Factorizations of Weighted Hardy Inequalities

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Abstract We present factorizations of weighted Lebesgue, Cesàro and Copson spaces, for weights satisfying the conditions which assure the boundedness of the Hardy's integral operator between weighted Lebesgue spaces. Our results enhance, among other, the best known forms of weighted Hardy inequalities.

Keywords Factorization of function spaces · Hardy averaging integral operator · Cesàro spaces · Copson spaces

Mathematics Subject Classification Primary 26D20; Secondary 46B25

1 Introduction

Let $p > 1$. The classical Hardy inequalities

$$\left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^n |a_k| \right)^p \right)^{1/p} \leq \frac{p}{p-1} \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \quad (1)$$

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and

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right)^{1/p} \leq \frac{p}{p-1} \left(\int_0^\infty |f(t)|^p \right)^{1/p} \quad (2)$$

(see e.g. Kufner et al. 2007) can be interpreted as inclusions between the Lebesgue space and Cesàro space of sequences (respectively functions). The Cesàro space of sequences is defined to be the set of all real sequences $a = (a_n)_{n \geq 1}$ that satisfy

$$\|a\|_{\text{ces}(p)} = \left(\sum_{n=1}^\infty \left(\frac{1}{n} \sum_{k=1}^n |a_k| \right)^p \right)^{1/p} < \infty$$

and the Cesàro space of functions is defined to be the set of all Lebesgue measurable real functions on $[0, \infty)$ such that

$$\|f\|_{\text{Ces}(p)} = \left(\int_0^\infty \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right)^{1/p} < \infty.$$

The same interpretation is valid if the Hardy operator is substituted by its dual. In his celebrated book, Bennett (1996) “enhanced” the classical Hardy inequality by substituting it with an equality, factorizing the Cesàro space of sequences, with the final aim to characterize its Köthe dual. He proved that a sequence x belongs to the Cesàro space of sequences $\text{ces}(p)$ if and only if it admits a factorization $x = y \cdot z$ with $y \in l^p$ and $z_1^{p'} + \dots + z_n^{p'} = O(n)$, where $p' = \frac{p}{p-1}$ is the conjugate index of p . This factorization gives also a better insight in the structure of Cesàro spaces. The answer to the question of the dual space of the Cesàro space was given for the first time by Jagers (1974). This problem was posed by the Dutch Academy of Sciences. His description of the dual space of the Cesàro space is complicated, given in terms of the least decreasing majorant, namely

$$\left\{ x : \sum_{n=1}^\infty \left(n(\tilde{x}_n - \tilde{x}_{n+1})^{p'} \right) < \infty \right\}$$

where $\tilde{x}$ is the least decreasing majorant of $|x|$, with the property that

$$\frac{\tilde{x}_m - \tilde{x}_n}{\frac{1}{m^p} + \dots + \frac{1}{(n-1)^p}}, \quad n > m$$

increase with n if m is fixed. By means of factorizations, a new isometric characterization, an alternative description of the dual Cesàro space of sequences, as being the space

$$d(p) = \left\{ x : \sum_{n=1}^\infty \sup_{k \geq n} |x_k|^p < \infty \right\}$$

was given by Bennett (1996). It was not proved yet directly, that Jagers and Bennet's characterizations are equivalent.

In the case of functions, the same factorization results as well as the dual space of Cesàro space are only mentioned in Bennett (1996, Ch. 20), for the unweighted spaces. A factorization result for the unweighted Cesàro function spaces was proved in Astashkin and Maligranda (2009) where also an isomorphic description of the dual space of the Cesàro space of functions was given. An isometric description for the general weighted case, in the spirit of Jagers was given in Kamińska and Kubiak (2012). Although the characterization is given for general weights, the condition which defines the dual space is difficult. The results of Bennett (1996) have a big impact in many parts of analysis but it seems that the corresponding results for weighted spaces are less studied. For a recent survey of results on classical Cesàro spaces see Astashkin and Maligranda (2014). In the newly, very interesting papers, Leśnik and Maligranda (2015), Kolwicz et al. (2014) similar results and motivations, in an abstract, very general setting were presented. However, in Carton-Lebrun and Heinig (2003) can be found the following factorization result which is a weighted integral analogue of a result obtained by Bennett in Bennett (1996) for the discrete Hardy operator in the unweighted case.

Theorem 1.1 (Carton-Lebrun and Heinig 2003) *Let $1 < p < \infty$ and w, v two weights such that $w > 0, v > 0$ a.e. and assume that h is a non-negative function on $[0, \infty)$. Then the function h belongs to $Ces_p(w)$ if and only if it admits a factorization $h = f \cdot g$, $f \geq 0, g > 0$ on $[0, \infty)$, with $f \in L^p(v)$ and g such that*

$$\|g\|_{w,v} = \sup_{t>0} \left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{1/p} \left(\int_0^t v^{1-p'}(s) g^{p'}(s) ds \right)^{1/p'} < \infty.$$

Moreover

$$\inf \|f\|_{L^p(v)} \|g\|_{w,v} \leq \|h\|_{Ces_p(w)} \leq 2(p')^{1/p'} p^{1/p} \inf \|f\|_{L^p(v)} \|g\|_{w,v},$$

where the infimum is taken over all possible factorizations.

Throughout this paper, we use standard notations and conventions. The letters $u, v, w, \dots$, are used for weight functions which are positive a.e. and locally integrable on $(0, \infty)$. The function f is real-valued and Lebesgue measurable on $(0, \infty)$. Also for a given weight v we write $V(t) = \int_0^t v(s) ds, 0 \leq t < \infty$. By the symbol χ_A we denote the characteristic function of the measurable set A . The symbol $C \cdot D$ stays for the set of products of measurable, real valued functions defined on $(0, \infty)$, $\{f \cdot g : f \in C \text{ and } g \in D\}$.

Observe that the best known form of the Hardy inequality does not follow from Theorem 1.1. The aim of this paper is to prove factorization results of the same type for the weighted Lebesgue, Cesàro and Copson spaces of functions, which enhance in the same manner the weighted Hardy inequality. The weights satisfy the natural conditions which assure the boundedness of the Hardy, respectively the dual Hardy operators as well as some reversed conditions.

We denote by P the Hardy operator and by Q its adjoint

$$Pf(t) = \frac{1}{t} \int_0^t f(x)dx; \quad Qf(t) = \int_t^\infty \frac{f(x)}{x}dx, \quad (t > 0). \quad (3)$$

For $p \geq 1$, it is known that P is bounded on the weighted Lebesgue space $L^p(w)$ if and only if $w \in M_p$ (see Muckenhoupt (1972)), where M_p is the class of weights for which there exists a constant $C > 0$ such that, for all $t > 0$ it holds

$$M_p : \left(\int_t^\infty \frac{v(x)}{x^p}dx \right)^{1/p} \left(\int_0^t v^{1-p'}(x)dx \right)^{1/p'} \leq C. \quad (4)$$

The least constant satisfying the condition M_p will be denoted by $[v]_{M_p}$. Similarly, we denote by m_p the class of weights satisfying the reverse inequality and by $[v]_{m_p}$ the biggest constant for which the reverse inequality holds.

The Hardy operator P is bounded on $L^1(v)$ if and only if there exists $C > 0$, such that

$$M_1 : \int_t^\infty \frac{v(x)}{x}dx \leq Cv(t), \quad \text{for every } t > 0. \quad (5)$$

We denote by $[v]_{M_1}$ the least constant for which the above inequality is satisfied. Similarly, $[v]_{m_1}$ is the biggest constant for which the reverse inequality of (5) is satisfied.

The corresponding condition for the boundedness of the adjoint operator Q on $L^p(v)$ (see Muckenhoupt (1972)) is given by

$$M_p^* : \left(\int_0^t v(x)dx \right)^{1/p} \left(\int_t^\infty \frac{v^{1-p'}(x)}{x^{p'}}dx \right)^{1/p'} \leq C, \quad \text{for every } t > 0. \quad (6)$$

The least constant satisfying the M_p^* condition will be denoted $[w]_{M_p^*}$. Similarly, we denote by m_p^* the class of weights satisfying the reverse inequality and by $[w]_{m_p^*}$ the biggest constant for which the reverse inequality holds.

The dual Hardy operator Q , (defined by (3)) is bounded on $L^1(v)$ if and only if there exists $C > 0$, such that

$$M_1^* : \frac{1}{t} \int_0^t v(x)dx \leq Cv(t), \quad \text{for all } t > 0. \quad (7)$$

We denote by $[v]_{M_1^*}$ the least constant for which the above inequality is satisfied. Similarly, $[v]_{m_1^*}$ is the biggest constant for which the reverse inequality of (7) is satisfied.

In Sect. 2 we prove a factorization result for the weighted Lebesgue spaces $L^p(v)$. This result is a natural extension of Theorem 3.8 from Bennett (1996).

In Sect. 3 we present some factorization theorems for the weighted Cesàro spaces in terms of weighted Lebesgue spaces and the spaces $G_p(v)$, for $p > 1$. We treat separately the case $p = 1$ which appears to be new. Moreover, our study is motivated

by similar factorization results established by Bennett (1996), in the unweighted case, for spaces of sequences and in Astashkin and Maligranda (2009, 2014), Kolwicz et al. (2014) in the unweighted, integral case or in abstract setting. Our study concentrates on the special weighted case of Cesàro spaces containing the Lebesgue spaces. As a consequence we recover the best known form of the Hardy inequality for weighted Lebesgue spaces.

We also present the optimal result for the power weights. Section 4 is devoted to the same problems but for Copson spaces.

2 The Spaces $D_p(v)$ and $G_p(v)$

For $0 < p < \infty$, the function spaces $G_p(v)$ and $D_p(v)$ are defined by

$$G_p(v) = \left\{ f : \sup_{t>0} \left(\frac{1}{V(t)} \int_0^t |f(x)|^p v(x) dx \right)^{1/p} < \infty \right\} \quad (8)$$

and

$$D_p(v) = \left\{ f : \left(\int_0^\infty \operatorname{esssup}_{t \geq x} |f(t)|^p v(x) dx \right)^{1/p} < \infty \right\}. \quad (9)$$

If $p = \infty$, we clearly have

$$G_\infty(v) = D_\infty(v) = L^\infty,$$

where L^∞ is the Lebesgue space of essentially bounded functions. The spaces $D_p(1)$ and $G_p(1)$ were introduced for the first time in Bennett (1996, page 124), where analogue integral results to the discrete ones were only formulated. Their weighted versions for $1 \leq p < \infty$, appeared firstly in Astashkin and Maligranda (2009, Remark 2).

Using standard arguments such as Minkowski inequality and Fatou's lemma (see Rudin 1987, Theorem 3.11), it is easy to see that $G_p(v)$ and $D_p(v)$ endowed with the norms

$$\|f\|_{G_p(v)} = \sup_{t>0} \left(\frac{1}{V(t)} \int_0^t |f(x)|^p v(x) dx \right)^{1/p}, \quad (10)$$

respectively

$$\|f\|_{D_p(v)} = \left(\int_0^\infty \operatorname{esssup}_{t \geq x} |f(t)|^p v(x) dx \right)^{1/p}$$

are Banach spaces, for $p \geq 1$.

We denote by $\widehat{f}(x) = \operatorname{esssup}_{t \geq x} |f(t)|$ the least decreasing majorant of the absolute value of the function f . Obviously, the function $f \in D_p(v)$ if and only if $\widehat{f} \in L^p(v)$ and that $\|f\|_{D_p(v)} = \|\widehat{f}\|_{L^p(v)}$.

In what follows we need the following two lemmas.

Lemma 2.1 (Hardy's lemma) *Let f, g be two nonnegative real-valued functions and h be a nonnegative decreasing function. If*

$$\int_0^t f(x)dx \leq \int_0^t g(x)dx, \quad \text{for any } t > 0$$

then

$$\int_0^\infty f(x)h(x)dx \leq \int_0^\infty g(x)h(x)dx.$$

Proof See Bennett and Sharpley (1988, Proposition 3.6). $\square$

Lemma 2.2 *Let h be a nonnegative measurable function on $(0, \infty)$, such that*

$$\lim_{x \rightarrow \infty} \frac{\int_0^x h(t)v(t)dt}{\int_0^x v(t)dt} = 0.$$

Then there exists a nonnegative decreasing function h° on $(0, \infty)$, called the level function of h with respect to the measure $v(x)dx$ satisfying the following conditions:

- (1) $\int_0^x h(t)v(t)dt \leq \int_0^x h^\circ(t)v(t)dt$;
- (2) *up to a set of measure zero, the set $\{x : h(x) \neq h^\circ(x)\} = \cup_{k=1}^\infty I_k$, where I_k are bounded disjoint intervals such that*

$$\int_{I_k} h^\circ(t)v(t)dt = \int_{I_k} h(t)v(t)dt$$

and h° is constant on I_k , i.e. $h^\circ(t) = \frac{\int_{I_k} h v}{\int_{I_k} v}$.

Proof For a proof see e.g. Barza et al. (2009) or Sinnamon (1994). $\square$

We are now ready to prove the main theorem of this section, which contains one of the possible factorizations of weighted Lebesgue spaces. Our proof of factorization of $L^p(v)$ is based on Hardy's lemma and some properties of the so-called "level function", and is different than that given in Astashkin and Maligranda (2009) for the unweighted case. This factorization is a natural extension to the weighted integral case of the discrete, unweighted version (Bennett 1996, Theorem 3.8) and of the unweighted integral case proved in Astashkin and Maligranda (2009, Proposition 2). Also we mention here that the statement of the next Theorem, without a proof, appears in Astashkin and Maligranda (2009, Remark 2).

Theorem 2.1 *If $0 < p \leq \infty$, then a function $h \in L^p(v)$ if and only if f admits a factorization $h = f \cdot g$ such that $f \in D_p(v)$ and $g \in G_p(v)$. Moreover,*

$$\|h\|_{L^p(v)} = \inf\{\|f\|_{D_p(v)}\|g\|_{G_p(v)}\},$$

where the infimum is taken over all possible factorizations $h = f \cdot g$ with $f \in D_p(v)$ and $g \in G_p(v)$.

Proof The case $p = \infty$ is trivial. Observe that the spaces $L^p(v)$, $D_p(v)$, $G_p(v)$ are homogeneous, namely $f \in L^p(v)$, $D_p(v)$ or $G_p(v)$ if and only if $f^p \in L^1(v)$, $D_1(v)$ or $G_1(v)$. Hence, by homogeneity (or p -convexification), it is sufficient to prove the theorem for $p = 1$.

We first prove that $D_1(v) \cdot G_1(v) \subseteq L^1(v)$. Suppose that h admits a factorization $h = f \cdot g$ with $f \in D_1(v)$, $g \in G_1(v)$. Then

$$\|h\|_{L^1(v)} = \int_0^\infty |f(x)g(x)|v(x)dx \leq \int_0^\infty \widehat{f}(x)|g(x)|v(x)dx.$$

From definition (10) we have the inequality

$$\int_0^t |g(x)|v(x)dx \leq \|g\|_{G_1(v)} \int_0^t v(x)dx, \quad \text{for any } t > 0$$

which together with Lemma 2.1 give

$$\int_0^\infty \widehat{f}(x)|g(x)|v(x)dx \leq \|g\|_{G_1(v)} \int_0^\infty \widehat{f}(x)v(x)dx = \|g\|_{G_1(v)}\|f\|_{D_1(v)}.$$

Thus we have that $D_1(v) \cdot G_p(v) \subseteq L^1(v)$ and that

$$\|h\|_{L^1(v)} \leq \inf\{\|g\|_{G_1(v)}\|f\|_{D_1(v)}\},$$

where the infimum is taken over all possible factorizations $h = f \cdot g$.

Conversely let h be a nonnegative function such that $h \in L^1(v)$. We set $f(x) = h^\circ(x)$, $x > 0$, where $h^\circ(x)$ is the level function of h with respect to the measure $v(x)dx$, as in Lemma 2.2. Since $h^\circ(x)$ is a decreasing function by the definition of the space $D_1(v)$ and by Lemma 2.2 we have that

$$\|f\|_{D_1(v)} = \|h^\circ\|_{D_1(v)} = \|h^\circ\|_{L^1(v)} = \|h\|_{L^1(v)}.$$

We define $g(x) = \frac{h(x)}{h^\circ(x)}$ on $\{x > 0 : h^\circ(x) \neq 0\} = [0, a)$, for some $a > 0$ and $g(x) = 0$ if $x > a$. If $t \in I_n$, for some n , we have

$$\begin{aligned} \frac{1}{V(t)} \int_0^t g(x)v(x)dx &= \frac{1}{V(t)} \int_0^t \frac{h(x)}{h^\circ(x)}v(x)dx \\ &= \frac{1}{V(t)} \left(\int_E \frac{h(x)}{h^\circ(x)}v(x)dx + \int_{\bigcup_{k=1}^{n-1} I_k} \frac{h(x)}{h^\circ(x)}v(x)dx \right. \\ &\quad \left. + \int_{a_n}^t \frac{h(x)}{h^\circ(x)}v(x)dx \right), \end{aligned}$$

where $E = \{x \in (0, t) : h(x) = h^\circ(x)\}$ and $I_k = (a_k, b_k)$ are the disjoint intervals from Lemma 2.2. Hence, by Lemma 2.2 we get that

$$\begin{aligned}\int_E \frac{h(x)}{h^\circ(x)} v(x) dx &= \int_E v(x) dx, \\ \int_{I_k} \frac{h(x)}{h^\circ(x)} v(x) dx &= \int_{I_k} v(x) dx\end{aligned}$$

and

$$\int_{a_n}^t \frac{h(x)}{h^\circ(x)} v(x) dx \leq \int_{I_n} v(x) dx.$$

Hence

$$\|g\|_{G_1(v)} \leq 1.$$

Since $h = f \cdot g$, with $f \in D_1(v)$ and $g \in G_1(v)$ we have that $L^1(v) \subseteq D_1(v) \cdot G_1(v)$ and

$$\|h\|_{L^1(v)} = \|f\|_{D_1(v)} \geq \|f\|_{D_1(v)} \cdot \|g\|_{G_1(v)} \geq \inf\{\|f\|_{D_1(v)} \cdot \|g\|_{G_1(v)}\},$$

where the infimum is taken over all possible factorizations $h = f \cdot g$. It is easy to see from this proof that the infimum is actually attained and this concludes the proof of the theorem. $\square$

3 Factorization of the Weighted Cesàro Spaces

In this section we present a factorization of the weighted Cesàro spaces $\text{Ces}_p(v)$. We treat separately the cases $p > 1$ and $p = 1$. The weighted Cesàro spaces of functions, $\text{Ces}_p(v)$ is defined to be the space of all Lebesgue measurable real functions on $[0, \infty)$ such that

$$\|f\|_{\text{Ces}_p(v)} = \left(\int_0^\infty \left(\frac{1}{x} \int_0^x |f(t)| dt \right)^p v(x) dx \right)^{1/p} < \infty.$$

These spaces are obviously Banach spaces, for $p \geq 1$ and if the weight v satisfies (4) we have that $L^p(v) \subseteq \text{Ces}_p(v)$. We denote by

$$!h!_{p,v} = \inf\{\|f\|_{L^p(v)} \|g\|_{G_{p'(v^{1-p'})}}\} \quad (11)$$

where the infimum is taken over all possible decompositions of $h = f \cdot g$, with $f \in L^p(v)$ and $g \in G_{p'(v^{1-p'})}$.

The following Theorem is an extension to the weighted case of Astashkin and Maligranda (2009, Proposition 1). The discrete, unweighted case was proved in Bennett (1996, Theorem 1.5).

Theorem 3.1 Let $p > 1$ and v belongs to the classes M_p and m_p . The function h belongs to $\text{Ces}_p(v)$ if and only if it admits a factorization $h = f \cdot g$, with $f \in L^p(v)$ and $g \in G_{p'}(v^{1-p'})$. Moreover

$$[v]_{m_p}!h!_{p,v} \leq \|h\|_{\text{Ces}_p(v)} \leq (p')^{1/p'} p^{1/p} [v]_{M_p}!h!_{p,v}.$$

Proof Let $f \in L^p(v)$ and $g \in G_{p'}(v^{1-p'})$. First we prove that the function $h = f \cdot g \in \text{Ces}_p(v)$ and the right-hand side inequality. Let u be an arbitrary decreasing function. By Hölder's inequality we get

$$\begin{aligned} \int_0^t |h(x)|dx &= \int_0^t |f(x)g(x)|dx \leq \left(\int_0^t |f(x)|^p v(x) u^{-p}(x) dx \right)^{1/p} \\ &\quad \cdot \left(\int_0^t |g(x)|^{p'} v^{1-p'}(x) u^{p'}(x) dx \right)^{1/p'}. \end{aligned} \quad (12)$$

On the other hand, by Lemma 2.1 we obtain

$$\begin{aligned} \left(\int_0^t |g(x)|^{p'} v^{1-p'}(x) u^{p'}(x) dx \right)^{1/p'} &\leq \|g\|_{G_{p'}(v^{1-p'})} \\ &\quad \cdot \left(\int_0^t v^{1-p'}(x) u^{p'}(x) dx \right)^{1/p'}. \end{aligned} \quad (13)$$

Hence, by (12) and (13), integrating from 0 to ∞ and by applying Fubini's theorem we have

$$\begin{aligned} \int_0^\infty \left(\frac{1}{t} \int_0^t |h(x)|dx \right)^p v(t) dt &\leq \|g\|_{G_{p'}(v^{1-p'})}^p \int_0^\infty \left(\int_0^t |f(x)|^p v(x) u^{-p}(x) dx \right) \\ &\quad \cdot \left(\int_0^t v^{1-p'}(x) u^{p'}(x) dx \right)^{p-1} t^{-p} v(t) dt \\ &= \|g\|_{G_{p'}(v^{1-p'})}^p \int_0^\infty |f(x)|^p v(x) u^{-p}(x) \\ &\quad \cdot \left(\int_x^\infty t^{-p} v(t) \left(\int_0^t v^{1-p'}(x) u^{p'}(x) dx \right)^{p-1} dt \right) dx. \end{aligned}$$

Taking $u(t) = \left(\int_0^t v^{1-p'}(s) ds \right)^{-1/(pp')}$, since $v \in M_p$ we get

$$\|h\|_{\text{Ces}_p(v)} \leq p^{1/p} (p')^{1/p'} [v]_{M_p} \|g\|_{G_{p'}(v^{1-p'})} \|f\|_{L^p(v)}.$$

Hence $h \in \text{Ces}_p(v)$ and

$$\|h\|_{\text{Ces}_p(v)} \leq p^{1/p} (p')^{1/p'} [v]_{M_p} \inf \{ \|f\|_{L^p(v)} \cdot \|g\|_{G_{p'}(v^{1-p'})} \},$$

where the infimum is taken over all possible factorizations of h . This completes the first part of the proof of the theorem.

For the reversed embedding, i.e. $\text{Ces}_p(v) \subseteq L^p(v) \cdot G_{p'}(v^{1-p'})$, let $h \in \text{Ces}_p(v)$. Since $v > 0$ a.e. for $t > 0$, we may assume, without loss of generality that $v(t) > 0$, for any $t > 0$. Set now

$$w(t) := \frac{1}{v(t)} \int_t^\infty \frac{v(x)}{x} \left(\frac{1}{x} \int_0^x |h(s)| ds \right)^{p-1} dx,$$

for $t > 0$. We define $f(t) = |h(t)|^{1/p} w^{1/p}(t) \text{sign } h(t)$ and $g(t) = |h(t)|^{1/p'} w^{-1/p'}(t)$. It is easy to see that

$$\|f\|_{L^p(v)} = \|h\|_{\text{Ces}_p(v)} < \infty. \quad (14)$$

By Hölder's inequality we have

$$\begin{aligned} \left(\int_0^t g^{p'}(x) v^{1-p'}(x) dx \right)^p &\leq \left(\int_0^t |h(x)| dx \right)^{p-1} \\ &\quad \cdot \left(\int_0^t |h(x)| w^{-p'}(x) v^{-p'}(x) dx \right). \end{aligned} \quad (15)$$

Multiplying the inequality (15) by $\int_t^\infty x^{-p} v(x) dx$ and using that $w(t)v(t)$ is a decreasing function we get

$$\begin{aligned} &\left(\int_t^\infty \frac{v(x)}{x^p} dx \right) \left(\int_0^t g^{p'}(x) v^{1-p'}(x) dx \right)^p \\ &\leq \left(\int_t^\infty \frac{v(x)}{x^p} \left(\int_0^x |h(s)| ds \right)^{p-1} dx \right) \\ &\quad \cdot \left(\int_0^t |h(x)| w^{-p'}(x) v^{-p'}(x) dx \right) \\ &= W(t)v(t) \left(\int_0^t |h(x)| w^{-p'}(x) v^{-p'}(x) dx \right) \\ &\leq \int_0^t |h(x)| w^{1-p'}(x) v^{1-p'}(x) dx. \end{aligned}$$

Since $g^{p'}(x) = |h(x)| w^{1-p'}(x)$ we obtain

$$\begin{aligned} \left(\frac{1}{\int_0^t v^{1-p'}(x) dx} \int_0^t g^{p'}(x) v^{1-p'}(x) dx \right)^{1/p'} &\leq \left(\int_0^t v^{1-p'}(x) dx \right)^{-1/p'} \\ &\quad \cdot \left(\int_t^\infty \frac{v(x)}{x^p} dx \right)^{-1/p}. \end{aligned}$$

Hence

$$\sup_{t>0} \left(\frac{1}{\int_0^t v^{1-p'}(x) dx} \int_0^t g^{p'}(x) v^{1-p'}(x) dx \right)^{1/p'} \leq \frac{1}{[v]_{m_p}}$$

which shows that g belongs to $G_{p'}(v^{1-p'})$ and

$$\|h\|_{\text{Ces}_p(v)} = \|f\|_{L^p(v)} \geq [v]_{m_p} \|f\|_{L^p(v)} \|g\|_{G_{p'}(v^{1-p'})}.$$

In this way, we get the left-hand side inequality. $\square$

If we take $g(x) = 1$, $x > 0$ the right-hand side inequality implies the best form of the weighted Hardy inequality for $1 < p < \infty$ namely

$$\|f\|_{\text{Ces}_p(v)} \leq (p')^{1/p'} p^{1/p} [v]_{M_p} \|f\|_{L^p(v)}$$

(see e.g. Kufner et al. 2007).

Observe also that the infimum is attained.

In particular, we denote by L_α^p the weighted Lebesgue space with the power weight $v(t) = t^\alpha$ and in a similar way the spaces $G_{p,\alpha}$ and $\text{Ces}_{p,\alpha}$. In analogy with the general case we also denote by

$$!h!_{p,\alpha} = \inf \|f\|_{L_\alpha^p} \|g\|_{G_{p',\alpha(1-p')}},$$

where the infimum is taken over all possible decompositions of $h = f \cdot g$, with $f \in L_\alpha^p$ and $g \in G_{p',\alpha(1-p')}$.

Corollary 3.2 *Let $p > 1$ and $-1 < \alpha < p - 1$. The function h belongs to $\text{Ces}_{p,\alpha}$ if and only if it admits a factorization $h = f \cdot g$, with $f \in L_\alpha^p$ and $g \in G_{p',\alpha(1-p')}$. Moreover*

$$\left(\frac{1}{p}\right)^{1/p} \left(\frac{1}{p'}\right)^{1/p'} \frac{p}{p-\alpha-1} !h!_{p,\alpha} \leq \|h\|_{\text{Ces}_{p,\alpha}} \leq \frac{p}{p-\alpha-1} !h!_{p,\alpha}.$$

Proof Take $v(t) = t^\alpha$ in Theorem 3.1. The constant in the right hand-side inequality is optimal since it is the best constant in Hardy's inequality with a power weight (see e.g. Kufner et al. 2007, p. 23). $\square$

For the sake of completeness, as well as for the independent interest we present separately the case $p = 1$, although the proof of the main result in this case follows the same ideas as for $p > 1$.

By L^∞ we denote, as usual, the space of all measurable functions which satisfy the condition

$$\|g\|_\infty := \text{esssup}_{x>0} |g(x)| < \infty.$$

As before,

$$\|h\|_{1,v} = \inf \|f\|_{L^1(v)} \|g\|_{\infty}$$

where the infimum is taken over all possible factorizations of $h = f \cdot g$, with $f \in L^1(v)$ and $g \in L^{\infty}$.

Theorem 3.3 *Let v belong to M_1 and m_1 . The function h belongs to $Ces_1(v)$ if and only if it admits a factorization $h = f \cdot g$, with $f \in L^1(v)$ and $g \in L^{\infty}$. Moreover*

$$[v]_{m_1} \|h\|_{1,v} \leq \|h\|_{Ces_1(v)} \leq [v]_{M_1} \|h\|_{1,v}.$$

Proof Let $f \in L^1(v)$ and $g \in L^{\infty}$. We prove that the function $h = fg$ belongs to $Ces_1(v)$. By Hölder's inequality and since $g \in L^{\infty}$ we get

$$\int_0^{\infty} \left(\frac{1}{t} \int_0^t h(s) ds \right) v(t) dt \leq \|g\|_{\infty} \int_0^{\infty} \left(\frac{1}{t} \int_0^t f(x) dx \right) v(t) dt. \quad (16)$$

By Fubini's theorem and taking into account that $v \in M_1$ we have that

$$\|h\|_{Ces_1(v)} \leq [v]_{M_1} \|g\|_{\infty} \|f\|_{L^1(v)},$$

for any f, g as above. Hence $h \in Ces_1(v)$ and

$$\|h\|_{Ces_1(v)} \leq [v]_{M_1} \inf \|g\|_{\infty} \|f\|_{L^1(v)},$$

where infimum is taken over all possible factorizations of h . This completes the first part of the proof.

Conversely, let $h \in Ces_1(v)$ and $w(t) = \frac{1}{v(t)} \int_t^{\infty} \frac{v(x)}{x} dx$. We may assume, without loss of generality that $v(t) > 0$, for all $t > 0$.

Let $f(t) = |h(t)|w(t) \operatorname{sign} h(t)$ and $g(x) = \frac{1}{w(x)}$. It is easy to see that

$$\|f\|_{L^1(v)} = \|h\|_{Ces_1(v)} < \infty.$$

Since $v \in m_1$, g belongs to L^{∞} and

$$\|g\|_{\infty} \leq \frac{1}{[v]_{m_1}}.$$

Moreover, $\|h\|_{Ces_1(v)} = \|f\|_{L^1(v)} \geq [v]_{m_1} \|f\|_{L^1(v)} \|g\|_{\infty}$ and we get the left-hand side inequality of the theorem. The proof is complete. $\square$

4 Factorization of the Weighted Copson Spaces

In the same manner, in this section we present the factorizations of the weighted Copson space, namely the space

$$\text{Cop}_p(v) = \left\{ f : \int_0^\infty \left(\int_t^\infty \frac{|f(x)|}{x} dx \right)^p v(t) dt < \infty \right\}.$$

Let

$$G_p^*(v) = \left\{ f : \sup_{t>0} \left(\frac{1}{\int_t^\infty v(x)x^{-p} dx} \int_t^\infty f(x)^p v(x)x^{-p} dx \right)^{1/p} < \infty \right\} \quad (17)$$

To prove the main result we need the following Lemma.

Lemma 4.1 *Let f, g be two non-negative real-valued functions and h be a non-negative increasing function. If*

$$\int_t^\infty f(x) dx \leq \int_t^\infty g(x) dx, \quad t > 0$$

then

$$\int_0^\infty f(x)h(x) dx \leq \int_0^\infty g(x)h(x) dx.$$

Proof The proof follows by a change of variable and Lemma 2.1. $\square$

We denote by

$$\|h\|_{p,v} = \inf \|f\|_{L^p(v)} \|g\|_{G_{p'}^*(v^{1-p'})} \quad (18)$$

where the infimum is taken over all possible factorizations of $h = f \cdot g$. The following theorem extends to the weighted case a result formulated without proof in Bennett (1996, Theorem 21.6). The discrete case is proved in Bennett (1996, Theorem 5.5).

Theorem 4.1 *Let $p > 1$ and v belong to the classes M_p^* and m_p^* .*

The function h belongs to $\text{Cop}_p(v)$ if and only if it admits a factorization $h = f \cdot g$, with $f \in L^p(v)$ and $g \in G_{p'}^(v^{1-p'})$. Moreover*

$$[v]_{m_p^*} \|h\|_{p,v} \leq \|h\|_{\text{Cop}_p(v)} \leq p^{1/p'} p^{1/p} [v]_{M_p^*} \|h\|_{p,v}.$$

Proof Let $f \in L^p(v)$ and $g \in G_{p'}^*(v^{1-p'})$.

We show first that the function $h = fg \in \text{Cop}_p(v)$. Let u be an arbitrary positive increasing function. Hölder's inequality gives

$$\begin{aligned} & \int_t^\infty \frac{f(x)g(x)}{x} dx \\ & \leq \left(\int_t^\infty f^p(x)v(x)u^{-p}(x)dx \right)^{1/p} \left(\int_t^\infty g^{p'}(x)\frac{v^{1-p'}(x)}{x^{p'}}u^{p'}(x)dx \right)^{1/p'}. \quad (19) \end{aligned}$$

By Hardy's Lemma 4.1 we obtain

$$\begin{aligned} & \left(\int_t^\infty g^{p'}(x)\frac{v^{1-p'}(x)}{x^{p'}}u^{p'}(x)dx \right)^{1/p'} \leq \|g\|_{G_{p'}^*(v^{1-p'})}^p \\ & \cdot \left(\int_t^\infty \frac{v^{1-p'}(x)}{x^{p'}}u^{p'}(x)dx \right)^{1/p'}. \end{aligned}$$

Hence, multiplying (19) by $v(t)$, raising to p and integrating from 0 to ∞ , we get

$$\begin{aligned} & \int_0^\infty \left(\int_t^\infty \frac{h(x)}{x} dx \right)^p v(t)dt \leq \|g\|_{G_{p'}^*(v^{1-p'})}^p \\ & \cdot \int_0^\infty v(t) \left(\int_t^\infty f^p(x)v(x)u^{-p}(x)dx \right) \left(\int_t^\infty \frac{v^{1-p'}(x)}{x^{p'}}u^{p'}(x)dx \right)^{p-1} dt. \end{aligned}$$

By Fubini's theorem we have

$$\begin{aligned} & \int_0^\infty \left(\int_t^\infty \frac{h(s)}{s} ds \right)^p v(t)dt \leq \|g\|_{G_{p'}^*(v^{1-p'})}^p \\ & \cdot \int_0^\infty f^p(x)v(x) \left(\int_0^x v(t) \left(\int_t^\infty \frac{v^{1-p'}(s)}{s^{p'}}u^{p'}(s)ds \right)^{p-1} dt \right) u^{-p}(x)dx. \end{aligned}$$

Taking $u(t) = \left(\int_t^\infty \frac{v^{1-p'}(x)}{x^{p'}} dx \right)^{-1/pp'}$, in the above inequality and since

$$\int_t^\infty \frac{v^{1-p'}(s)}{s^{p'}} u^{p'}(s)ds = p' \left(\int_t^\infty \frac{v^{1-p'}(s)}{s^{p'}}(s)ds \right)^{1/p'} dx$$

we have that

$$\begin{aligned} & \int_0^\infty f^p(x)v(x) \left(\int_0^x v(t) \left(\int_t^\infty \frac{v^{1-p'}(s)}{s^{p'}} u^{p'}(s) ds \right)^{p-1} dt \right) u^{-p}(x) dx \\ &= (p')^{p-1} \int_0^\infty f^p(x)v(x) \left(\int_0^x v(t) \left(\int_t^\infty \frac{v^{1-p'}(s)}{s^{p'}} ds \right)^{\frac{p-1}{p'}} dt \right) \\ & \quad \cdot \left(\int_x^\infty \frac{v^{1-p'}(s)}{s^{p'}} ds \right)^{1/p'}. \end{aligned}$$

By the definition of M_p^* we get

$$\begin{aligned} & \int_0^\infty \left(\int_t^\infty \frac{h(s)}{s} ds \right)^p v(t) dt \leq (p')^{p-1} \|v\|_{M_p^*}^{p-1} \\ & \quad \cdot \int_0^\infty f^p(x)v(x) \left(\int_0^x v(t) \left(\int_0^t v(s) ds \right)^{-1/p'} dt \right) \left(\int_x^\infty \frac{v^{1-p'}(s)}{s^{p'}} ds \right)^{1/p'} dx \\ &= (p')^{p-1} p \|v\|_{M_p^*}^{p-1} \int_0^\infty f^p(x)v(x) \left(\int_0^x v(t) \right)^{1/p} \left(\int_x^\infty \frac{v^{1-p'}(s)}{s^{p'}} ds \right)^{1/p'} dx \\ &\leq (p')^{p-1} p \|v\|_{M_p^*}^p \int_0^\infty f^p(x)v(x) dx, \end{aligned}$$

since

$$\frac{d}{dx} \left(\int_0^x v(t) dt \right)^{1/p} = \frac{1}{p} \left(\int_0^x v(t) dt \right)^{1/p-1} v(x).$$

Hence

$$\|h\|_{\text{Cop}_p(v)} \leq p^{1/p} p^{1/p'} [v]_{M_p^*} \|g\|_{G_{p', (v^{1-p'})}^*} \|f\|_{L^p(v)}.$$

for any f, g as above. Hence $h \in \text{Cop}_p(v)$ and

$$\|h\|_{\text{Cop}_p(v)} \leq p^{1/p} p^{1/p'} [v]_{M_p^*} \inf \|g\|_{G_{p', (v^{1-p'})}^*} \|f\|_{L^p(v)}$$

where the infimum is taken over all possible factorizations of h which gives the left-hand side inequality of the theorem.

For the reverse embedding, i.e. $\text{Cop}_p(v) \subset L^p(v)G_{p', (v^{1-p'})}^*$, let $h \in \text{Cop}_p(v)$ and

$$w(t) := \frac{1}{t v(t)} \int_0^t v(x) \left(\int_x^\infty \frac{h(s)}{s} ds \right)^{p-1} dx,$$

if $v \neq 0$ and $w(t) = 0$ on of Lebesgue measure possibly $v = 0$. Define $f(t) = |h|^{1/p}(t)w^{1/p}(t)\text{sign } h(t)$ and $g(t) = |h|^{1/p'}(t)w^{-1/p}(t)$. An easy application of Fubini theorem gives

$$\|f\|_{L^p(v)} = \|h\|_{\text{Cop}_p(v)} < \infty.$$

By Hölder's inequality and the definition of g we have

$$\left(\int_x^\infty g^{p'}(t) \frac{v^{1-p'}(t)}{t^{p'}} dt \right)^p \leq \left(\int_x^\infty \frac{|h(t)|}{t} dt \right)^{p-1} \int_x^\infty \frac{|h(t)|}{t} w^{-p'}(t) \frac{v^{-p'}(t)}{t^{p'}} dt. \quad (20)$$

We estimate first the right-hand side term of the inequality (20) multiplied by $\int_0^x v(t)dt$.

$$\begin{aligned} & \left(\int_0^x v(t)dt \right) \left(\int_x^\infty \frac{|h(t)|}{t} dt \right)^{p-1} \int_x^\infty \frac{|h(t)|}{t} w^{-p'}(t) \frac{v^{-p'}(t)}{t^{p'}} dt \\ & \leq \int_0^x v(t) \left(\int_t^\infty \frac{|h(s)|}{s} ds \right)^{p-1} dt \int_x^\infty \frac{|h(t)|}{t} w^{-p'}(t) \frac{v^{-p'}(t)}{t^{p'}} dt \\ & = xw(x)v(x) \left(\int_x^\infty h(t)w^{-p'}(t)v^{-p'}(t)dt \right) \\ & \leq \int_x^\infty h(t)w^{1-p'}(t)v^{1-p'}(t)t^{-p'} dt, \end{aligned}$$

since, by definition, $xw(x)v(x)$ is an increasing function. By using that $g^{p'}(x) = h(x)w^{1-p'}(x)$ we get

$$\begin{aligned} & \left(\frac{1}{\int_x^\infty \frac{v^{1-p'}(t)}{t^{p'}} dx} \int_x^\infty g^{p'}(t) \frac{v^{1-p'}(t)}{t^{p'}} dt \right)^{1/p'} \\ & \leq \frac{1}{\left(\int_0^x v(t)dt \right)^{1/p} \left(\int_x^\infty t^{-p'} v^{1-p'}(t)dt \right)^{1/p'}}. \end{aligned}$$

Hence

$$\sup_{t>0} \left(\frac{1}{\int_t^\infty v^{1-p'}(x)x^{-p'} dx} \int_t^\infty g^{p'}(x)v^{1-p'}(x)x^{-p'} dx \right)^{1/p'} \leq \frac{1}{[v]_{m_p^*}} \quad (21)$$

which means that g belongs to $G_{p'}^*(v^{1-p'})$. Moreover,

$\|h\|_{\text{Cop}_p(v)} = \|f\|_{L^p(v)} \geq [v]_{m_p^*} \|f\|_{L^p(v)} \|g\|_{G_{p'}^*(v^{1-p'})}$. In this way the left-hand side inequality is proved. $\square$

The space $\text{Cop}_{p,\alpha}$ is the Copson weighted space with the weight t^α . We have the following result for the case of a power weight.

Corollary 4.2 *Let $p > 1$ and $\alpha > -1$. The function h belongs to $Cop_{p,\alpha}$ if and only if it admits a factorization $h = f \cdot g$, with $f \in L_\alpha^p$ and $g \in G_{p',\alpha(1-p')}$. Moreover*

$$\frac{(p-1)^{1/p'}}{\alpha+1} !h!_{p,\alpha} \leq \|h\|_{p,\alpha} \leq \frac{p}{\alpha+1} !h!_{p,\alpha}.$$

The constants in both inequalities are optimal.

Proof Take $v(t) = t^\alpha$ in Theorem 4.1. The constant in the right hand-side inequality is optimal since it is the best constant in Hardy inequality (see e.g. Kufner et al. 2007) and the optimality of the constant in the left-hand side follows if we take $h(x) = \chi_{(a-\varepsilon, a+\varepsilon)}$ and let then $\varepsilon \rightarrow 0$ and $a \rightarrow \infty$. $\square$

We present now the case $p = 1$.

The dual Hardy operator Q , (defined by (3)) is bounded on $L^1(v)$ if and only if there exists $C > 0$, such that

$$M_1^* : \frac{1}{t} \int_0^t v(x) dx \leq C v(t). \quad (22)$$

We denote by $[v]_{M_1^*}$ the least constant for which the above inequality is satisfied. Similarly, $[v]_{m_1^*}$ is the biggest constant for which the reverse inequality of (22) is satisfied.

Theorem 4.3 *Let v belong to M_1^* and m_1^* . The function h belongs to $Cop_1(v)$ if and only if it admits a factorization $h = f \cdot g$, with $f \in L^1(v)$ and $g \in L^\infty$. Moreover*

$$[v]_{m_1^*} !h!_{1,v} \leq \|h\|_{Cop_1(v)} \leq [v]_{M_1^*} !h!_{1,v}.$$

and the constants are optimal.

Proof The proof is similar with that of Theorem 3.3. $\square$

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