Calc. Var. (2024) 63:131
https://doi.org/10.1007/s00526-024-02710-y

Calculus of Variations

Relationship between variational problems with norm constraints and ground state of semilinear elliptic equations in $\mathbb{R}^2$

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Received: 2 November 2023 / Accepted: 6 March 2024

Abstract
In this paper, we investigate variational problems in $\mathbb{R}^2$ with the Sobolev norm constraints and with the Dirichlet norm constraints. We focus on property of maximizers of the variational problems. Concerning variational problems with the Sobolev norm constraints, we prove that maximizers are ground state solutions of corresponding elliptic equations, while we exhibit an example of a ground state solution which is not a maximizer of corresponding variational problems. On the other hand, we show that maximizers of maximization problems with the Dirichlet norm constraints and ground state solutions of corresponding elliptic equations are the same functions, up to scaling, under suitable setting.

Mathematics Subject Classification Primary: 35B38 Critical points of functionals in context of PDEs (e.g., energy functionals); Secondary: 35A15 Variational methods applied to PDEs · 35B08 Entire solutions to PDEs · 35J15 Second-order elliptic equations · 35J60 Nonlinear elliptic equations

1 Introduction

We consider the following variational problems

$$C_{G,\mu,\alpha} := \sup\left\{ \int_{\mathbb{R}^2} G(u^2)dx \,\middle|\, u \in H^1(\mathbb{R}^2),\ \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \mu u^2\right) dx = \alpha \right\}$$

and

$$D_{G,\alpha} := \sup\left\{ \frac{\int_{\mathbb{R}^2} G(u^2)dx}{\int_{\mathbb{R}^2} u^2 dx} \,\middle|\, u \in H^1(\mathbb{R}^2),\ \int_{\mathbb{R}^2} |\nabla u|^2 dx = \alpha \right\},$$

where μ and α are positive constants and $G : [0, \infty) \to \mathbb{R}$ satisfies

Communicated by M. del Pino.

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Published online: 11 May 2024

(G1) $G(0) = 0$, $G \in C^1((0,\infty);\mathbb{R})$ and G is convex,
(G2) there exists a nonnegative constant m such that $\lim_{s\to+0} G(s)/s = m$ and $G(s) \not\equiv ms$,
(G3) $G(s) \le Ce^{Cs}$ holds for all $s > 0$ with some positive constant C.

In the case $G(s) = s^p$ with $p > 1$, problem $C_{G,\mu,\alpha}$ is the best constant for the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^{2p}(\mathbb{R}^2)$ and $D_{G,\alpha}$ is the best constant of the Gagliardo-Nirenberg-Sobolev inequality. It is known that for any μ and α there exists a function which attains $C_{G,\mu,\alpha}$ by the compactness of the embedding $H^1_{rad}(\mathbb{R}^2) \hookrightarrow L^{2p}(\mathbb{R}^2)$, and $D_{G,\alpha}$ is also attained. On the other hand, if $G(s) = s$, then $C_{G,\mu,\alpha}$ is the best constant for $H^1(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$ and the constant is not attained due to the non-compactness of the embedding $H^1_{rad}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$. Obviously, if $G(s) = s$, then $D_{G,\alpha} = 1$ and $D_{G,\alpha}$ is attained.

In the case $G(s) = e^s - 1$ and $\alpha \le 4\pi$, the constant $C_{G,\mu,\alpha}$ is the best constant of the Trudinger-Moser inequality, which boundedness is obtained by B. Ruf [40]. The existence of a maximizer for $C_{G,\mu,4\pi}$ is also proved in [40]. In addition to the existence result, it is shown by M. Ishiwata [16] that there exists a threshold $\alpha_* < 4\pi$ such that if $\alpha > \alpha_*$, then $C_{G,\mu,\alpha} > \alpha/\mu$ and $C_{G,\mu,\alpha}$ is attained, while if $\alpha < \alpha_*$, then $C_{G,\mu,\alpha} = \alpha/\mu$ and $C_{G,\mu,\alpha}$ is not attained. Concerning $D_{G,\alpha}$, it is shown by T. Ogawa [34] that there exists a positive constant C_0 such that $D_{G,1} \le C_0$ holds. Later, it is shown by Adachi and Tanaka [3] that $D_{G,\alpha} < \infty$ holds if and only if $\alpha < 4\pi$. In [20] and [7], the existence of a maximizer of $D_{G,\alpha}$ for any $\alpha < 4\pi$ is proved. Moreover, by Cassani, Sani and Tarsi [7], a sharp estimate of $D_{G,\alpha}$ with respect to α is obtained, and then it is proved that the boundedness of $D_{G,\alpha}$ for any $\alpha < 4\pi$ is equivalent to the boundedness of $C_{G,\mu,\alpha}$ for $\mu = 1$ and $\alpha = 4\pi$. For more about the existence of extremal functions for Trudinger-Moser inequality and its generalization, we refer reader to [1, 8, 9, 11, 13, 17, 21–23, 26, 27, 31–33, 35, 36] and references therein.

Maximizers of $C_{G,\mu,\alpha}$ and of $D_{G,\alpha}$ are solutions of elliptic equations of the form

$$-\Delta u + \omega u = \lambda u g(u^2) \quad \text{in} \quad \mathbb{R}^2$$

with positive constants ω and λ, where g satisfies $G(s) = \int_0^s g(t)dt$, and by proper scaling of solutions, the equation can be simplified to

$$-\Delta u + u = \Lambda u g(u^2) \quad \text{in} \quad \mathbb{R}^2 \tag{1}$$

with a positive constant Λ. Concerning more general equations, equation of the form

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{2}$$

has been extensively studied starting from the fundamental papers due to Berestycki and Lions [5] and to Berestycki, Gallouët and Kavian [6]. Equation (2) has the variational structure and solutions of (2) can be characterized as critical points of the functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u)dx,$$

where $F(s) = \int_0^s f(t)dt$. In [5] and [6], the authors establish the existence of ground state solution, namely, solutions of (2) which have least energy among all nontrivial critical points of I, through the minimization problems:

$$\inf \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx \;\middle|\; \int_{\mathbb{R}^N} F(u)dx = 1 \right\} \quad \text{for} \quad N \geq 3,$$
$$\inf \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 dx \;\middle|\; \int_{\mathbb{R}^2} F(u)dx = 0 \right\} \quad \text{for} \quad N = 2.$$

The uniqueness of ground state solution is studied in [2, 4, 10, 24, 25, 29, 30, 37–39, 42, 43]. In particular, if $f(s) = s^p - as^q - s$ with $a \geq 0$ and $1 < q < p < (N+2)/(N-2)$, then the ground state solution of (2) is unique.

In this paper, we investigate property of maximizers of $C_{G,\mu,\alpha}$ and of $D_{G,\alpha}$. More precisely, we study the relationship between these maximizers and ground state solutions of (1). As mentioned above, in the case $G(s) = s$, $C_{G,\mu,\alpha}$ is not attained and $D_{G,\alpha}$ is attained by any functions satisfying the constraint. Thus, it is natural to assume that $G(s) \not\equiv ms$ in (G2).

Concerning maximizers of $C_{G,\mu,\alpha}$ and ground state solutions of (1), we prove the following result.

Theorem 1 *Assume that $u_0 \in H^1(\mathbb{R}^2)$ is a maximizer of $C_{G,\mu,\alpha}$. Then, there exists a positive constant Λ_0 such that u_0 is a ground state solution of (1) with $\Lambda = \Lambda_0$, up to scaling.*

The proof of Theorem 1 relies on suitable scaling properties which investigated in [7], and we use the best constant $D_{G,\alpha}$ to specify the Lagrange multiplier. Moreover, we do not use any variational techniques to prove Theorem 1.

In general, ground state solution of (1) and maximizer of $C_{G,\mu,\alpha}$ are distinct. The next result is an example of a ground state solution which is not a maximizer of $C_{G,\mu,\alpha}$.

Theorem 2 *Assume that $G(s) = e^s - 1$ and w_Λ is a ground state solution of (1) for $\Lambda > 0$. Let $\alpha_\mu = \int_{\mathbb{R}^2} \left(|\nabla w_\Lambda|^2 + \mu w_\Lambda^2\right) dx$ for $\mu > 0$. Then, there exists $\Lambda_* \in (0, 1)$ such that for any $\Lambda \in (0, \Lambda_*)$ and $\mu > 0$, either $\alpha_\mu > 4\pi$ or $\int_{\mathbb{R}^2} G(w_\Lambda^2)dx < C_{G,\mu,\alpha_\mu}$ provided that $\alpha_\mu \leq 4\pi$.*

The existence of a ground state solution of (1) with $G(s) = e^s - 1$ and $\Lambda \in (0, 1)$ is guaranteed by the result of Ruf and Sani [41]. Theorem 2 asserts that a ground state solution w_Λ of (1) with small Λ is either a critical point of $\int_{\mathbb{R}^2}(e^{u^2} - 1)dx$ under the constraint $\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \mu u^2\right) dx \leq 4\pi$ except a maximizer, or a critical point of $\int_{\mathbb{R}^2}(e^{u^2} - 1)dx$ under the constraint $\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \mu u^2\right) dx > 4\pi$, though $C_{G,\mu,\alpha} = \infty$ for $\alpha > 4\pi$. Theorems 1 and 2 assert that equivalence of maximizers of $C_{G,\mu,\alpha}$ and ground state solutions of (1) does not hold in general.

To state our results regarding relationship between maximizers of variational problems $D_{G,\alpha}$ and ground state solutions of (1), we consider the next condition on G.

(G4) $D_{G,\alpha}$ is attained whenever $D_{G,\alpha} < \infty$.

We prove the following results.

Theorem 3 *Assume that G satisfies (G1)-(G3) and $v_0 \in H^1(\mathbb{R}^2)$ is a maximizer of $D_{G,\alpha}$. Then, v_0 is a ground state solution of (1) for $\Lambda = D_{G,\alpha}^{-1}$, up to scaling.*

Theorem 4 *Assume that G satisfies (G1)-(G4) and $w_0 \in H^1(\mathbb{R}^2)$ is a ground state solution of (1) for $\Lambda > 0$. Let $\alpha_0 = \int_{\mathbb{R}^2} |\nabla w_0|^2 dx$. Then,*

$$\Lambda = D_{G,\alpha_0}^{-1}$$

and w_0 is a maximizer of D_{G,α_0}.

As for the condition (G4), using results [3, 18, 19] and arguments to prove Theorem 1.1 in [7], we describe some sufficient conditions of (G4). Under the conditions (G1)-(G3), by the result of [18], if G satisfies

$$\lim_{s\to\infty} \frac{sG(s)}{e^{Ks}} = 0 \tag{3}$$

for some positive constant K provided that $\lim_{s\to\infty} G(s)/e^{K_- s} = \infty$ and $\lim_{s\to\infty} G(s)/e^{K_+ s} = 0$ for any $K_- < K < K_+$, then $D_{G,\alpha} < \infty$ if and only if $\alpha \le 4\pi/K$. Moreover, by the conditions (G1)-(G3) and the arguments of [7], we derive the existence of a maximizer of $D_{G,\alpha}$ for any $\alpha \in (0, 4\pi/K]$. If G satisfies

$$\lim_{s\to\infty} \frac{sG(s)}{e^{Ks}} = \infty \quad \text{and} \quad \lim_{s\to\infty} \frac{G(s)}{e^{Ks}} < \infty, \tag{4}$$

then $D_{G,\alpha} < \infty$ for $\alpha < 4\pi/K$ and $D_{G,\alpha} = \infty$ for $\alpha \ge 4\pi/K$ by the results of [3] and [18]. In the former case, there exists a maximizer of $D_{G,\alpha}$ for any $\alpha < 4\pi/K$ by the same reason as in the case (3). In the remaining case

$$0 < \lim_{s\to\infty} \frac{sG(s)}{e^{Ks}} < \infty, \tag{5}$$

the attainability of $D_{G,4\pi/K}$ depends on lower order perturbations included in G. Conditions of existence and non-existence of a maximizer of $D_{G,4\pi/K}$ are given by Theorem 1.1 in [19]. Thus, in addition to the subcritical case, G satisfies (G4) if the growth of G satisfies (3), (4) or (5) with an existence condition of Theorem 1.1 in [19]. In particular, functions $G(s) = e^s - 1$ and $G(s) = s^p$ with $p > 1$ satisfy (G4). It is shown in Corollary 1.3 in [19] that there exists a function G satisfying (5) for which there is no mountain pass solution of (1) with small Λ. Such function G does not satisfy (G4).

In the special case $G(s) = s^p$ with $p > 1$, a stronger result follows from the uniqueness result on positive solution of (1) by Kwong [25]. In the situation $G(s) = s^p$ for $p > 1$, maximizers of $C_{G,\mu,\alpha}$ and $D_{G,\alpha}$ are positive solutions of (1) with $\Lambda = 1$, up to dilation and multiplicative constant of the maximizers. Moreover, the existence of positive ground state solution of (1) with $\Lambda = 1$ is obtained in [6], and the uniqueness result on positive solution of (1) with $\Lambda = 1$ is proved in [25]. Thus, these results yield that any maximizers of $C_{G,\mu,\alpha}$ and $D_{G,\alpha}$ for any positive constants μ and α are the same as the unique positive ground state solution of (1) with $\Lambda = 1$, up to dilation and multiplicative constant.

Different from Theorem 2, any ground state solution of (1) attains a maximization problem $D_{G,\alpha}$ for some α under the additional condition (G4). By Theorems 3, 4 and a scaling property of (1), existence of a maximizer of $D_{G,\alpha}$ is equivalent to existence of a ground state solution of (1) with $\Lambda = D_{G,\alpha}^{-1}$ under the condition (G4), and the ground state level is $\alpha/2$ if a ground state solution exists.

This paper is organized as follows. In Sect. 2, we prove Theorems 1 and 3. We first prove Theorem 3, and then, using Theorem 3, we prove Theorem 1. The key argument to prove Theorems 3 is the characterization of ground state solutions of (2) given in [6] in the subcritical case. In order to prove Theorem 1, we show that a maximizer of $C_{G,\mu,\alpha}$ is also a maximizer of D_{G,α_1} for some $\alpha_1 < \alpha$. In Sect. 3, we prove Theorem 2. To prove Theorem 2, we estimate the Dirichlet norm of the ground state solution w_Λ for small Λ. We show that w_Λ concentrates at origin as $\Lambda \to 0$, unless $\alpha_\mu > 4\pi$. Then, under the assumption $\alpha_\mu \le 4\pi$, we apply blow-up analysis in [27] to w_Λ. In Sect. 4, we prove Theorem 4. In Sect. 5, we extend Theorems 1, 2, 3 and 4 to higher dimensional case $N \ge 3$ and $W^{1,N}(\mathbb{R}^N)$.

2 Proof of Theorems 1 and 3

In this section, we prove Theorems 1 and 3. In order to prove these theorems, we fix some notations. For a positive constant K, we define

$$D^*_{G,\alpha,K} := \sup\left\{\int_{\mathbb{R}^2} G(u^2)dx \,\middle|\, u \in H^1(\mathbb{R}^2),\ \int_{\mathbb{R}^2} |\nabla u|^2 dx = \alpha,\ \int_{\mathbb{R}^2} u^2 dx = K\right\}.$$

For G satisfying (G1)-(G3) we define a function g such that $G(s) = \int_0^s g(t)dt$. We define the energy functional $I_\Lambda : H^1(\mathbb{R}^2) \to \mathbb{R}$ corresponding to the equation (1) by

$$I_\Lambda(u) := \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) dx - \frac{\Lambda}{2}\int_{\mathbb{R}^2} G(u^2)dx.$$

Then, the ground state level is defined as

$$M_\Lambda := \inf\left\{I_\Lambda(u) \,\middle|\, u \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ is a solution of } (1)\right\}.$$

We summarize some properties of G. By the conditions (G1) and (G2), a lower estimate $G(s) \geq ms$ holds for any $s \geq 0$ and there exists $s_0 > 0$ such that $G(s_0) > ms_0$. Set

$$S_0 := \inf\left\{s_0 \geq 0 \,\middle|\, G(s_0) > ms_0\right\}.$$

Using the convexity of G again, we observe that

$$G(\kappa s) < \kappa G(s) \quad \text{for any} \quad s > S_0 \quad \text{and} \quad \kappa \in (0, 1). \tag{6}$$

Moreover, for the same constant S_0, we have

$$G(s) < sg(s) \quad \text{for any} \quad s > S_0. \tag{7}$$

Going back to the properties that $G(s) \geq ms$ holds for any $s \geq 0$ and $G(s_0) > ms_0$ holds for some s_0, we have

$$D_{G,\alpha} > m \tag{8}$$

for any $\alpha > 0$.

We first prove Theorem 3. Assume that a function G satisfies (G1)-(G3), $\alpha > 0$ and $v_0 \in H^1(\mathbb{R}^2)$ is a maximizer of $D_{G,\alpha}$. By the Lagrange multiplier theorem, v_0 satisfies

$$-\Lambda_0 \Delta v_0 = \frac{1}{\int_{\mathbb{R}^2} v_0^2 dx}\left(-D_{G,\alpha} v_0 + v_0 g(v_0^2)\right) \quad \text{in} \quad \mathbb{R}^2, \tag{9}$$

where $\Lambda_0 \in \mathbb{R}$ is the Lagrange multiplier. By (8), we see that $\|v_0\|_{L^\infty(\mathbb{R}^2)} > \sqrt{S_0}$, and thus by (7), we have

$$\int_{\mathbb{R}^2} v_0^2 g(v_0^2)dx > \int_{\mathbb{R}^2} G(v_0^2)dx.$$

Multiplying (9) by v_0 and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}\Lambda_0 \int_{\mathbb{R}^2} |\nabla v_0|^2 dx &= -D_{G,\alpha} + \frac{\int_{\mathbb{R}^2} v_0^2 g(v_0^2)}{\int_{\mathbb{R}^2} v_0^2 dx} \\ &> -D_{G,\alpha} + \frac{\int_{\mathbb{R}^2} G(v_0^2) dx}{\int_{\mathbb{R}^2} v_0^2 dx} \\ &= 0.\end{aligned}$$

Hence, it holds that $\Lambda_0 > 0$.

Set

$$w_0(x) = v_0(\theta x) \quad \text{with} \quad \theta := \sqrt{\frac{\Lambda_0 \int_{\mathbb{R}^2} v_0^2 dx}{D_{G,\alpha}}}. \tag{10}$$

Then, w_0 is a solution of

$$-\Delta w + w = D_{G,\alpha}^{-1} w g(w^2) \quad \text{in} \quad \mathbb{R}^2 \tag{11}$$

and it holds that

$$\int_{\mathbb{R}^2} |\nabla w_0|^2 dx = \alpha. \tag{12}$$

In [6], the Pohozaev identity was shown under the condition that g has a subcritical growth. We prove the same equality for G such that (G1)-(G3).

Proposition 5 *Assume that a function G satisfies the conditions (G1)-(G3). Then, any solution $u \in H^1(\mathbb{R}^2)$ of (1) with $\Lambda > 0$ satisfies*

$$\int_{\mathbb{R}^2} \left(\Lambda G(u^2) - u^2\right) dx = 0.$$

Proof By the convexity of G, we have

$$g(s_1) \le \frac{G(s_2) - G(s_1)}{s_2 - s_1}$$

for any positive constants s_1 and s_2 with $s_2 > s_1$. In particular, it holds that

$$g(s_1) \le \frac{G(2s_1)}{s_1}$$

for any s_1, and then by (G2) and (G3), there exists $L > 0$ such that

$$g(s) \le Le^{Ls}$$

for any $s \ge 0$. By the regularity theory, we derive that $u \in W^{2,q}_{loc}(\mathbb{R}^2)$ for any $q > 1$. Hence, applying the argument to prove Claim 5.3 in [14], we obtain the equality of the proposition. □

By Proposition 5, we can write

$$M_\Lambda = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 dx \,\middle|\, u \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ is a solution of (1)} \right\}. \tag{13}$$

Next, we prove the monotonicity of $D_{G,\alpha}$ with respect to α.

Proposition 6 *Assume that $\beta > 0$. Then, for any $v \in H^1(\mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} |\nabla v|^2 dx < \beta$, it holds that*

$$\frac{\int_{\mathbb{R}^2} G(v^2)dx}{\int_{\mathbb{R}^2} v^2 dx} < D_{G,\beta}.$$

Proof Let $v \in H^1(\mathbb{R}^2)$ be such that $\int_{\mathbb{R}^2} |\nabla v|^2 dx < \beta$ and put $\gamma := \int_{\mathbb{R}^2} |\nabla v|^2 dx$. We distinguish two cases:
Case 1.

$$\|v\|_{L^\infty(\mathbb{R}^2)} \leq \sqrt{S_0}.$$

In this case, $G(v(x)^2)$ coincides with $mv(x)^2$ for a.e. $x \in \mathbb{R}^2$. Thus, we have

$$\frac{\int_{\mathbb{R}^2} G(v^2)dx}{\int_{\mathbb{R}^2} v^2 dx} = m < D_{G,\beta}.$$

Hence, we obtain desired estimate.
Case 2.

$$\|v\|_{L^\infty(\mathbb{R}^2)} > \sqrt{S_0}.$$

We consider

$$v_\beta(x) = \sqrt{\frac{\beta}{\gamma}} v(x).$$

It is easy to check that $\int_{\mathbb{R}^2} |\nabla v_\beta|^2 dx = \beta$. Moreover, by the hypothesis and (6), we derive that

$$\int_{\{v > \sqrt{S_0}\}} G\left(v^2\right) dx < \frac{\gamma}{\beta} \int_{\{v > \sqrt{S_0}\}} G(v_\beta^2) dx.$$

Hence,

$$\frac{\int_{\mathbb{R}^2} G(v^2)dx}{\int_{\mathbb{R}^2} v^2 dx} < \frac{\int_{\mathbb{R}^2} G(v_\beta^2)dx}{\int_{\mathbb{R}^2} v_\beta^2 dx} \leq D_{G,\beta}.$$

Consequently, we conclude that Proposition 6 holds. □

Proof of Theorem 3 Propositions 5 and 6 give that a necessary condition of solutions of (11) is

$$\int_{\mathbb{R}^2} |\nabla w|^2 dx \geq \alpha.$$

The estimate and (13) yield the following lower bound of the ground state level:

$$M_{D_{G,\alpha}^{-1}} \geq \frac{\alpha}{2}.$$

Moreover, it holds that, by (12) and (13),

$$M_{D_{G,\alpha}^{-1}} \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla w_0|^2 dx = \frac{\alpha}{2}.$$

Hence, we derive that $M_{D_{G,\alpha}^{-1}} = \int_{\mathbb{R}^2} |\nabla w_0|^2/2$. Consequently, w_0 is a ground state solution of (11), and by (10), we conclude Theorem 3. □

We next prove Theorem 1. Assume that G satisfies (G1)-(G3), $\mu > 0$, $\alpha > 0$ and $u_0 \in H^1(\mathbb{R}^2)$ is a maximizer of $C_{G,\mu,\alpha}$. The maximizer is a solution of

$$-\Delta u + \mu u = \Lambda_1 u g(u^2) \quad \text{in} \quad \mathbb{R}^2,$$

where Λ_1 is the Lagrange multiplier characterized by

$$\Lambda_1 = \frac{\alpha}{\int_{\mathbb{R}^2} u_0^2 g(u_0^2) dx}.$$

Since $\alpha > 0$, we see that $\Lambda_1 > 0$. We define a constant by

$$\alpha_1 := \int_{\mathbb{R}^2} |\nabla u_0|^2 dx.$$

Then, we prove the following proposition.

Proposition 7 *The function $u_0 \in H^1(\mathbb{R}^2)$ is a maximizer of D_{G,α_1} and we have*

$$\Lambda_1 = \frac{\mu}{D_{G,\alpha_1}}. \tag{14}$$

Proof By the constraint of $C_{G,\mu,\alpha}$, we see that

$$\int_{\mathbb{R}^2} u_0^2 dx = \frac{\alpha - \alpha_1}{\mu}.$$

Then, it follows from the definitions of $C_{G,\mu,\alpha}$ and $D^*_{G,\beta,K}$ that

$$C_{G,\mu,\alpha} \geq D^*_{G,\alpha_1,(\alpha-\alpha_1)/\mu}.$$

Since u_0 is a maximizer of $C_{G,\mu,\alpha}$ and satisfies the constraint of $D^*_{G,\alpha_1,(\alpha-\alpha_1)/\mu}$, u_0 also attains the best constant $D^*_{G,\alpha_1,(\alpha-\alpha_1)/\mu}$.

Here, for any function $v \in H^1(\mathbb{R}^2)$ and positive constant K, we consider the following scaling

$$v_K(x) = v(\theta_K x) \quad \text{with} \quad \theta_K = \sqrt{\frac{\int_{\mathbb{R}^2} v^2 dx}{K}}.$$

Then, we observe that

$$\frac{D^*_{G,\beta,K}}{K} = D_{G,\beta} \tag{15}$$

for any G and $\beta > 0$, and hence, u_0 also attains D_{G,α_1}.

Next, we prove the equality (14). The same argument to prove Proposition 5 yields that

$$\int_{\mathbb{R}^2} \left(\Lambda_1 G(u_0^2) - \mu u_0^2\right) dx = 0,$$

and then we derive that

$$\Lambda_1 D^*_{G,\alpha_1,(\alpha-\alpha_1)/\mu} - (\alpha - \alpha_1) = 0,$$

or

$$\Lambda_1 = \frac{\alpha - \alpha_1}{D^*_{G,\alpha_1,(\alpha-\alpha_1)/\mu}}.$$

The equality with (15) gives the equality (14), and hence, Proposition 7 is proved. □

Proof of Theorem 1 Set

$$w_1(x) = u_0\left(x/\sqrt{\mu}\right).$$

By Proposition 7 and Theorem 3, w_1 is a ground state solution of

$$-\Delta w + w = D_{G,\alpha_1}^{-1} w g(w^2) \quad \text{in} \quad \mathbb{R}^2.$$

Consequently, the proof of Theorem 1 is complete. □

3 Proof of theorem 2

Proof of Theorem 2 Suppose that $G(s) = e^s - 1$ for $s \geq 1$. Let $\{\Lambda_n\}$ be a sequence of positive numbers such that $\Lambda_n \to 0$ as $n \to \infty$ and let $w_n \in H^1(\mathbb{R}^2)$ be a ground state solution of

$$-\Delta w + w = \Lambda_n w e^{w^2} \quad \text{in} \quad \mathbb{R}^2. \tag{16}$$

We note that w_n is positive and radially symmetric by the result of [15]. For $\mu > 0$, a constant $\alpha_{\mu,n}$ denotes $\int_{\mathbb{R}^2}\left(|\nabla w_n|^2 + \mu w_n^2\right)dx$ and in the following, we assume that $\alpha_{\mu,n} \leq 4\pi$. We first prove that w_n does not attain $C_{G,\mu,\alpha_{\mu,n}}$ for any $\mu \neq 1$. Assume on the contrary that $\int_{\mathbb{R}^2}\left(e^{w_n^2} - 1\right)dx = C_{G,\mu,\alpha_{\mu,n}}$ holds with $\mu \neq 1$. We observe that w_n is a solution of

$$-\Delta w + \mu w = \Lambda_\mu w e^{w^2} \quad \text{in} \quad \mathbb{R}^2 \tag{17}$$

with a Lagrange multiplier Λ_μ depending on n. Applying the argument in the proof of Proposition 5 to the above equation, we have

$$\int_{\mathbb{R}^2}\left[\mu w_n^2 - \Lambda_\mu\left(e^{w_n^2} - 1\right)\right]dx = 0.$$

On the other hand, by the characterization of ground state solutions of (16) given in [41], we have

$$\int_{\mathbb{R}^2}\left[w_n^2 - \Lambda_n\left(e^{w_n^2} - 1\right)\right]dx = 0. \tag{18}$$

The two equalities yield that $\Lambda_\mu = \mu\Lambda_n$. Then, since w_n is a solution of both (16) and (17) again, we have

$$\left(1 - \frac{1}{\mu}\right)\Delta w_n = 0,$$

which implies that $w_n \equiv 0$. This is a contradiction, and hence, w_n is not a maximizer of $C_{G,\mu,\alpha_{\mu,n}}$ for $\mu \neq 1$.

In the following, we assume that $\mu = 1$. For simplicity, we set $C_\alpha := C_{G,1,\alpha}$ and $\alpha_n := \alpha_{1,n}$. We will prove that a ground state solution w_n does not attain C_{α_n} for sufficiently large n. Going back to (18), we derive that

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}^2}\left(e^{w_n^2} - 1\right)dx}{\int_{\mathbb{R}^2} w_n^2 dx} = \infty,$$

which implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}|\nabla w_n|^2 dx \geq 4\pi$$

by the results in [3]. The lower bound and the assumption of α_n yield that $\lim_{n\to\infty}\alpha_n = 4\pi$, $\lim_{n\to\infty}\int_{\mathbb{R}^2}|\nabla w_n|^2dx = 4\pi$ and $\int_{\mathbb{R}^2} w_n^2dx = 0$. Hence, $\{w_n\}$ concentrates at the origin, that is it holds that $\lim_{n\to\infty} w_n(0) = \infty$ and that $\lim_{n\to\infty} w_n(x) = 0$ for all $x \in \mathbb{R}^2\backslash\{0\}$. Using the same arguments in [27] to prove the existence of maximizers of $C_{4\pi}$, we have, after passing to a subsequence,

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left(e^{w_n^2}-1\right)dx \le \pi e^{4\pi A} < C_{4\pi}$$

with an explicit constant A. Hence, by the continuity of the best constant C_α with respect to α, we derive that $\int_{\mathbb{R}^2}\left(e^{w_n^2}-1\right)dx < C_{\alpha_n}$ for large n.

Consequently, for sufficiently large n, it holds that $\int_{\mathbb{R}^2}\left(e^{w_n^2}-1\right)dx < C_{\alpha_n}$ unless $\alpha_n > 4\pi$. The proof of Theorem 2 is complete. □

4 Proof of theorem 4

Proof of Theorem 4 Assume that G satisfies (G1)-(G4) and $w_0 \in H^1(\mathbb{R}^2)$ is a ground state solution of (1) for $\Lambda > 0$. We first estimate Λ. Since G is convex and G satisfies (G2), by Proposition 5, we derive that

$$0 = \int_{\mathbb{R}^2}\left(\Lambda G(w_0^2) - w_0^2\right)dx > \int_{\mathbb{R}^2}\left(\Lambda m w_0^2 - w_0^2\right)dx,$$

and thus, we have $\Lambda^{-1} > m$.

Let $\alpha_0 = \int_{\mathbb{R}^2}|\nabla w_0|^2dx$. Then, we observe that

$$\frac{1}{\Lambda} = \frac{\int_{\mathbb{R}^2} G(w_0^2)dx}{\int_{\mathbb{R}^2} w_0^2dx} \le D_{G,\alpha_0}.$$

To prove $\Lambda^{-1} = D_{G,\alpha_0}$, assuming that, on the contrary

$$\frac{1}{\Lambda} < D_{G,\alpha_0},$$

we derive a contradiction. Since $D_{G,\alpha}$ is continuous with respect to α, $\lim_{\alpha\to 0} D_{G,\alpha} = m$ and $m < \Lambda^{-1}$, there exists $\beta \in (0, \alpha_0)$ such that

$$\frac{1}{\Lambda} = D_{G,\beta}.$$

By (G4), there exists $v_\beta \in H^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2}|\nabla v_\beta|^2dx = \beta$ and

$$\frac{\int_{\mathbb{R}^2} G(v_\beta^2)dx}{\int_{\mathbb{R}^2} v_\beta^2dx} = D_{G,\beta}.$$

Thus, by Theorem 3, v_β is another ground state solution of (1), up to scaling. Recalling the characterization of the ground state level given by (13), we have

$$M_\Lambda = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla v_\beta|^2dx = \frac{\beta}{2}.$$

However, since w_0 is also a ground state solution of (1), we have

$$M_\Lambda = \frac{1}{2}\int_{\mathbb{R}^2}|\nabla w_0|^2dx = \frac{\alpha_0}{2},$$

which contradicts that $\beta < \alpha_0$. Consequently, it holds that $\Lambda^{-1} = D_{G,\alpha_0}$ and w_0 is a maximizer of D_{G,α_0}. □

5 Higher dimensional case

In this section, we deal with $N \geq 3$ and $W^{1,N}(\mathbb{R}^N)$. We consider $G : [0, \infty) \to \mathbb{R}$ satisfies

(G1) $G(0) = 0$, $G \in C^1((0, \infty); \mathbb{R})$ and G is convex,
(G2) there exists a nonnegative constant m such that $\lim_{s\to+0} G(s)/s = m$ and $G(s) \not\equiv ms$,
(G3) $G(s) \leq Ce^{Cs^{\frac{1}{N-1}}}$ holds for all $s > 0$ with some positive constant C.

Set

$$\mathscr{C}_{G,\mu,\alpha} := \sup\left\{\int_{\mathbb{R}^N} G(|u|^N)dx \;\middle|\; u \in W^{1,N}(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\left(|\nabla u|^N + \mu|u|^N\right)dx = \alpha\right\}$$

and

$$\mathscr{D}_{G,\alpha} := \sup\left\{\frac{\int_{\mathbb{R}^N} G(|u|^N)dx}{\int_{\mathbb{R}^N} |u|^N dx} \;\middle|\; u \in W^{1,N}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |\nabla u|^N dx = \alpha\right\},$$

where μ and α are positive constants. Then, consider the condition

(G4) $\mathscr{D}_{G,\alpha}$ is attained whenever $\mathscr{D}_{G,\alpha} < \infty$.

It is worth noting that the results in [18] are extended to the case $N \geq 3$ by Masmoudi and Sani [28]. By the boundedness result in higher dimensional case, if G satisfies (G1)-(G3) and

$$\lim_{s\to\infty} \frac{s^{\frac{1}{N-1}} G(s)}{e^{Ks^{\frac{1}{N-1}}}} = 0 \tag{19}$$

for some positive constant K provided that $\lim_{s\to\infty} G(s)/e^{K_- s^{\frac{1}{N-1}}} = \infty$ and $\lim_{s\to\infty} G(s)/e^{K_+ s^{\frac{1}{N-1}}} = 0$ for any $K_- < K < K_+$, then $\mathscr{D}_{G,\alpha} < \infty$ if and only if $\alpha \leq (\alpha_N^* K)^{N-1}$, where $\alpha_N^* = N\omega_{N-1}^{1/(N-1)}$ and ω_{N-1} is the surface area of the unit sphere in $\mathbb{R}^N$. Moreover, $\mathscr{D}_{G,\alpha}$ is attained for any $\alpha \leq (\alpha_N^* K)^{N-1}$ by the compactness result in [28] and the arguments of [7] (see Remark 2.8 in [7]). If G satisfies (G1)-(G3),

$$\lim_{s\to\infty} \frac{s^{\frac{1}{N-1}} G(s)}{e^{Ks^{\frac{1}{N-1}}}} = \infty \quad \text{and} \quad \lim_{s\to\infty} \frac{G(s)}{e^{Ks^{\frac{1}{N-1}}}} < \infty, \tag{20}$$

then $\mathscr{D}_{G,\alpha} < \infty$ if and only if $\alpha < (\alpha_N^* K)^{N-1}$ by [3] and [28]. In the situation $\mathscr{D}_{G,\alpha} < \infty$, there exists a maximizer of $\mathscr{D}_{G,\alpha}$ by the same reason as in the case (19). In the case

$$0 < \lim_{s\to\infty} \frac{s^{\frac{1}{N-1}} G(s)}{e^{Ks^{\frac{1}{N-1}}}} < \infty,$$

different from the case $N = 2$, condition of existence of a maximizer for $\mathscr{D}_{G,(\alpha_N^* K)^{N-1}}$ is still open. Thus, if the growth of G satisfies at least (19), (20) or the subcritical growth, then G satisfies (G4) in the higher dimensional case.

Quasilinear elliptic equations related to variational problems $\mathscr{C}_{G,\mu,\alpha}$ and $\mathscr{D}_{G,\alpha}$ are of the form

$$-\Delta_N u + u^{N-1} = \Lambda u^{N-1} g(u^N), \quad u > 0 \quad \text{in} \quad \mathbb{R}^N \tag{21}$$

with positive constant Λ, where Δ_N is the usual N-Laplace operator defined by $\Delta_N u := \operatorname{div}\left(|\nabla u|^{N-2}\nabla u\right)$. If $u \in W^{1,N}(\mathbb{R}^N)$ is a solution of (21), then $u \in C^{1,\rho}_{loc}(\mathbb{R}^N)$ by the conditions (G1)-(G3) and the regularity result obtained by E. DiBenedetto [12]. Thus, by the same argument to prove Claim 5.3 in [14], we obtain that any solution $u \in W^{1,N}(\mathbb{R}^N)$ of (21) satisfies

$$\int_{\mathbb{R}^N} \left(\Lambda G(u^N) - u^N\right) dx = 0.$$

Consequently, we extend Theorems 1-4 to the following results.

Theorem 8 *Assume that $u_0 \in W^{1,N}(\mathbb{R}^N)$ is a maximizer of $\mathscr{C}_{G,\mu,\alpha}$. Then, there exists a positive constant Λ_0 such that u_0 is a ground state solution of (21) with $\Lambda = \Lambda_0$, up to scaling.*

Theorem 9 *Assume that*

$$G(s) = e^{s^{\frac{1}{N-1}}} - \sum_{j=0}^{N-2} \frac{s^{\frac{j}{N-1}}}{j!}$$

and w_Λ is a ground state solution of (21) for $\Lambda > 0$. Let $\alpha_\mu = \int_{\mathbb{R}^N} \left(|\nabla w_\Lambda|^N + \mu w_\Lambda^N\right) dx$ for $\mu > 0$. Then, there exists $\Lambda_* \in (0, (N-1)!)$ such that for any $\Lambda \in (0, \Lambda_*)$ and $\mu > 0$, either $\alpha_\mu > (\alpha_N^*)^{N-1}$ or $\int_{\mathbb{R}^2} G(w_\Lambda^N) < \mathscr{C}_{G,\mu,\alpha_\mu}$ provided that $\alpha_\mu \leq (\alpha_N^*)^{N-1}$.

Theorem 10 *Assume that G satisfies (G1)-(G3) and $v_0 \in W^{1,N}(\mathbb{R}^N)$ is a maximizer of $\mathscr{D}_{G,\alpha}$. Then, v_0 is a ground state solution of (21) for $\Lambda = \mathscr{D}_{G,\alpha}^{-1}$, up to scaling.*

Theorem 11 *Assume that G satisfies (G1)-(G4) and $w_0 \in W^{1,N}(\mathbb{R}^N)$ is a ground state solution of (21) for $\Lambda > 0$. Let $\alpha_0 = \int_{\mathbb{R}^2} |\nabla w_0|^N dx$. Then,*

$$\Lambda = \mathscr{D}_{G,\alpha_0}^{-1}$$

and w_0 is a maximizer of $\mathscr{D}_{G,\alpha_0}$.

Acknowledgements This work was supported by JSPS KAKENHI Grant Nos. 19K14571 and 23K13002. This work was partly supported by MEXT Promotion of Distinctive Joint Research Center Program JPMXP0723833165.

Funding Open Access funding provided by Osaka University.

Data availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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