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Calculus of Variations

Smooth approximation of Lipschitz domains, weak curvatures and isocapacitary estimates

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Abstract

We provide a novel approach to approximate bounded Lipschitz domains via a sequence of smooth, bounded domains. The flexibility of our method allows either inner or outer approximations of Lipschitz domains which also possess weakly defined curvatures, namely, domains whose boundary can be locally described as the graph of a function belonging to the Sobolev space $W^{2,q}$ for some $q \geq 1$. The sequences of approximating sets is also characterized by uniform isocapacitary estimates with respect to the initial domain Ω.

Mathematics Subject Classification 53A07 · 46E35 · 41A30 · 41A63

1 Introduction

In this paper we are concerned with inner and outer approximation of bounded Lipschitz domains Ω of the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Specifically, we construct two sequences of C^∞-smooth bounded domains $\{\omega_m\}$, $\{\Omega_m\}$ such that $\omega_m \Subset \Omega \Subset \Omega_m$ for all $m \in \mathbb{N}$, which also satisfy natural covergence properties like, for instance, in the sense of the Lebesgue measure and in the sense of Hausdorff to Ω.

Geometric quantities like a Lipschitz characteristic $\mathcal{L}_\Omega = (L_\Omega, R_\Omega)$ and the diameter d_Ω of the domain Ω are comparable to the corresponding ones of its approximating sets ω_m, Ω_m. Here, the constant R_Ω stands for the radius of the ball domains on which the boundary can be described as a function of $(n-1)$-variables– i.e. the local boundary chart– and L_Ω is their Lipschitz constant– see Sect. 2 for the precise definition of a Lipschitz characteristic of Ω.

Furthermore, the smooth charts locally describing the boundaries $\partial\omega_m$, $\partial\Omega_m$ are defined on the same reference systems as the local charts describing $\partial\Omega$, together with strong convergence in the Sobolev space $W^{1,p}$ for all $p \in [1, \infty)$.

If in addition the local charts describing $\partial\Omega$ belong to the Sobolev space $W^{2,q}$ for some $q \in [1, \infty)$, then we also have strong convergence in the $W^{2,q}$-sense. In a certain way, this

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means that the second fundamental forms $\mathcal{B}_{\omega_m}$ and $\mathcal{B}_{\Omega_m}$ of the regularized sets converge in L^q to the "weak" curvature $\mathcal{B}_\Omega$ of the initial domain Ω.

Smooth approximation of open sets, not necessarily having Lipschitzian boundary, has been object of study by many authors. To the best of our knowledge, the first author who provided an approximation of this kind is Nečas [20], followed by Massari & Pepe [15] and Doktor [6]. The underlying idea behind their proof is nowadays standard, and it is typically used to approximate sets of finite perimeter. This consists in regularizing the characteristic function of Ω via mollification and convolution, and then define the approximating set Ω_m as a suitable superlevel set of the mollified characteristic functions–see for instance [1, Theorem 3.42] or [14, Section 13.2]. We point out that Schmidt [21] and Gui, Hu & Li [8] constructed smooth approximating domains *strictly contained* in Ω under additional assumptions on the finite perimeter domain Ω, whereas an outer approximation via smooth sets is given by Doktor [6] when the domain Ω is endowed with a Lipschitz continuous boundary.

A different kind of approach, which makes use of Stein's regularized distance, has been recently developed by Ball & Zarnescu [4]. Here, the authors deal with C^0 domains, i.e. domains whose boundary can be locally described by merely continuous charts, and hence need not have finite perimeter. We mention that their regularized domains Ω_ε are defined as the ε-superlevel set of the regularized distance function, which in turn is obtained via mollification of the usual signed distance function. Here, the parameter ε can be taken either positive or negative, according to the preferred method of approximation, whether from the inside or outside of Ω.

The aforementioned techniques have thus been used to treat domains with "rough" boundaries; however, they do not seem suitable to approximate domains which possess weakly defined curvatures, even in the case of domains having bounded curvatures, e.g. $\partial\Omega \in C^{1,1}$. Namely, we do not recover any quantitative information or convergence property regarding the second fundamental forms $\mathcal{B}_{\Omega_m}$ from the original one $\mathcal{B}_\Omega$. This is because first-order estimates regarding Ω_m are proven by a careful pointwise analysis of the gradient of the local charts describing their boundaries. In order to obtain estimates about their second fundamental form $\mathcal{B}_{\Omega_m}$, such pointwise analysis needs to be extended to second-order derivatives, and this calls for the application of the implicit function theorem, for which Ω is required to be at least of class C^2.

This drawback is probably due to the fact that the above regularization procedures are global in nature, i.e. they are obtained via mollification of functions "globally" describing Ω, like its characteristic function or signed distance, whereas the second fundamental form of hypersurfaces of $\mathbb{R}^n$ is defined via local parametrizations.

Comparatively, our proof relies on techniques which, in a sense, can be deemed as local in nature, since the starting point of our method is the regularization of the functions of $(n-1)$-variables which locally describe $\partial\Omega$. Thus, our approach seems more suitable when dealing with weak curvatures, though at the cost of requiring Ω to have a Lipschitz continuous boundary.

Regarding its applications, approximation via a sequence of smooth bounded domains has proven to be a powerful tool especially when dealing with boundary value problems in Partial Differential Equations. Indeed, by tackling the same boundary value problem (or its suitable regularization) on smoother domains, accordingly one obtains smoother solutions, hence it is possible to perform all the desired computations and infer a priori estimates which do not depend on the full regularity of the approximating sets Ω_m, but only on their Lipschitz characteristics or other suitable quantities possibly depending on the second fundamental form $\mathcal{B}_{\Omega_m}$. For instance, various investigations such as [2, 3, 5, 17, 18] showed that global regularity of solutions to linear and quasilinear PDEs may depend on a weighted isocapacitary function for subsets $\partial\Omega$, the weight being the norm of the second fundamental form on $\partial\Omega$.

This function, which we denote by $\mathcal{K}_\Omega$, is defined as

$$\mathcal{K}_\Omega(r) = \sup_{\substack{E \subset B_r(x) \\ x \in \partial\Omega}} \frac{\int_{\partial\Omega\cap E} |\mathcal{B}_\Omega| d\mathcal{H}^{n-1}}{\operatorname{cap}(E, B_r(x))} \qquad \text{for } r > 0, \tag{1.1}$$

and it was first introduced in [5]. Above, $\operatorname{cap}(E, B_r(x))$ denotes the standard capacity of a compact set E relative to the ball $B_r(x)$, i.e.

$$\operatorname{cap}(E, B_r(x)) = \inf\left\{ \int_{B_r(x)} |\nabla v|^2 \, dx \; : \; v \in C_c^{0,1}(B_r(x)), \; v \geq 1 \text{ on } E \right\},$$

where $C_c^{0,1}(A)$ is the set of Lipschitz continuous functions with compact support in A.

We remark that, in order for $\mathcal{K}_\Omega(r)$ to be well defined, it suffices that $\partial\Omega$ is Lipschitz continuous and belongs to $W^{2,1}$, as it can be inferred from inequalities (2.10) below.

Plan of the paper

The rest of the paper is organized as follows: in Sect. 2, we explain some non-standard notation used throughout the paper, and provide the definitions of $\mathcal{L}_\Omega$-Lipschitz domain, of $W^{2,q}$-domain and of weak curvature.

In Sect. 3 we state in detail our main results, and we provide a few comments and an outline of their proofs.

In Sect. 4 we state and prove a useful convergence property of mollification and convolution, which will be used in the proof of the convergence properties of the approximating sets.

In Sect. 5 we introduce the notion of *transversality* of a unit vector $\mathbf{n}$ to a Lipschitz function ϕ, and we show a very interesting fact, i.e. this transversality property is equivalent to the graphicality of ϕ with respect to the coordinate system (y', y_n) having $\mathbf{n} = e_n$. We then close this section by showing that the transversality condition– hence the graphicality with respect to the reference system (y', y_n)– is inherited by the convoluted function $M_m(\phi)$.

As a byproduct, we will find an interesting, yet expected result: if $\partial\Omega \in W^{2,q}$, then any Lipschitz function locally describing $\partial\Omega$ is of class $W^{2,q}$. This means that second-order Sobolev regularity is an intrinsic property of the local charts describing $\partial\Omega$– see Corollary 1.

Finally, Sect. 6 is devoted to the proof of the main Theorem 1.

2 Basic notation and definitions

In this section, we provide the relevant definitions and notation of use throughout the rest of the paper.

- For $d \in \mathbb{N}$, $U \subset \mathbb{R}^d$ open, and a function $v : U \to \mathbb{R}$, we shall denote by ∇v its d-dimensional gradient, and $\nabla^2 v$ its hessian matrix. We will often use the short-hand notation for its level and sublevel sets

$$\{v < 0\} := \{z \in U : v(z) < 0\}.$$
$$\{v = 0\} := \{z \in U : v(z) = 0\}.$$

- We denote by $W^{k,p}(\Omega)$ the usual Sobolev space of $L^p(\Omega)$ weakly differentiable functions having weak k-th order derivatives in $L^p(\Omega)$. For any $\alpha \in (0, 1]$, the spaces $C^k(\Omega)$ and $C^{k,\alpha}(\Omega)$ will denote, respectively, the space of functions with continuous and α-Hölder continuous derivatives up to order $k \in \mathbb{N}$.
- Point of $\mathbb{R}^n$ will be written as $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We write $B_r(x)$ to denote the n-dimensional ball of radius $r > 0$ and centered at $x \in \mathbb{R}^n$. Also, $B'_r(x')$ will denote the $(n-1)$-dimensional ball of radius $r > 0$ and centered at $x' \in \mathbb{R}^{n-1}$—when the centers are omitted, the balls are assumed to be centered at the origin, i.e. $B_r := B_r(0)$ and $B'_r := B'_r(0')$.
- For $d \in \mathbb{N}$, and for a given matrix $X \in \mathbb{M}_{d\times d}$, we shall denote by $|X|$ its Frobenius Norm $|X| = \sqrt{\operatorname{tr}(X^t X)}$, where X^t is the transpose of X.
- Given a Lebesgue measurable set A, we shall write $|A|$ for its Lebesgue measure. Also, given two open bounded sets A, B, we will denote by $\operatorname{dist}_{\mathcal{H}}(A, B)$ their Hausdorff distance.
- For a given function $\phi : U \to \mathbb{R}$ with $U \subset \mathbb{R}^{n-1}$ open, we write G_ϕ and S_ϕ to denote its graph and subgraph in $\mathbb{R}^n$, i.e.

$$G_\phi = \{x = \big(x', \phi(x')\big) : x' \in U\} \quad \text{and} \quad S_\phi = \{x = \big(x', x_n\big) : x' \in U,\ x_n < \phi(x')\}.$$

- We will denote by $\rho = \rho(x')$ the standard convolution Kernel in $\mathbb{R}^{n-1}$, i.e.

$$\rho(x') = \begin{cases} \exp\left\{ -\dfrac{1}{1-|x'|^2} \right\} & \text{if } |x'| < 1 \\ 0 & \text{if } |x'| \geq 1, \end{cases}$$

  and we will write $\rho_m(x') = m^{n-1}\rho\big(m\, x'\big)$ for $m \in \mathbb{N}$. Given $h \in L^1_{loc}(\mathbb{R}^{n-1})$, the convolution operator $M_m(h)$ is defined as

$$M_m(h)(x') = h * \rho_m(x') = \int_{\mathbb{R}^{n-1}} h(y')\, \rho_m(x' - y')\, dy'.$$

In the following, we specify the definition of Lipschitz domain and of Lipschitz characteristic.

Definition 1 (Lipschitz characteristic of a domain) An open, connected set Ω in $\mathbb{R}^n$ is called a Lipschitz domain if there exist constants $L_\Omega > 0$ and $R_\Omega \in (0, 1)$ such that, for every $x_0 \in \partial\Omega$ and $R \in (0, R_\Omega]$ there exist an orthogonal coordinate system centered at $0 \in \mathbb{R}^n$ and an L_Ω-Lipschitz continuous function $\phi : B'_R \to (-\ell, \ell)$, where

$$\ell = R(1 + L_\Omega), \tag{2.1}$$

satisfying $\phi(0') = 0$, and

$$\begin{aligned} \partial\Omega \cap \big(B'_R \times (-\ell, \ell)\big) &= \{(x', \phi(x')) : x' \in B'_R\}, \\ \Omega \cap \big(B'_R \times (-\ell, \ell)\big) &= \{(x', x_n) : x' \in B'_R,\ -\ell < x_n < \phi(x')\}. \end{aligned} \tag{2.2}$$

Moreover, we set

$$\mathfrak{L}_\Omega = (L_\Omega, R_\Omega), \tag{2.3}$$

and call $\mathfrak{L}_\Omega$ a Lipschitz characteristic of Ω.

It is easily seen that the above definition coincides with the standard one for uniformly Lipschitz domains–see e.g. [9, Section 2.4]. Our definition has the advantage of pointing out $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$ which appears in the characterization of our approximation sets.

We also remark that, in general, a Lipschitz characteristic $\mathfrak{L}_\Omega = (L_\Omega, R_\Omega)$ is not uniquely determined. For instance, if $\partial\Omega \in C^1$, then L_Ω may be taken arbitrarily small, provided that R_Ω is chosen sufficiently small.

The function ϕ in definition 1 is typically called *local (boundary) chart*. By Rademacher's theorem, this function is differentiable for $\mathcal{H}^{n-1}$-almost every x', with gradient $\nabla\phi$ bounded by L_Ω. In particular, this implies that any Lipschitz domain Ω admits a tangent plane on $\mathcal{H}^{n-1}$-almost every point of its boundary.

The local chart ϕ naturally endows $\partial\Omega$ of a local parametrization

$$\iota_\phi(x') = \big(x', \phi(x')\big) \tag{2.4}$$

under which, whenever ϕ is differentiable at x', a basis of the tangent space at the point $(x', \phi(x'))$ is given by

$$\mathcal{E}_\phi = \left\{ e_i + \frac{\partial\phi(x')}{\partial x'_i} \right\}_{i=1,\dots,n-1} \tag{2.5}$$

where $e_i = (0, \dots, 1, \dots, 0)$ is the i-th canonical unit vector of $\mathbb{R}^n$.

Moreover, via such parametrization $\iota_\phi(x')$, the first fundamental form $g = \{g_{ij}\}_{i,j=1}^{n-1}$ can be computed as

$$g_{ij}(x') = \delta_{ij} + \frac{\partial\phi(x')}{\partial x'_i}\,\frac{\partial\phi(x')}{\partial x'_j}, \tag{2.6}$$

where δ_{ij} denotes the Kronecker's delta, and x' is a point of differentiability of ϕ. Then, the inverse matrix $g^{-1} = \{g^{ij}\}_{i,j=1}^{n-1}$ can be explictly computed:

$$g^{ij}(x') = \delta_{ij} - \frac{1}{1 + |\nabla\phi(x')|^2}\,\frac{\partial\phi(x')}{\partial x'_i}\,\frac{\partial\phi(x')}{\partial x'_j}. \tag{2.7}$$

For such points $x_0 = \big(x', \phi(x')\big) \in \partial\Omega$, we shall denote by $T_{x_0}\partial\Omega = T_{x'}\partial\Omega$ the tangent space at x^0. From the discussion above, $\partial\Omega$ admits a tangent plane $\mathcal{H}^{n-1}$-almost every point $x_0 \in \partial\Omega$, hence we may want to define a notion of weak second fundamental form which extends the classical one for C^∞-smooth domains of $\mathbb{R}^n$.

For this purpose, we need some additional regularity assumptions on ϕ, and in particular on its second-order derivatives.

Definition 2 ($W^{2,q}$ domains and weak curvature) Let $q \in [1, \infty)$. We say that a bounded Lipschitz domain Ω is of class $W^{2,q}$ if the local boundary chart ϕ satisfying (2.2) belongs to the Sobolev space $W^{2,q}(B'_R)$. If $\phi \in W^{2,\infty}(B'_R)$, we say that $\partial\Omega \in C^{1,1}$ (or $\partial\Omega \in W^{2,\infty}$).

If $\partial\Omega \in W^{2,1}$, the weak curvature $\mathcal{B}_\Omega$ of $\partial\Omega$ is a bilinear operator

$$\mathcal{B}_\Omega(x_0) : T_{x_0}\partial\Omega \times T_{x_0}\partial\Omega \to \mathbb{R}$$

defined for $\mathcal{H}^{n-1}$-almost every point $x_0 \in \partial\Omega$. With the choice of local parametrization ι_ϕ in (2.4), its components $\{\mathcal{B}_{ij}\}_{i,j=1}^{n-1}$ with respect to the basis $\mathcal{E}_\phi$ in (2.5) of $T_{x'}\partial\Omega$ are locally defined as

$$\mathcal{B}_{ij}(x') = -\frac{1}{\sqrt{1 + |\nabla\phi(x')|^2}}\,\frac{\partial^2\phi(x')}{\partial x'_i\partial x'_j}, \tag{2.8}$$

for $\mathcal{H}^{n-1}$-almost every points $x' \in B'_R$ of differentiability of ϕ. Its norm is then given by

$$|\mathcal{B}_\Omega(x')| = \frac{\sqrt{\operatorname{trace}\big((g^{-1}\,\nabla^2\phi)^2\big)}}{\sqrt{1+|\nabla\phi(x')|^2}}, \tag{2.9}$$

where g^{-1} is the inverse matrix of g given by (2.7).

The reader may verify that identities (2.6)-(2.9) concur with the usual ones when $\partial\Omega$ is a smooth hypersurface of $\mathbb{R}^n$–see e.g. [12, pp. 246-249]. However, these definitions also make sense when ϕ is merely Lipschitz continuous and belongs to the Sobolev space $W^{2,1}$. Indeed, the following inequalities hold true:

$$\frac{|\nabla^2\phi(x')|}{(1+L_\Omega^2)^{3/2}} \le |\mathcal{B}_\Omega(x')| \le |\nabla^2\phi(x')|. \tag{2.10}$$

In order to prove (2.10), we first recall that for all symmetric matrices X, Y, with X definite positive, we have the elementary linear algebra inequalities

$$\lambda_{\min}^2|Y|^2 \le \operatorname{tr}\big((XY)^2\big) \le \lambda_{\max}^2\,|Y|^2,$$

where $\lambda_{\min}, \lambda_{\max}$ denote the smallest and largest eigenvalues of X–see e.g. [2, Lemma 3.6] and its proof. Then, owing to (2.7), we observe that the largest and smallest eigenvalues of the matrix g^{-1} are respectively 1 and $(1+|\nabla\phi|^2)^{-1}$, and since $|\nabla\phi| \le L_\Omega$ we immediately infer (2.10). Inequalities (2.10) also show that (locally) second fundamental form $\mathcal{B}_\Omega$ is equivalent to the second-order derivatives of the local charts.

We close this section by pointing out that the above definitions can be easily extended to domains with boundary $\partial\Omega \in W^{k,q}$. Similarly, standard definitions follow for domains of class C^k and $C^{k,\alpha}$.

3 Main results

Having dispensed of the necessary definitions and notation, we can now give a precise statement of our main results. This is the content of this section, coupled with a few comments and an outline of the proofs. Our first main result reads as follows.

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded, Lipschitz domain, with Lipschitz characteristic $\mathcal{L}_\Omega = (L_\Omega, R_\Omega)$.*

(i) There exist sequences of bounded domains $\{\omega_m\}, \{\Omega_m\}$, such that $\partial\omega_m \in C^\infty$, $\partial\Omega_m \in C^\infty$, and

$$\omega_m \Subset \Omega \Subset \Omega_m \quad \text{for all } m \in \mathbb{N}.$$

Their diameters satisfy

$$d_{\Omega_m} \le c(n)\,d_\Omega, \quad d_{\omega_m} \le c(n)\,d_\Omega, \tag{3.1}$$

the following convergence property hold true

$$\lim_{m\to\infty} |\Omega_m \setminus \Omega| = 0, \quad \lim_{m\to\infty} |\Omega \setminus \omega_m| = 0, \tag{3.2}$$

the Hausdorff distances safisfy

$$\operatorname{dist}_{\mathcal{H}}(\omega_m, \Omega) + \operatorname{dist}_{\mathcal{H}}(\Omega_m, \Omega) \le \frac{12\,L_\Omega\sqrt{1+L_\Omega^2}}{m} \quad \text{for all } m \in \mathbb{N}, \tag{3.3}$$

and we may choose their Lipschitz characteristics $\mathcal{L}_{\Omega_m} = (L_{\Omega_m}, R_{\Omega_m})$ *and* $\mathcal{L}_{\omega_m} = (L_{\omega_m}, R_{\omega_m})$ *such that*

$$\begin{aligned} L_{\Omega_m} &\le c(n)(1+L_\Omega^2), \quad R_{\Omega_m} \ge R_\Omega / \big(c(n)(1+L_\Omega^2)\big) \\ L_{\omega_m} &\le c(n)(1+L_\Omega^2), \quad R_{\omega_m} \ge R_\Omega / \big(c(n)(1+L_\Omega^2)\big), \quad \text{for all } m \in \mathbb{N}. \end{aligned} \tag{3.4}$$

Moreover, the smooth boundaries $\partial\omega_m, \partial\Omega_m$ *are described with the help of the same co-ordinate systems as* $\partial\Omega$*, i.e. there exist finite number of local boundary charts* $\{\phi^i\}_{i=1}^N$, $\{\psi_m^i\}_{i=1}^N$ *and* $\{\varphi_m^i\}_{i=1}^N$ *which describe* $\partial\Omega$, $\partial\Omega_m$ *and* $\partial\omega_m$ *respectively, such that for each* $i = 1, \dots, N$ *the functions* $\psi_m^i, \varphi_m^i \in C^\infty$ *are defined on the same reference system as* ϕ^i*, and*

$$\psi_m^i \xrightarrow{m\to\infty} \phi^i \quad \text{and} \quad \varphi_m^i \xrightarrow{m\to\infty} \phi^i \quad \text{in } W^{1,p}(B'_{R_\Omega-\varepsilon_0}), \tag{3.5}$$

for all $p \in [1, \infty)$*, for all* $i = 1, \dots, N$*, and any fixed constant* $\varepsilon_0 \in (0, R_\Omega/2)$*.*
(ii) If in addition $\partial\Omega \in W^{2,q}$ *for some* $q \in [1, \infty)$*, then*

$$\psi_m^i \xrightarrow{m\to\infty} \phi^i \quad \text{and} \quad \varphi_m^i \xrightarrow{m\to\infty} \phi^i \quad \text{in } W^{2,q}(B'_{R_\Omega-\varepsilon_0}), \tag{3.6}$$

and there exists a constant $\widehat{c} = \widehat{c}(n, \mathcal{L}_\Omega, d_\Omega)$ *such that*

$$\mathcal{K}_{\Omega_m}(r) + \mathcal{K}_{\omega_m}(r) \le \begin{cases} \widehat{c}\Big\{\mathcal{K}_\Omega\big(\widehat{c}(r+\frac{1}{m})\big) + r\Big\} & \text{if } n \ge 3 \\ \\ \widehat{c}\Big\{\mathcal{K}_\Omega\big(\widehat{c}(r+\frac{1}{m})\big) + r\,\log(1+\frac{1}{r})\Big\} & \text{if } n = 2 \end{cases} \tag{3.7}$$

for all $m \in \mathbb{N}$ *and* $r \le r_0(n, \mathcal{L}_\Omega)$*.*

Let us briefly comment on our result. Part (i) of Theorem 1 is mostly analogous to [6, Theorem 5.1]; as expected from domains Ω with Lipschitz continuous boundary, the local charts of $\partial\Omega_m, \partial\omega_m$ converge to the corresponding local charts of $\partial\Omega$ in $W^{1,p}$ for all $p \in [1, \infty)$. In particular, by the classical Morrey-Sobolev's embedding Theorems, this entails an "almost Lipschitz convergence", i.e. the local charts ψ_m^i and φ_m^i converge to ϕ^i in every Hölder space $C^{0,\alpha}$ with $\alpha \in (0, 1)$.

The main novelty of our result is given in Part (ii), where information about the second fundamental forms $\mathcal{B}_{\omega_m}$ and $\mathcal{B}_{\Omega_m}$ (or equivalently $\nabla^2\varphi_m^i$ and $\nabla^2\psi_m^i$) is retrieved when $\partial\Omega$ is endowed with a weak curvature. For instance, by definition (2.8) and from the results of Theorem 1, via a standard covering argument it is easy to show that

$$\int_{\partial\Omega_m} |\mathcal{B}_{\Omega_m}|^q \, d\mathcal{H}^{n-1} \to \int_{\partial\Omega} |\mathcal{B}_\Omega|^q \, d\mathcal{H}^{n-1} \quad \text{and} \quad \int_{\partial\omega_m} |\mathcal{B}_{\omega_m}|^q \, d\mathcal{H}^{n-1} \to \int_{\partial\Omega} |\mathcal{B}_\Omega|^q \, d\mathcal{H}^{n-1}, \tag{3.8}$$

for all $q \in [1, \infty)$ such that $\partial\Omega \in W^{2,q}$.

Other than this, we obtain the isocapacitary estimate (3.7), where $\mathcal{K}_\Omega(r)$ and $\mathcal{K}_{\Omega_m}, \mathcal{K}_{\omega_m}$ are the functions defined in (1.1) relative to Ω, Ω_m and ω_m, respectively. In the proof of (3.7), we will also explicitly write the constant $\widehat{c}$ appearing therein.

Finally, the boundaries $\partial\Omega$, $\partial\Omega_m$ and $\partial\omega_m$ all share the same coordinate cylinders $\{K_{\varepsilon_0}^i\}_{i=1}^N$ which are, up to an isometry, equal to $B'_{R_\Omega-\varepsilon_0} \times (-\ell, \ell)$, with $\ell = (1 + L_\Omega)\, R_\Omega$.

This means that their local boundary charts, ϕ^i, ψ_m^i and φ_m^i respectively, are defined on the same $(n-1)$-dimensional ball $B'_{R_\Omega-\varepsilon_0}$, independently on $i = 1, \dots N$ and $m \in \mathbb{N}$—see Fig. f1 below.

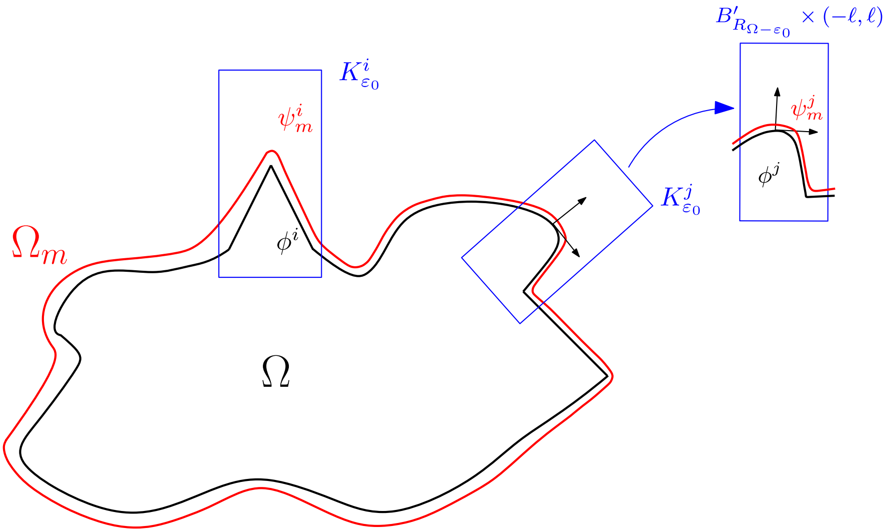

Fig. 1 The local boundary charts of $\partial\Omega$ and $\partial\Omega$, ϕ^i and ψ^i_m respectively, are defined on the same reference system

Here, the fixed parameter $\varepsilon_0 \in (0, R_\Omega/2)$, also appearing in (3.5) and (3.6), is purely technical and does not affect the validity of the convergence results. Indeed, from our construction in Sect. 6, the boundaries $\partial\Omega$, $\partial\Omega_m$ and $\partial\omega_m$ will be covered by smaller coordinate cylinders of the kind $B'_{R_\Omega/2} \times (-\ell, \ell)$.

Outline of the proof

We fix a covering of $\partial\Omega$, with corresponding partition of unity $\{\xi_i\}_i$ and local boundary charts $\{\phi^i\}_i$, which are L_Ω-Lipschitz continuous.

Then we regularize each function ϕ^i via convolution, and add (or subtract) a suitable constant, so that we obtain C^∞-smooth functions $\{\phi^i_m\}_i$ such that $\phi^i_m > \phi^i$ (or $\phi^i_m < \phi^i$).

However, in the original reference system, the graphs of these smooth functions $G_{\phi^i_m}$ are not "glued" together, and thus their union is not the boundary of a domain, unlike the graphs G_{ϕ^i} whose union describes $\partial\Omega$—see Fig. 2 below.

To overcome this problem, we define a suitable C^∞-smooth function F_m, built upon $\{\phi^i_m\}_i$ and $\{\xi_i\}_i$– see equation (6.14) below– and define the regularized set Ω_m as the sublevel set $\{F_m < 0\}$, so that

$$\partial\Omega_m = \{F_m = 0\},$$

and by construction we will have $\omega_m \Subset \Omega \Subset \Omega_m$.
The function F_m is called *boundary defining functions* of Ω_m—see [11, Section 5.4].

In order to show that $\partial\Omega_m$ is a smooth manifold, we prove that the gradient of F_m along the directions of graphicality of ϕ^i is greater than a positive constant depending on L_Ω—see estimate (6.20). This property of F_m will be proven by exploiting the so-called *transversality condition* of ϕ^i, which is inherited via convolution by ϕ^i_m as well. Therefore, F_m is strictly monotone along these directions, which entails that its zero-level set $\partial\Omega_m$ is a smooth manifold with local boundary charts ψ^i_m defined on the same reference system as ϕ^i.

Thanks to the properties of convolution, we show that F_m converge to the boundary defining function F of Ω built upon $\{\phi^i\}_i$ and $\{\xi_i\}_i$– see equations (6.9) and (6.10)– and thus ψ_m^i converge uniformly to ϕ^i.

Then, as in the proof of the implicit function theorem, we differentiate the identity $F_m\big(y', \psi_m^i(y')\big) = 0$, so that we may express the gradient $\nabla\psi_m^i$ (and its Hessian $\nabla^2\psi_m^i$) in terms of $\{\phi_m^j, \nabla\phi_m^j\}_j$ (and $\{\nabla^2\phi_m^j\}_j$), and then (3.4), (3.5) (and (3.6)) will be obtained by exploiting the convergence properties of convolution.

Finally, in order to get the isocapacitary estimate (3.7), we make use of the estimates on $|\nabla^2\psi_m^i|$ obtained in the previous steps, as to evaluate weighted Poincaré type quotients of the kind

$$\frac{\int_{\partial\Omega_m} v^2\, |\mathcal{B}_{\Omega_m}|\, d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla v|^2 dx}, \quad v \in C_c^\infty\big(B_r(x_m^0)\big),\ x_m^0 \in \partial\Omega_m$$

in terms of the corresponding quotient with weight $|\mathcal{B}_\Omega|$, and then (3.7) will follow from the celebrated isocapacitary equivalency Theorem of Maz'ya [16], [19, Theorem 2.4.1].

Our next and final result shows the flexibility of our approximation method, which takes into account even higher regularity of the domain Ω.

Theorem 2 *Under the same notations as Theorem 1, we have that*

(1) if $\partial\Omega \in C^k$ for some $k \in \mathbb{N}$, then

$$\psi_m^i \xrightarrow{m\to\infty} \phi^i \quad and \quad \varphi_m^i \xrightarrow{m\to\infty} \phi^i \quad in\ C^k(B'_{R_\Omega-\varepsilon_0});$$

(2) if $\partial\Omega \in C^{k,\alpha}$ for some $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, then

$$\psi_m^i \xrightarrow{m\to\infty} \phi^i \quad and \quad \varphi_m^i \xrightarrow{m\to\infty} \phi^i \quad in\ C^{k,\alpha'}(B'_{R_\Omega-\varepsilon_0}),$$

for all $0 < \alpha' < \alpha$;

(3) if $\partial\Omega \in W^{k,q}$ for some $k \in \mathbb{N}$ and $q \in [1, \infty)$, then

$$\psi_m^i \xrightarrow{m\to\infty} \phi^i \quad and \quad \varphi_m^i \xrightarrow{m\to\infty} \phi^i \quad in\ W^{k,q}(B'_{R_\Omega-\varepsilon_0}).$$

(4) if $\partial\Omega \in C^{k,1}$ for some $k \in \mathbb{N}$, then

$$\psi_m^i \xrightarrow{m\to\infty} \phi^i \quad and \quad \varphi_m^i \xrightarrow{m\to\infty} \phi^i \quad weakly\text{-} * \ in\ W^{k,\infty}(B'_{R_\Omega-\varepsilon_0}).$$

The proof of Theorem 2 can be easily carried out by extending the proof and estimates of Theorem 1 to higher order derivatives, and by using standard compactness theorems such as Ascoli-Arzelá's and weak-$*$ compactness. For this very reason, we decided to omit the proof.

4 Auxiliary results

In this section, we state and prove a useful convergence property regarding the convolution of functions composed with a suitable family of bi-Lipschitz maps.

Proposition 1 *Let $U \subset \mathbb{R}^{n-1}$ be a bounded domain, $K > 0$ be a constant, and $\{\Psi_m\}_{m\in\mathbb{N}}$ be a family of bi-Lipschitz maps on U such that*

$$\sup_{m\in\mathbb{N}} \|\nabla\Psi_m^{-1}\|_{L^\infty} \le K, \tag{4.1}$$

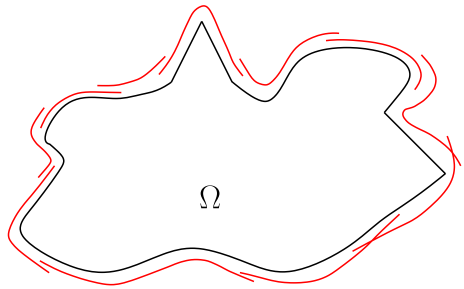

Fig. 2 In red: the graphs of the regularized local charts (up to isometry)

and there exists a bi-Lipschitz map $\Psi : U \to \Psi(U)$ *such that*

$$\|\Psi_m - \Psi\|_{L^\infty(U)} \leq \frac{K}{m} \quad \textit{for all } m \in \mathbb{N}. \tag{4.2}$$

Let $\mathcal{O} \subset \mathbb{R}^{n-1}$ *open be such that* $\Psi(U) \Subset \mathcal{O}$, *and* $\phi \in L^p(\mathcal{O})$ *for some* $p \in [1, \infty)$. *Then*

$$M_m(\phi) \circ \Psi_m \xrightarrow{m \to \infty} \phi \circ \Psi \quad \mathcal{H}^{n-1}\text{-a.e. in} U \text{ and in } L^p(U). \tag{4.3}$$

Proof Set

$$U_\phi := \left\{ x' \in U : \ \Psi(x') \text{ is a Lebesgue point of } \phi \right\}$$

By Lebesgue differentiation theorem and since Ψ is a bi-Lipschitz map, we have that U_ϕ is a subset of U with full measure. Also, thanks to (4.2) and the fact that $\Psi\big(U\big) \Subset \mathcal{O}$, we have that ϕ and $M_m(\phi)$ are well defined on a neighbourhood of $\Psi_m(U)$ for $m > m_0$ large enough. Then, for all $x' \in U_\phi$ we have

$$\begin{aligned} \left|M_m(\phi)\big(\Psi_m(x')\big) - \phi\big(\Psi(x')\big)\right| &= \left| \int_{B'_{\frac{1}{m}}(\Psi_m(x'))} \Big[\phi(z') - \phi\big(\Psi(x')\big)\Big] \rho_m\big(\Psi_m(x') - z'\big)\, dz \right| \\ &\leq \Big(\sup_{\mathbb{R}^{n-1}} \rho\Big)\, m^{n-1} \int_{B'_{\frac{(K+1)}{m}}(\Psi(x'))} \left|\phi(z') - \phi\big(\Psi(x')\big)\right| dz' \xrightarrow{m\to\infty} 0. \end{aligned}$$

Above we used the fact that $\Psi(x')$ is a Lebesgue point of ϕ, and $B'_{\frac{1}{m}}(\Psi_m(x')) \subset B'_{\frac{(K+1)}{m}}(\Psi(x'))$ as a consequence of (4.2).

Now fix $\varepsilon > 0$, and take a function $\widetilde{\phi} \in C_c^\infty(\mathbb{R}^{n-1})$ satisfying

$$\|\phi - \widetilde{\phi}\|^p_{L^p(\mathcal{O})} \leq \varepsilon. \tag{4.4}$$

Standard properties of convolutions ensure that

$$\|M_m(\widetilde{\phi}) - \widetilde{\phi}\|_{L^\infty(\mathcal{O})} \xrightarrow{m\to\infty} 0. \tag{4.5}$$

Then we have

$$\begin{aligned} \int_U \left|M_m(\phi)\big(\Psi_m(x')\big) - \phi\big(\Psi(x')\big)\right|^p dx' &\leq c(p) \int_U \left|M_m(\phi - \widetilde{\phi})\big(\Psi_m(x')\big)\right|^p dx' \\ &+ c(p) \int_U \left|M_m(\widetilde{\phi})\big(\Psi_m(x')\big) - \widetilde{\phi}\big(\Psi(x')\big)\right|^p dx' + c(p) \int_U \left|\widetilde{\phi}\big(\Psi(x') - \phi\big(\Psi(x')\big)\right|^p dx' \end{aligned} \tag{4.6}$$

By applying Jensen inequality, the change of variables $w' = \Psi_m(x') - z'$ and Fubini-Tonelli's Theorem we obtain

$$\begin{aligned}
&\int_U \big|M_m(\phi - \widetilde{\phi})\big(\Psi_m(x')\big)\big|^p\,dx' \\
&\le \int_U \int_{B'_{1/m}} \big|\phi\big(\Psi_m(x') - z'\big) - \widetilde{\phi}\big(\Psi_m(x') - z'\big)\big|^p \rho_m(z')\,dz'\,dx' \\
&\le c(n)\,K^{n-1} \int_{\mathbb{R}^{n-1}} \rho_m(z')\,dz' \int_{\mathcal{O}} \big|\phi(w') - \widetilde{\phi}(w')\big|^p\,dw' \le c(n)\,K^{n-1}\,\varepsilon,
\end{aligned}$$

where we also used estimates (4.1) and (4.4).

Then, by using (4.2) and (4.5), it is immediate to verify that

$$\lim_{m\to\infty} \int_U \big|M_m(\widetilde{\phi})\big(\Psi_m(x')\big) - \widetilde{\phi}\big(\Psi(x')\big)\big|^p\,dx' = 0,$$

and finally, via a change of variables $y' = \Psi(x')$, and (4.4) we get

$$\int_U \big|\widetilde{\phi}\big(\Psi(x') - \phi\big(\Psi(x')\big)\big|^p\,dx' \le c(n)\,\|\nabla\Psi^{-1}\|_{L^\infty}^{n-1}\,\varepsilon.$$

Henceforth, by plugging the last three estimates into (4.6), we find

$$\limsup_{m\to\infty} \int_U \big|M_m(\phi)\big(\Psi_m(x')\big) - \phi\big(\Psi(x')\big)\big|^p\,dx' \le c(n, p, L, \Psi)\,\varepsilon,$$

and thus (4.3) follows by the arbitrariness of ε. □

We close this section recalling a variant of Lebesgue dominated convergence Theorem which will be useful later on. Since we could not find a precise reference, we provide a proof.

Theorem 3 *(Dominated convergence Theorem) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of measurable functions on $E \subset \mathbb{R}^{n-1}$ such that*

(i) $f_k \to f$ almost everywhere on E;
(ii) $|f_k| \le g_k$ almost everywhere on E, with $g_k \in L^q(E)$ for some $q \in [1, \infty)$;
(iii) there exists $g \in L^q(E)$ such that $g_k \to g$ a.e. on E, and $\int_E g_k^q\,dx \to \int_E g^q\,dx$.

Then $f \in L^q(E)$, and

$$\int_E |f_k - f|^q\,dx \to 0.$$

Proof Set

$$F_k = |f_k - f|^q, \quad F = 0, \quad G_k = 2^{q-1}\{g_k + g\}, \quad \text{and} \quad G = 2^q\,g.$$

Observe that, by hypothesis, $F_k \to F$ and $G_k \to G$ almost everywhere on E as $k \to \infty$, $0 \le F_k \le G_k$ almost everywhere, with $G_k, G \in L^1(E)$, and

$$\int_E G_k\,dx \to \int_E G\,dx.$$

The thesis then follows from a standard generalization of dominated convergence theorem– see for instance [7, Exercise 20, pp. 59]. □

5 Transversality and graphicality

Throughout this section, we shall consider an isometry T of $\mathbb{R}^n$, such that

$$Tx = \mathcal{R}x + x^0, \quad x \in \mathbb{R}^n, \tag{5.1}$$

where $\mathcal{R} = \{\mathcal{R}_{ij}\}_{i,j=1}^n$ is an orthogonal matrix of $\mathbb{R}^n$, and $x^0 \in \mathbb{R}^n$. Let

$$\mathbf{n} = \mathcal{R}^t e_n \in \mathbb{S}^{n-1},$$

where e_n denotes the n-th canonical vector of $\mathbb{R}^n$, i.e. $e_n = (0, \dots, 0, 1)$, $\mathcal{R}^t$ is the transpose matrix of $\mathcal{R}$, and $\mathbb{S}^{n-1}$ is the unit sphere on $\mathbb{R}^n$.

Here we introduce the geometric notion of *transversality*, which was already used in [10] in a wider sense. The definition given here suffices to our purposes.

Definition 3 (Transversality) Let $\phi : U \to \mathbb{R}$ be a Lipschitz continuous function on $U \subset \mathbb{R}^{n-1}$ open. We say that a unit vector $\mathbf{n} \in \mathbb{S}^{n-1}$ is transversal to ϕ if there exists $\kappa > 0$ such that

$$\mathbf{n} \cdot \nu(x') \geq \kappa \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in U,$$

where ν denotes the outward normal to G_ϕ with respect to the subgraph S_ϕ.

The next proposition shows a very interesting feature: the transversality of $\mathbf{n} \in \mathbb{S}^{n-1}$ to a Lipschitz function ϕ is equivalent to the graphicality (and subgraphicality) of ϕ with respect to any reference system having $e_n = \mathbf{n}$, that is after performing a rotation of the axes through $\mathcal{R}$, the graph and subgraph of ϕ are mapped onto the graph and subgraph of another function ψ– see identities (5.2) below.

Proposition 2 *Let $U \subset \mathbb{R}^{n-1}$ be open, $\phi : U \to \mathbb{R}$ be a Lipschitz function, let T be an isometry of the form (5.1), and let $\mathbf{n} = \mathcal{R}^t e_n$.*

(i) If there exists an L-Lipschitz function $\psi : V \to \mathbb{R}$ such that

$$TG_\phi = G_\psi \quad \textit{and} \quad TS_\phi = S_\psi \cap T(U \times \mathbb{R}), \tag{5.2}$$

then we have the transversality condition

$$\mathbf{n} \cdot \nu(x') \geq \frac{1}{\sqrt{1 + L^2}} \quad \textit{for } \mathcal{H}^{n-1}\textit{-a.e. } x' \in U. \tag{5.3}$$

(ii) Viceversa, if $\phi \in C^k(U)$ for some $k \in \mathbb{N}$ and (5.3) holds, then there exist $V \subset \mathbb{R}^{n-1}$ open, and a function $\psi \in C^k(V)$ such that $\|\nabla\psi\|_{L^\infty(V)} \leq L$ and (5.2) holds true.

Let us comment on this result. Part (i) states that if G_ϕ and S_ϕ are, respectively, the graph and subgraph of an L-Lipschitz function ψ with respect to the reference system $z = (z', z_n)$ having $\mathbf{n} = e_n$, then the quantitative transversality estimate (5.3) holds true.

Part (ii) states the opposite in the C^k case: the transversality condition (5.3) implies the graphicality and subgraphicality of ϕ with respect to the coordinate system $z = (z', z_n)$, and it also provides a Lipschitz estimate to ψ.

Before starting the proof, we need to introduce the so-called *transition map* $\mathcal{C}$ from ϕ to ψ. Under the same notation as Proposition 2, the transition map $\mathcal{C} : U \to V$ is defined as

$$\mathcal{C}x' := \Pi\, T\big(x', \phi(x')\big).$$

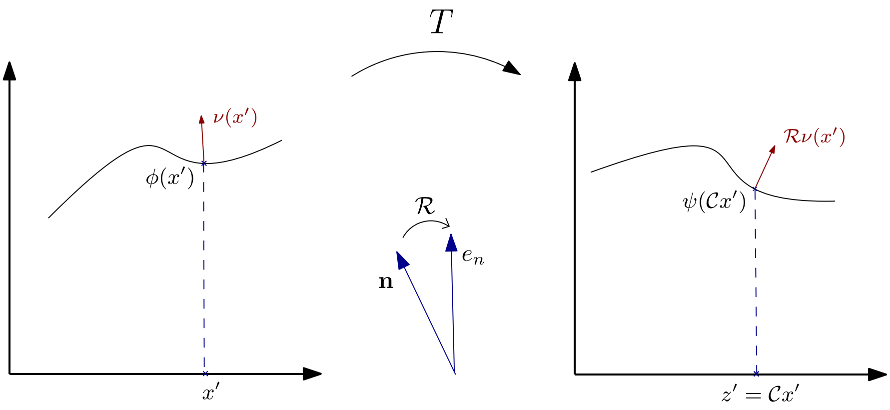

Fig. 3 The new graph after the rigid motion T

Here $\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection map $\Pi(x', x_n) = x'$. Observe that, when identities (5.2) hold true, by the very definition of $\mathcal{C}$ we have the equation

$$T\big(x', \phi(x')\big) = \big(\mathcal{C}x', \psi\big(\mathcal{C}x'\big)\big)$$

In particular, this implies that $\mathcal{C}$ is a bijection, with inverse function $\mathcal{C}^{-1} : V \to U$ given by

$$\mathcal{C}^{-1}z' = \Pi\, T^{-1}\big(z', \psi(z')\big).$$

Also, since ϕ, ψ are Lipschitz continuous, then $\mathcal{C}$ is a bi-Lipschitz tranformation from U to V.

Proof of Proposition 2 (i) By Rademacher's theorem, the normal vector ν to G_ϕ outward with respect to S_ϕ is well defined $\mathcal{H}^{n-1}$-almost everywhere, and thanks to (5.2) and the definition of $\mathcal{C}$, we may write (Fig. 3)

$$\nu(x') = \frac{(-\nabla\phi(x'), 1)}{\sqrt{1+|\nabla\phi(x')|^2}} = \mathcal{R}^t\left(\frac{(-\nabla\psi(\mathcal{C}x'), 1)}{\sqrt{1+|\nabla\psi(\mathcal{C}x')|^2}}\right) \quad \mathcal{H}^{n-1}\text{-a.e. } x' \in U. \tag{5.4}$$

Therefore, since $\mathcal{R}\mathbf{n} = e_n$ and $|\nabla\psi| \le L$, from (5.4) we infer

$$\mathbf{n}\cdot\nu(x') = e_n\cdot\mathcal{R}\nu(x') = \frac{1}{\sqrt{1+|\nabla\psi(\mathcal{C}x')|^2}} \ge \frac{1}{\sqrt{1+L^2}} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in U. \tag{5.5}$$

(ii) Assume $\phi \in C^k(U)$ and that (5.3) is in force.

Consider the C^k-function $f : U \times \mathbb{R} \to \mathbb{R}$, defined as $f(x) := x_n - \phi(x')$, so that

$$\{f = 0\} = G_\phi \quad \text{and} \quad \{f < 0\} = S_\phi. \tag{5.6}$$

Now let $\tilde{f} : T(U \times \mathbb{R}) \to \mathbb{R}$ be the function defined as $\tilde{f}(z) = f(x)$ for $z = Tx$. Recalling $\mathcal{R}\mathbf{n} = e_n$, via the chain rule we compute

$$\frac{\partial\tilde{f}(z)}{\partial z_n} = \mathcal{R}_{nn} - \sum_{k=1}^{n-1}\frac{\partial\phi(x')}{\partial x'_k}\mathcal{R}_{nk} = (-\nabla\phi(x'), 1)\cdot\mathbf{n}. \tag{5.7}$$

Thus, from expression (5.4) of $\nu(x')$ and estimate (5.3), we obtain

$$\frac{\partial\tilde{f}(z)}{\partial z_n} = \sqrt{1+|\nabla\phi(x')|^2}\,\nu(x')\cdot\mathbf{n} \ge \frac{1}{\sqrt{1+L^2}} \quad \text{for } z = Tx. \tag{5.8}$$

Therefore, owing to (5.8) and the implicit function theorem, we immediately infer the existence of a function $\psi \in C^k(V)$, with $V \subset \mathbb{R}^{n-1}$ open, such that

$$\{\tilde{f} = 0\} = G_\psi \quad \text{and} \quad \{\tilde{f} = 0\} = S_\psi \cap T(U \times \mathbb{R}).$$

Thereby, (5.2) follows from the very definition of $\tilde{f}$ and (5.6).

Finally, by using (5.5) we infer that $|\nabla\psi(\mathcal{C}x')| \le L$ for all $x' \in U$, whence $\|\nabla\psi\|_{L^\infty(V)} \le L$ since the transition map $\mathcal{C}$ is a bijection.

Remark 1 We point out that inequality (5.8), when evaluated at points $z = T\big(x', \phi(x')\big)$, holds true if ϕ and ψ are merely Lipschitz continuous and satisfy (5.2).

Indeed, since $\mathcal{C}$ is a bi-Lipschitz map, by Rademacher's Theorem and the chain rule we may perform the same computations as (5.7)-(5.8) and get

$$\mathcal{R}_{nn} - \sum_{k=1}^{n-1} \frac{\partial\phi(x')}{\partial x'_k} \mathcal{R}_{nk} \ge \frac{1}{\sqrt{1+L^2}} \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x' \in U. \tag{5.9}$$

By making use of this information, we now show that the transversality condition (5.3) is inherited by the regularized function $M_m(\phi)$. This is the content of the following proposition

Proposition 3 *Let $U, V \subset \mathbb{R}^{n-1}$ be open bounded, let T be an isometry of the form (5.1), and $\mathbf{n} = \mathcal{R}^t e_n$. Let $\phi : U \to \mathbb{R}$ and $\psi : V \to \mathbb{R}$ be L-Lipschitz functions satisfying (5.2). If we set*

$$U_m := \left\{x' \in U : \operatorname{dist}(x', \partial U) > \tfrac{1}{m}\right\}$$

and for some sequence $\{c_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ we define

$$\phi_m(x') := M_m(\phi)(x') + c_m \quad \textit{for } x' \in U_m,$$

then ϕ_m is L-Lipschitz continuous on U_m and

$$\|\phi_m - \phi\|_{L^\infty(U_m)} \le \frac{L}{m} + |c_m|. \tag{5.10}$$

In addition, we have the transversality condition

$$\mathcal{R}_{nn} - \sum_{k=1}^{n-1} \frac{\partial\phi_m}{\partial x'_k}(x')\mathcal{R}_{nk} = \big(-\nabla\phi_m(x'), 1\big)\cdot\mathbf{n} \ge \frac{1}{\sqrt{1+L^2}} \quad \textit{for all } x' \in U_m, \tag{5.11}$$

and

$$\mathbf{n}\cdot\nu_m(x') \ge \frac{1}{1+L^2} \quad \textit{for all } x' \in U_m, \tag{5.12}$$

where ν_m is the outward unit normal to G_{ϕ_m} with respect to the subgraph S_{ϕ_m}.

Proof Let $x'_0 \in U_m$. By multiplying (5.9) with $\rho_m(x'_0 - x')$ and integrating in x' we immediately obtain

$$\mathcal{R}_{nn} - \sum_{k=1}^{n-1} \frac{\partial M_m(\phi)(x'_0)}{\partial x'_k}\mathcal{R}_{nk} \ge \frac{1}{\sqrt{1+L^2}} \quad \text{for all } x'_0 \in U_m,$$

and (5.11) holds true.

Next, from the L-Lipschitz continuity of ϕ, we have

$$\begin{aligned}\left|M_m(\phi)(x') - M_m(\phi)(y')\right| &\le \int_{\mathbb{R}^{n-1}} \left|\phi(x'-z') - \phi(y'-z')\right| \rho_m(z')\,dz' \\ &\le L\,|x'-y'| \int_{\mathbb{R}^{n-1}} \rho_m(z')\,dz' = L\,|x'-y'|\end{aligned}$$

for all $x', y' \in U_m$, hence ϕ_m is L-Lipschitz continuous as well. From this and (5.11), we get

$$\mathbf{n}\cdot\nu_m(x') = \mathbf{n}\cdot\frac{\big(-\nabla M_m(\phi)(x'),1\big)}{\sqrt{1+|\nabla M_m(\phi)(x')|^2}} \ge \frac{1}{1+L^2} \quad \text{for all } x' \in U_m,$$

that is (5.12). Next, since ρ_m is radially symmetric and ϕ is L-Lipschitz continuous, for all $x' \in U_m$ we get

$$\begin{aligned}\left|M_m(\phi)(x') - \phi(x')\right| &\le \int_{B'_{1/m}} \left|\phi(x'+y') - \phi(x')\right| \rho_m(y')\,dy' \\ &\le \int_{B'_{1/m}} L\,|y'|\,\rho_m(y')\,dy' \le \frac{L}{m},\end{aligned}$$

and thus (5.10) follows. □

Since we have proven that the regularized function $M_m(\phi)$ satisfies the transversality condition, Part (ii) of Proposition 2 entails its "graphicality" with respect to the coordinate system having $\mathbf{n} = e_n$.

Proposition 4 *Under the same assumptions of Proposition 3, there exist $V_m \subset \mathbb{R}^{n-1}$ open bounded such that*

$$\mathrm{dist}_{\mathcal{H}}(V_m, V) \le \frac{2\sqrt{1+L^2}}{m} + |c_m|, \tag{5.13}$$

and a function $\psi_m \in C^\infty(V_m)$ satisfying

$$\|\nabla\psi_m\|_{L^\infty(V_m)} \le 2(1+L^2), \tag{5.14}$$

$$TG_{\phi_m} = G_{\psi_m} \quad \textit{and} \quad TS_{\phi_m} = S_{\psi_m} \cap T\big(U_m\times\mathbb{R}\big). \tag{5.15}$$

If in addition $V_m \cap V \ne \emptyset$, then

$$\|\psi_m - \psi\|_{L^\infty(V_m\cap V)} \le \frac{L(1+L)}{m} + (1+L)\,|c_m|, \tag{5.16}$$

and if $\mathcal{C}_m$ is the transition map of ϕ_m, we have that

$$\|\mathcal{C}_m - \mathcal{C}\|_{L^\infty(U_m)} + \|\mathcal{C}_m^{-1} - \mathcal{C}^{-1}\|_{L^\infty(V_m\cap V)} \le c(n)\,(1+L^2)\Big(\frac{1}{m} + |c_m|\Big). \tag{5.17}$$

Proof From the results of Part (ii) of Proposition 2 and (5.12), there exist $V_m \subset \mathbb{R}^{n-1}$ open bounded, and a function $\psi_m \in C^\infty(V_m)$ such that (5.15) holds. Also, owing to (5.3), we immediately obtain (5.14).

Now we recall that the transition map of ϕ_m is the function $\mathcal{C}_m : U_m \to V_m$ defined as $\mathcal{C}_m x' = \Pi\,T\big(x', \phi_m(x')\big)$, and for all $x' \in U_m$ we have

$$T\big(x', \phi(x')\big) = \big(\mathcal{C}x', \psi(\mathcal{C}x')\big) \quad \text{and} \quad T\big(x', \phi_m(x')\big) = \big(C_m x', \psi_m(\mathcal{C}_m x')\big),$$

so that from (5.10) we infer

$$
\begin{aligned}
|c_m| + \frac{L}{m} &\geq |\phi_m(x') - \phi(x')| \\
&= \big|\big(x', \phi_m(x')\big) - \big(x', \phi(x')\big)\big| = \big|\big(\mathcal{C}_m x', \psi_m(\mathcal{C}_m x')\big) - \big(\mathcal{C}x', \psi(\mathcal{C}x')\big)\big|,
\end{aligned}
$$

for all $x' \in U_m$. In particular

$$
\begin{cases}
|\mathcal{C}_m x' - \mathcal{C}x'| \leq \dfrac{L}{m} + |c_m| \\
\big|\psi_m\big(\mathcal{C}_m x'\big) - \psi\big(\mathcal{C}x'\big)\big| \leq \dfrac{L}{m} + |c_m|
\end{cases}
\quad \text{for all } x' \in U_m \tag{5.18}
$$

The first inequality in (5.18) entails $\mathrm{dist}_{\mathcal{H}}\big(V_m, \mathcal{C}(U_m)\big) \leq \frac{L}{m} + |c_m|$.

On the other hand, by definition of U_m, for any $x' \in U$ we may find $x'_m \in U_m$ such that $|x' - x'_m| \leq \frac{1}{m}$. Since Π and T are 1-Lipschitz continuous, and ϕ is L-Lipschitz continuous, it follows that

$$
|\mathcal{C}x' - \mathcal{C}x'_m| \leq \big|\big(x', \phi(x')\big) - \big(x'_m, \phi(x'_m)\big)\big| \leq \frac{\sqrt{1+L^2}}{m},
$$

which implies $\mathrm{dist}_{\mathcal{H}}\big(\mathcal{C}(U_m), V\big) \leq \frac{\sqrt{1+L^2}}{m}$ since $\mathcal{C}(U) = V$. Hence, by using the triangle inequality we get

$$
\mathrm{dist}_{\mathcal{H}}\big(V_m, V\big) \leq \mathrm{dist}_{\mathcal{H}}\big(V_m, \mathcal{C}(U_m)\big) + \mathrm{dist}_{\mathcal{H}}\big(\mathcal{C}(U_m), V\big) \leq \frac{2\sqrt{1+L^2}}{m} + |c_m|,
$$

that is (5.13).

Next, on assuming that $V_m \cap V \neq \emptyset$, and $\mathcal{C}_m$ being a bijection between U_m and V_m, we may take a point $y' \in V_m \cap V$ such that $y' = \mathcal{C}_m x'$ for some $x' \in U_m$ From (5.18) we find

$$
|\mathcal{C}_m x' - \mathcal{C}x'| = |y' - \mathcal{C}\mathcal{C}_m^{-1} y'| \leq \frac{L}{m} + |c_m|,
$$

and

$$
\big|\psi\big(\mathcal{C}x'\big) - \psi_m\big(\mathcal{C}_m x'\big)\big| = \big|\psi\big(\mathcal{C}\mathcal{C}_m^{-1} y'\big) - \psi_m(y')\big| \leq \frac{L}{m} + |c_m|.
$$

By using these two estimates and the L-Lipschitz continuity of ψ, we obtain

$$
\begin{aligned}
|\psi(y') - \psi_m(y')| &\leq |\psi(y') - \psi(\mathcal{C}\mathcal{C}_m^{-1} y')| + |\psi(\mathcal{C}\mathcal{C}_m^{-1} y') - \psi_m(y')| \\
&\leq L\,|y' - \mathcal{C}\mathcal{C}_m^{-1} y'| + \frac{L}{m} + |c_m| \leq \frac{L(1+L)}{m} + (1+L)\,|c_m| \quad \text{for all } y' \in V_m \cap V,
\end{aligned}
$$

that is (5.16). Finally, by making use of (5.16) and a similar argument as in the proof of (5.18), we obtain (5.17). □

The next proposition shows that if $\phi \in W^{2,q}$, then $\psi \in W^{2,q}$ as well. Namely, graphicality preserves Sobolev second-order regularity for Lipschitz functions.

Proposition 5 *Under the same assumptions of Propositions 3-4, if in addition* $\phi \in W^{2,q}_{loc}(U)$ *for some* $q \in [1, \infty]$, *then* $\psi \in W^{2,q}_{loc}(V)$.

Proof In the following proof, we will make use of Propositions 3-4 with $c_m \equiv 0$.

Fix $U_0 \Subset U$ open, and set $V_0 = \mathcal{C}(U_0)$. Since $\mathrm{dist}_{\mathcal{H}}(V_m, V) \to 0$ due to (5.13), from [9, Proposition 2.2.17] we may find $m_0 > 0$ large enough such that

$$V_0 \Subset V \cap V_m \quad \text{for all } m > m_0.$$

Now let

$$f_m(x) = x_n - M_m(\phi)(x') \quad \text{for } x \in U_m \times \mathbb{R},$$

and set $\widetilde{f}_m(y) \equiv f_m(x)$ for $y = Tx$. Then owing to (5.15), we have that $\widetilde{f}_m\big(y', \psi_m(y')\big) = 0$ for all $y' \in V_m$. By differentiating this expression, we obtain

$$\frac{\partial \psi_m}{\partial y_k'}(y') = -\left(\frac{\partial \widetilde{f}_m}{\partial y_n}\big(y', \psi_m(y')\big)\right)^{-1}\left(\frac{\partial \widetilde{f}_m}{\partial y_k'}\big(y', \psi_m(y')\big)\right), \tag{5.19}$$

and from the chain rule, equation $\mathbf{n} = \mathcal{R}^t e_n$, the definition of $\mathcal{C}_m^{-1}$ and (5.11), we have

$$\begin{aligned} \frac{\partial \widetilde{f}_m}{\partial y_k'}\big(y', \psi_m(y')\big) &= \mathcal{R}_{kn} - \sum_{l=1}^{n-1} \frac{\partial M_m(\phi)}{\partial x_l'}(\mathcal{C}_m^{-1} y')\, \mathcal{R}_{kl} \\ \frac{\partial \widetilde{f}_m}{\partial y_n}\big(y', \psi_m(y')\big) &= \mathcal{R}_{nn} - \sum_{l=1}^{n-1} \frac{\partial M_m(\phi)}{\partial x_l'}(\mathcal{C}_m^{-1} y')\mathcal{R}_{nl} \geq \frac{1}{\sqrt{1+L^2}}, \end{aligned} \tag{5.20}$$

Moreover, thanks to (5.14) and the L-Lipschitz continuity of $M_m(\phi)$, the maps $\mathcal{C}_m$ are uniformly bi-Lipschitz, i.e.

$$\|\nabla \mathcal{C}_m\|_{L^\infty} + \|\nabla \mathcal{C}_m^{-1}\|_{L^\infty} \leq C(n, L).$$

Thanks to this piece of information and (5.17), we may apply Proposition 1 and get

$$\nabla M_m(\phi)(\mathcal{C}_m^{-1} y') \to \nabla \phi(\mathcal{C}^{-1} y') \quad \text{for} \mathcal{H}^{n-1}\text{-a.e.} y' \in V_0 \tag{5.21}$$

By combining (5.19)-(5.21), and by using dominated convergence theorem, we find that $\nabla \psi_m$ converges in $L^p(V_0)$ to some vector-valued function G for all $p \in [1, \infty)$. It then follows from (5.16) and the uniqueness of the distributional limit that $G = \nabla \psi$, hence

$$\nabla \psi_m \to \nabla \psi \quad \mathcal{H}^{n-1}\text{ -a.e. in } V_0 \text{ and in } L^p(V_0). \tag{5.22}$$

Next, we differentiate twice identity $\widetilde{f}_m\big(y', \psi_m(y')\big) = 0$, and for $k, r = 1, \ldots, n-1$ we obtain

$$\begin{aligned} &\frac{\partial^2 \psi_m}{\partial y_k' \partial y_r'}(y') = -\left(\frac{\partial \widetilde{f}}{\partial y_n}\big(y', \psi_m(y')\big)\right)^{-1} \\ &\left\{\frac{\partial^2 \widetilde{f}}{\partial y_k' \partial y_r'}\big(y', \psi_m(y')\big) + \frac{\partial^2 \widetilde{f}}{\partial y_k' \partial y_n}\big(y', \psi_m(y')\big)\, \frac{\partial \psi_m}{\partial y_r'}(y') \right. \\ &\quad + \frac{\partial^2 \widetilde{f}}{\partial y_r' \partial y_n}\big(y', \psi_m(y')\big)\, \frac{\partial \psi_m}{\partial y_k'}(y') \\ &\quad \left. + \frac{\partial^2 \widetilde{f}}{\partial y_n \partial y_n}\big(y', \psi_m(y')\big)\, \frac{\partial \psi_m}{\partial y_k'}(y')\, \frac{\partial \psi_m}{\partial y_r'}(y') \right\}, \end{aligned} \tag{5.23}$$

while from the chain rule and the properties of $\mathcal{C}_m$, we obtain

$$\frac{\partial^2 \widetilde{f}}{\partial y_k' \partial y_r'}\big(y', \psi_m(y')\big) = -\sum_{l,t=1}^{n-1} \frac{\partial^2 M_m(\phi)}{\partial x_l' \partial x_t'}(\mathcal{C}_m^{-1} y')\, \mathcal{R}_{kl}\mathcal{R}_{rt}. \tag{5.24}$$

Then, another application of Proposition 1 entails that

$$\nabla^2 M_m(\phi)(\mathcal{C}_m^{-1} y') \to \nabla^2 \phi(\mathcal{C}^{-1} y') \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y' \in V_0 \text{ and in } L^q(V_0),$$

in the Case $q \in [1, \infty)$. From this, (5.20), (5.22)-(5.24) and by using dominated convegence Theorem 3, we find that $\nabla^2 \psi_m$ converges in $L^q(V_0)$ to some matrix valued function H. Whence $H = \nabla^2 \psi$ due to the uniqueness of the distributional limit, and the proof in the Case $q \in [1, \infty)$ is complete due to the arbitrariness of U_0.

In the Case $q = \infty$, from (5.20), (5.23) and (5.24) we infer that $\{\psi_m\}_m$ is a sequence uniformly bounded in $W^{2,\infty}(V_0)$ with respect to m. Therefore, up to a subsequence, we have that ψ_m weakly-$*$ converge in $W^{2,\infty}(V_0)$ to ψ, thus completing the proof. □

At last, we close this section with the following intrinsic property of $W^{2,q}$ domains.

Corollary 1 *Let Ω be a bounded Lipschitz domains such that $\partial\Omega \in W^{2,q}$ for some $q \in [1, \infty]$. Then any Lipschitz local chart ψ of $\partial\Omega$ is of class $W^{2,q}$.*

Proof From Definition 2, there exists a Lipschitz local chart $\phi \in W^{2,q}$ and an isometry T such that (5.2) holds. The thesis then follows from Proposition 5. □

As a final remark, let us mention that both Proposition 5 and Corollary 1 can be easily extended to the $W^{k,q}$ Case.

6 Proof of Theorem 1

This section is devoted to the proof of Theorem 1, which is divided into a few steps.
From here onward, m_0 and k_0 will denote positive integers, possibly changing from line to line.

6.1 Covering of $\partial\Omega$

By Definition 1, for any $x_0 \in \partial\Omega$, we may find an L_Ω-Lipschitz function $\phi^{x_0} : B'_{R_\Omega} \to \mathbb{R}$, and an isometry T^{x_0} of $\mathbb{R}^n$ such that $T^{x_0} x_0 = 0$, and

$$T^{x_0}\partial\Omega \cap \big(B'_{R_\Omega} \times (-\ell, \ell)\big) = \big\{(y', \phi^{x_0}(y')) : \ y' \in B'_{R_\Omega}\big\},$$
$$T^{x_0}\Omega \cap \big(B'_{R_\Omega} \times (-\ell, \ell)\big) = \big\{(y', y_n) : x' \in B'_{R_\Omega}, \ -\ell < y_n < \phi^{x_0}(y')\big\},$$

where $\ell = R_\Omega(1 + L_\Omega)$. Let us consider the open covering $\{B_{R_\Omega/8}(x_0)\}_{x_0 \in \partial\Omega}$ of $\partial\Omega$.[1] By compactness, we may find a finite sequence of points $\{x^i\}_{i=1}^N \subset \partial\Omega$ such that

$$\partial\Omega \Subset \bigcup_{i=1}^N B_{\frac{R_\Omega}{8}}(x^i), \tag{6.1}$$

[1] Any other open covering is allowed, as long as its sets are strictly contained in the coordinate cylinders $B'_{R_\Omega} \times (-\ell, \ell)$. The open covering here chosen helps simplifying a few computations, especially in the isocapacitary estimate (3.7).

as well as L_Ω-Lipschitz functions ϕ^i and isometries T^i satisfying

$$\begin{aligned} T^i\partial\Omega\cap\left(B'_{R_\Omega}\times(-\ell,\ell)\right)&=\left\{(y',\phi^i(y')):\ y'\in B'_{R_\Omega}\right\},\\ T^i\Omega\cap\left(B'_{R_\Omega}\times(-\ell,\ell)\right)&=\left\{(y',y_n):\ y'\in B'_{R_\Omega},\ -\ell<y_n<\phi^i(y')\right\}. \end{aligned}\tag{6.2}$$

We denote by $\mathcal{R}^i$ the orthogonal matrix of T^i, i.e. T^i can be written as

$$T^ix=\mathcal{R}^i(x-x^i)\quad x\in\mathbb{R}^n.$$

Notice also that the cardinality N of this covering of $\partial\Omega$ may be chosen satisfying

$$N\le c(n)\left(\frac{d_\Omega}{R_\Omega}\right)^n.\tag{6.3}$$

We then set

$$\Omega_t:=\{x\in\Omega:\ \mathrm{dist}(x,\partial\Omega)>t\},$$

so that by (6.1) we have

$$\overline{\Omega}\Subset W:=\bigcup_{i=1}^N B_{\frac{R_\Omega}{8}}(x^i)\cup\Omega_{\frac{R_\Omega}{32}}.\tag{6.4}$$

Starting from this point, we construct a suitable partition of unity: let

$$\eta_i:=\tilde\rho_{\frac{R_\Omega}{32}}*\chi_{B_{\frac{3R_\Omega}{16}}(x^i)}\quad\text{and}\quad\eta_0:=\tilde\rho_{\frac{R_\Omega}{64}}*\chi_{\Omega_{\frac{3R_\Omega}{64}}},$$

where $\tilde\rho_t$ is the standard, radially symmetric convolution kernel on $\mathbb{R}^n$, and χ_A denotes the indicator function of a set A.
Standard properties of convolution ensure that $\eta_i\in C_c^\infty(B_{\frac{R_\Omega}{4}}(x^i))$, $\eta_0\in C_c^\infty(\Omega_{\frac{R_\Omega}{16}})$, $0\le\eta_i\le 1$,

$$\eta_i\ge 1\quad\text{on } B_{\frac{R_\Omega}{8}}(x^i),\quad \eta_0\ge 1\quad\text{on }\Omega_{\frac{R_\Omega}{32}},$$

and

$$|\nabla^k\eta_i|\le\frac{c(n,k)}{R_\Omega^k},\quad\text{for all } k\in\mathbb{N}.$$

Therefore, by defining $\xi_i:\ W\to[0,1]$ as

$$\xi_i:=\frac{\eta_i}{\sqrt{\sum_{j=0}^N\eta_j}},\quad i=0,\dots,N,$$

then we have that $\xi_i\in C_c^\infty(B_{\frac{R_\Omega}{4}}(x^i))$ for $i=1,\dots,N$, $\xi_0\in C_c^\infty(\Omega_{\frac{R_\Omega}{16}})$,

$$\sum_{i=0}^N\xi_i(x)=1\quad\text{for all } x\in W,\tag{6.5}$$

and

$$|\nabla^k\xi_i|\le\frac{c(n,k)}{R_\Omega^k}\quad\text{on } W,\ \text{for all } k\in\mathbb{N}.\tag{6.6}$$

6.2 Boundary defining function

Starting from the partition of unity $\{\xi_i\}_{i=0}^N$, and the local charts $\{\phi^i\}_{i=1}^N$, we can construct the boundary defining function of $\partial\Omega$ as in [11, Proposition 5.43].

For any $\varepsilon \in [0, R_\Omega)$ and $j = 1, \dots, N$, we define the rotated cylinders

$$K_\varepsilon^j := (T^j)^{-1}\big(B'_{R_\Omega-\varepsilon} \times (-\ell, \ell)\big), \tag{6.7}$$

where $\ell = R_\Omega(1 + L_\Omega)$. Let $f^j : K_0^j \to \mathbb{R}$ be the functions defined as

$$f^j(x) := z_n - \phi^j(z'), \quad z = T^j x,$$

and observe that from (6.2) we have

$$\begin{aligned} \{f^j = 0\} &= \partial\Omega \cap K_0^j \\ \{f^j < 0\} &= \Omega \cap K_0^j \end{aligned} \tag{6.8}$$

A boundary defining function of $\overline{\Omega}$ is the function $F : W \to \mathbb{R}$ defined as

$$F(x) := \sum_{j=1}^N f^j(x)\,\xi_j(x) - \xi_0(x), \tag{6.9}$$

where the product $f^j(x)\,\xi_j(x)$ is set equal to zero if $x \notin \operatorname{supp}\xi_j$. Since each f^j is Lipschitz continuous, so is the function F.

Thanks to the properties of $\{\xi_j\}_{j=0}^N$, (6.2) and (6.8), it is easily seen that

$$\Omega = \{x \in W : F(x) < 0\} \quad \text{and} \quad \partial\Omega = \{x \in W : F(x) = 0\}. \tag{6.10}$$

6.3 Regularization and definition of the smooth approximating sets ω_m, Ω_m

For $i = 1, \dots, N$, we can define the smooth functions $\phi_m^i, \widetilde{\phi}_m^i : B'_{R_\Omega - \frac{1}{m}} \to \mathbb{R}$ as

$$\begin{aligned} \phi_m^i &:= M_m(\phi^i) + \|M_m(\phi^i) - \phi^i\|_{L^\infty(B'_{R_\Omega-1/m})} + \frac{L_\Omega}{m} \\ &\text{and} \\ \widetilde{\phi}_m^i &:= M_m(\phi^i) - \|M_m(\phi^i) - \phi^i\|_{L^\infty(B'_{R_\Omega-1/m})} - \frac{L_\Omega}{m}\,. \end{aligned} \tag{6.11}$$

From the results of Proposition 3, we deduce that $\phi_m^i, \widetilde{\phi}_m^i \in C^\infty$ are L_Ω-Lipschitz functions, and

$$\begin{aligned} \frac{L_\Omega}{m} &\le \phi_m^i(y') - \phi^i(y') \le \frac{3\,L_\Omega}{m} \\ \frac{L_\Omega}{m} &\le \phi^i(y') - \widetilde{\phi}_m^i(y') \le \frac{3\,L_\Omega}{m}, \end{aligned} \tag{6.12}$$

for all $y' \in B'_{R_\Omega-1/m}$ and $i = 1, \dots, N$. Taking inspiration from (6.8) and (6.10), we are led to define the functions

$$\begin{aligned} f_m^j(x) &:= z_n - \phi_m^j(z') \\ \tilde{f}_m^j(x) &:= z_n - \widetilde{\phi}_m^j(z'), \quad z = T^j x \in B'_{R_\Omega - \frac{1}{m}} \times (-\ell, \ell), \end{aligned} \tag{6.13}$$

and functions $F_m, \widetilde{F}_m : W \to \mathbb{R}$ defined as

$$\begin{aligned} F_m(x) &:= \sum_{j=1}^{N} f_m^j(x)\,\xi_j(x) - \xi_0(x) \\ \widetilde{F}_m(x) &:= \sum_{j=1}^{N} \tilde{f}_m^j(x)\,\xi_j(x) - \xi_0(x), \end{aligned} \tag{6.14}$$

where the products $f_m^j(x)\,\xi_j(x)$ and $\tilde{f}_m^j(x)\,\xi_j(x)$ have to be interpreted equal to zero when $x \notin \operatorname{supp} \xi_j$.

Clearly, F_m and $\widetilde{F}_m$ are C^∞-smooth functions on W, and since

$$\frac{L_\Omega}{m} \le f^j(x) - f_m^j(x) < \frac{3\,L_\Omega}{m}, \quad \frac{L_\Omega}{m} \le \tilde{f}_m^j(x) - f^j(x) < \frac{3\,L_\Omega}{m} \tag{6.15}$$

for all $x \in K_{1/m}^j$ thanks to (6.12), we then have

$$\frac{L_\Omega}{m} \le F(x) - F_m(x) \le \frac{3\,L_\Omega}{m}, \quad \frac{L_\Omega}{m} \le \widetilde{F}_m(x) - F(x) \le \frac{3\,L_\Omega}{m} \quad \text{for all } x \in W. \tag{6.16}$$

The approximating open sets Ω_m, ω_m are thus defined as follows

$$\Omega_m := \{x \in W : F_m(x) < 0\} \quad \text{and} \quad \omega_m := \{x \in W : \widetilde{F}_m(x) < 0\}, \tag{6.17}$$

with boundaries

$$\partial\Omega_m = \{x \in W : F_m(x) = 0\} \quad \text{and} \quad \partial\omega_m = \{x \in W : \widetilde{F}_m(x) = 0\}. \tag{6.18}$$

In particular, since $F_m(x) < F(x) < \widetilde{F}_m(x)$ for all $x \in W$, owing to (6.10) we have

$$\omega_m \Subset \Omega \Subset \Omega_m \quad \text{for all } m \in \mathbb{N}.$$

We now proceed to prove the remaining properties of Theorem 1 for the outer sets Ω_m. The proofs for the inner sets ω_m are analogous.

6.4 $\partial\Omega_m$, $\partial\omega_m$ are smooth manifolds

Let us show that $\partial\Omega_m$ is a smooth manifold, with local charts $\{\psi_m^i\}_{i=1}^N$ defined on the same coordinate systems as $\{\phi^i\}_{i=1}^N$.

We fix a constant $\varepsilon_0 \in (0, R_\Omega/4)$, and for all $i = 1, \dots, N$ we set

$$F^i(y) = F(x) \quad \text{and} \quad F_m^i(y) = F_m(x) \quad \text{for } y = T^i x, x \in W.$$

Namely $F^i = F \circ (T^i)^{-1}$ and $F_m^i = F_m \circ (T^i)^{-1}$.

Owing to (6.2) we have

$$\begin{gathered} \partial\Omega \cap K_0^i \cap K_0^j = (T^i)^{-1} G_{\phi^i} \cap K_0^j = (T^j)^{-1} G_{\phi^j} \cap K_0^i \\ \text{and} \\ \Omega \cap K_0^j \cap K_0^i = (T^i)^{-1} S_{\phi^i} \cap K_0^j \cap K_0^i = (T^j)^{-1} S_{\phi^j} \cap K_0^i \cap K_0^j, \end{gathered} \tag{6.19}$$

whenever $\partial\Omega \cap K_0^i \cap K_0^j \neq \emptyset$.

This piece of information will allow us to use the transversality property. Specifically, thanks to (6.19) we may apply Propositions 2-3 with functions $\phi = \phi^j$, $\psi = \phi^i$, isometry $T = T^i(T^j)^{-1}$, and defining set

$$U = U^{j,i} = \Pi\Big(G_{\phi^j} \cap T^j K_0^i\Big) \subset B'_{R_\Omega}.$$

Claim 1. There exists $m_0 > 0$ such that, for all $i = 1, \dots, N$, for all $m \geq m_0$ and all $x \in \big\{\frac{-3L_\Omega}{m_0} \leq F \leq \frac{3L_\Omega}{m_0}\big\} \cap K_{\varepsilon_0}^i$, we have

$$\frac{\partial F_m^i}{\partial y_n}(y) \geq \frac{1}{2\sqrt{1+L_\Omega^2}}, \quad \text{for all } y = T^i x \in B'_{R_\Omega-\varepsilon_0} \times (-\ell, \ell). \tag{6.20}$$

Suppose by contradiction this is false; then for every $k \in \mathbb{N}$, we may find $m_k \geq k$ and a sequence $x^k \in \big\{-\frac{3L_\Omega}{k} \leq F \leq \frac{3L_\Omega}{k}\big\}$ such that $y^k = T^i x^k \in B'_{R_\Omega-\varepsilon_0} \times (-\ell, \ell)$ and

$$\frac{\partial F_{m_k}^i}{\partial y_n}(y^k) < \frac{1}{2\sqrt{1+L_\Omega^2}}, \quad \text{for all } k \in \mathbb{N} \tag{6.21}$$

By compactness, we may extract a subsequence, still labeled as x^k, such that $x^k \to x^0$, and in particular $x^0 \in \overline{K_0^i}$ and $F(x^0) = 0$, hence $x^0 \in \partial\Omega \cap \overline{K_0^i}$ due to (6.10).

Then, by the chain rule we have

$$\frac{\partial f_m^i}{\partial y_n}(x) = 1 \quad \text{and} \quad \frac{\partial f_m^j}{\partial y_n}(x) = \big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{nn} - \sum_{s=1}^{n-1} \frac{\partial \phi_m^j}{\partial z'_s}(z')\,\big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{sn}, \tag{6.22}$$

if $x \in \operatorname{supp} \xi_j$, where $z' = \Pi\, T^j x$. We now distinguish two cases:
(i) $j \in \{1, \dots, N\}$ is such that $x^0 \notin \operatorname{supp} \xi_j$. Then $\operatorname{dist}\big(x^0, \operatorname{supp} \xi_j\big) > 0$, hence $x^k \notin \operatorname{supp} \xi_j$ for all $k \geq k_0$ large enough.
(ii) $j \in \{1, \dots, N\}$ is such that $x^0 \in \operatorname{supp} \xi_j$. In this case, it follows that $x^0 \in \partial\Omega \cap K_0^i \cap B_{\frac{R_\Omega}{4}}(x^j)$, so that from (6.19) we have $T^j x^0 \in G_{\phi^j} \cap B_{\frac{R_\Omega}{4}} \cap T^j \overline{K_{\varepsilon_0}^i}$. By setting $(z^k)' = \Pi\, T^j x^k$, we thus have

$$B'_{\frac{1}{m_k}}\big((z^k)'\big) \Subset \Pi\Big(G_{\phi^j} \cap T^j K_0^i\Big),$$

for all $k \geq k_0$ large enough. Recalling the remarks after (6.19), by applying Proposition 3, and in particular the transversality property (5.11) in (6.22), we infer

$$\frac{\partial f_{m_k}^j}{\partial y_n}\big(x^k\big) = \big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{nn} - \sum_{s=1}^{n-1} \frac{\partial \phi_{m_k}^j}{\partial z'_s}\big((z^k)'\big)\,\big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{sn} \geq \frac{1}{\sqrt{1+L_\Omega^2}},$$

provided $k \geq k_0$ is large enough.

In both cases, we have found that

$$\frac{\partial f_{m_k}^j}{\partial y_n}(x^k)\,\xi_j\big(x^k\big) \geq \frac{\xi_j(x^k)}{\sqrt{1+L_\Omega^2}} \quad \text{for all } j = 1, \dots, N \text{ and } k \geq k_0. \tag{6.23}$$

Also, owing to (6.15) and (6.8) we have

$$|f^j_{m_k}(x^k)|\left|\frac{\partial\xi_j(x^k)}{\partial y_n}\right| \le |f^j_{m_k}(x^k)-f^j(x^k)|\,|\nabla\xi_j(x^k)|+|f^j(x^k)|\,|\nabla\xi_j(x^k)|$$
$$\le \frac{1}{m_k}+|f^j(x^k)|\,|\nabla\xi_j(x^k)| \xrightarrow{k\to\infty} |f^j(x^0)|\,|\nabla\xi_j(x^0)|=0,$$

and $|\nabla\xi_0(x^k)|\to|\nabla\xi_0(x^0)|=0$ since $x^0\in\partial\Omega$. By coupling this piece of information with (6.5), (6.21) and (6.23), we finally obtain

$$\frac{1}{2\sqrt{1+L_\Omega^2}} > \frac{\partial F^i_{m_k}}{\partial y_n}(y^k)=\sum_{j=1}^N \frac{\partial f^j_{m_k}}{\partial y_n}(x^k)\,\xi_j(x^k)+\sum_{j=1}^N f^j_{m_k}(x^k)\,\frac{\partial\xi_j}{\partial y_n}(x^k)-\frac{\partial\xi_0}{\partial y_n}(x^k)$$
$$\ge \sum_{j=1}^N \frac{\xi_j(x^k)}{\sqrt{1+L_\Omega^2}}+\sum_{j=1}^N f^j_{m_k}(x^k)\,\frac{\partial\xi_j}{\partial y_n}(x^k)-\frac{\partial\xi_0}{\partial y_n}(x^k)$$
$$\xrightarrow{k\to\infty} \sum_{j=1}^N \frac{\xi_j(x^0)}{\sqrt{1+L_\Omega^2}}=\frac{1}{\sqrt{1+L_\Omega^2}},$$

which is a contradiction, and thus (6.20) holds true.

Claim 2. There exists $m_0>0$ such that $\forall y'\in B'_{R_\Omega-\varepsilon_0}$, $\forall m\ge m_0$, $\exists y_n\in(-\ell,\ell)$ with $y=(y',y_n)=T^ix\in T^iW$ satisfying $F^i_m(y)\ge 0$.

Again, assume by contradiction this is false. Then for all $k\in\mathbb{N}$, we may find sequences $m_k\ge k$ and $(y^k)'\in B'_{R_\Omega-\varepsilon_0}$ such that

$$F^i_{m_k}\big((y^k)',y_n\big)<0 \quad \text{for all } y_n\in(-\ell,\ell) \text{ such that } \big((y^k)',y_n\big)\in T^iW. \tag{6.24}$$

By compactness, we may find a subsequence, still labeled as $(y^k)'$, satisfying $(y^k)'\to(y^0)'\in\overline{B}'_{R_\Omega-\varepsilon_0}$. Fix $w_n\in(-\ell,\ell)$ such that $\big((y^0)',w_n\big)\in T^iW$, and let $\{w_n^k\}_{k\in\mathbb{N}}\subset\mathbb{R}$ be a sequence satisfying $w_n^k\xrightarrow{k\to\infty}w_n$. Then $\big((y^k)',w_n^k\big)\to\big((y^0)',w_n\big)$, so that $\big((y^k)',w_n^k\big)\in T^iW$ for $k\ge k_0$ large enough being W open, and from (6.24) we have $F^i_{m_k}\big((y^k)',w_n^k\big)<0$. By using (6.16) and the Lipschitz continuity of F, it is readily shown that

$$\lim_{k\to\infty}F^i_{m_k}\big((y^k)',w_n^k\big)=F^i\big((y^0)',w_n\big),$$

whence $F^i\big((y^0)',w_n\big)\le 0$ for all w_n as above, but this contradicts the fact that $F^i\big((y^0)',w_n\big)>0$ whenever $w_n>\phi^i\big((y^0)'\big)$ due to (6.10), hence Claim 2 is proven.

Now let $y'\in B'_{R_\Omega-\varepsilon_0}$; by (6.16) and since $F^i\big(y',\phi^i(y')\big)=0$, we have $F^i_m\big(y',\phi^i(y')\big)<0$. Thus, owing to Claim 2 we may find y_n such that $F^i_m(y',y_n)=0$.

The monotonicity property (6.20) of Claim 1, and the fact that $\partial\Omega_m=\{F_m=0\}\subset\{\frac{L_\Omega}{m}\le F\le\frac{3L_\Omega}{m}\}$ due to (6.16) ensure that such point y_n is unique for all $y'\in B'_{R_\Omega-\varepsilon_0}$. This entails the existence of a function $\psi^i_m: B'_{R_\Omega-\varepsilon_0}\to\mathbb{R}$ such that

$$F^i_m\big(y',\psi^i_m(y')\big)=0 \quad \text{for all } y'\in B'_{R_\Omega-\varepsilon_0}. \tag{6.25}$$

Owing to (6.10) and (6.16), we also have that $\psi^i_m(y')>\phi^i(y')$ for all $y'\in B'_{R_\Omega-\varepsilon_0}$.

Then, since F^i_m are C^∞-smooth, thanks to (6.20) and (6.25), we may repeat the proof of the implicit function theorem in order to obtain $\psi^i_m\in C^\infty\big(B'_{R_\Omega-\varepsilon_0}\big)$.

Moreover, via a compactness argument as in Claim 1-2 and (6.1), one can prove that

$$\begin{aligned}&\left\{-\frac{3L_\Omega}{m}\le F\le\frac{3L_\Omega}{m}\right\}\subset\bigcup_{i=1}^N B_{\frac{R}{8}}(x^i)\\&\left\{-\frac{3L_\Omega}{m}\le F\le\frac{3L_\Omega}{m}\right\}\cap\operatorname{supp}\xi_0=\emptyset,\quad\text{for all } m>m_0,\end{aligned}\tag{6.26}$$

so that, in particular, the cylinders $\left\{K^i_{2\varepsilon_0}\right\}_{i=1}^N$ are an open cover of $\partial\Omega_m$, and $\partial\Omega_m\cap\operatorname{supp}\xi_0=\emptyset$ provided $m>m_0$ is large enough

We have thus proven that $\partial\Omega_m$ is a C^∞-smooth manifold for $m>m_0$, with local boundary charts $\{\psi^i_m\}_{i=1}^N$ defined on the same coordinate cylinders as $\{\phi^i\}_{i=1}^N$, that is (see Fig. 1 above).

$$\begin{aligned}&T^i\partial\Omega_m\cap\left(B'_{R_\Omega-\varepsilon_0}\times(-\ell,\ell)\right)=\left\{(y',\psi^i_m(y')):\ y'\in B'_{R_\Omega-\varepsilon_0}\right\},\\&T^i\Omega_m\cap\left(B'_{R_\Omega-\varepsilon_0}\times(-\ell,\ell)\right)=\left\{(y',y_n):\ y'\in B'_{R_\Omega-\varepsilon_0},\ -\ell<y_n<\psi^i_m(y')\right\}.\end{aligned}\tag{6.27}$$

6.5 Approximation properties

First, we show that there exists $m_0>0$ such that

$$\|\psi^i_m-\phi^i\|_{L^\infty(B'_{R_\Omega-2\varepsilon_0})}\le\frac{6\,L_\Omega\sqrt{1+L_\Omega^2}}{m}\quad\text{for all } m>m_0.\tag{6.28}$$

Assume by contradiction this is false; then we may find sequences $m_k\uparrow\infty$ and $(y^k)'\in B'_{R_\Omega-2\varepsilon_0}$ such that

$$\psi^i_{m_k}\left((y^k)'\right)-\phi^i\left((y^k)'\right)>\frac{6\,L_\Omega\sqrt{1+L_\Omega^2}}{m_k}\tag{6.29}$$

Up to a subsequence, we have $(y^k)'\to(y^0)'\in\overline{B}'_{R_\Omega-2\varepsilon_0}$, and $\psi^i_{m_k}\left((y^k)'\right)\to\ell_0\in\mathbb{R}$. Furthermore, since $\left((y^k)',\psi^i_m\left((y^k)'\right)\right)\in\{F^i_{m_k}=0\}\subset T^i\{\frac{L_\Omega}{m_k}\le F\le\frac{3L_\Omega}{m_k}\}$, we readily infer that $F^i\left((y^0)',\ell_0\right)=0$, whence $\ell_0=\phi^i\left((y')^0\right)$ due to (6.10) and (6.2). By continuity we also have $\phi^i\left((y^k)'\right)\to\phi^i\left((y^0)'\right)$, which implies that

$$\psi^i_{m_k}\left((y^k)'\right)-\phi^i\left((y^k)'\right)\xrightarrow{k\to\infty}0.$$

Then, for all $t\in[0,1]$, we have

$$\begin{aligned}&\left|F^i\left((y^k)',t\,\psi^i_{m_k}\left((y^k)'\right)+(1-t)\phi^i\left((y^k)'\right)\right)-F^i\left((y^k)',\phi^i\left((y^k)'\right)\right)\right|\\&\qquad\le L_{F^i}\,t\,|\psi^i_{m_k}\left((y^k)'\right)-\phi^i\left((y^k)'\right)|\xrightarrow{k\to\infty}0,\end{aligned}$$

where L_{F^i} denotes the Lipschitz constant of F^i– recall that $F^i=F\circ(T^i)^{-1}$, and F is Lipschitz continuous. This implies that for all $k\ge k_0$ large enough, the line segment

$$\{(y^k)'\}\times\left[\phi^i\left((y^k)'\right),\psi^i_{m_k}\left((y^k)'\right)\right]\subset T^i\left\{-\frac{3\,L_\Omega}{m_0}\le F\le\frac{3\,L_\Omega}{m_0}\right\}.$$

Therefore, by using (6.2), (6.10) (6.16), (6.20) and (6.29), we obtain

$$\begin{aligned}
\frac{3\,L_\Omega}{m_k} &\geq F^i\Big((y^k)', \phi^i\big((y^k)'\big)\Big) - F^i_{m_k}\Big((y^k)', \phi^i\big((y^k)'\big)\Big) = -F^i_{m_k}\Big((y^k)', \phi^i\big((y^k)'\big)\Big) \\
&= F^i_{m_k}\Big((y^k)', \psi^i_{m_k}\big((y^k)'\big)\Big) - F^i_{m_k}\Big((y')^k, \phi^i\big((y')^k\big)\Big) \\
&= \Big(\int_0^1 \frac{\partial F^i_{m_k}}{\partial y_n}\Big((y^k)', t\,\psi^i_{m_k}\big((y^k)'\big) + (1-t)\,\phi^i\big((y^k)'\big)\Big)\,dt\Big)\Big[\psi^i_{m_k}\big((y^k)'\big) - \phi^i\big((y^k)'\big)\Big] \\
&> \frac{1}{2\sqrt{1+L^2_\Omega}}\,\frac{6\,L_\Omega\sqrt{1+L^2}}{m_k} = \frac{3\,L_\Omega}{m_k}, \quad \text{for all } k \geq k_0 \text{ large enough,}
\end{aligned}$$

which is a contradiction, hence (6.28) holds true.

Now, recalling that $\{K^j_{2\varepsilon_0}\}_{j=1}^N$ is an open cover of $\partial\Omega$ and $\partial\Omega_m$, from (6.2), (6.27) and (6.28), one can easily obtain that

$$\mathrm{dist}_{\mathcal{H}}\big(\partial\Omega_m, \partial\Omega\big) \leq \frac{6\,L_\Omega\sqrt{1+L^2_\Omega}}{m}.$$

This convergence property in the sense of Hausdorff immediately implies that $d_{\Omega_m} \leq c(n)\,d_\Omega$, and $\lim_{m\to\infty} |\Omega_m \setminus \Omega| = 0$—see for instance [9, Proposition 2.2.23]—and thus (3.1), (3.2) and (3.3) are proven.

Let us now introduce the transition maps related to the local charts of $\partial\Omega$ and $\partial\Omega_m$.

First of all, note that thanks to (6.27), we have

$$\begin{gathered}
\partial\Omega_m \cap K^i_{\varepsilon_0} \cap K^j_{\varepsilon_0} = (T^i)^{-1} G_{\psi^i_m} \cap K^j_{\varepsilon_0} = (T^j)^{-1} G_{\psi^j_m} \cap K^i_{\varepsilon_0} \\
\text{and} \\
\Omega_m \cap K^j_{\varepsilon_0} \cap K^i_{\varepsilon_0} = (T^i)^{-1} S_{\psi^i_m} \cap K^j_{\varepsilon_0} \cap K^i_{\varepsilon_0} = (T^j)^{-1} S_{\psi^j_m} \cap K^i_{\varepsilon_0} \cap K^j_{\varepsilon_0},
\end{gathered} \tag{6.30}$$

whenever $\partial\Omega_m \cap K^i_{\varepsilon_0} \cap K^j_{\varepsilon_0} \neq \emptyset$.

For all $i \in \{1, \ldots, N\}$, we define the set of indexes

$$\mathcal{I}_i := \big\{ j \in \{1, \ldots, N\} : \partial\Omega \cap K^i_{2\varepsilon_0} \cap K^j_{2\varepsilon_0} \neq \emptyset \big\}. \tag{6.31}$$

If $j \in \mathcal{I}_i$, then owing to (6.2) there exists $y' \in B'_{R_\Omega - 2\varepsilon_0}$ such that $(T^i)^{-1}\big(y', \phi^i(y')\big) \in \partial\Omega \cap K^j_{2\varepsilon_0}$. Since ϕ^j is L_Ω-Lipschitz continuous and $\phi^j(0') = 0$, we have $|\phi^j(z')| \leq L_\Omega\,|z'|$, so it follows from (6.19), (6.27) and (6.28) that $(T^i)^{-1}\big(y', \psi^i_m(y')\big) \in \partial\Omega_m \cap K^i_{\varepsilon_0} \cap K^j_{\varepsilon_0}$ for all $m \geq m_0$ large enough.

Henceforth, for all $j \in \mathcal{I}_i$, (6.19) and (6.30) allow us to define the transition maps $\mathcal{C}^{i,j}$, $\mathcal{C}^{i,j}_m$ from ϕ^i to ϕ^j and from ψ^i_m to ψ^j_m respectively, i.e.

$$\begin{aligned}
\mathcal{C}^{i,j}\,y' &= \Pi\,T^j(T^i)^{-1}\big(y', \phi^i(y')\big) \\
\mathcal{C}^{i,j}_m\,y' &= \Pi\,T^j(T^i)^{-1}\big(y', \psi^i_m(y')\big),
\end{aligned} \tag{6.32}$$

which are defined on the open sets

$$U^{i,j} = \Pi\left(G_{\phi^i} \cap T^i\,K^j_0\right) \quad \text{and} \quad U^{i,j}_m = \Pi\left(G_{\psi^i_m} \cap T^i\,K^j_{\varepsilon_0}\right).$$

In particular, by their definitions and the arguments of Sect. 5, we may write

$$x = (T^i)^{-1}\big(y', \phi^i(y')\big) = (T^j)^{-1}\big(\mathcal{C}^{i,j} y', \phi^j(\mathcal{C}^{i,j} y')\big) \quad \text{for } x \in \partial\Omega \cap K_0^i \cap K_0^j$$
$$x^m = (T^i)^{-1}\big(y', \psi_m^i(y')\big) = (T^j)^{-1}\big(\mathcal{C}_m^{i,j} y', \psi_m^j(\mathcal{C}_m^{i,j} y')\big) \quad \text{for } x^m \in \partial\Omega_m \cap K_{\varepsilon_0}^i \cap K_{\varepsilon_0}^j. \tag{6.33}$$

and their inverse functions are $(\mathcal{C}^{i,j})^{-1} = \mathcal{C}^{j,i}$ and $(\mathcal{C}_m^{i,j})^{-1} = \mathcal{C}_m^{j,i}$. Observe also that $\mathcal{C}^{i,i} = \mathcal{C}_m^{i,i} = \mathrm{Id}$.

Furthermore, since $\operatorname{supp} \xi_j \Subset B_{R_\Omega/4}(x^j) \Subset K_{2\varepsilon_0}^j$, it follows from the definition of $\mathcal{I}_i$ and (6.28) that

$$\nabla^k \xi_j\big((T^i)^{-1}(y', \phi^i(y'))\big) = \nabla^k \xi_j\big((T^i)^{-1}(y', \psi_m^i(y'))\big) = 0 \quad \text{if } j \notin \mathcal{I}_i, \tag{6.34}$$

for all $k \in \mathbb{N}$, for all $y' \in B'_{R_\Omega - \varepsilon_0}$, and all $m \geq m_0$.

We now claim that for all $j \in \mathcal{I}_i$, there exists an open set $V^{i,j} \subset B'_{R_\Omega - 2\varepsilon_0}$ for which we have

$$\xi_j\big((T^i)^{-1}(y', \phi^i(y'))\big) = \xi_j\big((T^i)^{-1}(y', \psi_m^i(y'))\big) = 0 \quad \text{if } y' \notin V^{i,j}, \tag{6.35}$$

and such that $V^{i,j} \subset U^{i,j} \cap U_m^{i,j}$ for all $m > m_0$. This in particular implies that both $\mathcal{C}^{i,j}$ and $\mathcal{C}_m^{i,j}$ are defined on $V^{i,j}$.

To this end, let

$$V^{i,j} := \Pi\Big(G_{\phi^i} \cap T^i K_{2\varepsilon_0}^j\Big) \cap B'_{R_\Omega - 2\varepsilon_0}.$$

Then, owing to (6.28) it is immediate to verify that

$$B'_{R_\Omega - 2\varepsilon_0} \cap \bigg(\Pi\Big(G_{\phi^i} \cap T^i B_{R_\Omega/4}(x^j)\Big) \cup \Pi\Big(G_{\psi_m^i} \cap T^i B_{R_\Omega/4}(x^j)\Big)\bigg) \Subset V^{i,j}, \tag{6.36}$$

whenever $m > m_0$ is large enough, and thus (6.35) is satisfied by our choice of set $V^{i,j}$.

Clearly $V^{i,j} \subset U^{i,j}$, so we are left to verify that $V^{i,j} \subset U_m^{i,j}$. To this end, let $y' \in V^{i,j}$; then by (6.30) and (6.33) we may write

$$\begin{aligned} T^j (T^i)^{-1}(y', \phi^i(y')) = (\mathcal{C}^{i,j} y', \phi^j(\mathcal{C}^{i,j} y')) &\in B'_{R_\Omega - 2\varepsilon_0} \\ &\times\big(-L_\Omega(R_\Omega - 2\varepsilon_0), L_\Omega(R_\Omega - 2\varepsilon_0)\big), \end{aligned}$$

where in the latter inclusion we made use of the inequality $|\phi^j(z')| \leq L_\Omega |z'|$. Therefore, thanks to (6.28), for $m > m_0$ we have $(T^i)^{-1}(y', \psi_m^i(y')) \in \partial\Omega_m \cap K_{\varepsilon_0}^i \cap K_{2\varepsilon_0}^j$, hence $y' \in U_m^{i,j}$ by (6.30) and the definition of $U_m^{i,j}$, so the claim is proven.

We now prove that

$$\bigcup_{j \in \mathcal{I}_i} V^{i,j} = B'_{R_\Omega - 2\varepsilon_0}. \tag{6.37}$$

Since $\{K_{2\varepsilon_0}^j\}_{j=1}^N$ is a cover of $\partial\Omega$, from (6.2) and by the definition of $\mathcal{I}_i$ (6.31), we have that $\{T^i K_{2\varepsilon_0}^j\}_{j \in \mathcal{I}_i}$ is an open cover of $G_{\phi^i} \cap K_{2\varepsilon_0}^i$. We now exploit that the projection map Π is a homeomorphism from G_{ϕ^i} (with the induced topology) to B'_{R_Ω}. More precisely, we have that

$$\Pi : G_{\phi^i} \cap K_{2\varepsilon_0}^i \to B'_{R_{\Omega - 2\varepsilon_0}}$$

is a homeomorphism by definition of $K^i_{2\varepsilon_0}$ and (6.2). From these two observations, it follows that

$$\bigcup_{j\in\mathcal{I}_i} V^{i,j} = \bigcup_{j\in\mathcal{I}_i} \Pi\Big(G_{\phi^i}\cap T^i K^j_{2\varepsilon_0}\Big)\cap B'_{R_\Omega-2\varepsilon_0} = \Pi\Big(\bigcup_{j\in\mathcal{I}_i}\big(G_{\phi^i}\cap K^i_{2\varepsilon_0}\cap T^i K^j_{2\varepsilon_0}\big)\Big)$$
$$= \Pi\Big(G_{\phi^i}\cap K^i_{2\varepsilon_0}\Big) = B'_{R_{\Omega-2\varepsilon_0}},$$

that is (6.37).

Then, owing to (6.28) and by proceeding as in the derivation of (5.18), we obtain

$$\|\mathcal{C}^{i,j}_m - \mathcal{C}^{i,j}\|_{L^\infty(V^{i,j})} \le \frac{6\,L_\Omega\sqrt{1+L^2_\Omega}}{m} \quad \text{for all } m > m_0. \tag{6.38}$$

Our next goal is to obtain estimates on $\nabla\psi^i_m$. To this end, we differentiate equation $F^i_m(y',\psi^i_m(y')) = 0$ with respect to y'_k, for $k = 1,\dots,n-1$, and recalling (6.34) we find

$$\frac{\partial\psi^i_m}{\partial y'_k}(y') = -\left(\frac{\partial F^i_m(y',\psi^i_m(y'))}{\partial y_n}\right)^{-1}\sum_{j\in\mathcal{I}_i}\left\{\frac{\partial f^j_m(x^m)}{\partial y'_k}\,\xi_j(x^m) + f^j_m(x^m)\,\frac{\partial\xi_j(x^m)}{\partial y'_k}\right\}, \tag{6.39}$$

where $x^m = (T^i)^{-1}\big(y',\psi^i_m(y')\big)$, $y'\in B'_{R_{\Omega-2\varepsilon_0}}$.

For all $l = 1,\dots,n$, by using the chain rule and recalling the definition of $\mathcal{C}^{i,j}_m$, we find

$$\begin{aligned}&\frac{\partial f^i_m}{\partial y'_l}(x^m) = -\frac{\partial\phi^i_m}{\partial y'_l}(y') \quad\text{and}\quad \frac{\partial f^i_m}{\partial y_n}(x^m) = 1\\ &\frac{\partial f^j_m}{\partial y_l}(x^m) = \big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{nl} - \sum_{r=1}^{n-1}\frac{\partial\phi^j_m}{\partial z'_r}(\mathcal{C}^{i,j}_m y')\,\big(\mathcal{R}^j(\mathcal{R}^i)^t\big)_{rl},\end{aligned} \tag{6.40}$$

for all $j\in\mathcal{I}_i$ such that $x^m\in\operatorname{supp}\xi_j$. Since ϕ^j_m are L_Ω-Lipschitz continuous, from (6.40) it follows that

$$\sum_{l=1}^{n}\left|\frac{\partial f^j_m(x^m)}{\partial y_l}\right| \le c(n)(1+L_\Omega), \quad \text{for all } j\in\mathcal{I}_i. \tag{6.41}$$

Moreover, from (6.15), (6.28) and (6.8), we find that $f^j_m(x^m)\,|\nabla\xi_j(x^m)| \xrightarrow{m\to\infty} f^j(x^0)\,|\nabla\xi_j(x^0)| = 0$, where $x^0 = (T^i)^{-1}\big(y',\phi^i(y')\big)\in\partial\Omega$.

By making use of this piece of information, (6.41) and (6.20), from (6.39) we finally obtain the gradient estimate

$$|\nabla\psi^i_m(y')| \le c(n)\big(1+L^2_\Omega\big), \quad \text{for all } y'\in B'_{R_{\Omega-2\varepsilon_0}}, \tag{6.42}$$

for all $i = 1,\dots,N$ and $m > m_0$ large enough. In particular, owing to (6.28), (6.27) and (6.42), it is readily seen that Ω_m are $\mathcal{L}_{\Omega_m}$-Lipschitz domains, with

$$L_{\Omega_m} \le c(n)\big(1+L^2_\Omega\big) \quad\text{and}\quad R_{\Omega_m} \ge \frac{R_\Omega}{c(n)\,\big(1+L^2_\Omega\big)},$$

and (3.4) is proven.

Next, the definition of $\mathcal{C}^{i,j}$ and $\mathcal{C}_m^{i,j}$, (6.42) and the L_Ω-Lipschitz continuity of ϕ^i imply

$$\sup_{i=1,\dots,N}\ \sup_{j\in\mathcal{I}_i}\left\{\|\nabla\mathcal{C}^{i,j}\|_{L^\infty}+\|\nabla\mathcal{C}_m^{i,j}\|_{L^\infty}\right\}\le c(n)(1+L_\Omega^2)\quad\text{for all } m>m_0,\tag{6.43}$$

and in particular $\mathcal{C}^{i,j}$ and $\mathcal{C}_m^{i,j}$ are uniformly bi-Lipschitz transformations.

Hence, thanks to (6.38) and (6.43), we are in the position to apply Proposition 1 and get

$$\frac{\partial\phi_m^j}{\partial z_r'}(\mathcal{C}_m^{i,j}y')\xrightarrow{m\to\infty}\frac{\partial\phi^j}{\partial z_r'}(\mathcal{C}^{i,j}y')\quad\text{for } \mathcal{H}^{n-1}\text{-a.e. } y'\in V^{i,j}.\tag{6.44}$$

From this, (6.20), (6.35), (6.37), (6.40) and identity (6.39) we find

$$\nabla\psi_m^i(y')\xrightarrow{m\to\infty}G(y')\quad\text{for } \mathcal{H}^{n-1}\text{-a.e. } y'\in B'_{R_\Omega-2\varepsilon_0},$$

where G is a bounded vector valued function which can be explictly written. From (6.42) and on applying dominated convergence theorem, we get that $\nabla\psi_m^i\xrightarrow{m\to\infty}G$ in $L^p(B'_{R-2\varepsilon_0})$ for all $p\in[1,\infty)$. On the other hand, (6.28) and the uniqueness of the distributional limit imply that $G=\nabla\phi^i$, hence (3.5) is proven.

6.6 Curvature convergence

Assume now that $\partial\Omega\in W^{2,q}$ for some $q\in[1,\infty)$. Then the local charts $\phi^i\in W^{2,q}(B'_{R_\Omega})$.

Let $y'\in B'_{R_\Omega-2\varepsilon_0}$, and differentiate twice the identity $F_m^i(y',\psi_m^i(y'))=0$ with respect to $y_k'\,y_l'$ for $k,l=1,\dots n-1$, as to find

$$\begin{aligned}\frac{\partial^2\psi_m^i}{\partial y_k'\partial y_l'}(y')=&-\left(\frac{\partial F_m^i(y',\psi_m^i(y'))}{\partial y_n}\right)^{-1}\\&\left\{\frac{\partial^2F_m^i(y',\psi_m^i(y'))}{\partial y_k'\partial y_l'}+\frac{\partial^2F_m^i(y',\psi_m^i(y'))}{\partial y_l'\partial y_n}\frac{\partial\psi_m^i}{\partial y_k'}(y')+\right.\\&+\frac{\partial^2F_m^i(y',\psi_m^i(y'))}{\partial y_k'\partial y_n}\frac{\partial\psi_m^i}{\partial y_l'}(y')+\\&\left.+\frac{\partial^2F_m^i(y',\psi_m^i(y'))}{\partial y_n\partial y_n}\frac{\partial\psi_m^i}{\partial y_k'}(y')\frac{\partial\psi_m^i}{\partial y_l'}(y')\right\}.\end{aligned}\tag{6.45}$$

By differentiating twice $F_m^i=F_m\circ(T^i)^{-1}$, and recalling Definition (6.14), for all $l,r=1,\dots n$ we get

$$\begin{aligned}\frac{\partial^2F_m^i}{\partial y_r\partial y_l}(y',\psi_m^i(y'))=\sum_{j\in\mathcal{I}_i}&\left\{\frac{\partial^2f_m^j}{\partial y_r\partial y_l}(x^m)\,\xi_j(x^m)+\frac{\partial f_m^j}{\partial y_r}(x^m)\frac{\partial\xi_j}{\partial y_l}(x^m)\right.\\&\left.+\frac{\partial f_m^j}{\partial y_l}(x^m)\frac{\partial\xi_j}{\partial y_r}(x^m)+f_m^j(x^m)\frac{\partial^2\xi_j}{\partial y_r\partial y_l}(x^m)\right\},\end{aligned}\tag{6.46}$$

where $x^m=(T^i)^{-1}(y',\psi_m^i(y'))$. The above summation is only over the set of indices $\mathcal{I}_i$, since $\nabla^k\xi_j(x^m)=0$ owing to (6.35).

We also have

$$\frac{\partial^2f_m^j}{\partial y_r\partial y_l}(x^m)=-\sum_{s,t=1}^{n-1}\frac{\partial^2\phi_m^j}{\partial z_s'\partial z_t'}(\mathcal{C}_m^{i,j}y')\left(\mathcal{R}^j(\mathcal{R}^i)^t\right)_{sr}\left(\mathcal{R}^j(\mathcal{R}^i)^t\right)_{tl}\tag{6.47}$$

for all $j \in \mathcal{I}_i$ such that $x^m \in \operatorname{supp} \xi_j$.

Thanks to (6.15), (6.28) and (6.8), we readily find that $f_m^j(x^m)\,|\nabla \xi_j(x^m)| \to 0$ and $f_m^j(x^m)\,|\nabla^2 \xi_j(x^m)| \to 0$. From this, and by using (6.6), (6.20), (6.41), (6.42) and (6.45)-(6.47), we obtain

$$|\nabla^2 \psi_m^i(y')| \le c(n)(1+L_\Omega^5) \sum_{j\in\mathcal{I}_i} \left\{ |\nabla^2 \phi_m^j|(\mathcal{C}_m^{i,j} y')\, \xi_j\big((T^i)^{-1}(y', \psi_m^i(y')\big) + \frac{(1+L_\Omega)}{R_\Omega} \right\}, \tag{6.48}$$

for all $y' \in B'_{R_\Omega - 2\varepsilon_0}$, provided $m > m_0$ is large enough.

Then again, thanks to (6.38) and(6.43), we may apply Proposition 1 and infer

$$\nabla^2 \phi_m^j(\mathcal{C}_m^{i,j} y') \to \nabla^2 \phi^j(\mathcal{C}^{i,j} y') \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } y' \in V^{i,j} \text{ and in } L^q(V^{i,j}). \tag{6.49}$$

Finally, recalling (6.35) and (6.37), we may exploit the properties (6.20), (6.28), (6.40), (6.44), (6.45)-(6.49) in order to apply dominated convergence Theorem 3 with dominating functions

$$g_m = c(n, L_\Omega, R_\Omega) \sum_{j\in\mathcal{I}_i} \left\{ |\nabla^2 \phi_m^j|(\mathcal{C}_m^{i,j} y')\, \xi_j\big((T^i)^{-1}(y', \psi_m^i(y')\big) + 1 \right\}$$

$$g = c(n, L_\Omega, R_\Omega) \sum_{j\in\mathcal{I}_i} \left\{ |\nabla^2 \phi^j|(\mathcal{C}^{i,j} y')\, \xi_j\big((T^i)^{-1}(y', \phi^i(y')\big) + 1 \right\}$$

This entails

$$\nabla^2 \psi_m^i \to M, \quad \mathcal{H}^{n-1}\text{-a.e. on } B'_{R_\Omega - 2\varepsilon_0} \text{ and in } L^q(B'_{R_\Omega - 2\varepsilon_0}), \tag{6.50}$$

for some matrix valued function M, which can be explictly written in terms of ϕ^j, $\nabla\phi^j$, $\nabla^2\phi^j$ and ξ_j. On the other hand, (6.28) and the uniqueness of the distributional limit imply that $M = \nabla^2 \phi^i$, hence (3.6) is proven.

6.7 Proof of the isocapacitary estimate (3.7)

In the following subsection, we will denote by $\widetilde{M}_m(h)$ the convolution of a function $h \in L^1_{loc}(\mathbb{R}^n)$ with respect to the first $(n-1)$-variables, i.e.

$$\widetilde{M}_m(h)(z', z_n) = \int_{\mathbb{R}^{n-1}} h(x', z_n)\, \rho_m(z' - x')\, dx'.$$

We then have the following elementary lemma, which will be useful later.

Lemma 1 *Let $v \in C_c^\infty(\mathbb{R}^n)$. Then, if we set*

$$\widetilde{v}_m := \sqrt{\widetilde{M}_m(v^2)},$$

we have that $\widetilde{v}_m$ is Lipschitz continuous on $\mathbb{R}^n$, and

$$|\nabla \widetilde{v}_m| \le c(n) \sqrt{\widetilde{M}_m\big(|\nabla v|^2\big)} \quad \text{a.e. on } \mathbb{R}^n. \tag{6.51}$$

Proof By Hölder's inequality, for $k = 1, \dots, n$ we have

$$\left|\frac{\partial \widetilde{M}_m(v^2)}{\partial x_k}\right| = \left|\widetilde{M}_m\left(\frac{\partial v^2}{\partial x_k}\right)\right| = 2\left|\widetilde{M}_m\left(v\,\frac{\partial v}{\partial x_k}\right)\right| \le 2\sqrt{\widetilde{M}_m(v^2)}\sqrt{\widetilde{M}_m\left(\left|\frac{\partial v}{\partial x_k}\right|^2\right)}.$$

Therefore, on setting $\widetilde{v}_{\varepsilon,m} := \sqrt{\varepsilon^2 + \widetilde{M}_m(v^2)}$, for all $\varepsilon \in (0,1)$ we have that

$$|\nabla \widetilde{v}_{\varepsilon,m}| = \frac{\left|\nabla \widetilde{M}_m(v^2)\right|}{2\sqrt{\varepsilon^2 + \widetilde{M}_m(v^2)}} \le c(n)\,\frac{\sqrt{\widetilde{M}_m(v^2)}\,\sqrt{\widetilde{M}_m(|\nabla v|^2)}}{\sqrt{\varepsilon^2 + \widetilde{M}_m(v^2)}} \le c(n)\sqrt{\widetilde{M}_m\left(|\nabla v|^2\right)}. \tag{6.52}$$

Thus, the sequence $\{\widetilde{v}_{\varepsilon,m}\}_{\varepsilon\in(0,1)}$ is uniformly bounded in $C^{0,1}_c(\mathbb{R}^n)$, and since $\widetilde{v}_{\varepsilon,m} \xrightarrow{\varepsilon\to 0^+} \widetilde{v}_m$ on $\mathbb{R}^n$, we deduce that $\widetilde{v}_m \in C^{0,1}_c(\mathbb{R}^n)$ by weak-$*$ compactness, and the thesis follows by letting $\varepsilon \to 0$ in (6.52) and by Rademacher's Theorem. □

Now let $x^0_m \in \partial\Omega_m$; then owing to (6.26) and (6.16), there exists $i \in \{1, \dots, N\}$ such that $x^0_m \in B_{R_\Omega/8}(x^i)$. Therefore, we may write $x^0_m = (T^i)^{-1}\Big((y^0)', \psi^i_m\big((y^0)'\big)\Big)$ for some $(y^0)' \in B'_{R_\Omega/8}$, and we also set $x^0 := (T^i)^{-1}\Big((y^0)', \phi^i\big((y^0)'\big)\Big) \in \partial\Omega$. Let

$$r_0 := \frac{R_\Omega}{C(n)\left(1 + L^2_\Omega\right)},$$

for some fixed constant $C(n) > 1$ large enough, and consider $r \le r_0$, and $v \in C^\infty_c\big(B_r(x^0_m)\big)$. Then, since $B_r(x^0_m) \Subset B_{R_\Omega/4}(x^i) \Subset K^i_{2\varepsilon_0}$, we have

$$\int_{\partial\Omega_m} v^2\,|\mathcal{B}_{\Omega_m}|\,d\mathcal{H}^{n-1} = \int_{B'_{R_\Omega/4}} v^2\Big((T^i)^{-1}\big(y', \psi^i_m(y')\big)\Big)|\mathcal{B}_{\Omega_m}(y')|\sqrt{1 + |\nabla\psi^i_m(y')|^2}\,dy'.$$

Consider the new set of indices

$$\mathbb{J}^{x^0_m}_r := \Big\{j \in \mathcal{I}_i \,:\, B_r(x^0_m) \cap \operatorname{supp}\xi_j \ne \emptyset\Big\}.$$

Owing to (2.10), (6.33), (6.35), (6.42) and the Hessian estimate (6.48), we obtain

$$\begin{aligned}
&\int_{\partial\Omega_m} v^2\,|\mathcal{B}_{\Omega_m}|\,d\mathcal{H}^{n-1} \le \sqrt{1 + L^2_\Omega}\int_{B'_{R_\Omega/4}} v^2\Big((T^i)^{-1}\big(y', \psi^i_m(y')\big)\Big)\,|\nabla^2\psi^i_m(y')|\,dy' \\
&\le c(n)\,(1 + L^6_\Omega)\sum_{j\in\mathbb{J}^{x^0_m}_r}\int_{V^{i,j}}\Big\{v^2\Big((T^j)^{-1}\big(\mathcal{C}^{i,j}_m y', \psi^j_m(\mathcal{C}^{i,j}_m y')\big)\Big)\times \\
&\times \xi_j\Big((T^j)^{-1}\big(\mathcal{C}^{i,j}_m y', \psi^j_m(\mathcal{C}^{i,j}_m y')\big)\Big)\,M_m\big(|\nabla^2\phi^j|\big)(\mathcal{C}^{i,j}_m y')\Big\}\,dy' \\
&\qquad + c(n)\,\frac{(1 + L^7_\Omega)}{R_\Omega}\,|\mathbb{J}^{x^0_m}_r|\int_{B'_{R/4}} v^2\Big((T^i)^{-1}\big(y', \psi^i_m(y')\big)\Big)dy'.
\end{aligned} \tag{6.53}$$

By using $|\mathbb{J}_r^{x_m^0}| \le N$, (6.3), (3.4) and the results of [2, Corollary 6.6], we get

$$\frac{(1+L_\Omega^7)}{R_\Omega}\,|\mathbb{J}_r^{x_m^0}| \int_{B'_{R_\Omega/4}} v^2\Big((T^i)^{-1}\big(y', \psi_m^j(y')\big)\Big)dy' \le c(n)\frac{(1+L_\Omega^7)\,d_\Omega^n}{R_\Omega^{n+1}} \int_{\partial\Omega_m} v^2\, d\mathcal{H}^{n-1}$$
$$\le \begin{cases} c'(n)\,\dfrac{(1+L_\Omega^{25})\,d_\Omega^n}{R_\Omega^{n+1}}\left(\displaystyle\int_{\mathbb{R}^n} |\nabla v|^2\,dx\right) r & \text{if } n \ge 3 \\ c\,\dfrac{(1+L_\Omega^{31})\,d_\Omega^n}{R_\Omega^{n+1}}\left(\displaystyle\int_{\mathbb{R}^2} |\nabla v|^2\,dx\right) r\,\log\left(1+\dfrac{1}{r}\right) & \text{if } n = 2. \end{cases} \tag{6.54}$$

On the other hand, via the change of variables $z' = \mathcal{C}_m^{i,j}\,y'$, by making use of (6.43), (6.36), and observing that $B_r(x_m^0) \Subset K_{2\varepsilon_0}^i \cap K_{2\varepsilon_0}^j$ for all $j \in \mathbb{J}_r^{x_m^0}$, $x_m^0 \in \partial\Omega_m$ and $r \le r_0$, we find

$$\int_{V^{i,j}} \Big\{ v^2\Big((T^j)^{-1}\big(\mathcal{C}_m^{i,j}y', \psi_m^j(\mathcal{C}_m^{i,j}y')\big)\Big)\,\xi_j\Big((T^j)^{-1}\big(\mathcal{C}_m^{i,j}y', \psi_m^j(\mathcal{C}_m^{i,j}y')\big)\Big)\,M_m\big(|\nabla^2\phi^j|\big)(\mathcal{C}_m^{i,j}y')\Big\}\,dy'$$
$$\le c(n)(1+L_\Omega^{(n-1)}) \int_{W^{i,j}} w_{j,m}^2(z',0)\,M_m\big(|\nabla^2\phi^j|\big)(z')\,dz', \tag{6.55}$$

for some open set $W^{i,j} \Subset \mathcal{C}^{i,j}(U^{i,j})$, where we also set

$$w_{j,m}(z', z_n) := v\Big((T^j)^{-1}\big(z', z_n + \psi_m^j(z')\big)\Big).$$

Since $v \in C_c^\infty(B_r(x_m^0))$ and $x_m^0 = (T^j)^{-1}\Big(\mathcal{C}_m^{i,j}\big((y^0)'\big), \psi_m^j\big((y^0)'\big)\Big)$ for all $j \in \mathbb{J}_r^{x_m^0}$, by using (6.42) it is readily seen that

$$w_{j,m} \in C_c^\infty\Big(B_{c(n)(1+L_\Omega^2)\,r}\Big(\mathcal{C}_m^{i,j}\big((y^0)'\big), 0\Big)\Big),$$

and from the chain rule we find

$$|\nabla w_{j,m}(z', z_n)| \le c(n)(1+L_\Omega^2)\,\Big|\nabla v\Big((T^j)^{-1}\big(z', z_n + \psi_m^j(z')\big)\Big)\Big| \tag{6.56}$$

Next, by using Fubini-Tonelli's Theorem we obtain

$$\int_{W^{i,j}} w_{j,m}^2(z',0)\,M_m\big(|\nabla^2\phi^j|\big)(z')\,dz' = \int_{W^{i,j}} w_{j,m}^2(z',0) \int_{B'_{1/m}(z')} |\nabla^2\phi^j(\tilde z')|\,\rho_m(z'-\tilde z')\,d\tilde z'\,dz'$$
$$\le \int_{W^{i,j}+B'_{1/m}} |\nabla^2\phi^j(\tilde z')|\Big(\int_{B'_{1/m(\tilde z')}} w_{j,m}^2(z',0)\,\rho_m(\tilde z'-z')\,dz'\Big)\,d\tilde z'.$$

We have thus found that

$$\int_{W^{i,j}} w_{j,m}^2(z',0)\,M_m\big(|\nabla^2\phi^j|\big)(z')\,dz' \le \int_{\widetilde W^{i,j}} \widetilde M_m(w_{j,m}^2)(z',0)\,|\nabla^2\phi^j(z')|\,dz', \tag{6.57}$$

for some open set $\widetilde W^{i,j} \Subset \mathcal{C}^{i,j}(U^{i,j})$, provided $m > m_0$ is large enough.

Thanks to Lemma 1 and inequality (6.38), we easily infer

$$\sqrt{\widetilde M_m(w_{j,m}^2)} \in C_c^{0,1}\Big(B_{c(n)(1+L_\Omega^2)(r+\frac{1}{m})}\Big(\mathcal{C}^{i,j}\big((y^0)'\big), 0\Big)\Big),$$

and

$$\left|\nabla\sqrt{\widetilde{M}_m(w_{j,m}^2)}\right| \le c(n)\sqrt{\widetilde{M}_m\left(|\nabla w_{j,m}|^2\right)} \quad \text{a.e. on } \mathbb{R}^n. \tag{6.58}$$

Finally, set

$$\tilde{h}_{j,m}(x', x_n) := \sqrt{\widetilde{M}_m(w_{j,m}^2)}\Big(T^j\left(x', x_n - \phi^j(x')\right)\Big)$$

so that $\tilde{h}_{j,m}$ is Lipschitz continuous on $\mathbb{R}^n$. Moreover, thanks to (6.28), for all $j \in \mathbb{J}_r^{x_m^0}$, we have that

$$B_{c(n)(1+L_\Omega^3)(r+\frac{1}{m})}(x^0) \Subset K_{2\varepsilon_0}^i \cap K_{2\varepsilon_0}^j$$

for all $m > m_0$ sufficiently large and all $r \le r_0$, and thus we may write $x^0 = (T^j)^{-1}\Big(\mathcal{C}^{i,j}\left((y^0)'\right), \phi^j\left((y^0)'\right)\Big)$ due to (6.33). Recalling that ϕ^j is L_Ω-Lipschitz continous, it follows that

$$\tilde{h}_{j,m} \in C_c^{0,1}\Big(B_{c(n)(1+L_\Omega^3)(r+\frac{1}{m})}(x^0)\Big),$$

and from the chain rule

$$\left|\nabla\tilde{h}_{j,m}(x', x_n)\right| \le c(n)(1+L_\Omega)\left|\nabla\sqrt{\widetilde{M}_m(w_{j,m}^2)}(x', x_n - \phi^j(x'))\right| \quad \text{for a.e. } x. \tag{6.59}$$

Owing to (2.10) and the definition of $\widetilde{h}_{j,m}$, we have

$$\begin{aligned}
\int_{\widetilde{W}^{i,j}} \widetilde{M}_m(w_{j,m}^2)(z',0)\,|\nabla^2\phi^j(z')|\,dz' &= \int_{\widetilde{W}^{i,j}} \tilde{h}_{j,m}^2\Big((T^j)^{-1}(z',\phi^j(z'))\Big)\,|\nabla^2\phi^j(z')|\,dz' \\
&\le c(n)(1+L_\Omega^3)\int_{\widetilde{W}^{i,j}} \tilde{h}_{j,m}^2\Big((T^j)^{-1}(z',\phi^j(z'))\Big)\,\left|\mathcal{B}_\Omega(z')\right|\sqrt{1+|\nabla\phi^j(z')|^2}\,dz' \\
&= c(n)(1+L_\Omega^3)\int_{\partial\Omega} \tilde{h}_{j,m}^2\left|\mathcal{B}_\Omega\right|\,d\mathcal{H}^{n-1} \\
&\le c(n)(1+L_\Omega^3)\left(\sup \frac{\int_{\partial\Omega} h^2\left|\mathcal{B}_\Omega\right|\,d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n}|\nabla h|^2\,dx}\right)\int_{\mathbb{R}^n}|\nabla\tilde{h}_{j,m}|^2\,dx,
\end{aligned} \tag{6.60}$$

where the supremum above is taken over all functions $h \in C_c^{0,1}\Big(B_{c(n)(1+L_\Omega^3)(r+\frac{1}{}m)}(x^0)\Big)$.

Henceforth, by coupling (6.3) and estimates (6.53)-(6.60), for all $v \in C_c^\infty\big(B_r(x_m^0)\big)$ we obtain

$$\begin{aligned}
&\int_{\partial\Omega_m} v^2\left|\mathcal{B}_{\Omega_m}\right|d\mathcal{H}^{n-1} \le c(n)\,(1+L_\Omega^{n+4})\left(\sup\frac{\int_{\partial\Omega} h^2\left|\mathcal{B}_\Omega\right|d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n}|\nabla h|^2\,dx}\right)\sum_{j\in\mathbb{J}_r^{x_m^0}}\int_{\mathbb{R}^n}\widetilde{M}_m\big(|\nabla w_{j,m}|^2\big)\,dx \\
&+\tilde{c}\int_{\mathbb{R}^n}|\nabla v|^2dx \\
&\le c(n)\,(1+L_\Omega^{n+4})\left(\sup\frac{\int_{\partial\Omega} h^2\left|\mathcal{B}_\Omega\right|d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n}|\nabla h|^2\,dx}\right)\sum_{j\in\mathbb{J}_r^{x_m^0}}\int_{\mathbb{R}^n}|\nabla w_{j,m}|^2\,dx + \tilde{c}\int_{\mathbb{R}^n}|\nabla v|^2dx \\
&\le c(n)\,(1+L_\Omega^{n+8})\,N\left(\sup\frac{\int_{\partial\Omega} h^2\left|\mathcal{B}_\Omega\right|d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n}|\nabla h|^2\,dx}\right)\int_{\mathbb{R}^n}|\nabla v|^2\,dx + \tilde{c}\int_{\mathbb{R}^n}|\nabla v|^2dx \\
&\le c'(n)\,(1+L_\Omega^{n+8})\frac{d_\Omega^n}{R_\Omega^n}\left(\sup\frac{\int_{\partial\Omega} h^2\left|\mathcal{B}_\Omega\right|d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n}|\nabla h|^2\,dx}\right)\int_{\mathbb{R}^n}|\nabla v|^2dx + \tilde{c}\int_{\mathbb{R}^n}|\nabla v|^2dx,
\end{aligned}$$

where in the second inequality we made use of Fubini-Tonelli's Theorem, the supremum above is taken over all $h \in C_c^{0,1}\left(B_{c(n)(1+L_\Omega^3)(r+\frac{1}{m})}(x^0)\right)$, and we set

$$\tilde{c} = \tilde{c}(n, L_\Omega, R_\Omega, d_\Omega, r) = \begin{cases} c(n)\dfrac{(1+L_\Omega^{25})\, d_\Omega^n}{R_\Omega^{n+1}}\, r & \text{if } n \geq 3 \\ c(n)\dfrac{(1+L_\Omega^{31})\, d_\Omega^n}{R_\Omega^{n+1}}\, r\, \log\left(1+\dfrac{1}{r}\right) & \text{if } n = 2. \end{cases} \tag{6.61}$$

Therefore, for all $x_m^0 \in \partial\Omega_m$, $r \leq r_0$, we have found

$$\sup_{v \in C_c^\infty(B_r(x_m^0))} \frac{\int_{\partial\Omega_m} v^2 \left|\mathcal{B}_{\Omega_m}\right| d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla v|^2\, dx} \leq \frac{c(n)\,(1+L_\Omega^{n+8})\, d_\Omega^n}{R_\Omega^n} \left(\sup_{\substack{x^0 \in \partial\Omega \\ v \in C_c^{0,1}\left(B_{c(n)(1+L_\Omega^3)(r+1/m)}(x^0)\right)}} \frac{\int_{\partial\Omega} v^2 \left|\mathcal{B}_\Omega\right| d\mathcal{H}^{n-1}}{\int_{\mathbb{R}^n} |\nabla v|^2\, dx} \right) + \tilde{c}.$$

From this, (6.61) and the isocapacitary equivalence [19, Theorem 2.4.1], we finally obtain the desired estimate

$$\mathcal{K}_{\Omega_m}(r) \leq \begin{cases} \dfrac{c(n)\,(1+L_\Omega^{n+8})\, d_\Omega^n}{R_\Omega^n}\, \mathcal{K}_\Omega\left(c(n)(1+L_\Omega^3)(r+\frac{1}{m})\right) + \dfrac{c(n)\,(1+L_\Omega^{25})\, d_\Omega^n}{R_\Omega^{n+1}}\, r & \text{if } n \geq 3 \\ \dfrac{c(n)\,(1+L_\Omega^{n+8})\, d_\Omega^n}{R_\Omega^n}\, \mathcal{K}_\Omega\left(c(n)(1+L_\Omega^3)(r+\frac{1}{m})\right) + \dfrac{c(n)\,(1+L_\Omega^{31})\, d_\Omega^n}{R_\Omega^{n+1}}\, r\, \log\left(1+\dfrac{1}{r}\right) & \text{if } n = 2, \end{cases} \tag{6.62}$$

for all $r \leq r_0$ and $m > m_0$, and the proof is complete.

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