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Calculus of Variations

Ergodic mean field games: existence of local minimizers up to the Sobolev critical case

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Abstract

We investigate the existence of solutions to viscous ergodic Mean Field Games systems in bounded domains with Neumann boundary conditions and local, possibly aggregative couplings. In particular we exploit the associated variational structure and search for constrained minimizers of a suitable functional. Depending on the growth of the coupling, we detect the existence of global minimizers in the mass subcritical and critical case, and of local minimizers in the mass supercritical case, notably up to the Sobolev critical case.

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1 Introduction

In this work we investigate the existence of solutions to the following system arising in the theory of viscous ergodic Mean Field Games, with Neumann boundary conditions and local coupling

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x, m(x)) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial m}{\partial n} + m \nabla H(\nabla u) \cdot n = 0 & \text{on } \partial\Omega \\ \int_\Omega m = 1, \qquad \int_\Omega u = 0, \end{cases} \tag{1.1}$$

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and their minimality properties in a suitable variational framework. Throughout the paper, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with boundary of class C^3. To simplify some computations, we suppose that $|\Omega| = 1$, though all the arguments work for $|\Omega| \neq 1$. On the Hamiltonian H, we assume that

$$\begin{aligned} &C_H^{-1}\,|p|^{\gamma} - K_H \leq H(p) \leq C_H\,|p|^{\gamma} + K_H \\ &\nabla H(p)\cdot p - H(p) \geq C_H^{-1}\,|p|^{\gamma} - C_H \\ &|\nabla H(p)| \leq C_H\,|p|^{\gamma-1} + K_H \end{aligned} \tag{1.2}$$

for some $C_H > 1, K_H > 0, \gamma > 1$. As for the coupling, we suppose that $f : \overline{\Omega} \times [0, +\infty) \longrightarrow \mathbb{R}$ is Lipschitz with respect to both variables on bounded sets of $\overline{\Omega} \times [0, +\infty)$, and

$$-C_f m^{q-1} - K_f \leq f(x, m) \leq C_f m^{q-1} + K_f \tag{1.3}$$

for some $C_f > 0$, $K_f > 0$, $q > 1$.

Mean-Field Games (MFG) have been introduced in the seminal papers by Lasry and Lions [22] and Huang, Caines and Malhamè, [21], with the aim of describing Nash equilibria in differential games with infinitely many indistinguishable agents. The system (1.1) characterizes these equilibria in an ergodic game, where the cost of a typical agent is averaged over an infinite-time horizon. Neumann boundary conditions come from the assumption that agents' trajectories are constrained to Ω by normal reflection at the boundary (as in [13, 15, 25]).

Though systems of type (1.1) have been widely investigated over the last decade [3, 5, 7, 10, 11, 19, 20, 29], it is not yet known the existence of classical solutions in the full range $q > 1$. While known restrictions on q may be artificial, they are sometimes *structural*. Let us briefly describe the purely quadratic case $H(p) = |p|^2$ to clarify this point. By the classical Hopf-Cole transformation $\phi = e^{-u}/\int e^{-u} = \sqrt{m}$, (1.1) boils down to

$$\begin{cases} -\Delta\phi = \lambda\phi - f(x, \phi^2)\phi & \text{on } \Omega \\ \frac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_\Omega \phi^2 \mathrm{d}x = 1, \end{cases}$$

which, for $F' = f$, are the Euler-Lagrange equations of the functional defined on $W^{1,2}(\Omega)$

$$\mathcal{F}(\phi) = \int_\Omega |\nabla\phi|^2 + F(\phi^2)\mathrm{d}x \qquad \text{constrained to } \int_\Omega \phi^2 \mathrm{d}x = 1.$$

In the model case $F(m) = C_f m^q$, it is well known (by the Gagliardo-Nirenberg inequality) that $2q \leq 2 + 4/N$ is necessary for $\mathcal{F}$ to be bounded from below when constrained to $\int \phi^2 = 1$, while $2q < 2 + 4/(N-2)$ is needed, in dimension $N \geq 3$, for the compact embedding of $W^{1,2}(\Omega)$ into $L^{2q}(\Omega)$, which is crucial to set up variational methods. There are therefore two critical values that determine three regimes, each one of exhibiting different properties and difficulties (see also [27] and references therein). Note that when $C_f > 0$ the previous functional is convex, but we are not assuming here this property. In fact, we will be mainly interested in a model case $C_f < 0$, that corresponds to a MFG where aggregation between agents is enforced.

In the general nonquadratic case, (1.1) borrows a variational structure associated to the following functional and constraint (see for example [8, 30] and references therein for further details). Let

$$\mathcal{E}(m,w) := \begin{cases} \displaystyle\int_\Omega mL\left(-\frac{w}{m}\right) + F(x,m)\,dx & \text{if } (m,w)\in\mathcal{K} \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

where

$$F(x,m) := \begin{cases} \displaystyle\int_0^m f(x,n)\,dn & \text{if } m\ge 0 \\ +\infty & \text{if } m<0, \end{cases} \tag{1.5}$$

and

$$\begin{aligned} \mathcal{K} := \Big\{ &(w,m)\in L^{\frac{\gamma' q}{\gamma'+q-1}}(\Omega)\cap W^{1,r}(\Omega)\times L^1(\Omega) \text{ s.t.} \\ &\int_\Omega \nabla m\cdot\nabla\phi\,dx = \int_\Omega w\cdot\nabla\phi\,dx \;\; \forall\phi\in C^\infty(\overline{\Omega}), \\ &\int_\Omega m\,dx = 1, m\ge 0 \text{ a.e.}\Big\}, \qquad \text{where } \frac{1}{r} := \frac{1}{\gamma'}+\frac{1}{\gamma q}. \end{aligned} \tag{1.6}$$

It has been first observed in [11] that for any $\gamma > 1$, two critical values of q can be identified:

$$\bar{q} = 1+\frac{\gamma'}{N} \text{ (mass critical)}, \qquad q_c = \begin{cases} 1+\frac{\gamma'}{N-\gamma'} & \text{if } \gamma' < N \\ +\infty & \text{if } \gamma'\ge N \end{cases} \text{ (Sobolev critical)}.$$

As in the quadratic case, $q \le \bar{q}$ is necessary for $\mathcal{E}$ to be bounded from below, while $q < q_c = r^*$ guarantees the compact embedding of $W^{1,r}$ into L^q (this explains why we call this exponent "Sobolev critical").

Note that if $\gamma = 2$, $\bar{q}$ and q_c agree with the critical exponents mentioned in the previous paragraph.

Most of the analysis on systems of type (1.1) has been carried out in the case $q < q_c$. By means of fixed point methods, solutions have been shown to exist in [11] in the range $\bar{q} \le q < q_c$ under the further assumption that C_f be small enough (for problems which are set on the flat torus). No results are known when $q = q_c$, and if $q > q_c$ solutions may even fail to exist, at least when the system is set on the whole euclidean space.

The main goal of the present paper is to show the existence of solutions that are *local* minimizers of $\mathcal{E}$, in particular in the range $\bar{q} \le q \le q_c$, up to the critical exponent q_c. The further property of (local) minimality obtained here may be an important feature in the study of their *stability*, that is, their ability to capture the long-time behavior of the parabolic version of (1.1). Our main result reads as follows.

Theorem 1.1 *Assume that (1.2) and (1.3) hold. Suppose that either*

1. $1 < q < \bar{q}$, *or*
2. $q = \bar{q}$ *and* $C_f \le C_{\bar{q}}$, *or*
3. $\bar{q} < q < q_c$ *and* $C_f \le C_q$, *or*
4. $q = q_c$ *and* $C_f, K_f, K_H \le C_{crit}$.

Then there exists a solution (u, λ, m) *to the system (1.1), and* $m > 0$.

Moreover, the pair $(m, -m\nabla H(\nabla u))$ *is a global minimizer of* $\mathcal{E}$ *on* $\mathcal{K}$ *in cases 1 and 2, or a minimizer of* $\mathcal{E}$ *on* $\mathcal{K}\cap\{m : \|m\|_{L^q}^q < \bar{\alpha}\}$*, for a suitable* $\bar{\alpha} > 0$*, in cases 3 and 4.*

A solution to the system (1.1) is a triple $(u, \lambda, m) \in C^{2,\theta}(\overline{\Omega}) \times \mathbb{R} \times W^{1,p}(\Omega)$ for all $\theta \in (0, 1)$ and all $p > 1$ such that (u, λ) is a classical solution to the Hamilton-Jacobi equation and m is a weak solution to the Fokker-Planck equation. The constants $C_{\bar{q}}$, C_q and

C_{crit} are explicitly calculated throughout Sect. 3. We mention here that they depend on the data q, C_H, γ and also on some regularity and embedding constants C_S, C_E, C_q, δ_0, which in turn depend on q, Ω, N.

The model Hamiltonian $H(p) = |p|^\gamma$, $\gamma > 1$ clearly falls into our set of assumptions, and we can allow for a general growth of type $|p|^\gamma$ with different coefficients from above and below. We just need to be careful when $q = q_c$: H has to be small enough for small values of $|p|$. As for f, we include the model case

$$f(x, m) = C_f a(x) m^{q-1} + K_f b(x),$$

where a, b are smooth functions (with no sign condition). When $\bar{q} \le q < q_c$ we just need C_f to be small enough, while $q = q_c$ requires further smallness conditions on K_f.

The way local minima are identified is inspired by [26, 28]. The functional $\mathcal{E}$ is minimized first on the intersection between the constraint $\mathcal{K}$ and the ball $\{m : \|m\|^q_{L^q} \le \bar{\alpha}\}$. Estimates based on the Gagliardo-Nirenberg inequality for competitors belonging to $\mathcal{K}$, which involves a differential constraint of Fokker-Planck type, allow to choose $\bar{\alpha}$ in a way that these minimizers lie in fact in the open ball $\{m : \|m\|^q_{L^q} < \bar{\alpha}\}$, and therefore they give rise to solutions to the optimality conditions (1.1). There are a few technical obstacles to produce a solution of (1.1) from a local minimizer of $\mathcal{E}$. These are worked out following a strategy that involve a regularization of the functional and a linearization which allows to use convex duality methods. This strategy is detailed for example in [8]. We mention that an alternative approach to derive the MFG system from minimizers of $\mathcal{E}$, based on the analysis of its subdifferential, has been developed in [24, 25].

Note that the critical case requires further smallness assumptions on the coefficients. There is an interesting connection between this endpoint case and another phenomenon of criticality arising in Hamilton-Jacobi equations. If m is in a ball of L^{q_c}, then $f(m)$, which is the right-hand side of the Hamilton-Jacobi equation in (1.1), will be bounded in $L^{N/\gamma'}$. Recent works [12, 16, 18] on the so-called maximal regularity of Hamilton-Jacobi equations underlined the crucial role of the exponent N/γ': if the right-hand side $f(m)$ is just bounded in $L^{N/\gamma'}$, there are counterexamples (see e.g. [12, Rmk. 1]) showing that $H(\nabla u)$, Δu cannot be controlled separately in $L^{N/\gamma'}$, and therefore there is no hope to deduce further regularity via bootstrap arguments. In this sense, N/γ' is critical. But, if the $L^{N/\gamma'}$-norm of $f(m)$ is *small enough*, then $H(\nabla u)$, Δu can be controlled in the same Lebesgue space, which leads to further regularity. Additional smallness assumptions should be then expected when dealing with the endpoint case $q = q_c$.

A natural question concerns the uniqueness of solutions to (1.1), which is not expected in general. We discuss here a few examples, limiting to the model nonlinearities

$$H(p) = \frac{1}{\gamma}|p|^\gamma, \qquad L(q) = \frac{1}{\gamma'}|q|^{\gamma'}, \qquad f(x, m) = f(m) = \pm C_f m^{q-1}.$$

In this setting, clearly the system (1.1) admits the trivial solution $(u_{tr}, \lambda_{tr}, m_{tr}) \equiv (0, f(1), 1)$, which has a corresponding energy $\mathcal{E}(m_{tr}, -m_{tr}\nabla H(\nabla u_{tr})) = \mathcal{E}(1, 0) = F(1)$.

First of all, if $f(m) = C_f m^{q-1}$, $C_f > 0$, it is well known that (1.1) admits a unique solution, which must coincide with the trivial one (see for example [22]). Instead, let us assume $f(m) = -C_f m^{q-1}$, $C_f > 0$: using Theorem 1.1, we infer that the existence of multiple solutions follows whenever we can build an appropriate couple (m, w) in $\mathcal{K}$ (or in $\mathcal{K} \cap B_{\bar{\alpha}}$) satisfying $\mathcal{E}(m, w) < F(1)$; indeed, in such case the solution found in Theorem 1.1 cannot be the trivial one. To this aim, take any $\phi \in C^2(\overline{\Omega})$ such that $\int_\Omega \phi\, dx = 0$, and consider $m = 1 + \varepsilon\phi$ and $w = \nabla m$. For $0 \le \varepsilon \le \bar{\varepsilon}$ small enough, we have that $(m, w) \in K \cap B_{\bar{\alpha}}$ and $f'(1 - \bar{\varepsilon}\|\phi\|_\infty) \le -CC_f < 0$. We obtain, for some $0 < \xi < \bar{\varepsilon}$,

$$
\begin{aligned}
\mathcal{E}(m, w) &= \int_\Omega mL\left(-\frac{w}{m}\right) dx + \int_\Omega F(m)\, dx \\
&= \frac{1}{\gamma'} \int_\Omega (1+\varepsilon\phi(x))^{1-\gamma'} |\varepsilon\nabla\phi(x)|^{\gamma'}\, dx + \int_\Omega F(1+\varepsilon\phi(x))\, dx \\
&\le C\varepsilon^{\gamma'} + F(1) + \varepsilon f(1)\int_\Omega \phi(x)\, dx + \frac{\varepsilon^2}{2}\int_\Omega f'(1+\xi\phi(x))\phi^2(x)\, dx \\
&\le F(1) + C\varepsilon^{\gamma'} - C' C_f \varepsilon^2.
\end{aligned}
$$

Now, this last quantity is strictly smaller than $F(1)$ either for $\gamma' > 2$ and ε small enough, or for $\gamma' \le 2$, $C_f > C\bar{\varepsilon}^{-(2-\gamma')}/C'$ and $\varepsilon = \bar{\varepsilon}$. We deduce multiplicity of solutions when

$$
\text{either } \gamma < 2, \qquad \text{or } \gamma \ge 2,\ 1 < q < \bar{q} \text{ and } C_f \text{ sufficiently large.}
$$

We may obtain obtain the same result when $\gamma \ge 2$ and $q \ge \bar{q}$, in case the largeness condition on C_f here were compatible with the smallness one in Theorem 1.1. On the other hand, if $\gamma \ge 2$ and C_f is small enough, then uniqueness may be expected, see [14].

In case of multiplicity of solutions, a further question concerns the uniqueness of the minimizers. This is an interesting question, which will be the object of subsequent studies.

The rest of the paper is organized as follows: in Sect. 2 we recall some notions about the Legendre transform and some results about the Fokker-Plank and Hamilton-Jacobi equations. Moreover, for both equations we prove some a priori bounds in the critical endpoint case. In Sect. 3 we prove the main result. This is done in several steps. First, we regularize the problem, associate a variational structure and prove the existence of (local) minimizers. Once we have a minimum point for the energy, by a duality argument we deduce the existence of a regular solution. Lastly, the solution to the initial problem is found by an appropriate limit procedure.

Notations. For $k \in \mathbb{N}$ and $p \ge 1$ we denote by $\|u\|_p$ and $\|u\|_{k,p}$ the usual $L^p(\Omega)$ and $W^{k,p}(\Omega)$ norm respectively. For $p \ge 1$, the exponent p' is the conjugate exponent of p, $p' = \frac{p}{p-1}$. C, C' and so on denote non-negative universal constants, which we need not to specify, and which may vary from line to line.

2 Preliminaries

2.1 The Lagrangian

The Legendre-Fenchel transform L_H of H is necessary for the construction of the energy associated with the system:

$$
L = L_H(q) := \sup_{p \in \mathbb{R}^N} [p \cdot q - H(p)]
$$

Under our assumptions on H, the following properties of L_H are standard:

Proposition 2.1 *There exists $C_L > 0$ such that for all $p, b \in \mathbb{R}^N$*

1. $L_H \in C^2\left(\mathbb{R}^N \setminus \{0\}\right)$ *and it is strictly convex.*
2. $0 \le C_L\, |q|^{\gamma'} \le L_H(q) \le C_L^{-1}(|q|^{\gamma'} + 1)$
3. $\nabla L_H(q) \cdot q - L_H(q) \ge C_L\, |q|^{\gamma'} - C_L^{-1}$
4. $C_L\, |q|^{\gamma'-1} - C_L^{-1} \le |\nabla L_H(q)| \le C_L^{-1}(|q|^{\gamma'-1} + 1)$

Proof See for example [9]. □

The energy functional $\mathcal{E}$ involves the following Lagrangian term.

Proposition 2.2 *The function*

$$(m, w) \longrightarrow mL\left(-\frac{w}{m}\right) = \begin{cases} mL(-\frac{w}{m}) & \text{if } m > 0 \\ 0 & \text{if } m = 0, w = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

is convex, and strictly convex if restricted to $m > 0$*. We have,*

$$mH(p) = \begin{cases} \sup_{w \in \mathbb{R}^N} \left[-p \cdot w - mL\left(-\frac{w}{m}\right)\right] & \text{if } m \neq 0 \\ 0 & \text{if } m = 0. \end{cases} \tag{2.1}$$

Moreover

$$C_L \frac{|w|^{\gamma'}}{m^{\gamma'-1}} \leq mL\left(-\frac{w}{m}\right) \leq C_L^{-1} \frac{|w|^{\gamma'}}{m^{\gamma'-1}} + C_L^{-1} m. \tag{2.2}$$

Proof Equation (2.1) is standard, see for example [8]. Estimate (2.2) comes directly from Proposition 2.1. □

2.2 Fokker–Planck equations

We deal here with the Fokker–Planck equation

$$\begin{cases} -\Delta m - \operatorname{div}(mb) = 0 & \text{on } \Omega \\ \frac{\partial m}{\partial n} + mb \cdot n = 0 & \text{on } \partial\Omega \\ \int_\Omega m = 1 \end{cases} \tag{2.3}$$

where $b : \mathbb{R} \to \mathbb{R}^N$ will be (at least) in $L^s(\Omega; \mathbb{R}^N)$, for some $s > N$. Solutions $m \in W^{1,2}(\Omega)$ will be in the standard weak sense:

$$\int_\Omega \nabla m \cdot \nabla \phi \, dx = \int_\Omega bm \cdot \nabla \phi \, dx \qquad \forall \phi \in W^{1,2}(\Omega), \tag{2.4}$$

with $\int_\Omega m = 1$. The following existence result for b in L^∞ is classical.

Theorem 2.3 *Let* $b \in L^\infty(\Omega; \mathbb{R}^N)$*. Then there exist an unique weak solution* m *to (2.3),* $m \in W^{1,p}(\Omega)$ *for every* p *and*

$$\|m\|_{1,p} \leq C = C(\|b\|_\infty, p, N, \Omega).$$

Moreover, $m \in C^{0,\alpha}(\overline{\Omega})$ *for all* $\alpha \in (0, 1)$ *and there exists* $c = c(\|b\|_\infty, p, N, \Omega) > 0$ *such that*

$$c^{-1} \leq m(x) \leq c$$

for all $x \in \Omega$*.*

Proof see [4, Th. II.4.4, II.4.5, II.4.7]. □

We investigate further regularity properties, and recall the following proposition, which is an useful $W^{1,p}$ regularity result for linear equations in divergence form.

Proposition 2.4 *Let $\rho \in L^p(\Omega)$, with $p > 1$. Suppose that*

$$\left| \int_\Omega \rho \Delta \phi \, dx \right| \le K \, \|\nabla \phi\|_{p'} \tag{2.5}$$

for all $\phi \in C^\infty(\overline{\Omega})$, $\frac{\partial \phi}{\partial n} = 0$ for some $K > 0$. Then $\rho \in W^{1,p}(\Omega)$ and there exists $C_E = C_E(N, \Omega, p)$ such that

$$\|\rho\|_{1,p} \le C_E(K + \|\rho\|_p). \tag{2.6}$$

Moreover, the same estimate holds in a local form, that is, for every $B_{2R} \subset \Omega$,

$$\|\rho\|_{W^{1,p}(B_R)} \le C_{E,R}(K + \|\rho\|_{L^p(B_{2R})}). \tag{2.7}$$

Proof See [1, Theorems 7.1 and 8.1]. □

As a straightforward consequence, given $w \in L^p(\Omega; \mathbb{R}^N)$, $p > 1$, any weak solution $m \in W^{1,p}(\Omega)$ to

$$\begin{cases} -\Delta m - \operatorname{div} w = 0 & \text{on } \Omega \\ \frac{\partial m}{\partial n} + w \cdot n = 0 & \text{on } \partial\Omega \\ \int_\Omega m = 1. \end{cases} \tag{2.8}$$

satisfies

$$\|m\|_{1,p} \le C_E(\|w\|_p + \|m\|_p). \tag{2.9}$$

The previous estimate, combined with the Gagliardo-Nirenberg inequality, yields the following crucial result.

Proposition 2.5 *Let $(m, w) \in L^1 \cap W^{1,r}(\Omega) \times L^1(\Omega)$ be a solution of (2.8), and E be defined by*

$$E := \int_\Omega \frac{|w|^{\gamma'}}{m^{\gamma'-1}} \, dx,$$

which is assumed to be finite. Then there exists $C_q > 0$ such that

$$\|m\|_{1,r} \le C_q(E + 1) \tag{2.10}$$

$$\|m\|_q \le C_q(E + 1). \tag{2.11}$$

Moreover, if $q < \bar{q}$, there exists also $\delta > 0$ such that

$$\|m\|_q^{q(1+\delta)} \le C_q(E + 1). \tag{2.12}$$

Finally, if $q = \bar{q}$, then (2.12) holds with $\delta = 0$.

Proof The proof can be found for example in [8], and its adaption to the problem with Neumann conditions is straightforward. □

To conclude the section, we present a sharper estimate which will be useful in an endpoint case of our analysis, and requires b to be controlled in $L^N(\Omega)$ only.

Proposition 2.6 *Let $b \in L^\infty(\Omega; \mathbb{R}^N)$ such that $\|b\|_N \le \frac{1}{2C_E C_S}$, where C_E is defined in (2.6) and C_S is the p Sobolev Embedding constant. Then for any $p < N$ the solution m of (2.3) satisfies.*

$$\|m\|_{1,p} \le C = C(p, N, \Omega) \tag{2.13}$$

Proof By (2.9) we have that

$$\|m\|_{1,p} \le C_E(\|bm\|_p + \|m\|_p).$$

By Hölder inequality, Sobolev Embedding and Interpolation

$$\|bm\|_p + \|m\|_p \le \|b\|_N \|m\|_{p^*} + \|m\|_1^{1-\theta} \|m\|_{p^*}^{\theta} \le C_S \|b\|_N \|m\|_{1,p} + \|m\|_{1,p}^{\theta},$$

for some $\theta = \theta(p, N) < 1$, from which we can deduce the thesis using the bound assumed on $\|b\|_N$. □

2.3 Hamilton–Jacobi equations

We now consider the Hamilton–Jacobi equation

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x) & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_\Omega u = 0. \end{cases} \tag{2.14}$$

A solution of (2.14) is a pair $(u, \lambda) \in C^2(\overline{\Omega}) \times \mathbb{R}$ that satisfies the equation pointwise.

Theorem 2.7 *Let $f \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$. Then there exists an unique constant $\lambda \in \mathbb{R}$ such that (2.14) has a unique solution in $C^{2,\alpha}(\overline{\Omega})$ and*

$$\lambda = \sup\left\{c \in \mathbb{R} \text{ s.t. } \exists u \in C^2(\bar{\Omega}), \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega : -\Delta u + H(\nabla u) + c \le f\right\}. \tag{2.15}$$

Moreover, the following estimates hold:

$$\|\nabla u\|_\infty \le K_1 = K_1(N, \Omega, \|f\|_\infty),$$
$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \le K_2 = K_2(N, \Omega, \alpha, \|f\|_{C^\alpha(\overline{\Omega})}).$$

Note that the estimate holds also locally, that is, for every $B_{2R} \subset \Omega$,

$$\|\nabla u\|_{L^\infty(B_R)} \le K_1 = K_1(N, \Omega, R, \|f\|_{L^\infty(B_{2R})}).$$

The proof of this theorem is well-known in ergodic control theory, and it is typically obtained via a limiting procedure involving a discounted problem. The crucial gradient estimate which allows to pass to the limit in the procedure can be derived using the Bernstein method, see for example [8, 10, 23] and references therein. Though we are not going to use directly the characterization (2.15) of λ here, it is in fact a key step in the existence argument of Theorem 3.7, which follows some standard lines involving convex duality.

We are now interested in finding additional regularity results, which will be used in our critical endpoint case. In particular, we are interested in finding bounds for $|\nabla u|^\gamma$ in $L^{N/\gamma'}$ depending on f in $L^{N/\gamma'}$. This is a delicate endpoint case of the so-called L^q-maximal regularity for Hamilton-Jacobi equations, which has been recently discussed in [13, 16] for $q \ge \frac{N}{\gamma'}$. We provide here a simple proof which exploits a smallness condition on f.

We start by introducing the following lemma for linear equations, and provide its standard proof for the reader's convenience.

Lemma 2.8 *Let $f \in L^p(\Omega)$ for $p > 1$ and let $(u, \lambda) \in W^{2,p}(\Omega) \times \mathbb{R}$ be a solution of*

$$\begin{cases} -\Delta u + \lambda = f \text{ on } \Omega \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \\ \int_\Omega u = 0. \end{cases} \tag{2.16}$$

Then there exists $C = C(N, \Omega, p) > 0$ (independent of λ) such that

$$\|u\|_{2,p} \leq C \|f\|_p . \tag{2.17}$$

Proof Using elliptic regularity (which holds up to the boundary by homogeneous Neumann boundary conditions, see e.g. [31, Ch. 3]), we have that

$$\|u\|_{2,p} \leq C(\|f - \lambda\|_p + \|u\|_p). \tag{2.18}$$

If we test (2.16) with a constant function, we get that necessarily

$$\lambda = \frac{1}{|\Omega|} \int_\Omega f \,\mathrm{d}x, \tag{2.19}$$

hence we can suppose that $\|f - \lambda\|_p \leq C \|f\|_p$. Let us now claim that

$$\|u\|_p \leq C \|f\|_p . \tag{2.20}$$

Indeed, suppose by contradiction that there exists a sequence (u_n, f_n) satisfying (2.16) such that

$$\|u_n\|_p > n \|f_n\|_p . \tag{2.21}$$

By (2.18) we get

$$n \|f_n\|_p \leq C(\|f_n\|_p + \|u_n\|_p), \tag{2.22}$$

hence we can conclude that $\frac{\|f_n\|_p}{\|u_n\|_p} \to 0$. Now let $v_n = \frac{u_n}{\|u_n\|_p}$. Clearly $\|v_n\|_p = 1$, $\int_\Omega v_n = 0$ and

$$\|v_n\|_{2,p} \leq C \left(\frac{\|f_n\|_p}{\|u_n\|_p} + 1 \right) \leq C.$$

Hence we have that there exists a subsequence $v_n \to v$ strongly in $L^p(\Omega)$ and weakly in $W^{2,p}(\Omega)$. Moreover, $\|v\|_p = 1$, $\int_\Omega v = 0$. Now since u_n satisfies (2.16) we have for all $\phi \in C^\infty(\overline{\Omega})$

$$\int_\Omega \nabla v_n \nabla \phi \, dx = \frac{1}{\|u_n\|_p} \int_\Omega (f_n - \lambda)\phi \, dx \leq \frac{C \|f_n\|_p}{\|u_n\|_p} \|\phi\|_{p'} .$$

Passing to the limit we get that $\int_\Omega \nabla v \nabla \phi \, dx = 0$ for all $\phi \in C^\infty(\overline{\Omega})$, hence $\nabla v = 0$ which implies $v = K$ with $K = 0$ by the constraint. This is a contradiction with $\|v\|_p = 1$. □

Using this estimate the idea is to construct a "barrier" for ∇u which we employ in a topological fixed point argument. This means finding a value M such that no solutions exists with $\|\nabla u\|_{N(\gamma-1)} = M$.

Proposition 2.9 *Under the hypothesis of Theorem 2.7, let (u, λ) be the unique solution of (2.14). Then there exists δ_0 depending on N, Ω, γ, p, H such that if*

$$K_H + \|f\|_{\frac{N}{\gamma'}} \leq \delta \leq \delta_0,$$

then

$$\text{either } \|\nabla u\|_{N(\gamma-1)} < M \quad \text{ or } \|\nabla u\|_{N(\gamma-1)} > M \tag{2.23}$$

for some M depending on δ (and $N, \Omega, \gamma, p, C_H$). Moreover, $M = M(\delta) \to 0$ as $\delta \to 0$.

Proof Let $p = \frac{N}{\gamma'}$, so $p^* = p\gamma$, and assume that $K_H + \|f\|_{\frac{N}{\gamma'}} \leq \delta \leq \delta_0$, with δ_0 to be chosen below. Hence by Sobolev embedding, Lemma 2.8 and assumptions on H,

$$0 \geq y := \|\nabla u\|_{p\gamma} \leq \|u\|_{1,p\gamma} \leq C\,\|u\|_{2,p} \leq C(\|\nabla u\|_{p\gamma}^{\gamma} + K_H + \|f\|_p) \leq Cy^{\gamma} + C'\delta. \tag{2.24}$$

We now choose δ_0 such that for every $0 < \delta \leq \delta_0$ there exist two positive solutions $y_1(\delta) < y_2(\delta)$ to $y = Cy^{\gamma} + C'\delta$. In this way (2.24) yields

$$\text{either } y \leq y_1(\delta) \leq y_1(\delta_0) \qquad \text{or } y \geq y_2(\delta) \geq y_2(\delta_0),$$

and we can choose $M = M(\delta)$ such that $y_1(\delta) < M < y_2(\delta)$. Moreover, $y_1(\delta)$ can be made arbitrarily small if $\delta \leq \delta_0$ is chosen small enough, hence also $M(\delta) > 0$ can be made arbitrarily small. □

Now that we have the barrier, we can use Leray-Schauder fixed point theorem to deduce the following a priori estimate (which is in fact also an existence result).

Theorem 2.10 *Under the hypothesis of Theorem 2.7, there exists $\delta_0 > 0$ such that if $K_H + \|f\|_{\frac{N}{\gamma'}} \leq \delta \leq \delta_0$ then*

$$\|\nabla u\|_{N(\gamma-1)} < M. \tag{2.25}$$

Moreover, $M = M(\delta) \to 0$ as $\delta \to 0$ (M, δ, δ_0 are as in the previous proposition).

Proof Let δ_0 be as in Proposition 2.23, and assume $K_H + \|f\|_{\frac{N}{\gamma'}} \leq \delta \leq \delta_0$. We use Leray-Schauder fixed point theorem (see for instance [2, Thm. 4.3.4]). Let $U = \{w \in C^2(\overline{\Omega}) : \|\nabla u\|_{N(\gamma-1)} < M\}$ and consider the operator $T : \overline{U} \to C^2(\overline{\Omega})$ where $u = T(w)$ is defined by the solution of the system

$$\begin{cases} -\Delta u + H(\nabla w) + \lambda = f(x) & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u = 0. \end{cases}$$

We claim that T is continuous and compact. Under this condition, by homotopy with the identity, Leray-Schauder theorem asserts that either T has a fixed point in U, or there exists $s \in (0, 1]$ and $u \in \partial U$ such that $u = sT(u)$. However, the latter possibility is excluded by Proposition 2.9: indeed, $u = sT(u)$ yields

$$\begin{cases} -\Delta u + sH(\nabla u) + s\lambda = sf(x) & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_{\Omega} u = 0, \end{cases}$$

and, being $0 < s \leq 1$,

$$K_{sH} + \|sf\|_{\frac{N}{\gamma'}} = sK_H + s\,\|f\|_{\frac{N}{\gamma'}} \leq s\delta \leq \delta;$$

whence Proposition 2.9 applies, replacing $H(\nabla u), f$ with $sH(\nabla u), sf$ respectively, and we infer $\|\nabla u\|_{N(\gamma-1)} \neq M$, in contradiction with $u \in \partial U$. Hence T has a fixed point in U, which by Theorem 2.7 is the unique solution of (2.14), from which we can deduce the desired estimate.

Thus we are left to prove that T is continuous and compact. Clearly $T_1 : C^2(\overline{\Omega}) \to C^\alpha(\overline{\Omega})$ defined by $u = H(\nabla u)$ is continuous by our assumptions on H. Moreover let $T_2 : C^\alpha(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ be the operator defined by $u = T_2 z$ solving

$$\begin{cases} -\Delta u + z + \lambda = f & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \int_\Omega u = 0. \end{cases}$$

This is well defined and continuous by elliptic regularity and because we supposed $f \in C^\alpha$. Finally, by Ascoli-Arzelà the immersion $T_3 : C^{2,\alpha}(\overline{\Omega}) \to C^2(\overline{\Omega})$ is continuous and compact. Hence we conclude that $T = T_3 \circ T_2 \circ T_1$ is continuous and compact. □

3 Existence of a solution

We are now ready prove Theorem 1.1. The proof of this theorem consists of several steps, which are explained in detail in the following sections. Since we are dealing with local couplings, the duality procedure to get a solution to the system (1.1) is rather delicate (as the dual problem to the minimization of $\mathcal{E}$ would be related to solutions of an Hamilton-Jacobi equation with rough right-hand side). We will introduce a family of regularized problem with smoothing couplings, associate their energies $\mathcal{E}_\varepsilon$ and then prove the existence of (local) minimizers. Once we have a minimum point for the regularized energy, we will perform the convex duality argument to deduce the existence of a regular solution. The solution to the initial problem will be then found by an appropriate limit procedure on the regularized sequence of solutions.

3.1 Regularization

Let us consider, for $\varepsilon > 0$, the following regularized system

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f_\varepsilon[m](x) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m\nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \\ \frac{\partial m}{\partial n} + m\nabla H(\nabla u)\cdot n = 0 & \text{on } \partial\Omega \\ \int_\Omega m = 1, \qquad \int_\Omega u = 0, \end{cases} \tag{3.1}$$

where

$$f_\varepsilon[m](x) := f(\cdot, m * \chi_\varepsilon(\cdot)) * \chi_\varepsilon(x) = \int_{\mathbb{R}^N} \chi_\varepsilon(x-y) f\left(y, \int_{\mathbb{R}^N} m(z)\chi_\varepsilon(y-z)\,dz\right) dy \tag{3.2}$$

and χ_ε is a sequence of standard symmetric mollifiers approximating the unit (f and m are extended to 0 outside Ω). We notice that given

$$F_\varepsilon[m] := \int_\Omega F(x, m * \chi_\varepsilon(x))\,dx \tag{3.3}$$

we have that it holds

$$F_\varepsilon[m'] - F_\varepsilon[m] = \int_0^1 \int_\Omega f_\varepsilon[(1-t)m + tm'](x)(m'-m)(x)\,dx \tag{3.4}$$

for $m, m' \in L^1(\Omega)$ and $\int_\Omega m = \int_\Omega m' = 1$. This means that the regularized problem also admits a potential. Let us observe that using the properties of mollifiers and the assumptions on f, the following estimates hold:

$$-\frac{C_f}{q}\,\|m\|_q^q - K_f \le F_\varepsilon[m] \le \frac{C_f}{q}\,\|m\|_q^q + K_f, \tag{3.5}$$

and

$$-\frac{C_f}{q}\sup_\Omega \chi_\varepsilon^q - K_f \sup_\Omega \chi_\varepsilon \le F(x, m * \chi_\varepsilon(x)) \le \frac{C_f}{q}\sup_\Omega \chi_\varepsilon^q + K_f \sup_\Omega \chi_\varepsilon. \tag{3.6}$$

We now introduce the energy of the approximated system:

$$\mathcal{E}_\varepsilon(m, w) := \begin{cases} \int_\Omega mL\left(-\frac{w}{m}\right)dx + F_\varepsilon[m] & \text{if } (m, w) \in \mathcal{K} \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

3.2 Minimization of the regularized functional

Our goal now is to find a minimizer for the energy of the regularized system. Here, the exponent $\bar{q}$ comes into play. If $q < \bar{q}$, the energy can be proven to admit a global minimum. This is the case addressed in [8]. If $q = \bar{q}$, a global minimum can be found under some condition on the coefficients. If $q > \bar{q}$, no global minima is present in general and we have to look for local minima. Following the idea introduced in [26, 28], let us introduce, for $\alpha \ge 1$

$$B_\alpha := \{(m, w) \in \mathcal{K} : \|m\|_q^q \le \alpha\} \tag{3.8}$$

and

$$U_\alpha := \{(m, w) \in \mathcal{K} : \|m\|_q^q = \alpha\}. \tag{3.9}$$

Let us also define, for $\varepsilon > 0$ fixed,

$$c_\alpha = \inf_{(m,w)\in\mathcal{K}\cap B_\alpha} \mathcal{E}_\varepsilon(m, w) \tag{3.10}$$

and

$$\hat{c}_\alpha = \inf_{(m,w)\in\mathcal{K}\cap U_\alpha} \mathcal{E}_\varepsilon(m, w). \tag{3.11}$$

We start by proving the existence of a minimum in the sets B_α, and also adapt the arguments to prove the existence of a global minimum under the stricter assumptions.

Lemma 3.1 *For all $\alpha \ge 1$, c_α is achieved. Moreover, if $q < \bar{q}$ or $q = \bar{q}$ and $C_f < C_{\bar{q}} = q\frac{C_L}{C_q}$, then $\mathcal{E}_\varepsilon$ has a global minimum on $\mathcal{K}$.*

Proof Let us begin by bounding $\mathcal{E}_\varepsilon$ by below. Using (2.2), (2.11) and (3.5) we have:

$$\mathcal{E}_\varepsilon(m, w) \ge \frac{C_L}{C_q}\,\|m\|_q - C_L - \frac{C_f}{q}\,\|m\|_q^q - K_f \ge K', \tag{3.12}$$

since we know that $\|m\|_q^q \le \alpha$ in B_α. Consider now a minimizing sequence (m_n, w_n) for c_α. Eventually, $\mathcal{E}_\varepsilon(m_n, w_n) \le c_\alpha + 1$. Hence, again by (2.2) and (3.5) we have

$$\int_\Omega \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}} \le C_L^{-1}\left(c_\alpha + 1 - F_\varepsilon[m_n]\right) \le C_L^{-1}\left(c_\alpha + 1 + K_f + \frac{C_f}{q}\alpha\right) \tag{3.13}$$

which implies that $\left(\int_\Omega \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}\right)$ is bounded. Using (2.10), we have that $\|m_n\|_{1,r} \le C$. We can use Sobolev embeddings to conclude that up to subsequences

$$m_n \to m \text{ a.e. on } \Omega, \quad m_n \to m \text{ on } L^1(\Omega), \quad m_n \rightharpoonup m \text{ on } W^{1,r}(\Omega).$$

Then, using Hölder inequality

$$\int_\Omega |w_n|^{\frac{\gamma' q}{\gamma'+q-1}}\, dx \le \left(\int_\Omega \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}\right)^{\frac{q}{\gamma'+q-1}} \|m_n\|_q^{\frac{\gamma'-1}{q(\gamma'+q-1)}},$$

hence w_n is equibounded in $L^{\frac{\gamma' q}{\gamma'+q-1}}(\Omega)$ and so $w_n \rightharpoonup w$ in $L^{\frac{\gamma' q}{\gamma'+q-1}}(\Omega)$. By $L^1(\Omega)$ convergence of m_n we can conclude in a standard way that $m \ge 0$ and that $\int_\Omega m = 1$. Moreover, the convergences are strong enough to pass to the limit in the constraint $\mathcal{K}$, that is, $(m, w) \in \mathcal{K}$. Fatou's lemma also implies that $m \in B_\alpha$.

To conclude, it is known that $\int_\Omega mL\left(-\frac{w}{m}\right)dx$ is lower-semicontinuous with respect to the weak convergence of $W^{1,r}(\Omega) \times L^{\frac{\gamma' q}{\gamma'+q-1}}(\Omega)$ (indeed, one can exploit its convexity and adapt classical results that connect convexity and lower semicontinuity, see for instance [17, Th. 2.2.1]). Moreover, using (3.6) and the Dominated Convergence Theorem we deduce that F_ε is strongly continuous with respect to the $L^1(\Omega)$ convergence. Hence,

$$\mathcal{E}_\varepsilon(m, w) \le \liminf_n \int_\Omega m_n L\left(-\frac{w_n}{m_n}\right) dx + \lim_n F_\varepsilon[m_n] \le \liminf_n \mathcal{E}_\varepsilon(m_n, w_n) = c_\alpha.$$

Now suppose that $q < \bar{q}$. Then (2.12) holds. The proof of the existence of a minimizer is completely analogous as before, but there is no need to restrict the set B_α. Indeed, instead of (3.12), we can directly infer using (2.12) that

$$\mathcal{E}_\varepsilon(m, w) \ge \frac{C_L}{C_q}\|m\|_q^{q(1+\delta)} - C_L - \frac{C_f}{q}\|m\|_q^q - K_f \ge K'. \tag{3.14}$$

Moreover, we can set $e = \inf_{(m,w)\in\mathcal{K}} \mathcal{E}_\varepsilon(m, w)$ and argue as in (3.13), using (2.12) to conclude that

$$\int_\Omega \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}} \le C_L^{-1}\left(e + 1 + K_f + \frac{C_f}{q}\left(C_q \int_\Omega \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}dx + 1\right)^{\frac{1}{1+\delta}}\right)$$

which again implies that $\int_\Omega \frac{|w_n|^{\gamma'}}{m_n^{\gamma'-1}}$ is bounded. Finally, if $q = \bar{q}$ the previous steps are justified provided that $C_f < q\frac{C_L}{C_q}$ and a global minimum exists. □

Remark 3.2 Let $(m_\varepsilon, w_\varepsilon)$ be a minimizer for $\mathcal{E}_\varepsilon$ as in the previous lemma. Then, there exists $C > 0$ independent of ε such that

$$\|m_\varepsilon\|_q \le C, \qquad \|m_\varepsilon\|_{1,r} \le C \tag{3.15}$$

and

$$\|w_\varepsilon\|_{\frac{\gamma' q}{\gamma'+q-1}} \le C. \tag{3.16}$$

Indeed, when $q \le \bar{q}$ we can use the fact that

$$\mathcal{E}_\varepsilon(m_\varepsilon, w_\varepsilon) \le \mathcal{E}_\varepsilon(1, 0) \le C_L^{-1} + \frac{C_f}{q} + K_f \tag{3.17}$$

to conclude that inequalities (3.15) and (3.16) hold with a constant independent on ε. If $q > \bar{q}$, the same results hold, the proof being immediate since $m_\varepsilon \in B_\alpha$ is only a local minimum.

We see in the above lemma the role played by $\bar{q}$. When $q > \bar{q}$, $\mathcal{E}_\varepsilon$ is not globally bounded from below and no global minima exist (see the remark below). To show that a local minimum exists in B_α, we are left to prove that the candidate obtained in the previous lemma does not belong to U_α. To this aim, we look for $\bar{\alpha} > 1$ such that $c_{\bar{\alpha}} < \hat{c}_{\bar{\alpha}}$.

Remark 3.3 Let us notice that if $q \geq \bar{q}$, $\mathcal{E}$ (and also $\mathcal{E}_\varepsilon$) could indeed be unbounded.

Let $f(x, m(x)) = -C_f m(x)^{q-1} - K_f$ and $q > 1 + \frac{\gamma'}{N}$. Then, there exists (m_n, w_n) such that $\mathcal{E}(m_n, w_n) \to -\infty$. Choose $m_0 \in C_0^\infty(B_1(0))$ non-negative, such that $\int_{B_1(0)} m_0\, dx = 1$ and $\int_{B_1(0)} \frac{|\nabla m_0|^{\gamma'}}{m_0^{\gamma'-1}}\, dx$ is finite. Now pick $x_0 \in \Omega$ and define

$$m_\lambda(x) = \lambda^N m_0(\lambda(x - x_0)) \qquad w_\lambda(x) = \nabla m_\lambda(x) = \lambda^{N+1} \nabla m_0(\lambda(x - x_0)).$$

We notice that for $\lambda > \frac{1}{\mathrm{dist}(x_0, \partial\Omega)}$, then $\mathrm{supp}(m_\lambda) \subset B_{\frac{1}{\lambda}}(x_0)$ and $(m_\lambda, w_\lambda) \in \mathcal{K}$. Moreover $\|m_\lambda\|_q^q = \lambda^{N(q-1)} \|m_0\|_q^q$. Now we have, using our assumption and (2.2),

$$\mathcal{E}(m_\lambda, w_\lambda) = \int_\Omega m_\lambda L\left(-\frac{w_\lambda}{m_\lambda}\right) + F(x, m_\lambda)\, dx \leq$$

$$C_L^{-1} \int_\Omega \frac{|w_\lambda|^{\gamma'}}{m_\lambda^{\gamma'-1}}\, dx - \frac{C_f}{q} \|m_\lambda\|_q^q - C$$

$$= \lambda^{\gamma'} C_L^{-1} \int_{B_1(0)} \frac{|\nabla m_0|^{\gamma'}}{m_0^{\gamma'-1}}\, dt - \lambda^{N(q-1)} \frac{C_f}{q} \|m_0\|_q^q - C$$

and we see that under our assumptions on q for $\lambda \to +\infty$, the right-hand side goes to $-\infty$. By the above computations, we can also see that $\mathcal{E}$ may be unbounded in the case $q = 1 + \frac{\gamma'}{N}$, provided that C_f is large enough.

Let us go back to the existence of a local minimum in the case $q > \bar{q}$. To do so, we need to impose some restriction on the coefficient C_f, so that an α such that $c_\alpha < \hat{c}_\alpha$ can be found.

Theorem 3.4 *Let $q > \bar{q}$,*

$$\bar{\alpha} = \left(\frac{C_L}{C_f C_q}\right)^{q'}, \tag{3.18}$$

and suppose that

$$C_f < \min\left\{\frac{C_L}{C_q}, \left(K' q'\right)^{1-q} \left(\frac{C_L}{C_q}\right)^q\right\}, \tag{3.19}$$

where

$$K' := C_L + C_L^{-1} + 2K_f + \frac{C_L}{q C_q}. \tag{3.20}$$

Then, there exists $(m_\varepsilon, w_\varepsilon) \in \mathcal{K} \cap B_{\bar{\alpha}}$ such that $(m_\varepsilon, w_\varepsilon)$ is a local minimum for $\mathcal{E}_\varepsilon$. Moreover it holds

$$\mathcal{E}_\varepsilon(m_\varepsilon, w_\varepsilon) = c_{\bar{\alpha}-\delta} = c_{\bar{\alpha}} \tag{3.21}$$

for all δ less than some $\bar{\delta} > 0$ small enough.

Proof Let us start the by the simple observation that if we find $\alpha_2 > \alpha_1$ such that $\hat{c}_{\alpha_2} > \hat{c}_{\alpha_1}$, then we have:

$$c_{\alpha_2} = \min_{\alpha}\{\hat{c}_\alpha : 0 \le \alpha \le \alpha_2\} \le \hat{c}_{\alpha_1} < \hat{c}_{\alpha_2}$$

and so we can conclude the existence of an interior minimum in B_{α_2}. We choose $\alpha_1 = 1$ and $\alpha_2 = \bar{\alpha}$. Since $C_f < C_L/C_q$, we are sure that $\bar{\alpha} > 1$.
We now consider estimates for $\hat{c}_{\alpha_1}$. We begin by noticing that by Hölder inequality we have

$$1 = \|m\|_1 \le \|m\|_q \, |\Omega|^{\frac{1}{q'}} = 1$$

which implies $m = 1$ a.e. and therefore $(m, w) \equiv (1, 0) \in \mathcal{K} \cap U_{\alpha_1}$. Thus, using (2.2) and (3.5) we find

$$\hat{c}_{\alpha_1} \le \mathcal{E}_\varepsilon\,(1, 0) \le C_L^{-1} + \frac{C_f}{q} + K_f < C_L^{-1} + \frac{C_L}{qC_q} + K_f. \tag{3.22}$$

thanks to our assumptions on C_f. Next, we rewrite (3.12) with $\alpha = \bar{\alpha}$ as

$$\hat{c}_{\bar{\alpha}} \ge \frac{C_L}{C_q}\bar{\alpha}^{\frac{1}{q}} - C_L - \frac{C_f}{q}\bar{\alpha} - K_f. \tag{3.23}$$

To conclude that $\hat{c}_{\bar{\alpha}} > \hat{c}_1$, we need to check that

$$\phi(\alpha) := \frac{C_L}{C_q}\alpha^{\frac{1}{q}} - \frac{C_f}{q}\alpha > C_L + C_L^{-1} + 2K_f + \frac{C_L}{qC_q} = K', \tag{3.24}$$

and the previous inequality holds again by the assumptions on C_f (notice that $\bar{\alpha}$ maximizes ϕ). Moreover since ϕ is continuous, we get that some δ small enough exists so that also $\hat{c}_{\bar{\alpha}-\delta} > \hat{c}_{\alpha_1}$. □

In the previous construction, the assumptions on C_f are chosen to have the largest possible α such that the energy admits an interior minimizer in $B_{\bar{\alpha}}$, and $\bar{\alpha}$ depends on the value of C_f. To treat the case $q = q_c$, we will need to find a minimizer in $B_{\bar{\alpha}}$ independent of C_f. This is possible under different assumptions on C_f.

Theorem 3.5 *Let*

$$\hat{\alpha} = \left(\frac{C_q}{C_L}K'' + 1\right)^q \tag{3.25}$$

where

$$K'' = C_L + C_L^{-1} + 2K_f$$

and suppose

$$C_f < \frac{qC_L}{C_q(\hat{\alpha} + 1)} \tag{3.26}$$

then the results of Theorem 3.4 hold.

Proof The proof is analogous to the one of Theorem 3.4. The only differences are the choice of α when enforcing (3.24) ($\hat{\alpha}$ no longer maximizes $\phi(\alpha)$) and the last inequality in (3.22), which is skipped. □

3.3 Convex duality

We now employ some convex duality arguments to obtain, from the (local) minimizer constructed in the previous section, a solution to the MFG system (3.1). We follow the usual route (see e.g. [8] and references therein), which requires first to linearize the functional (which is not convex by the presence of the possibly nonconvex F_ε) around the minimizer that we found. Given $(m_\varepsilon, w_\varepsilon)$, which is a global minimizer of $\mathcal{E}_\varepsilon$ on $\mathcal{K}$, or a local minimizer on $\mathcal{K} \cap B_{\bar{\alpha}}$ when $q > \bar{q}$, let us introduce the following linearized functional:

$$J_\varepsilon(m, w) = \int_\Omega mL\left(-\frac{w}{m}\right) + f_\varepsilon[m_\varepsilon](x)m\, dx. \tag{3.27}$$

We notice that this functional is convex. We now prove that this functional admits the same minimizer as $\mathcal{E}_\varepsilon$.

Proposition 3.6 *Let $(m_\varepsilon, w_\varepsilon)$ be a global minimizer of $\mathcal{E}_\varepsilon$ on $\mathcal{K}$ or a local minimizer on $\mathcal{K} \cap B_{\bar{\alpha}}$ as constructed above. Then*

$$\min_{(m,w)\in\mathcal{K}} J_\varepsilon(m, w) = J_\varepsilon(m_\varepsilon, w_\varepsilon). \tag{3.28}$$

Proof Let $(m, w) \in \mathcal{K}$ and consider for $0 < \lambda < 1$

$$m_\lambda = \lambda m + (1-\lambda)m_\varepsilon$$

If $(m_\varepsilon, w_\varepsilon)$ is a local minimum, since by (3.21) $m_\varepsilon \in B_{\bar{\alpha}-\delta}$ for some positive δ, we can conclude that for λ small enough, $m_\lambda \in B_{\bar{\alpha}}$. If $(m_\varepsilon, w_\varepsilon)$ is a global minimum this argument holds for all λ. Hence, by minimality and convexity

$$\begin{aligned} F_\varepsilon[m_\varepsilon] - F_\varepsilon[m_\lambda] &\leq \int_\Omega m_\lambda L\left(-\frac{w_\lambda}{m_\lambda}\right) dx - \int_\Omega m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) dx \\ &\leq \lambda \int_\Omega mL\left(-\frac{w}{m}\right) dx + (1-\lambda)\int_\Omega m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) dx \\ &\quad - \int_\Omega m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) dx \\ &= \lambda\left(\int_\Omega mL\left(-\frac{w}{m}\right) dx - \int_\Omega m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) dx\right). \end{aligned}$$

Now, by (3.4), we have

$$\begin{aligned} F_\varepsilon[m_\varepsilon] - F_\varepsilon[m_\lambda] &= \int_0^1 \int_\Omega f_\varepsilon[(1-t)m_\lambda + tm_\varepsilon](x)(m_\varepsilon - m_\lambda)(x)\, dx \\ &= -\lambda \int_0^1 \int_\Omega f_\varepsilon[m_\varepsilon + \lambda(1-t)(m - m_\varepsilon)](x)(m - m_\varepsilon)(x)\, dx. \end{aligned}$$

Combining the two expressions, we can use Lipschitz estimates for $f_\varepsilon[\cdot](x)$ near m_ε and send λ to 0 to conclude that

$$\int_\Omega mL\left(-\frac{w}{m}\right) dx - \int_\Omega m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) dx \geq -\int_\Omega f_\varepsilon[m_\varepsilon](x)(m_\varepsilon - m)(x)\, dx,$$

which is equivalent to the minimality of J_ε (globally on $\mathcal{K}$). □

Now that we have global minimizer of a convex functional, we can construct a solution of (3.1).

Theorem 3.7 *Let $(m_\varepsilon, w_\varepsilon)$ be a minimizer of J_ε constructed above. Then $m_\varepsilon \in W^{1,p}(\Omega)$ for all $p > 1$ and there exists $\lambda_\varepsilon \in \mathbb{R}$ and $u_\varepsilon \in C^2(\overline{\Omega})$ such that $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ is a solution to (3.1). Moreover,*

$$w_\varepsilon = -m_\varepsilon \nabla H(\nabla(u_\varepsilon)), \tag{3.29}$$

and there exists $C > 0$ independent of ε such that

$$\|m_\varepsilon\|_q \le C, \qquad \|m_\varepsilon\|_{1,r} \le C, \tag{3.30}$$

and

$$|\lambda_\varepsilon| \le C. \tag{3.31}$$

Proof The proof is like [8, Th. 4] with minor modifications. □

3.4 Passage to the limit

We now wish to let $\varepsilon \to 0$, and to do so we need some a priori estimate. We distinguish two cases: if $q < q_c$, we can use a blow-up argument to deduce an a priori L^∞ bound on m_ε (that by a bootstrap procedure yields further estimates on u, m and their derivatives). If $q = q_c$, the argument fails and we need to require some extra smallness on C_f to obtain such bound.

The blow up argument follows the lines of [8], but we need an extra care for the presence of the Neumann boundary conditions.

Proposition 3.8 *Let $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ be a solution to (3.1) constructed above, and suppose that $q < q_c$. Then there exist $C > 0$ independent of ε such that*

$$\|m_\varepsilon\|_\infty \le C. \tag{3.32}$$

Proof Suppose by contradiction that

$$M_\varepsilon = \max_\Omega m_\varepsilon = m_\varepsilon(x_\varepsilon) \to +\infty.$$

Define

$$\mu_\varepsilon := M_\varepsilon^{-\beta} \qquad \beta := (q-1)\frac{\gamma-1}{\gamma}.$$

We have that $\mu_\varepsilon \to 0$. Define the following rescaling

$$\begin{cases} v_\varepsilon(x) &= \mu_\varepsilon^{\frac{2-\gamma}{\gamma-1}} u_\varepsilon(\mu_\varepsilon x + x_\varepsilon) \\ n_\varepsilon(x) &= M_\varepsilon^{-1} m_\varepsilon(\mu_\varepsilon x + x_\varepsilon). \end{cases}$$

We notice that $n_\varepsilon(0) = 1$ and that $0 \le n_\varepsilon(x) \le 1$. Define also

$$H_\varepsilon(q) = \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} H(\mu_\varepsilon^{\frac{1}{1-\gamma}} q) \qquad \nabla H_\varepsilon(q) = \mu_\varepsilon \nabla H(\mu_\varepsilon^{\frac{1}{1-\gamma}} q) \tag{3.33}$$

By (1.2) we get

$$C_H^{-1} |p|^\gamma - K_H \le H_\varepsilon(p) \le C_H |p|^\gamma + K_H \qquad |\nabla H_\varepsilon(p)| \le C_H |p|^{\gamma-1} + K_H. \tag{3.34}$$

Then, define

$$\tilde{f}_\varepsilon(x) := \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} f_\varepsilon[m_\varepsilon](x_\varepsilon + \mu_\varepsilon x). \tag{3.35}$$

Since $m_\varepsilon(x) \le M_\varepsilon$, we can use (1.3) to get that

$$\begin{aligned}\left\|\tilde{f}_\varepsilon[m_\varepsilon]\right\|_\infty &\le \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} \|f(\cdot, m * \chi_\varepsilon(\cdot)) * \chi_\varepsilon\|_\infty \le \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} \|f(\cdot, m_\varepsilon * \chi_\varepsilon)\|_\infty \\ &\le \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} C\left(\|m_\varepsilon * \chi_\varepsilon\|_\infty^{q-1} + 1\right) \le \mu_\varepsilon^{\frac{\gamma}{\gamma-1}}\left(C\|m_\varepsilon\|_\infty^{q-1} + C\right) \\ &\le C + CM_\varepsilon^{q-1-\beta\frac{\gamma}{\gamma-1}} \le C\end{aligned}$$

by our definitions of μ_ε and β. Lastly, define

$$\tilde{\lambda}_\varepsilon = \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} \lambda_\varepsilon. \tag{3.36}$$

Clearly by (3.31), $\left|\tilde{\lambda}_\varepsilon\right| \le C$. Now after some computations we have

$$\begin{cases}\Delta v_\varepsilon(x) = \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} \Delta u_\varepsilon(x\mu_\varepsilon + x_\varepsilon) \\ H_\varepsilon(\nabla v_\varepsilon(x)) = \mu_\varepsilon^{\frac{\gamma}{\gamma-1}} H(\nabla u_\varepsilon(x\mu_\varepsilon + x_\varepsilon)) \\ \Delta n_\varepsilon(x) = \mu_\varepsilon^{\frac{1}{\beta}+2} \Delta m_\varepsilon(x\mu_\varepsilon + x_\varepsilon) \\ \nabla H_\varepsilon(\nabla v_\varepsilon(x)) = \mu_\varepsilon \nabla H(\nabla u_\varepsilon(x\mu_\varepsilon + x_\varepsilon)) \\ \operatorname{div}(n_\varepsilon \nabla H_\varepsilon(\nabla v_\varepsilon(x))) = \mu_\varepsilon^{\frac{1}{\beta}+2} \operatorname{div}(m_\varepsilon(x\mu_\varepsilon + x_\varepsilon)\nabla H(\nabla u_\varepsilon(x\mu_\varepsilon + x_\varepsilon))) \\ \frac{\partial v_\varepsilon(x)}{\partial n} = \mu_\varepsilon^{\frac{1}{\gamma-1}} \frac{\partial u_\varepsilon(\mu_\varepsilon x + x_\varepsilon)}{\partial n}(\mu_\varepsilon x + x_\varepsilon) \\ \frac{\partial n_\varepsilon}{\partial n} = \mu_\varepsilon^{\frac{1}{\beta}+1} \frac{\partial m_\varepsilon(\mu_\varepsilon x + x_\varepsilon)}{\partial n}.\end{cases}$$

from which we deduce that $(v_\varepsilon, \tilde{\lambda}_\varepsilon, n_\varepsilon)$ is a solution of

$$\begin{cases}-\Delta v_\varepsilon + H_\varepsilon(\nabla v_\varepsilon) + \tilde{\lambda}_\varepsilon = \tilde{f}_\varepsilon(x) & \text{on } \Omega_\varepsilon \\ -\Delta n_\varepsilon - \operatorname{div}(n_\varepsilon \nabla H_\varepsilon(\nabla v_\varepsilon)) = 0 & \text{on } \Omega_\varepsilon \\ \frac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega_\varepsilon \\ \frac{\partial n_\varepsilon}{\partial n} + n_\varepsilon \nabla H_\varepsilon(\nabla v_\varepsilon) \cdot n = 0 & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} n_\varepsilon = M_\varepsilon^{-1-\beta}, \qquad \int_\Omega v_\varepsilon = 0. \end{cases} \tag{3.37}$$

where $\Omega_\varepsilon = \{x : \mu_\varepsilon x + x_\varepsilon \in \Omega\}$. We now have to distinguish two cases. Suppose first that

$$\lim_{\varepsilon \to 0} \frac{d(x_\varepsilon, \partial\Omega)}{\mu_\varepsilon} = +\infty.$$

From that we have that $\Omega_\varepsilon \uparrow \mathbb{R}^N$, hence for ε small enough we have that $\Omega_\varepsilon \supset B_{4R}(0)$ for an $R > 0$ independent on ε. We know that $\tilde{\lambda}_\varepsilon$ and $\tilde{f}_\varepsilon(x)$ are uniformly bounded, thus using Theorem 2.7 (and the remark below) we can conclude that there exists C independent of ε such that $\|\nabla v_\varepsilon\|_\infty \le C$ on B_{2R}. Now, using (3.34) we can deduce that $\|n_\varepsilon \nabla H_\varepsilon(\nabla v_\varepsilon)\|_\infty \le C$. Hence, by Proposition 2.4, for ε small enough n_ε is equibounded in $W^{1,p}(B_R(0))$ for all $p > 1$, and by Sobolev Embedding also in $C^\theta(\overline{B}_R(0))$, for all $\theta < 1$. We know that $n_\varepsilon(0) = 1$, therefore using equiboundness in $C^\theta(\overline{B}_R(0))$. we can deduce that there exists $\delta > 0$ and $r < R$ such that $\int_{B_r(0)} n_\varepsilon^q(x)\,dx > \delta > 0$. Thus we have that

$$0 < \delta \le \int_{B_r(0)} n_\varepsilon^q(x)\,dx \le \|n_\varepsilon\|_q^q = M_\varepsilon^{-q}\mu_\varepsilon^{-N}\|m_\varepsilon\|_q^q = M_\varepsilon^{-q+\beta N}\|m_\varepsilon\|_q^q. \tag{3.38}$$

Since $q < q_c$ then $-q + \beta N < 0$, and using (3.30) we have that

$$0 < \delta \le M_\varepsilon^{-q+\beta N} \|m_\varepsilon\|_q^q \le C M_\varepsilon^{-q+\beta N} \to 0$$

which is a contradiction.

Suppose now that

$$\lim_{\varepsilon \to 0} \frac{d(x_\varepsilon, \partial\Omega)}{\mu_\varepsilon} \le C.$$

Up to subsequences we can suppose that $x_\varepsilon \to \bar{x} \in \partial\Omega$ as $\varepsilon \to 0$. Moreover, up to an affine transformation we can assume $\bar{x} = 0 \in \partial\Omega$ and $n(0) = -e_N$. Define $x' = (x_1, \dots, x_{N-1})$. By the smoothness of Ω there exists $U \subset \mathbb{R}^N$, $\Gamma \subset \mathbb{R}^{N-1}$ and $\phi(x') \in C^{2,\alpha}(\Gamma)$ such that

$$\begin{aligned} &\phi(0) = 0, \qquad \nabla\phi(0) = 0, \\ &\partial\Omega \cap U = \{(x', x_N) : x_N = \phi(x')\}, \\ &\Omega \cap U = \{(x', x_N) : x_N > \phi(x')\}. \end{aligned}$$

Let us now define a diffeomorfism $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ that "straightens" the boundary. We set

$$y_i = (\Psi(x))_i := \begin{cases} x_i - x_N \frac{\partial\phi}{\partial x_i}(x') & \text{for } 1 \le i \le N-1 \\ x_N - \phi(x') & \text{for } i = N. \end{cases} \tag{3.39}$$

We can see that Ψ is invertible in a neighborhood of 0. We now extend with an even reflection v_ε and m_ε. We set

$$w_\varepsilon(y) = v_\varepsilon\left(\frac{\Psi^{-1}(y', |y_N|) - x_\varepsilon}{\mu_\varepsilon}\right) \tag{3.40}$$

$$\rho_\varepsilon(y) = n_\varepsilon\left(\frac{\Psi^{-1}(y', |y_N|) - x_\varepsilon}{\mu_\varepsilon}\right). \tag{3.41}$$

Due to the homogeneous Neumann boundary conditions, with some calculations it is possible to show that $\frac{\partial w_\varepsilon}{\partial y_N}|_{\{y_N=0\}} = 0$. Moreover, one can derive that w_ε, ρ_ε satisfy an equation similar to (3.37) in a fixed neighborhood of the boundary point p independent of ε (with coefficients that converge to the identity as $\varepsilon \to 0$). From this, we can repeat the above argument and reach a contradiction. □

We can see in this proof the criticality of the case $q = q_c$. Looking at (3.38) and the lines below, in the case $q = q_c$ it is not possible to reach a contradiction, since $M_\varepsilon^{-q+\beta N}$ does not vanish. To tackle this problem, the idea is to obtain additional regularity using finer estimates for both the Fokker-Planck and the Hamilton-Jacobi equation. Once we find uniform bounds for m_ε in some L^p with $p > q_c$, the above arguments can be used to conclude again that we have an uniform L^∞ bound. This procedure requires additional assumptions on C_f. Moreover we need to have a value for $\bar{\alpha}$ which is independent from C_f, as we have constructed in Theorem 3.5.

Below, C_S is the Sobolev Embedding constant for $W^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, where p^* is chosen so that

$$p^* < N \quad \text{and} \quad p^* > 1 + \frac{\gamma'}{N - \gamma'}.$$

Proposition 3.9 *Let* $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ *be a solution to (3.1) and suppose that* $q = q_c$. *Then*

$$\|\nabla u_\varepsilon\|_{N(\gamma-1)} \leq \frac{1}{(4C_E C_S C_H)^{\frac{1}{\gamma-1}}}, \tag{3.42}$$

where C_E *is defined in Proposition 2.4, provided that* C_f *and* K_f *are small enough (that is, smaller than some positive constant depending on* Ω, N, γ, q*).*

Proof We use Theorem 2.10, choosing δ small enough so that $M(\delta) \leq (4C_E C_S C_H)^{\frac{-1}{\gamma-1}}$. Let us compute the norm of f_ε in $L^{\frac{N}{\gamma'}}(\Omega)$. Using (1.3), convolution properties and the definition of q_c we have

$$\|f_\varepsilon\|_{\frac{N}{\gamma'}} = \|f(x, m * \chi_\varepsilon(x))\|_{\frac{N}{\gamma'}} \leq C_f \left\|m_\varepsilon^{q-1}\right\|_{\frac{N}{\gamma'}} + K_f = C_f(\|m_\varepsilon\|_q^q)^{\frac{1}{q'}} + K_f.$$

Now using Theorem 3.5, we know that

$$\|m\|_q^q \leq \hat{\alpha} = \left(\frac{C_q}{C_L}K'' + 1\right)^q.$$

Hence if

$$C_f \hat{\alpha}^{\frac{1}{q'}} + K_f \leq \delta$$

we have that $\|f_\varepsilon\|_{\frac{N}{\gamma'}} \leq \delta$. Thus, we can apply Theorem 2.10 to conclude. □

Corollary 3.10 *Let* $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ *be a solution to (3.1) and suppose that* $q = q_c$. *Under the assumptions of the previous proposition, supposing in addition that*

$$K_H \leq \frac{1}{4C_E C_S}, \tag{3.43}$$

then there exist $C > 0$ *independent of* ε *such that*

$$\|m_\varepsilon\|_\infty \leq C.$$

Proof Using Proposition 3.9 we have that $\|\nabla u_\varepsilon\|_{N(\gamma-1)} \leq \frac{1}{(2C_E C_S C_H)^{\frac{1}{\gamma-1}}}$. Hence

$$\|\nabla H(\nabla u_\varepsilon)\|_N \leq \frac{1}{4C_E C_S} + K_H$$

and we can use Proposition 2.6 to conclude that m_ε are uniformly bounded in $W^{1,p}(\Omega)$ for the chosen above; in this way, by Sobolev Embeddings m_ε are uniformly bounded in $L^q(\Omega)$ for some $q > q_c$. Once we have this bound, we can proceed as in Proposition 3.8 to conclude that m_ε is bounded in L^∞. □

Now everything is ready to prove the main result. Thanks to the uniform bounds, we use a bootstrap procedure to obtain the regularity which is necessary to pass to the limit into the equations.

Proof of Theorem 1.1 We set ourselves in the assumptions of the previous propositions and theorems, in particular we require C_f, K_f and K_H to be possibly small enough (see in particular Lemma 3.1, Eqs. (3.19), (3.26), Proposition 3.9). We first prove the existence of a solution by a bootstrap and limit procedure and then we show it is a minimum for the energy, using a Γ-convergence argument.

Existence of a solution. Let $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ be the sequence of solutions constructed in Theorem 3.7. We show that we can pass to the limit as $\varepsilon \to 0$ and obtain a solution of (1.1); in this we need to treat separately the Sobolev subcritical and critical cases. Let us first suppose $q < q_c$. By (3.31) we have that, up to subsequences, $\lambda_\varepsilon \to \lambda$. Moreover, by Proposition 3.8 we have that $\|m_\varepsilon\|_\infty \le C$. Hence by properties of mollifiers and (1.3) we get

$$\begin{aligned}\|f_\varepsilon[m_\varepsilon]\|_\infty = \; & \|f(\cdot, m * \chi_\varepsilon(\cdot)) * \chi_\varepsilon\|_\infty \le \|f(\cdot, m_\varepsilon * \chi_\varepsilon)\|_\infty \\ & \le C\,\|m_\varepsilon * \chi_\varepsilon\|_\infty^{q-1} + C \le C\,\|m_\varepsilon\|_\infty^{q-1} + C \le C\end{aligned}$$

Thus, we can conclude by Theorem 2.7, that $\|\nabla u_\varepsilon\|_\infty \le K$ for some $K > 0$ independent of ε. Using the estimates on H and elliptic regularity in the Hamilton-Jacobi equation we can conclude that $\|u_\varepsilon\|_{1,p} \le C$ for all $p > 1$. Moreover, by Sobolev embeddings, u_ε is equibounded in $C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$. Since $\nabla H(\nabla u_\varepsilon) \le C$, we now use Theorem 2.3 to conclude that $\|m_\varepsilon\|_{1,p} \le C$ for all $p > 1$, and therefore by Sobolev embedding m_ε is equibounded in $C^\theta(\overline{\Omega})$ for all $\theta \in (0,1)$. Hence, we get that, up to subsequences, $m_\varepsilon \rightharpoonup m$ in $W^{1,p}$ for all $p > 1$ and $m_\varepsilon \to m$ uniformly. We can then go back to the Hamilton-Jacobi equation and, with a similar reasoning, conclude that $f_\varepsilon[m_\varepsilon](x)$ is equibounded in $C^\theta(\Omega)$ for all $\theta \in (0,1)$. Hence u_ε is equibounded in $C^{2,\theta}(\overline{\Omega})$ for all $\theta \in (0,1)$. Finally, we can conclude that up to subsequences $u_\varepsilon \to u$ in $C^2(\overline{\Omega})$. Now the convergences are strong enough to pass to the limit in the equations, so we can conclude that (u, λ, m) is a solution of (1.1), with the positivity of m coming from Theorem 2.3 and pointwise convergence.

When $q = q_c$, we argue in the very same way, starting from Corollary 3.10.

Minimality. We are left to prove that the solutions we found are (local) minimizers of $\mathcal{E}$. We will use the fundamental theorem of $\Gamma-$convergence (see e.g. [6]). This says that if $\mathcal{E}_\varepsilon$ $\Gamma-$converges to $\mathcal{E}$, than any converging sequence of minima for $\mathcal{E}_\varepsilon$ converges to a minimum for $\mathcal{E}$. Notice that, since we know a priori that the sequence of minima converges, we do not need to prove an equicoercivity result. Let us show that $\mathcal{E}_\varepsilon$ $\Gamma-$converges to $\mathcal{E}$ on the space $X = L^q(\Omega) \cap W^{1,r}(\Omega) \times L^1(\Omega)$. Suppose that $(m_\varepsilon, w_\varepsilon) \to (m, w)$ in X. By properties of mollifiers and continuity of the convolution we have $m_\varepsilon * \chi_\varepsilon \to m$ in $L^q(\Omega)$. We already remarked the semicontinuity of the Lagrangian term in $\mathcal{E}_\varepsilon$, moreover by (1.3) we have strong $L^q(\Omega)$ continuity of $m \to \int_\Omega \int_0^m f(x,n)\,dn\,dx$. Thus

$$\begin{aligned}\liminf_\varepsilon \mathcal{E}_\varepsilon(m_\varepsilon, w_\varepsilon) = \; & \liminf_\varepsilon \int_\Omega m_\varepsilon L\left(-\frac{w_\varepsilon}{m_\varepsilon}\right) dx + \lim_\varepsilon F_\varepsilon[m_\varepsilon] \\ & \ge \int_\Omega m L\left(-\frac{w}{m}\right) dx + \int_\Omega F(x,m)\,dx = \mathcal{E}(m,w).\end{aligned}$$

As for the recovery sequence, it suffices to choose $(m_\varepsilon, w_\varepsilon) = (m, w)$ for all $\varepsilon > 0$ and we clearly have that $\mathcal{E}_\varepsilon(m,w) \to \mathcal{E}(m,w)$ by the properties of mollifiers and again by the strong $L^q(\Omega)$ continuity. To finish, we know by (3.7) that a minimum $(m_\varepsilon, w_\varepsilon)$ of $\mathcal{E}_\varepsilon$ yields a solution $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ of (3.1) and that the relation $w_\varepsilon = -m_\varepsilon \nabla H(\nabla(u_\varepsilon))$ holds. Since we know that $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ converges in $C^2(\overline{\Omega}) \times \mathbb{R} \times W^{1,p}(\Omega)$ for all p to a solution (u, λ, m) of the original problem, we get that $(m_\varepsilon, w_\varepsilon)$ converges in X to $(u, -m\nabla H(\nabla u))$. Hence, we can conclude that the solution (u, λ, m) is such that $(m, -m\nabla H(\nabla u))$ is a minimum of $\mathcal{E}$ (possibly restricted to $B_{\hat{\alpha}}$ when $q > \bar{q}$). □

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