e_2^{(71)} &= (-4, 0, 1, 0), & e_3^{(71)} &= (-7, 0, 0, 1). \end{aligned}$$

Define polytopes Δ_1 and Δ_2 by

$$\begin{aligned} \Delta_1 &:= \Delta^{(9)} = \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (4, -1, -1), (-1, 3, -1) \right\}, \\ \Delta_2 &:= \text{Conv}\{(-1, -1, 1), (-1, -1, -1), (4, -1, -1), (-1, 1, 0)\}. \end{aligned}$$

Since the polar dual polytope Δ_1^* of Δ_1 is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-4, -5, -10)$, and the linear maps of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 3 & 5 \end{pmatrix}$ gives an isomorphism from Δ_2 to Δ_1^* , the relation $\Delta_1^* \simeq \Delta_2$ holds.

No. 22. The Newton polytope of $F = F'$, which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(4, -1, -1)$, and $(-1, 1, 0)$, coincides with Δ_2 . Therefore, a pair $(\Delta_1, \Delta_{F'} = \Delta_2)$ is polytope-dual.

No. 23. The Newton polytope of F , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(4, -1, -1)$, and $(-1, 1, 0)$, coincides with Δ_2 . The Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, 0, -1)$, $(4, -1, -1)$, and $(-1, 1, 0)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains faces Γ_1 spanned by $(-1, -1, 1)$, $(-1, 0, -1)$, and $(4, -1, -1)$, and Γ_2 spanned by $(-1, 1, 0)$, $(-1, 0, -1)$, and $(4, -1, -1)$, of which the polar duals are non-integral vertices $1/7(2, 10, 5)$, and $1/4(-1, -5, 5)$, respectively. We have to replace these faces by unions of some sections. For Γ_1 , we consider the following try-outs:

First try-out Consider a section Γ'_1 spanned by $(-1, -1, 1)$, $(-1, 0, -1)$, and $(a, -1, -1)$ with $a = 3, 2, 1, 0, -1$. The polar dual of Γ'_1 is a vertex $v_1 = 1/(a+3)(2, 2(a+1), a+1)$. The vertex v_1 is a lattice point if and only if $a = -1$.

Second try-out Consider a section Γ''_1 spanned by $(-1, -1, 0)$, $(-1, 0, -1)$, and $(4, -1, -1)$. The polar dual of Γ''_1 is a non-integral vertex $1/6(1, 5, 5)$. Thus, it is impossible to use this section.

For Γ_2 , consider a section Γ'_2 spanned by $(-1, 1, 0)$, $(4, -1, -1)$, and $(-1, b, -1)$ with $b = 3, 2, 1$. The polar dual of the section Γ'_2 is a vertex $v = 1/(b-4)(b+1, 5, 5(b-1))$. The vertex v is a lattice point if and only if $b = 3$.

Therefore, a pair $(\Delta_F = \Delta_2, \Delta_1)$ is polytope-dual.

No. 24. The Newton polytope of $F = F'$, which is the convex hull of vertices $(-1, -1, 1)$, $(-1, 0, -1)$, $(4, -1, -1)$, and $(-1, 1, 0)$, is not reflexive. It is observed that one might take a polytope with vertices $(-1, 2, -1)$, $(0, 2, -1)$ and $(-1, -1, -1)$ instead of faces spanned by vertices $(-1, 1, 0)$, $(-1, 0, -1)$, $(4, -1, -1)$, and $(-1, -1, 1)$, $(-1, 0, -1)$, $(4, -1, -1)$, namely, $\Delta^{(71)}$.

In the polytope $\Delta^{(71)}$, the vertex $(-1, -1, -1)$ is adjacent to three other vertices: vertex $(-1, -1, 1)$ with an edge e_1 , vertex $(-1, 2, -1)$ with an edge e_2 , and vertex $(4, -1, -1)$ with an edge e_3 . On the edges e_1, e_2, e_3 , respectively, there are one, two, and four inner lattice points. Thus, the polar dual polytope $\Delta^{(71)*}$ must contain a triangle as a two-dimensional face that is adjacent to other two-dimensional faces with inner lattice points one, two, and four. However, it is easily observed that there is no such a configuration in the polytope $\Delta^{(71)*}$. Thus, $\Delta^{(71)}$ is not self-dual, and there does not exist a polytope-dual pair.

5.8 No. 25

We claim that there exist two polytope-dual pairs. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 37, 58$ by

$$\begin{aligned} e_1^{(37)} &= (-3, 1, 0, 0), & e_2^{(37)} &= (-4, 0, 1, 0), & e_3^{(37)} &= (-8, 0, 0, 1), \\ e_1^{(58)} &= (-4, 1, 0, 0), & e_2^{(58)} &= (-5, 0, 1, 0), & e_3^{(58)} &= (-6, 0, 0, 1). \end{aligned}$$

Define polytopes $\Delta_1, \Delta'_1, \Delta_2$ and Δ'_2 by

$$\begin{aligned} \Delta_1 &:= \text{Conv} \left\{ (-1, 1, 0), (-1, -1, 0), \right. \\ &\quad \left. (-1, -1, -1), (3, -1, -1), (0, -1, 1) \right\}, \\ \Delta'_1 &:= \Delta^{(37)} = \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (4, -1, -1), (3, 0, -1), (-1, 3, -1) \right\}, \\ \Delta_2 &:= \text{Conv} \left\{ (-1, 1, 0), (-1, -1, 1), \right. \\ &\quad \left. (-1, -1, -1), (3, -1, -1), (0, -1, 1) \right\}, \\ \Delta'_2 &:= \text{Conv} \left\{ (-1, -1, 1), (-1, -1, -1), \right. \\ &\quad \left. (3, -1, -1), (3, 0, -1), (-1, 3, -1) \right\}. \end{aligned}$$

Since the polar dual polytopes Δ_1^* and Δ_2^* of Δ_1 and Δ_2 are the convex hulls of vertices $(1, 0, 0), (0, 1, 0), (0, -1, 2), (1, 0, -1)$, and $(-4, -5, -6)$, respectively, $(1, 0, 0), (0, 1, 0), (0, -1, 2), (0, -1, -2)$, and $(-4, -5, -6)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 3 \end{pmatrix}$ gives isomorphisms from Δ'_1 to Δ_1^* , and from Δ'_2 to Δ_2^* , the relations $\Delta_1^* \simeq \Delta'_1$ and $\Delta_2^* \simeq \Delta'_2$ hold.

The Newton polytopes of F , which is the convex hull of vertices $(0, -1, 1), (-1, -1, -1), (3, -1, -1)$, and $(-1, 1, 0)$, is not reflexive. Indeed, the polytope Δ_F contains a face Γ spanned by $(-1, 1, 0), (-1, -1, -1)$, and $(0, -1, 1)$ of which the polar dual is a non-integral vertex $1/3(4, 1, -2)$. The face Γ has to be replaced by a union of other faces. In order to do this, consider a section Γ' spanned by $(-1, 1, 0), (0, -1, 1)$, and $(-1, -1, a)$ with $a = 0, 1$. The polar dual of Γ' is a vertex $v = 1/(a-2)(2(a-1), a, 2)$. The vertex v is a lattice point if and only if $a = 0, 1$. Moreover, we note that it is also possible to use a vertex $(-1, a', -1)$ with $a' = 1, 2$ instead of a face spanned by vertices $(0, -1, 1), (-1, -1, -1)$, and $(-1, 1, 0)$ which follows from a direct computation.

The Newton polytopes of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(3, 0, -1)$, and $(-1, 1, 0)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains faces Γ_1 spanned by $(-1, -1, 1)$, $(-1, -1, -1)$, and $(3, 0, -1)$, and Γ_2 spanned by $(-1, 1, 0)$, $(-1, -1, -1)$, and $(3, 0, -1)$, of which the polar duals are respectively non-integral vertices $1/3(-1, 4, 0)$ and $1/5(1, -4, 8)$. The faces Γ_1 and Γ_2 have to be replaced by unions of other sections.

For Γ_1 , consider a section Γ'_1 spanned by $(-1, -1, -1)$, $(3, 0, -1)$, and $(b, -1, -1)$ with $b = 4, 3, 2, 1, 0$. The polar dual of Γ'_1 is a vertex $v_1 = 1/(b - 5)(2, 2(b - 3), b + 1)$. The vertex v_1 is a lattice point if and only if $b = 3, 4$.

For Γ_2 , consider a section Γ'_2 spanned by $(-1, 1, 0)$, $(3, 0, -1)$, and $(-1, b', -1)$ with $b' = 3, 2, 1, 0$. The polar dual of Γ'_2 is a vertex $v_1 = 1/(b' - 4)(b', 4, 4(b' - 1))$. The vertex v_2 is a lattice point if and only if $b' = 0, 3, 2$.

Therefore, there are two polytope-dual pairs, that is, if $(a, b, b') = (0, 4, 3)$, then, (Δ_1, Δ'_1) , and if $(a, b, b') = (1, 3, 3)$, then, (Δ_2, Δ'_2) .

5.9 No. 26

We claim that there exist four polytope-dual pairs. Take a basis $\{e_1^{(87)}, e_2^{(87)}, e_3^{(87)}\}$ of a lattice M_{87} by

$$e_1^{(87)} = (-3, 1, 0, 0), \quad e_2^{(37)} = (-4, 0, 1, 0), \quad e_3^{(37)} = (-5, 0, 0, 1).$$

Define polytopes Δ_i, Δ'_i for $i = 1, 2, 3, 4$ by

$$\begin{aligned} \Delta_1 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, 1, 0), (0, -1, 1), \\ (-1, -1, -1), (2, -1, -1), (2, 0, -1), (-1, 0, -1) \end{array} \right\}, \\ \Delta'_1 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, 1, 0), (0, -1, 1), \\ (-1, -1, -1), (2, -1, -1), (2, 0, -1), (-1, 2, -1) \end{array} \right\}, \\ \Delta_2 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 0), (-1, 1, 0), (0, -1, 1), \\ (-1, -1, -1), (3, -1, -1), (2, 0, -1), (-1, 0, -1) \end{array} \right\}, \\ \Delta'_2 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 0), (-1, 1, 0), (0, -1, 1), \\ (-1, -1, -1), (3, -1, -1), (2, 0, -1), (-1, 2, -1) \end{array} \right\}, \\ \Delta_3 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 0), (-1, 1, 0), (0, -1, 1), \\ (-1, -1, -1), (2, -1, -1), (2, 0, -1), (-1, 2, -1) \end{array} \right\}, \\ \Delta'_3 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, 1, 0), (0, -1, 1), \\ (-1, -1, -1), (3, -1, -1), (2, 0, -1), (-1, 0, -1) \end{array} \right\}, \\ \Delta_4 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 0), (-1, 1, 0), (0, -1, 1), \\ (-1, -1, -1), (2, -1, -1), (2, 0, -1), (-1, 0, -1) \end{array} \right\}, \\ \Delta'_4 &:= \Delta^{(87)} = \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, 1, 0), (0, -1, 1), \\ (-1, -1, -1), (3, -1, -1), (2, 0, -1), \\ (-1, 2, -1) \end{array} \right\}. \end{aligned}$$

Since the polar dual polytopes Δ_i^* , $i = 1, 2, 3, 4$ are respectively given as follows:

$$\begin{aligned}\Delta_1^* &:= \text{Conv} \left\{ (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, -1, -2), \right. \\ &\quad \left. (-1, 0, -1), (0, -1, 1), (-3, -4, -5) \right\}, \\ \Delta_2^* &:= \text{Conv} \left\{ (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, -1), \right. \\ &\quad \left. (-2, -2, -3), (0, -1, 1), (-3, -4, -5) \right\}, \\ \Delta_3^* &:= \text{Conv} \left\{ (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, -1), \right. \\ &\quad \left. (-1, 0, -1), (-2, -3, -3), (-3, -4, -5) \right\}, \\ \Delta_4^* &:= \text{Conv} \left\{ (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, -1), \right. \\ &\quad \left. (-1, 0, -1), (0, -1, 1), (-3, -4, -5) \right\},\end{aligned}$$

and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$ gives isomorphisms from Δ'_i to Δ_i^* , $i = 1, 2, 3, 4$, the relations $\Delta_i^* \simeq \Delta'_i$, $i = 1, 2, 3, 4$ hold.

The Newton polytope of $F = F'$, which is the convex hull of vertices $(0, -1, 1)$, $(-1, -1, -1)$, $(2, 0, -1)$, and $(-1, 1, 0)$, is not reflexive. Indeed, the polytope Δ_F contains faces Γ_1 spanned by $(-1, 1, 0)$, $(-1, -1, -1)$, and $(0, -1, 1)$, Γ_2 spanned by $(0, -1, 1)$, $(-1, -1, -1)$, and $(2, 0, -1)$, and Γ_3 spanned by $(-1, 1, 0)$, $(-1, -1, -1)$, and $(2, 0, -1)$, of which the polar duals are respectively non-integral vertices $1/3(4, 1, -2)$, $1/5(-2, 6, 1)$, and $1/4(1, -3, 6)$. These faces have to be replaced by unions of other sections.

For Γ_1 , consider a section Γ'_1 spanned by $(-1, 1, 0)$, $(0, -1, 1)$, and $(-1, -1, a)$ with $a = 0, 1$. The polar dual of Γ'_1 is a vertex $v_1 = 1/(a-2)(2(a-1), a, 2)$. The vertex v_1 is a lattice point if and only if $a = 0, 1$.

For Γ_2 , consider a section Γ'_2 spanned by $(2, 0, -1)$, $(0, -1, 1)$, and $(b, -1, -1)$ with $b = 3, 2, 1, 0$. The polar dual of Γ'_2 is a vertex $v_2 = 1/(b-4)(2, 2(b-2), b)$. The vertex v_2 is a lattice point if and only if $b = 2, 3$.

For Γ_3 , consider a section Γ'_3 spanned by $(-1, 1, 0)$, $(2, 0, -1)$, and $(-1, c, -1)$ with $c = 2, 1, 0$. The polar dual of Γ'_3 is a vertex $v_3 = 1/(c-3)(c, 3, 3(c-1))$. The vertex v_3 is a lattice point if and only if $c = 0, 2$.

Therefore, there are four polytope-dual pairs, that is, if $(a, b, c) = (1, 2, 0)$ for one side, and $(a, b, c) = (1, 2, 2)$ for the other, then, (Δ_1, Δ'_1) ; if $(a, b, c) = (0, 3, 0)$ for one side, and $(a, b, c) = (0, 3, 2)$ for the other, then, (Δ_2, Δ'_2) ; if $(a, b, c) = (0, 2, 2)$ for one side, and $(a, b, c) = (1, 3, 0)$ for the other, then, (Δ_3, Δ'_3) ; and if $(a, b, c) = (0, 2, 0)$ for one side, and $(a, b, c) = (1, 3, 2)$ for the other, then, (Δ_4, Δ'_4) .

5.10 No. 27

We claim that there does not exist a polytope-dual pair. Take a basis $\{e_1^{(4)}, e_2^{(4)}, e_3^{(4)}\}$ of a lattice M_4 by

$$e_1^{(4)} = (-3, 1, 0, 0), \quad e_2^{(4)} = (-4, 0, 1, 0), \quad e_3^{(4)} = (-4, 0, 0, 1).$$

By a direct computation, one has $\Delta^{(4)*} \not\simeq \Delta^{(4)}$, which we say *not self-dual*.

It is observed that there exist four invertible projectivisations

$$F_1 = F'_1 = X^4 + Y^3 + Z^3 + W^{12}, \quad F_2 = F'_2 = X^4 + Y^3 + Z^3 + W^8 Z, \\ F_3 = F'_3 = X^4 + Y^3 + Z^3 + W^8 Y, \quad F_4 = F'_4 = X^4 + Y^3 + Z^3 + W^9 X.$$

The Newton polytope of $F_1 = F'_1$, which is the convex hull of vertices $(-1, -1, 2)$, $(-1, -1, -1)$, $(3, -1, -1)$, and $(-1, 2, -1)$, coincide with $\Delta^{(4)}$.

Since F_2 and F_3 are symmetric in variables Y and Z , we only treat with F_2 . The Newton polytope of $F_2 = F'_2$, which is the convex hull of vertices $(-1, -1, 2)$, $(-1, -1, 0)$, $(3, -1, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope Δ_{F_2} contains a face Γ spanned by $(-1, -1, 0)$, $(-1, 2, -1)$, and $(3, -1, -1)$ of which the polar dual is non-integral vertex $1/7(3, 4, 12)$. The face Γ has to be replaced by a union of some sections.

First try-out Consider a face Γ' spanned by $(-1, -1, 0)$, $(3, -1, -1)$, and $(-1, a, -1)$ with $a = 1, 0, -1$. The polar dual of Γ' is a vertex $v' = 1/(a+5)(a+1, 4, 4(a+1))$. The vertex v' is a lattice point if and only if $a = -1$.

Second try-out Consider a face Γ'' spanned by $(-1, -1, 0)$, $(-1, 2, -1)$, and $(b, -1, -1)$ with $b = 2, 1, 0, -1$. The polar dual of Γ'' is a vertex $v'' = 1/(b+4)(3, b+1, 3(b+1))$. The vertex v'' is a lattice point if and only if $b = -1$.

Third try-out Consider a face Γ''' spanned by $(-1, -1, 0)$, $(-1, a, -1)$, and $(b, -1, -1)$ with $a = 1, 0, -1$ and $b = 2, 1, 0, -1$. Note that $a = -1$ and $b = -1$ simultaneously occur. If $a = b = -1$, then, the resulting polytope is none other than $\Delta^{(4)}$. Otherwise, the polar dual of Γ''' is a vertex $v''' = 1/(a+b+2)(a+1, b+1, (a+1)(b+1))$, which is never a lattice point.

Therefore, the only reflexive polytope Δ such that $\Delta_{F_2} \subset \Delta$ is $\Delta^{(4)}$.

The Newton polytope of $F_4 = F'_4$, which is the convex hull of vertices $(-1, -1, 2)$, $(0, -1, -1)$, $(3, -1, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope Δ_{F_4} contains a face Γ spanned by $(-1, -1, 2)$, $(0, -1, -1)$, and $(-1, 2, -1)$, of which the polar dual is a non-integral vertex $1/2(3, 1, 1)$. The face Γ has to be replaced by a union of other sections. Consider a section Γ' spanned by $(-1, -1, 2)$, $(0, -1, -1)$, and $(-1, a, -1)$ with $a = 1, 0, -1$. The polar dual of Γ' is a vertex $v' = 1/(a+4)(3(a+1), 3, a+1)$. The vertex v' is a lattice point if and only if $a = -1$. Therefore, the only reflexive polytope Δ with $\Delta_{F_4} \subset \Delta$ is $\Delta^{(4)}$.

5.11 No. 28–No. 29

We claim that there exist two polytope-dual pairs for Nos. 28 and 29. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 12, 27$ and 49 by

$$e_1^{(12)} = (-2, 1, 0, 0), \quad e_2^{(12)} = (-6, 0, 1, 0), \quad e_3^{(12)} = (-9, 0, 0, 1), \\ e_1^{(27)} = -(1, -7, 1, 1), \quad e_2^{(27)} = -(1, 1, -2, 1), \quad e_3^{(27)} = -(0, 1, 1, -1), \\ e_1^{(49)} = -(0, -7, 1, 1), \quad e_2^{(49)} = -(1, 1, -2, 1), \quad e_3^{(49)} = -(1, 1, 1, -1).$$

In Kobayashi and Mase (2012), it is proved that the polytopes $\Delta^{(27)}$ and $\Delta^{(49)}$ are isomorphic to the polytope $\Delta_{(27,49)}$ with vertices $(0, 0, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(-3, -8, -12)$.

Define polytopes Δ_1 , Δ'_1 , Δ_2 and Δ'_2 by

$$\begin{aligned}\Delta_1 &:= \text{Conv}\{(0, 0, 1), (1, 0, 0), (0, 1, 0), (-2, -6, -9)\}, \\ \Delta'_1 &:= \Delta^{(12)} = \text{Conv}\left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), \\ (8, -1, -1), (-1, 2, -1) \end{array} \right\}, \\ \Delta_2 &:= \Delta^{(49)} = \text{Conv}\{(0, 0, 1), (1, 0, 0), (0, 1, 0), (-3, -8, -12)\}, \\ \Delta'_2 &:= \text{Conv}\{(-1, -1, 1), (-1, -1, -1), (7, -1, -1), (-1, 2, -1)\}.\end{aligned}$$

It is direct to see that the relations $\Delta_1^* \simeq \Delta'_1$ and $\Delta_2^* \simeq \Delta'_2$ hold.

No. 28. The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-2, -6, -9)$, coincides with Δ'_1 . The Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, 0)$, $(7, -1, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains a face Γ spanned by $(-1, -1, 0)$, $(7, -1, -1)$, and $(-1, 2, -1)$ of which the polar dual is a non-integral vertex $1/11(3, 8, 24)$. The face Γ has to be replaced by a union of other sections.

First try-out Consider a section Γ' spanned by $(-1, -1, 0)$, $(7, -1, -1)$, and $(-1, a, -1)$ with $a = 1, 0, -1$. The polar dual of Γ' is a vertex $v' = 1/(a+9)(a+1, 8, 8(a+1))$. The vertex v' is a lattice point if and only if $a = -1$.

Second try-out Consider a section Γ'' spanned by $(-1, -1, 0)$, $(-1, 2, -1)$, and $(b, -1, -1)$ with $b = 6, 5, 4, 3, 2, 1, 0, -1$. The polar dual of Γ'' is a vertex $v'' = 1/(b+4)(3, b+1, 3(b+1))$. The vertex v'' is a lattice point if and only if $b = -1$.

Third try-out Consider a section Γ''' spanned by $(-1, -1, 0)$, $(-1, a, -1)$, and $(b, -1, -1)$ with $a = 1, 0, -1$ and $b = 6, 5, 4, 3, 2, 1, 0, -1$. Note that $a = -1$ and $b = -1$ simultaneously occur. If $a = b = -1$, then, the resulting sections have lattice dual vertex. Otherwise, the polar dual of Γ''' is a vertex $v''' = 1/(a+b+2)(a+1, b+1, (a+1)(b+1))$, which is never a lattice point.

Therefore, there are two polytope-dual pairs: $(\Delta_F = \Delta_1, \Delta'_1)$, and (Δ_2, Δ'_2) .

No. 29. The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-2, -6, -9)$, coincides with Δ'_1 . The Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(0, -1, -1)$, $(7, -1, -1)$, and $(-1, 2, -1)$, is not reflexive.

Indeed, the polytope $\Delta_{F'}$ contains a face Γ spanned by $(-1, -1, 1)$, $(0, -1, -1)$, and $(-1, 2, -1)$, of which the polar dual is a non-integral vertex $1/5(6, 2, 3)$. The face Γ has to be replaced by a union of other sections. Consider a section Γ' spanned by $(-1, -1, 1)$, $(0, -1, -1)$, and $(-1, a, -1)$ with $a = 1, 0, -1$. The polar dual of Γ' is a vertex $v' = 1/(a+3)(2(a+1), 2, a+1)$. The vertex v' is a lattice point if and only if $a = -1$.

Note that the polytope $\Delta_{F'}$ contains a face spanned by $(-1, -1, 1)$, $(7, -1, -1)$, and $(-1, 2, -1)$, with which is possible to be replaced by

$$\text{Conv}\left\{ \begin{array}{l} (-1, -1, 1), (7, -1, -1), \\ (8, -1, -1) \end{array} \right\} + \text{Conv}\left\{ \begin{array}{l} (-1, -1, 1), (-1, 2, -1), \\ (8, -1, -1) \end{array} \right\}$$

or

$$\text{Conv} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} \right\} + \text{Conv} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} \right\}$$

or

$$\text{Conv} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\} + \text{Conv} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

In these cases, one has to take Δ to be a cone of a hexagon with Δ^* to have a configuration of edges that form a lattice of type $T_{9,3,4}$. However, the maximal number of lattice points in an edge of $\Delta^{(12)}$ is 8, and if we use this edge, we only construct a configuration of edges that form a lattice of type $T_{9,4,4}$. Thus, in such a case, it is impossible to find out a corresponding polytope-dual partner Δ' .

Therefore, there are two polytope-dual pairs, that is, $(\Delta_F = \Delta_1, \Delta'_1)$, and (Δ_2, Δ'_2) .

5.12 No. 30

We claim that there exist two polytope-dual pairs. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 40, 81$ by

$$\begin{aligned} e_1^{(40)} &= (-2, 1, 0, 0), & e_2^{(40)} &= (-4, 0, 1, 0), & e_3^{(40)} &= (-7, 0, 0, 1), \\ e_1^{(81)} &= (0, 7, -1, -1), & e_2^{(81)} &= (0, -1, 2, -1), & e_3^{(81)} &= (-1, -1, -1, 1). \end{aligned}$$

Define polytopes $\Delta_1, \Delta'_1, \Delta_2, \Delta'_2$ by

$$\begin{aligned} \Delta_1 &:= \text{Conv} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -6 \\ -9 \end{pmatrix} \right\}, \\ \Delta'_1 &:= \Delta^{(40)} = \text{Conv} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\}, \\ \Delta_2 &:= \text{Conv} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -6 \\ -9 \end{pmatrix} \right\}, \\ \Delta'_2 &:= \text{Conv} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}. \end{aligned}$$

Since the polar dual polytopes Δ_1^* and Δ_2^* of Δ_1 and Δ_2 are the convex hulls of vertices $(-1, -1, 1), (1, -1, -1), (-2, 2, -1), (-1, 2, -1), (8, -1, -1)$, resp., $(-1, -1, 1), (1, -1, -1), (-1, 1, -1), (-1, 2, -1), (8, -1, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ gives isomorphisms from Δ'_i to Δ_i^* for $i = 1, 2$, the relations $\Delta_1^* \simeq \Delta'_1$ and $\Delta_2^* \simeq \Delta'_2$ hold.

The Newton polytope of F , which is the convex hull of vertices $(1, 1, 1), (0, 1, 0), (0, 0, 1)$, and $(-2, -6, -9)$, is not reflexive. Indeed, the polytope Δ_F contains a face

spanned by $(0, 0, 1)$, $(1, 1, 1)$, and $(-1, -6, -9)$, of which the polar dual is a non-integral vertex $1/2(-5, 5, -2)$. The face should be replaced by a sum of some sections.

Consider a section spanned by $(0, 0, 1)$, $(1, 1, 1)$, and $(-2 + a, -6 + 2a, -9 + 3a)$ with $a = 1, 2, 3$. The polar dual of this section is a vertex $v = 1/(a - 4)(-3a + 10, 3a - 10, a - 4)$. Thus, the vertex v is a lattice point if and only if $a = 2, 3$. Namely, one can take a reflexive polytope Δ to be Δ_1 , i.e., $a = 2$ and Δ_2 , i.e., $a = 3$.

Besides, the Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, 0, -1)$, $(5, -1, -1)$, and $(0, 2, -1)$, is not reflexive. According to the observation for Δ , we must take two polytopes which are isomorphic to Δ_1^* and Δ_2^* . Indeed, we may take Δ_1' and Δ_2' that clearly contain $\Delta_{F'}$ as a subpolytope.

Therefore, there are two polytope-dual pairs: (Δ_1, Δ_1') , and (Δ_2, Δ_2') .

Note that taking cones over a hexagon as a polytope Δ is impossible since in each candidate cases, the configurations of lattice points in their polar dual is impossible as a subpolytope of $\Delta^{(40)}$.

5.13 No. 31

We claim that there exist five polytope-dual pairs. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 11, 24$ by

$$\begin{aligned} e_1^{(11)} &= (2, 7, -1, -1), & e_2^{(11)} &= -(1, 1, -2, 1), & e_3^{(11)} &= -(1, 1, 1, -1), \\ e_1^{(24)} &= (-2, 1, 0, 0), & e_2^{(24)} &= (-4, 0, 1, 0), & e_3^{(24)} &= (-5, 0, 0, 1). \end{aligned}$$

Define polytopes $\Delta_i, \Delta_i', i = 1, 2, 3, 4$ by

$$\begin{aligned} \Delta_1 &:= \Delta^{(11)} = \text{Conv}\{(0, 1, 0), (0, 0, 1), (2, 2, 3), (-3, -8, -12)\}, \\ \Delta_1' &:= \text{Conv}\{(0, -1, 1), (-1, -1, -1), (4, -1, -1), (-1, 2, -1)\}, \\ \Delta_2 &:= \text{Conv}\{(0, 1, 0), (0, 0, 1), (2, 2, 3), (-2, -6, -9)\}, \\ \Delta_2' &:= \text{Conv}\{(0, -1, 1), (-1, -1, -1), (5, -1, -1), (-1, 2, -1)\}, \\ \Delta_3 &:= \text{Conv}\{(0, 1, 0), (0, 0, 1), (1, 1, 2), (0, -2, -3), (-2, -6, -9)\}, \\ \Delta_3' &:= \Delta^{(24)} = \text{Conv} \left\{ \begin{array}{l} (0, -1, 1), (-1, -1, -1), \\ (5, -1, -1), (-1, -1, 1), (-1, 2, -1) \end{array} \right\}, \\ \Delta_4 &:= \text{Conv} \left\{ \begin{array}{l} (0, 1, 0), (0, 0, 1), (1, 1, 2), \\ (0, -2, -3), (-2, -6, -9), (-1, -2, -4) \end{array} \right\}, \\ \Delta_4' &:= \text{Conv} \left\{ \begin{array}{l} (0, -1, 1), (-1, -1, -1), (4, -1, -1), \\ (3, 0, -1), (-1, 2, -1), (-1, -1, 1) \end{array} \right\}, \\ \Delta_5 &:= \text{Conv} \left\{ \begin{array}{l} (0, 1, 0), (0, 0, 1), (1, 1, 2), \\ (0, -2, -3), (-2, -6, -9), (-2, -5, -8) \end{array} \right\}, \\ \Delta_5' &:= \text{Conv} \left\{ \begin{array}{l} (0, -1, 1), (-1, -1, -1), (4, -1, -1), \\ (1, 1, -1), (-1, 2, -1), (-1, -1, 1) \end{array} \right\}. \end{aligned}$$

Since the polar dual polytopes Δ_i^* of Δ_i , $i = 1, 2, 3, 4, 5$ are respectively the convex hulls of vertices $(-1, -1, 1)$, $(2, -1, -1)$, $(-1, 2, -1)$, and $(7, -1, -1)$, resp., $(-1, -1, 1)$, $(2, -1, -1)$, $(-1, 2, -1)$, and $(8, -1, -1)$, resp., $(-1, -1, 1)$, $(2, -1, -1)$, $(-1, 2, -1)$, $(-2, -1, 1)$, and $(8, -1, -1)$, resp., $(-1, -1, 1)$, $(2, -1, -1)$, $(-1, 2, -1)$, $(-2, -1, 1)$, $(5, 0, -1)$, and $(7, -1, -1)$, resp. $(-1, -1, 1)$, $(2, -1, -1)$, $(-1, 2, -1)$, $(-2, -1, 1)$, $(2, 1, -1)$, and $(7, -1, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ gives isomorphisms from Δ'_i to Δ_i^* for $i = 1, 2, 3, 4, 5$, the relations $\Delta_i^* \simeq \Delta'_i$, $i = 1, 2, 3, 4, 5$ hold.

The Newton polytope of F , which is the convex hull of vertices $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 2)$, $(-2, -6, -9)$, is not reflexive. Indeed, the polytope Δ_F contains a face spanned by $(1, 1, 2)$, $(0, 1, 0)$, and $(-2, -6, -9)$, of which the polar dual is a non-integral vertex $1/5(-14, -5, 7)$. Consider a section spanned by $(1, 1, 2)$, $(0, 1, 0)$, and $(-2 + a, -6 + 2a, -9 + 3a)$ with $a = 1, 2, 3, 4$. The polar dual of this section is a vertex $v = 1/(a - 5)(2(2a - 7), a - 5, 2a - 7)$. The vertex v is a lattice point if and only if $a = 2, 4$. Note that we can use a vertex $(-1, -2, -4)$ or $(-2, -5, -8)$ and replace the face spanned by $(0, 01)$, $(0, 1, 0)$ and $(-2, -6, -9)$ by either the sum

$$\text{Conv} \left\{ \begin{pmatrix} 0, 0, 1 \\ -1, -2, -4 \end{pmatrix}, \begin{pmatrix} -2, -6, -9 \\ -1, -2, -4 \end{pmatrix} \right\} + \text{Conv} \left\{ \begin{pmatrix} 0, 0, 1 \\ -1, -2, -4 \end{pmatrix}, \begin{pmatrix} 0, 1, 0 \\ -1, -2, -4 \end{pmatrix} \right\}$$

or

$$\text{Conv} \left\{ \begin{pmatrix} 0, 0, 1 \\ -2, -5, -8 \end{pmatrix}, \begin{pmatrix} -2, -6, -9 \\ -2, -5, -8 \end{pmatrix} \right\} + \text{Conv} \left\{ \begin{pmatrix} 0, 0, 1 \\ -2, -5, -8 \end{pmatrix}, \begin{pmatrix} 0, 1, 0 \\ -2, -5, -8 \end{pmatrix} \right\}.$$

However, it is impossible to consider as Δ to be

$$\text{Conv} \{(0, 01), (-2, -6, -9), (2, 2, 3), (0, 1, 0), v\},$$

where $v = (-1, -2, -4)$ or $(-2, -5, -8)$ because the polar dual polytope should contain three incident edges of type $T_{2,3,6}$ without an edge with two lattice points, which is impossible.

Therefore, by the observation, there are five polytope-dual pairs, that is, $(\Delta_1, \Delta_{F'} = \Delta'_1)$, (Δ_2, Δ'_2) , (Δ_3, Δ'_3) , (Δ_4, Δ'_4) , and (Δ_5, Δ'_5) .

5.14 No. 32–No. 34

We claim that there exist two polytope-dual pairs for Nos. 32 to 34. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n with for $n = 6, 26, 34$, and 76 by

$$\begin{aligned} e_1^{(6)} &= (-2, 1, 0, 0), & e_2^{(6)} &= (-2, 0, 1, 0), & e_3^{(6)} &= (-5, 0, 0, 1), \\ e_1^{(26)} &= -(1, -4, 1, 1), & e_2^{(26)} &= -(1, 1, -3, 1), & e_3^{(26)} &= -(0, 1, 1, -1), \\ e_1^{(34)} &= -(1, -4, 1, 1), & e_2^{(34)} &= -(0, 1, -3, 1), & e_3^{(34)} &= -(1, 1, 1, -1), \\ e_1^{(76)} &= (1, 1, 1, -1), & e_2^{(76)} &= (4, 1, 0, -1), & e_3^{(76)} &= (9, -1, 0, -1). \end{aligned}$$

In Kobayashi and Mase (2012), it is proved that the polytopes $\Delta^{(26)}$ and $\Delta^{(34)}$ are isomorphic to the polytope with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-4, -5, -10)$.

Define polytopes Δ_1 , $\tilde{\Delta}_1$, Δ'_1 , and Δ_2 , $\tilde{\Delta}_2$, Δ'_2 by

$$\begin{aligned}\Delta_1 &:= \text{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-2, -2, -5)\}, \\ \tilde{\Delta}_1 &:= \text{Conv}\{(0, 0, -1), (2, 0, -1), (2, -1, 0), (-3, 1, 3)\}, \\ \Delta'_1 &:= \Delta^{(6)} = \text{Conv}\left\{\begin{array}{l} (-1, -1, 1), (-1, -1, -1), \\ (4, -1, -1), (-1, 4, -1) \end{array}\right\}, \\ \Delta_2 &:= \text{Conv}\left\{\begin{array}{l} (1, 0, 0), (0, 1, 0), (0, 0, 1), \\ (-2, -2, -5), (-a, -(a+1), -2(a+1)) \end{array}\right\}, \\ \tilde{\Delta}_2 &:= \text{Conv}\left\{\begin{array}{l} (0, 0, -1), (2, 0, -1), (2, -1, 0), \\ (-3, 1, 3), (a-2, 1, -a+2) \end{array}\right\}, \\ \Delta'_2 &:= \text{Conv}\left\{\begin{array}{l} (-1, -1, 1), (-1, -1, -1), (4, -1, -1), \\ (-1, 3, -1), (a, 3-a, -1) \end{array}\right\}, \\ &\text{here, } a = 3, 2, 1, 0.\end{aligned}$$

Since the polar dual polytopes Δ_1^* and $\tilde{\Delta}_1^*$ of Δ_1 and $\tilde{\Delta}_1$ are the convex hulls of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(4, -1, -1)$, and $(-1, 4, -1)$, respectively, $(-1, -1, -1)$, $(0, 1, 1)$, $(0, -4, 1)$, and $(5, 11, 1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} -1 & -3 & 0 \\ -1 & -2 & 0 \\ -3 & -6 & -1 \end{pmatrix}$ gives an isomorphism from Δ'_1 to $\tilde{\Delta}_1^*$, the relations $\Delta_1^* = \Delta'_1 \simeq \tilde{\Delta}_1^*$ hold. Moreover, Since the polar dual polytopes Δ_2^* and $\tilde{\Delta}_2^*$ of Δ_2 and $\tilde{\Delta}_2$ are convex hulls of the vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(4, -1, -1)$, $(a, 3-a, -1)$, and $(-1, 3, -1)$, respectively, $(-1, -1, -1)$, $(0, 1, 1)$, $(5, 11, 1)$, $(1, -1, 1)$, and $(0, a-3, 1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} -1 & -2 & 0 \\ -1 & -3 & 0 \\ -3 & -6 & -1 \end{pmatrix}$ gives an isomorphism from Δ'_2 to $\tilde{\Delta}_2^*$, the relations $\Delta_2^* = \Delta'_2 \simeq \tilde{\Delta}_2^*$ hold.

No. 32. The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-2, -2, -5)$, coincides with Δ_1 . Besides, the Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 0)$, $(-1, 3, -1)$, $(-1, -1, 1)$, and $(4, -1, -1)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains a face spanned by $(-1, -1, 1)$, $(-1, -1, 0)$, and $(-1, 3, -1)$, of which the polar dual is a non-integral vertex $1/9(4, 5, 20)$. Thus, we consider a section spanned by $(-1, -1, 0)$, $(4, -1, -1)$, and $(-1, a, -1)$ with $a = 1 - 0, 1, 2$. The polar dual of this section is a vertex $v = 1/(a+6)(a+1, 5, 5(a+1))$. Thus, the vertex v is a lattice point if and only if $a = -1$. Therefore, there are two polytope-dual pairs, that is, $(\Delta_F = \Delta_1, \Delta'_1)$, and (Δ_2, Δ'_2) .

No. 33. The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-2, -2, -5)$, coincides with Δ_1 . Besides, the Newton polytope of F' , which is the convex hull of vertices $(-1, 0, -1)$, $(-1, 3, -1)$, $(-1, -1, 1)$, and $(4, -1, -1)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains a face spanned by $(-1, -1, 1)$, $(-1, 0, -1)$, and $(4, -1, -1)$, of which the polar dual is a non-integral vertex $1/7(2, 10, 5)$. Thus, we consider a section spanned by $(-1, -1, 1)$,

$(-1, 0, -1)$, and $(a, -1, -1)$ with $a = 1 - 1, 0, 1, 2, 3$. The polar dual of this section is a vertex $v = 1/(a+3)(2, 2(a+1), a+1)$. Thus, the vertex v is a lattice point if and only if $a = -1$. Therefore, there are two polytope-dual pairs, that is, $(\Delta_F = \Delta_1, \Delta'_1)$, and (Δ_2, Δ'_2) .

No. 34. The Newton polytope of F , which is the convex hull of vertices $(0, 0, -1)$, $(2, 0, -1)$, $(2, -1, 0)$, and $(-3, 1, 3)$, coincides with $\tilde{\Delta}_1$. Besides, the Newton polytope of F' , which is the convex hull of vertices $(-1, 0, -1)$, $(3, -1, -1)$, $(0, 3, -1)$, and $(-1, -1, 1)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains two faces Γ_1 spanned by $(-1, -1, 1)$, $(-1, 0, -1)$, and $(3, -1, -1)$, of which the polar dual is $1/3(1, 4, 2)$, and Γ_2 spanned by $(-1, -1, 1)$, $(-1, 0, -1)$, and $(0, 3, -1)$, of which the polar dual is $1/5(6, -2, -1)$. The face Γ_1 is replaced by a sum of other sections as is discussed in No.33. For the face Γ_2 , consider a section spanned by $(-1, -1, 1)$, $(0, 3, -1)$, and $(-1, a, -1)$ with $a = 1, 2, 3, 4$. The polar dual of this section is a vertex $v = 1/(a-5)(2(a-3), 2, a+1)$. The vertex v is a lattice point if and only if $a = 3, 4$. Therefore, there are two polytope-dual pairs, that is, $(\Delta_F = \tilde{\Delta}_1, \Delta'_1)$, and $(\tilde{\Delta}_2, \Delta'_2)$.

5.15 No. 35–No. 37

We claim that there exists a unique polytope-dual pair for Nos. 35 to 37. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ lattices M_n for $n = 10, 46, 65$, and 80 by

$$\begin{aligned} e_1^{(10)} &= (-1, 1, 0, 0), & e_2^{(10)} &= (-4, 0, 1, 0), & e_3^{(10)} &= (-6, 0, 0, 1), \\ e_1^{(46)} &= (1, 1, 1, -1), & e_2^{(46)} &= (2, 2, -1, 0), & e_3^{(46)} &= (11, 0, -1, -1), \\ e_1^{(65)} &= (1, 0, 1, -1), & e_2^{(65)} &= (2, 1, -1, 0), & e_3^{(65)} &= (10, -1, -1, -1), \\ e_1^{(80)} &= (1, 1, 1, -1), & e_2^{(80)} &= (2, 1, -1, 0), & e_3^{(80)} &= (10, -1, -1, -1). \end{aligned}$$

In Kobayashi and Mase (2012), it is proved that the polytopes $\Delta^{(46)}$, $\Delta^{(65)}$ and $\Delta^{(80)}$ are isomorphic to the polytope $\Delta_{(46,65,80)}$ with vertices $(-1, 0, 0)$, $(1, -1, 0)$, $(0, 0, 1)$, and $(2, 4, -1)$. Let a polytope Δ_1 be $\Delta_{(46,65,80)}$ and Δ'_1 be the polytope $\Delta^{(10)}$ which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(11, -1, -1)$, and $(-1, 2, -1)$. Since the polar dual polytope Δ_1^* of Δ_1 is the convex hull of vertices $(1, -1, -1)$, $(1, 2, 11)$, $(1, 2, -1)$, and $(-1, 0, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -4 \\ -1 & -1 & -6 \end{pmatrix}$ gives an isomorphism from Δ'_1 to Δ_1^* , the relation $\Delta_1^* \simeq \Delta'_1$ holds.

For each case of Nos. 35, 36, 37, the Newton polytope of F , which is the convex hull of vertices $(-1, 0, 0)$, $(1, -1, 0)$, $(0, 0, 1)$, and $(2, 4, -1)$, coincides with $\Delta_1 = \Delta_{(46,65,80)}$. Thus, there exists only one reflexive polytope Δ such that $\Delta_F \subset \Delta \subset \Delta_{(46,65,80)}$, which is none other than Δ_1 . The polytope Δ'_1 no doubt satisfies $\Delta_{F'} \subset \Delta'_1 = \Delta^{(10)}$. Therefore, there exists only one polytope-dual pair (Δ_1, Δ'_1) .

5.16 No. 38–No. 40

We claim that there exist two polytope-dual pairs for Nos. 38 and 40, and that none for No. 39. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 42, 68, 83$, and 92 by

$$\begin{aligned} e_1^{(42)} &= (-1, 1, 0, 0), & e_2^{(42)} &= (-3, 0, 1, 0), & e_3^{(42)} &= (-5, 0, 0, 1), \\ e_1^{(68)} &= (1, 0, 1, -1), & e_2^{(68)} &= (3, 1, 0, -1), & e_3^{(68)} &= (9, -1, -1, -1), \\ e_1^{(83)} &= (1, 1, 1, -1), & e_2^{(83)} &= (3, 3, 0, -1), & e_3^{(83)} &= (10, 1, -1, -1), \\ e_1^{(92)} &= (1, 1, 1, -1), & e_2^{(92)} &= (3, 2, 0, -1), & e_3^{(92)} &= (10, 0, -1, -1). \end{aligned}$$

Define polytopes Δ_1 , Δ'_1 , Δ_2 , Δ'_2 , $\tilde{\Delta}_3$, Δ_3 , and Δ'_3 by

$$\begin{aligned} \Delta_1 &:= \text{Conv}\{(-1, 0, 0), (0, 0, 1), (-1, 2, 0), (-1, 3, -1), (2, -1, 0)\}, \\ \Delta'_1 &:= \Delta^{(42)} = \text{Conv}\left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), \\ (9, -1, -1), (0, 2, -1), (-1, 2, -1) \end{array} \right\}, \\ \Delta_2 &:= \Delta^{(68)} = \text{Conv}\left\{ \begin{array}{l} (-1, 0, 0), (0, 0, 1), \\ (-2, 4, -1), (-1, 3, -1), (2, -1, 0) \end{array} \right\}, \\ \Delta'_2 &:= \text{Conv}\left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), \\ (9, -1, -1), (0, 2, -1), (-1, 1, -1) \end{array} \right\}, \\ \tilde{\Delta}_3 &:= \text{Conv}\{(-1, 0, 0), (1, -1, 1), (-1, 2, 0), (-2, 4, -1), (2, -1, 0)\}, \\ \Delta_3 &:= \Delta^{(83)} = \text{Conv}\left\{ \begin{array}{l} (-1, 0, 0), (0, 0, 1), \\ (-2, 4, -1), (2, -1, 0), (1, -1, 1) \end{array} \right\}, \\ \Delta'_3 &:= \text{Conv}\left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), \\ (9, -1, -1), (3, 1, -1), (-1, 2, -1) \end{array} \right\}. \end{aligned}$$

Since the polar dual polytopes Δ_1^* and Δ_2^* of Δ_1 and Δ_2 , and $\tilde{\Delta}_3^*$ and Δ_3^* of $\tilde{\Delta}_3$ and Δ_3 are respectively the convex hulls of vertices $(-1, -1, -1)$, $(1, 0, 0)$, $(1, 3, 9)$, $(1, 3, -1)$, and $(1, 0, -1)$, resp. $(-1, -1, -1)$, $(1, 1, 3)$, $(1, 3, 9)$, $(1, 3, -1)$, and $(1, 0, -1)$, resp. $(-1, -1, -1)$, $(1, 0, -1)$, $(1, 3, 11)$, $(1, 3, 1)$, and $(1, 0, -2)$, resp. $(-1, -1, -1)$, $(1, 0, -1)$, $(1, 3, 11)$, $(1, 3, 1)$, and $(1, 1, -1)$, and the linear maps of $\mathbb{R}^3$ determined by matrices $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -3 \\ -1 & -2 & -5 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & -4 \\ -1 & -2 & -6 \end{pmatrix}$ give isomorphisms from

Δ'_1 to Δ_1^* and from Δ'_2 to Δ_2^* , respectively from Δ'_1 to $\tilde{\Delta}_3^*$ and from Δ'_3 to Δ_3^* , the relations $\Delta_1^* \simeq \Delta'_1$ and $\Delta_2^* \simeq \Delta'_2$, and $\tilde{\Delta}_3^* \simeq \Delta'_1$ and $\Delta_3^* \simeq \Delta'_3$ hold.

No. 38. The Newton polytope of F , which is the convex hull of vertices $(-1, 0, 0)$, $(0, 0, 1)$, $(-1, 3, -1)$, and $(2, -1, 0)$, is not reflexive. Indeed, the polytope Δ_F contains a face spanned by $(-1, 0, 0)$, $(0, 0, 1)$, and $(-1, 3, -1)$, of which the polar dual is a non-integral vertex $1/3(-3, -1, -3)$. Since the complement of the set of lattice points in edges of Δ_F in that of $\Delta^{(68)}$ consists of two lattice points $(-1, 2, 0)$ and $(-2, 4, -1)$, there are only two polytopes possibly to be considered:

Case 1 The polytope Δ is spanned by $(-1, 0, 0)$, $(0, 0, 1)$, $(-1, 2, 0)$, $(-1, 3, -1)$, and $(2, -1, 0)$, namely $\Delta = \Delta_1$. In this case, relations $\Delta_{F'} \subset \Delta_1^* = \Delta'_1 \subseteq \Delta^{(42)}$ clearly hold.

Case 2 The polytope Δ is spanned by $(-1, 0, 0)$, $(0, 0, 1)$, $(-2, 4, -1)$, $(-1, 3, -1)$, and $(2, -1, 0)$, namely $\Delta = \Delta_2 = \Delta^{(68)}$. In this case, relations $\Delta_{F'} \subset \Delta_2^* = \Delta'_2 \subset \Delta^{(42)}$ clearly hold.

Therefore there are two polytope-dual pairs : $(\Delta_1, \Delta^{(42)} = \Delta'_1)$, and $(\Delta^{(68)} = \Delta_2, \Delta'_2)$.

No. 39. The Newton polytope of F , which is the convex hull of vertices $(-1, 0, 0)$, $(1, -3, 1)$, $(-2, 4, -1)$, and $(2, -1, 0)$, is easily seen to be reflexive. Besides, the Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(0, -1, -1)$, $(6, 0, -1)$, and $(-1, 2, -1)$, is not reflexive. It is observed that for any reflexive polytope Δ such that $\Delta_F \subset \Delta \subset \Delta^{(92)}$, the polar dual polytope Δ^* should be a tetrahedron of which each edge contains 3 lattice points. However, there does not exist a reflexive polytope Δ' such that $\Delta_{F'} \subset \Delta' \subset \Delta^{(42)}$ that contains a vertex which is adjacent to three vertices between which the edges contain 3 lattice points. Thus, in this case, no pair is polytope-dual.

No. 40. The Newton polytope of F , which is the convex hull of vertices $(-1, 0, 0)$, $(1, -1, 1)$, $(-2, 4, -1)$, and $(2, -1, 0)$, is not reflexive. Indeed, the polytope Δ_F contains a face spanned by $(-1, 0, 0)$, $(1, -1, 1)$, and $(-2, 4, -1)$, of which the polar dual is a non-integral vertex $1/3(3, -1, -7)$. Since the complement of the set of lattice points in edges of Δ_F in that of $\Delta^{(68)}$ consists of two lattice points $(-1, 2, 0)$ and $(0, 0, 1)$, there are only two polytopes possibly to be considered:

Case 1 The polytope Δ is spanned by $(-1, 0, 0)$, $(1, -1, 1)$, $(-1, 2, 0)$, $(-2, 4, -1)$, and $(2, -1, 0)$, namely $\Delta = \Delta_1$. In this case, relations $\Delta_{F'} \subset \Delta_1^* = \Delta'_1 \subseteq \Delta^{(42)}$ clearly hold.

Case 2 The polytope Δ is spanned by $(-1, 0, 0)$, $(0, 0, 1)$, $(-2, 4, -1)$, $(2, -1, 0)$, and $(1, -1, 1)$, namely $\Delta = \Delta_3 = \Delta^{(83)}$. In this case, relations $\Delta_{F'} \subset \Delta_3^* = \Delta'_3 \subset \Delta^{(42)}$ clearly hold.

Therefore, there are two polytope-dual pairs : $(\tilde{\Delta}_1, \Delta^{(42)} = \Delta'_1)$, and $(\Delta^{(83)} = \Delta_3, \Delta'_3)$.

5.17 No. 41–No. 43

We claim that there exist two polytope-dual pairs for Nos. 41 and 43, and that a unique pair for No. 42. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n with for $n = 25, 43, 48$, and 88 by

$$\begin{aligned} e_1^{(25)} &= (-1, 1, 0, 0), & e_2^{(25)} &= (-3, 0, 1, 0), & e_3^{(25)} &= (-4, 0, 0, 1), \\ e_1^{(43)} &= (-1, -8, 1, 1), & e_2^{(43)} &= (0, -1, 2, -1), & e_3^{(43)} &= -(1, 1, 1, -1), \\ e_1^{(48)} &= (0, 8, -1, -1), & e_2^{(48)} &= -(1, 1, -2, 1), & e_3^{(48)} &= -(1, 1, 1, -1), \\ e_1^{(88)} &= (0, 4, -1, -1), & e_2^{(88)} &= (-1, -1, 2, -1), & e_3^{(88)} &= (-1, 0, -1, 1). \end{aligned}$$

Define polytopes Δ_1 , Δ'_1 , Δ_2 , and Δ'_2 by

$$\begin{aligned}\Delta_1 &:= \text{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-2, -6, -9)\}, \\ \Delta'_1 &:= \text{Conv}\{(-1, -1, 1), (-1, -1, -1), (8, -1, -1), (-1, 2, -1)\}, \\ \Delta_2 &:= \text{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, -2, -3), (-1, -3, -4)\}, \\ \Delta'_2 &:= \Delta^{(25)} = \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), \\ (8, -1, -1), (0, -1, 1), (-1, 2, -1) \end{array} \right\}.\end{aligned}$$

Since the polar dual polytopes Δ_i^* for $i = 1, 2$ are the convex hulls of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(8, -1, -1)$, and $(-1, 2, -1)$, respectively, $(-1, -1, 1)$, $(-1, -1, -1)$, $(8, -1, -1)$, $(0, -1, 1)$, and $(-1, 2, -1)$, the relations $\Delta_1^* = \Delta'_1$ and $\Delta_2^* = \Delta'_2$ clearly hold.

Nos. 41 and 43. The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-1, -3, -4)$, is not reflexive. Indeed, the polytope Δ_F contains a face spanned by $(1, 0, 0)$, $(0, 1, 0)$, and $(-1, -3, -4)$, of which the polar dual is a non-integral vertex $(-1, -1, 5/4)$. Since the complement of the set of lattice points on edges of Δ_F in that of $\Delta^{(43)}$ are three points $(1 - a, -2a, -3a)$ with $a = 1, 2, 3$, we consider a section Γ spanned by $(0, 1, 0)$, $(-1, -3, -4)$, and $(1 - a, -2a, -3a)$ with $a = 1, 2, 3$. The polar dual of the section Γ is a vertex $v = 1/(a - 4)(-4(a - 1), -(a - 4), 2a - 5)$. The vertex v is a lattice point if and only if $a = 1, 3$. Note that if $a = 3$, the polytope is Δ_1 , and if $a = 1$, it is Δ_2 . The Newton polytope of F' , which is the convex hull of vertices $(-1, -1, 1)$, $(-1, 0, -1)$, $(8, -1, -1)$, and $(-1, 2, -1)$, is not reflexive. Indeed, the polytope $\Delta_{F'}$ contains a face spanned by $(-1, -1, 1)$, $(-1, 0, -1)$, and $(8, -1, -1)$, of which the polar dual is a non-integral vertex $1/11(2, 18, 9)$. It is clear that both Δ'_1 and Δ'_2 are reflexive polytopes that satisfy relations $\Delta_{F'} \subset \Delta' \subset \Delta^{(25)}$. Therefore, there are two polytope-dual pairs: $(\Delta^{(43)} = \Delta_1, \Delta'_1)$, and (Δ_2, Δ'_2) .

No. 42. The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-1, -3, -4)$, is not reflexive. Indeed, the polytope Δ_F contains a face spanned by $(1, 0, 0)$, $(0, 1, 0)$, and $(-1, -3, -4)$ of which the polar dual is a non-integral vertex $(-1, -1, 5/4)$. Since the complement of the set of lattice points on edges of Δ_F in those of $\Delta^{(88)}$ are three points $(-1, -3, -5)$, $(1 - a, -2a, -3a)$ with $a = 1, 2$, consider the following sections:

Case 1 Section spanned by $(-1, -3, -4)$, $(1, 0, 0)$, and $(-1, -3, -5)$ has the polar dual $(-1, 2/3, 0)$, which is not integral.

Case 2 Section spanned by $(-1, -3, -4)$, $(0, 1, 0)$, and $(1 - a, -2a, -3a)$ with $a = 1, 2$ has the polar dual $v = 1/(a - 4)(-4(a - 1), -(a - 4), 2a - 5)$. The vertex v is a lattice point if and only if $a = 1$. Therefore, there are two possibilities: $\Delta^{(88)}$ or Δ_2 . The polar dual of $\Delta^{(88)}$ contains a vertex $(9, -1, -1)$. However, the polytope $\Delta^{(25)}$ does not contain the vertex. Therefore, $\Delta^{(88)}$ cannot be a polar-dual. It is directly checked that relations $\Delta_{F'} \subset \Delta'_2 \subset \Delta^{(25)}$ hold. Therefore, the pair (Δ_2, Δ'_2) is polytope-dual.

5.18 No. 44

We claim that there exists a unique polytope-dual pair. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 7$ and 64 by

$$\begin{aligned} e_1^{(7)} &= (-1, 1, 0, 0), & e_2^{(7)} &= (-2, 0, 1, 0), & e_3^{(7)} &= (-4, 0, 0, 1), \\ e_1^{(64)} &= (-1, 5, -1, -1), & e_2^{(64)} &= (0, -1, 2, -1), & e_3^{(64)} &= (-1, -1, 1, 0). \end{aligned}$$

Define polytopes Δ_1 and Δ'_1 by

$$\begin{aligned} \Delta_1 &:= \text{Conv}\{(1, 0, 0), (0, -1, 1), (0, 0, 1), (-1, 2, -6)\}, \\ \Delta'_1 &:= \Delta^{(7)} = \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), \\ (7, -1, -1), (-1, 3, -1) \end{array} \right\}. \end{aligned}$$

Since the polar dual polytope Δ_1^* is the convex hull of vertices $(-1, 2, 1)$, $(-1, 0, -1)$, $(7, 0, -1)$, and $(-1, -4, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ gives an isomorphism from Δ'_1 to Δ_1^* , the relation $\Delta_1^* = \Delta'_1$ holds.

The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, -1, 1)$, $(0, 0, 1)$, and $(-1, 2, -6)$, coincides with Δ_1 . Since the complement of the set of lattice points on edges of Δ_F in those of $\Delta^{(64)}$ consists of the only point $(0, 1, 0)$, the possible Δ is either Δ_1 or $\Delta^{(64)}$. The polar dual of $\Delta^{(64)}$ contains a vertex $(9, -1, -1)$, which does not belong to $\Delta^{(7)}$. Therefore, $\Delta = \Delta_1$. It is directly checked that the polar dual $\Delta^* = \Delta'_1$ coincides with $\Delta^{(7)}$. Therefore, the pair $(\Delta_F = \Delta_1, \Delta'_1)$ is polytope-dual.

5.19 No. 45

We claim that there exists a unique polytope-dual pair. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 35$ and 66 by

$$\begin{aligned} e_1^{(35)} &= (7, 0, -1, -1), & e_2^{(35)} &= (6, -1, 0, -1), & e_3^{(35)} &= (-1, -1, -1, 1), \\ e_1^{(66)} &= (-1, 1, 0, 0), & e_2^{(66)} &= (-2, 0, 1, 0), & e_3^{(66)} &= (-3, 0, 0, 1). \end{aligned}$$

Define polytopes Δ_1 and Δ'_1 by

$$\begin{aligned} \Delta_1 &:= \text{Conv}\{(0, 0, 1), (0, 1, 0), (2, -2, -1), (3, -4, -2), (-2, 2, -1)\}, \\ \Delta'_1 &:= \text{Conv} \left\{ \begin{array}{l} (-1, 1, 0), (0, -1, 1), \\ (-1, -1, 1), (-1, -1, -1), (6, -1, -1) \end{array} \right\}. \end{aligned}$$

Since the polar dual polytope Δ_1^* is the convex hull of vertices $(-1, -1, 1)$, $(0, -1, -1)$, $(-2, -1, -1)$, $(-1, 0, -1)$, and $(7, 6, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} -1 & -1 & 0 \\ -2 & -2 & 1 \\ -4 & -3 & 0 \end{pmatrix}$ gives an isomorphism from Δ'_1 to Δ_1^* , the relation $\Delta_1^* = \Delta'_1$ holds.

The Newton polytope of F , which is the convex hull of vertices $(0, 0, 1)$, $(0, 1, 0)$, $(3, -4, -2)$, and $(-1, 1, 0)$, is not reflexive. Indeed, the polytope contains two faces Γ_1 spanned by $(-1, 1, 0)$, $(0, 1, 0)$, and $(3, -4, -2)$, of which the polar dual is a non-integral vertex $(0, -1, 2/5)$, and Γ_2 spanned by $(0, 0, 1)$, $(0, 1, 0)$, and $(3, -4, -2)$, of which the polar dual is a non-integral vertex $(-7/3, -1, -1)$. For Γ_1 , one has to make a union of two sections by adding the vertex $(-2, 2, -1)$. For Γ_2 , consider the section spanned by $(0, 1, 0)$, $(0, 0, 1)$, and $(2, -2, -1)$. The polar dual of this section is $(-2, -1, -1)$, which does not belong to $\Delta^{(66)}$. Therefore, the only possibility is $\Delta = \Delta_1$. It is directly checked that $\Delta_{F'} \subset \Delta'_1 \subset \Delta^{(35)}$. Therefore, the pair (Δ_1, Δ'_1) is polytope-dual.

5.20 No. 46–No. 47

We claim that there exists a unique polytope-dual pair for Nos. 46 and 47. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 21, 30$ and 86 by

$$\begin{aligned} e_1^{(21)} &= (-1, 1, 0, 0), & e_2^{(21)} &= (-1, 0, 1, 0), & e_3^{(21)} &= (-2, 0, 0, 1), \\ e_1^{(30)} &= (0, 4, -1, -1), & e_2^{(30)} &= (-1, -1, 4, -1), & e_3^{(30)} &= -(1, 1, 1, -1), \\ e_1^{(86)} &= (4, 0, -1, -1), & e_2^{(86)} &= (3, -1, -1, 0), & e_3^{(86)} &= (0, -1, 2, -1). \end{aligned}$$

Define polytopes Δ_1 , Δ'_1 , Δ_2 and Δ'_2 by

$$\begin{aligned} \Delta_1 &:= \text{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 1, 0), (2, -3, -1)\}, \\ \Delta'_1 &:= \text{Conv} \left\{ \begin{array}{l} (-1, -1, 1), (-1, -1, -1), (4, -1, -1), \\ (-1, 4, -1), (-1, 2, 0), (2, -1, 0) \end{array} \right\}, \\ \Delta_2 &:= \text{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-2, -2, -5)\}, \\ \Delta'_2 &:= \text{Conv}\{(-1, -1, 1), (-1, -1, -1), (4, -1, -1), (-1, 4, -1)\}. \end{aligned}$$

Since the polar dual polytopes Δ_1^* and Δ_2^* are the convex hulls of vertices $(0, -1, -1)$, $(-1, -1, -1)$, $(-1, -1, 2)$, $(0, -1, 4)$, $(5, 4, -1)$, $(-1, 0, -1)$, res., $(-1, -1, -1)$, $(-1, -1, 1)$, $(4, -1, -1)$, $(-1, 4, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 0 \\ -3 & -2 & 0 \end{pmatrix}$ gives an isomorphism from Δ'_1 to Δ_1^* , the relations $\Delta_1^* \simeq \Delta'_1$ and $\Delta_2^* = \Delta'_2$ hold.

No. 46. The Newton polytope of F , which is the convex hull of vertices $(0, 0, 1)$, $(0, 1, 0)$, $(2, -3, -1)$, and $(-1, 1, 0)$, is not reflexive. Indeed, the polytope Δ_F contains a face spanned by $(0, 1, 0)$, $(0, 0, 1)$, and $(2, -3, -1)$, of which the polar dual is $(-5/2, -1, -1)$. Since the complement of the set of lattice points on edges of Δ_F in those of $\Delta^{(86)}$ consists of the only $(1, 0, 0)$, the only possible Δ is $\Delta^{(86)} = \Delta_1$. It is directly checked that relations $\Delta_{F'} \subset \Delta'_1 = \Delta^{(86)}$. Therefore, the pair $(\Delta^{(86)} = \Delta_1, \Delta'_1)$ is polytope-dual.

No. 47. The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-1, -1, -2)$, is not reflexive. Indeed, the polytope Δ_F contains a face spanned by $(1, 0, 0)$, $(0, 1, 0)$, and $(-1, -1, -2)$, of which the polar

dual is $(-1, -1, 3/2)$. Since the complement of the set of lattice points on edges of Δ_F in those of $\Delta^{(30)}$ consists of the only $(-2, -2, -5)$, the only possible Δ is $\Delta^{(30)} = \Delta_2$. It is directly checked that relations $\Delta_{F'} \subset \Delta'_2 \subset \Delta^{(21)}$. Therefore, the pair $(\Delta^{(30)} = \Delta_2, \Delta'_2)$ is polytope-dual.

5.21 No. 48–No. 49

We claim that there exists a unique polytope-dual pair. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 5, 56$ and 73 by

$$\begin{aligned} e_1^{(5)} &= (-1, 1, 0, 0), & e_2^{(5)} &= (-1, 0, 1, 0), & e_3^{(5)} &= (-3, 0, 0, 1), \\ e_1^{(56)} &= (-1, 0, 2, -1), & e_2^{(56)} &= (-1, -1, 0, 1), & e_3^{(56)} &= (5, -1, -1, -1), \\ e_1^{(73)} &= (-1, -1, 4, -1), & e_2^{(73)} &= (-1, -1, -1, 1), & e_3^{(73)} &= (5, 0, -1, -1). \end{aligned}$$

In Kobayashi and Mase (2012), it is proved that the polytopes $\Delta^{(56)}$ and $\Delta^{(73)}$ are isomorphic to the polytope $\Delta_{(56,73)}$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-1, -3, -1)$, under the above choice of basis. Let Δ_1 be $\Delta_{(56,73)}$, and Δ'_1 be the polytope $\Delta^{(5)}$ which is the convex hull of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(5, -1, -1)$, and $(-1, 5, -1)$. Since the polar dual polytope Δ_1^* of Δ_1 is the convex hull of vertices $(-1, -1, -1)$, $(-1, 1, -1)$, $(5, -1, -1)$, and $(-1, -1, 5)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ gives an isomorphism from Δ'_1 to Δ_1^* , the relation $\Delta_1^* \simeq \Delta'_1$ holds.

Nos. 48 and 49. In both cases, the Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-1, -3, -1)$, coincides with Δ_1 . It is directly checked that the Newton polytope of F' is a subpolytope of Δ'_1 . Therefore, the pair $(\Delta_F = \Delta_1, \Delta'_1)$ is the only polytope-dual.

5.22 No. 50

We claim that there exists a unique polytope-dual pair. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of lattices M_n for $n = 1$ and 52 by

$$\begin{aligned} e_1^{(1)} &= (-1, 1, 0, 0), & e_2^{(1)} &= (-1, 0, 1, 0), & e_3^{(1)} &= (-1, 0, 0, 1), \\ e_1^{(52)} &= (-1, 2, -1, 0), & e_2^{(52)} &= (-1, -1, 3, -1), & e_3^{(52)} &= (-1, -1, -1, 2). \end{aligned}$$

Define polytopes Δ_1 and Δ'_1 by

$$\begin{aligned} \Delta_1 &:= \Delta^{(52)} = \text{Conv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)\}, \\ \Delta'_1 &:= \Delta^{(1)} = \text{Conv}\{(-1, -1, 3), (-1, -1, -1), (3, -1, -1), (-1, 3, -1)\}. \end{aligned}$$

Since the polar dual polytope Δ_1^* of Δ_1 is the convex hull of vertices $(-1, -1, 3)$, $(-1, -1, -1)$, $(3, -1, -1)$, and $(-1, 3, -1)$, the relation $\Delta_1^* = \Delta'_1$ clearly holds.

The Newton polytope of F , which is the convex hull of vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(-1, -1, -1)$, coincides with Δ_1 . Thus, $\Delta = \Delta_F = \Delta^{(52)} = \Delta_1$ is the only possible reflexive polytope. The polar dual Δ'_1 coincides with $\Delta^{(1)}$ and thus the Newton polytope $\Delta_{F'}$ of F' is a subpolytope of Δ'_1 . Therefore, the pair $(\Delta_F = \Delta_1, \Delta^{(1)} = \Delta'_1)$ is polytope-dual.

5.23 No. 51

We claim that there exist sixteen polytope-dual pairs. Take a basis $\{e_1^{(n)}, e_2^{(n)}, e_3^{(n)}\}$ of the lattice M_{32} by

$$e_1^{(32)} = (-1, 0, 3, -1), \quad e_2^{(32)} = (0, -1, 3, -1), \quad e_3^{(32)} = (-1, -1, -1, 1).$$

Define polytopes $\Delta_{i,a}$ and $\Delta_{i',a}$ with $i = 1, 2$ by

$$\begin{aligned} \Delta_{1,a} &:= \text{Conv} \left\{ (0, 0, 1), (1, 0, 0), (0, 1, 0), (-4, 3, -2), \right. \\ &\quad \left. (-a + 3, a - 4, -2), (2, -2, -1) \right\}, \\ \Delta_{1',a} &:= \text{Conv} \left\{ (0, 0, 1), (2, -2, -1), (0, 1, 0), (-2, 2, -1), \right. \\ &\quad \left. (-a + 3, a - 4, -2), (3, -4, -2) \right\}, \\ \Delta_{2,a} &:= \text{Conv} \left\{ (0, 0, 1), (1, 0, 0), (-2, 2, -1), (-4, 3, -2), \right. \\ &\quad \left. (-a + 3, a - 4, -2), (2, -2, -1) \right\}, \\ \Delta_{2',a} &:= \text{Conv} \left\{ (0, 0, 1), (1, 0, 0), (0, 1, 0), (-2, 2, -1), \right. \\ &\quad \left. (-a + 3, a - 4, -2), (3, -4, -2) \right\}, \end{aligned}$$

where $a = 0, 1, \dots, 7$.

Since the polar dual polytopes $\Delta_{1,a}^*$ and $\Delta_{2,a}^*$ of $\Delta_{1,a}$ and $\Delta_{2,a}$ are respectively the convex hulls of vertices $(-1, -1, 1)$, $(-1, -1, -1)$, $(0, -1, -1)$, $(3, 3, -1)$, $(a - 1, a, -1)$, and $(-1, 0, -1)$; and $(-1, -1, 1)$, $(-1, -2, -1)$, $(0, -1, -1)$, $(3, 3, -1)$, $(a - 1, a, -1)$ and $(-1, 0, -1)$, and the linear map of $\mathbb{R}^3$ determined by a matrix $\begin{pmatrix} -1 & -2 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ gives an isomorphism from $\Delta'_{i,a}$ to $\Delta_{i,a}^*$, the relation $\Delta_{i,a}^* \simeq \Delta'_{i,a}$ for $i = 1, 2$ holds.

The Newton polytope of $F = F'$, which is the convex hull of vertices $(1, 0, 0)$, $(-4, 3, -2)$, $(0, 0, 1)$, and $(2, -2, -1)$, is not reflexive. Indeed, the polytope Δ_F contains a face spanned by $(1, 0, 0)$, $(0, 0, 1)$, and $(-4, 3, -2)$, of which the polar dual is a non-integral vertex $(-1, -7/3, -1)$.

Since the complement of the set of lattice points on edges of Δ_F in those of $\Delta^{(32)}$ consists of $(2, -2, -1)$, $(1, 0, 0)$, and $(-a + 3, a - 4, -2)$ with $a = 0, 1, \dots, 7$, it is clear that we just need to consider polytopes of form $\Delta_{1,a}$ and $\Delta_{2,a}$ with $a = 0, 1, \dots, 7$. And for each a , we can find a polytope-dual partner.

Therefore, the claims are verified. $\square$

6 Closing Remarks

As is mentioned in Ebeling (2006), almost all mirror symmetric pairs of weight systems in the sense of Belcastro (2002) are also strongly coupled. We would like to study not only the full families of $K3$ surfaces, but families of $K3$ surfaces associated to reflexive polytopes Δ and Δ' obtained in Theorem 5.1. More precisely, it is interesting to study a relation with a lattice duality (Dolgachev 1996) and the coupling.

Acknowledgements Open Access funding provided by Projekt DEAL. The author thanks to Professor Wolfgang Ebeling for his suggestion of the study and discussions.

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