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Calculus of Variations

Higher differentiability and integrability for some nonlinear elliptic systems with growth coefficients in *BMO*

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Abstract

We consider local solutions u of nonlinear elliptic systems of the type

$$\operatorname{div} A(x, Du) = \operatorname{div} F \qquad \text{in} \quad \Omega \subset \mathbb{R}^n,$$

where $u : \Omega \to \mathbb{R}^N$ is in a weighted $W^{1,p}_{loc}$ space, with $p \geq 2$, F is in a weighted $W^{1,2}_{loc}$ space and $x \to A(x, \xi)$ has growth coefficients in the space of functions with bounded mean oscillation. We prove higher differentiability of u in the sense that the nonlinear expression of its gradient $V_\mu(Du) := (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du$, with $0 < \mu \leq 1$, is weakly differentiable with $D(V_\mu(Du))$ in a weighted L^2_{loc} space. Moreover we derive some local Calderón–Zygmund estimates when the source term is not necessarily differentiable. Global estimates for a suitable Dirichlet problem are also available.

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Contents

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1 Introduction

We consider nonlinear elliptic systems of the type

$$\operatorname{div} A(x, Du(x)) = \operatorname{div} F(x) \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, $n > 2$, and with $u : \Omega \to \mathbb{R}^N$, $N > 1$. We suppose that the vector field $A : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ is a Carathéodory function, i.e.

- $x \to A(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^{N\times n}$,
- $\xi \to A(x, \xi)$ is continuous for a.e. $x \in \Omega$.

Furthermore, we assume that there exist a function $b(x) \geq \lambda_0 > 0$, belonging to the space BMO, and a function $K(x)$, belonging to the Marcinkiewicz space $L^{n,\infty}(\Omega)$, such that $F \in W^{1,2}_{loc}(b, \Omega; \mathbb{R}^{N\times n})$ and, for a.e. $x, y \in \Omega$,

$$|A(x, \xi) - A(x, \eta)| \leq k b(x) |\xi - \eta| \, (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}, \tag{1.2}$$

$$\frac{1}{k} b(x) |\xi - \eta|^2 \, (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle, \tag{1.3}$$

$$|A(x, \eta) - A(y, \eta)| \leq |x - y| \, [K(x) + K(y)] \, (\mu^2 + |\eta|^2)^{\frac{p-1}{2}} \tag{1.4}$$

$$A(x, 0) = 0 \tag{1.5}$$

$$|b(x) - b(y)| \leq |x - y| \, [K(x) + K(y)], \tag{1.6}$$

where k is a positive constant, $\mu \in (0, 1]$, $p \geq 2$, ξ and η are arbitrary elements of $\mathbb{R}^{N\times n}$. For the definition of weighted Sobolev spaces, see Sect. 2 below. Note that, by virtue of a characterization of the Sobolev functions due to Hajlasz [43], the conditions (1.4) and (1.6) describe a weak form of continuity with respect to the x-variable since the function K may blow up at some points.

In the account of the typical functions of BMO and $L^{n,\infty}$ respectively, the functions

$$\begin{aligned} b(x) &= \frac{e^{-|x|}}{\Lambda} - \Lambda \log |x| \\ K(x) &= \frac{e^{-|x|}}{\Lambda} + \Lambda \frac{1}{|x|}, \end{aligned} \tag{1.7}$$

defined for a positive Λ with $x \in B(0, 1) = \{y \in \mathbb{R}^n : 0 < |y| < 1\}$, satisfy assumption (1.6).

A vector field u in the Sobolev space $W^{1,r}_{loc}(b, \Omega; \mathbb{R}^N)$, $r > \frac{2n}{n+2}$, is a local solution of (1.1) if it verifies

$$\int_{\operatorname{supp}\phi} \langle A(x, Du(x)), D\phi(x) \rangle \, dx$$

$$= \int_{\operatorname{supp}\phi} \langle F(x), D\phi(x)\rangle\, dx \qquad \forall \phi \in C_0^\infty(\Omega, \mathbb{R}^N). \tag{1.8}$$

In this paper our first goal is to study regularity properties of local solutions to (1.1) for r close to p. The existence of second derivatives is not clear due to the degeneracy of the problem; anyway, although the first derivatives of the solutions may not be differentiable, the higher differentiability of solutions holds in the sense that the nonlinear expressions $V_\mu(Du) := (\mu^2+|Du|^2)^{\frac{p-2}{4}} Du$ of their gradients, with $\mu \in (0, 1]$, are weakly differentiable. Therefore, the main result will be the following.

Theorem 1.1 *Let Ω be a regular domain, $A(x, \xi)$ a mapping verifying assumptions (1.2)–(1.5), and $F \in W^{1,2}_{loc}(b, \Omega; \mathbb{R}^{N\times n})$, with $b(x)$ as in (1.6). There exist $0 < \varepsilon_1 < \frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, and $\alpha_1 > 0$, depending on p, n, λ_0, μ and k, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b, \Omega; \mathbb{R}^N)$, with $0 \le \varepsilon < \varepsilon_1$, is a local solution of (1.1) and*

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1, \tag{1.9}$$

then $D(V_\mu(Du)) \in L^2_{loc}(b, \Omega)$ and the following estimate holds:

$$\int_{B_R} |D(V_\mu(Du))|^2 b\, dx \le c \int_{B_{2R}} \left(\left(1 + \frac{1}{R^2}\right) (\mu^2 + |Du|^2)^{\frac{p}{2}} + (\mu^2 + |DF|^2) \right) b\, dx,$$

for every ball $B_{2R} \subset\subset \Omega$ and for a constant c depending on p, k, λ_0, n, μ and $\mathscr{D}_K$.

For the definition of regular domain, see the Sect. 2 below. Anyway, balls and cubes of $\mathbb{R}^n$ are regular domains. The novelty of Theorem 1.1 is to consider nonlinear systems with growth coefficients in BMO and not uniformly continuous in the spatial variable, whose feature is that they are allowed to be very irregular. Moreover we deal with local solutions u to (1.1) lying in $W^{1,r}$ with $r \le p$. In this case the energy functional

$$\int_\Omega \langle A(x, Du(x)), Du(x)\rangle\, dx \tag{1.10}$$

could not be bounded. We refer to such a solution as a *very weak solution* as stated by Iwaniec and Sbordone [49]. We explicitly remark that, thanks to the embedding Theorem 2.1 below our results apply if the growth coefficients lie in $W^{1,n}$. The condition (1.9) is clearly satisfied if the derivatives of $A(x, \xi)$ with respect to x belong to any subspace of $L^{n,\infty}$ in which L^∞ is dense, and then, in particular, if they belong to $L^{n,q}$ with $1 < q < \infty$. On the contrary, L^∞ is not dense in $L^{p,\infty}$ for any $p > 1$. We point out that condition (1.9) does not in general imply the smallness of the norm of $K(x)$ in $L^{n,\infty}$. In fact, if $K(x)$ is the function in (1.7), an elementary calculation shows that it reduces to consider the constant $\Lambda < \alpha_1 \omega_n^{-\frac{1}{n}}$, where ω_n denotes the measure of unit ball in $\mathbb{R}^n$. For a more complete treatment about condition (1.9), we refer to [31, 32]. In these papers a bound similar to (1.9) turned out to be useful in proving the existence of solutions to noncoercive PDEs having singularities in the coefficients of lower order terms. We also mention the similar conditions in [11, 40, 41].

In the linear case, the study of the second order regularity of solutions to equations with discontinuous coefficients goes back to Miranda who considered in [63, 64] equations with coefficients in the Sobolev class $W^{1,n}$. Subsequently, a complete regularity theory for equations in nondivergence form was developed by assuming coefficients in the vanishing mean oscillation space VMO (see e.g. [23, 24]). Regularity results of Schauder type in the class of Hölderian funtions are proved by Campanato [18, 20]). More recently, in connection with the regularity of minimizers of functionals of the Calculus of Variations [1], the study of higher differentiability for solutions to problems of the type (1.1) had a remarkable development. In particular, in the vectorial case $N > 1$, estimates as in Theorem 1.1 are important elements to prove partial regularity properties of solutions to nonlinear elliptic systems with Uhlenbeck structure. We refer to [53, 54, 59] and reference therein for an almost complete treatment. In [68] Stroffolini studied the Dirichlet problem for very weak solutions to a linear system with coefficients in BMO. Recently in [65], assuming a condition similar to (1.9), a higher differentiability result has been proved for this class of systems. The case of a nonlinear system with $b(x) \in L^\infty(\Omega)$ has been considered in [35]. Optimal second order regularity properties of solutions to nonlinear p-Laplacean systems are given in [25], when the datum in the right hand side of (1.1) is not in divergence form.

We point out that for local solutions of homogeneous systems

$$\operatorname{div} A(x, Du) = 0,$$

Theorem 1.1 also applies in the degenerate case, i.e. $\mu = 0$, with constants independent of μ (see Proposition 5.1). As a consequence, in Sect. 6 we establish certain local Calderón and Zygmund type estimates without assuming any differentiability condition on the datum. More precisely, for $G \in L^p_{loc}(b, \Omega; \mathbb{R}^{N\times n})$ we consider the problem

$$\operatorname{div} A(x, Du(x)) = \operatorname{div} |G|^{p-2} G \quad \text{in} \quad \Omega. \tag{1.11}$$

Then we prove the following result:

Theorem 1.2 *Let Ω be a regular domain and $A(x, \xi)$ a mapping verifying assumptions (1.2)–(1.5), with $b(x)$ as in (1.6). There exists $\alpha_2 > 0$, depending on p, n, λ_0 and k, such that, if $u \in W^{1,p}_{loc}(b, \Omega; \mathbb{R}^N)$ is a local solution of (1.11) and*

$$\mathscr{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_2, \tag{1.12}$$

then

$$G \in L^q_{loc}(b, \Omega; \mathbb{R}^{N\times n}) \quad \Longrightarrow \quad Du \in L^q_{loc}(b, \Omega; \mathbb{R}^{N\times n})$$

for any $q \in (p, s)$, where $s := \frac{np}{n-1} + \delta$ for a suitable $\delta > 0$, depending on p, k, λ_0, n, $\mathscr{D}_K$ and the BMO-norm of b. Moreover, for every cube $Q_{2R} \subset\subset \Omega$ and $\mu \in [0, 1]$, we have

$$\left(\fint_{Q_R} (\mu^2 + |Du|^2)^{\frac{q}{2}}\, b\, dx \right)^{\frac{1}{q}} \le c \left(\fint_{Q_{2R}} (\mu^2 + |Du|^2)^{\frac{p}{2}}\, b\, dx \right)^{\frac{1}{p}} + c \left(\fint_{Q_{2R}} (\mu^2 + |G|^2)^{\frac{q}{2}}\, b\, dx \right)^{\frac{1}{q}},$$

where c depends on p, $s - q$, k, λ_0, n, $\mathscr{D}_K$ and the BMO-norm of b and is independent of μ.

This type of result was previously established, in the case of the p-Laplacean equation with $p > 2$, in the fundamental paper by Iwaniec [45]. Let us remark that such kind of estimate is relevant to provide upper bounds for the Hausdorff dimension of the singular set of minima of general variational integrals (see [54, 55, 62]). Additionally, the a priori knowledge of higher integrability of the gradient allows to implement better schemes in the numerical treatment of problems modeled by energies like (1.10), as e.g. electrorheological fluids. Subsequently

Iwaniec's results were generalized to systems by DiBenedetto and Manfredi [27]. Regarding equations of the type

$$\operatorname{div}\left((A(x)\,\nabla u\cdot\nabla u)^{\frac{p-2}{2}}\,A(x)\,\nabla u\right)=\operatorname{div}|G|^{p-2}G, \tag{1.13}$$

with $A(x):\Omega\to\mathbb{R}^{n\times n}$ symmetric, local and global estimates for the gradient of a solution were considered by Iwaniec and Sbordone [47, 48] and by Kinnunen and Zhou [52] when the coefficients of $A(x)$ are bounded and in VMO. The condition about $A(x)$ in VMO is relaxed to a small BMO condition in [14, 15]. Recently local and global estimates for degenerate equations of the type (1.13) are given in weighted spaces in [6, 8] assuming a smallness condition for the BMO norm of $\log A(x)$ depending on the exponent q. This result is not strictly comparable with Theorem 1.2, where the exponent depends on the BMO norm and on the bound on the distance $\mathscr{D}_K$ in (1.12). Moreover, as mentioned earlier, we require no condition of smallness of the norm.

New main estimates for the development of nonlinear Calderón–Zygmund theory for equations and systems are due to Mingione, starting from pioneering papers [58, 61]. Regarding systems, weaker results are available unless in the case of the p-Laplacean system (see [27, 71]). Indeed, some bounds on exponent q is necessary even in the case of Uhlenbeck-type systems according to the example exhibited in [69]. If the vector field $A(\xi)$ is sufficiently regular, then CZ-estimates survive for $q\in\left(p,\frac{np}{n-2}\right)$ (see [29, 60] and references therein). A significant extension of CZ-theory to non-uniformly elliptic operators shaped on the $p(x)$-Laplacean [2, 21] and to the double-phase problems [26] has also been established, following the fundamental paper [57].

Now, let us spend some words on the strategy of the proof. In Theorem 1.1 we deal with very weak solutions and so, a priori, we cannot use in (1.8) test functions proportional to a solution u. Then, in Sect. 3, we first achieve a higher integrability result. Following [46, 49], via a weighted version of Hodge decomposition [22] and connectedness arguments, we construct suitable test functions and in Lemma 3.1 we prove a reverse Hölder inequality for Du. The statement of this lemma does not require assumptions (1.4) and (1.6) and extends a result proved in [34]. For another approach to treat very weak solutions see [56]. Once acquired the higher integrability of Du, in Sect. 4 we prove an a priori estimate by using the classic difference quotient method (for details see for example [1, 37, 39]). Finally, in Sect. 5, the proof of Theorem 1.1 follows by constructing appropriate approximating boundary value problems, whose solvability is known and for which the a priori estimate applies. In order to prove Theorem 1.2, the main difficulty is the interplay of the nonlinearity and the presence of a weight which does not allow us to follow the scheme of previous papers, based on comparing a solution w to the initial problem with the solution to homogeneous systems with frozen coefficients, i.e. $\operatorname{div} A(x_0, Dw)=0$. In order to deal with such a peculiarity, we first compare a local solution to (1.11) with the solution to a related homogeneous Dirichlet problem for which higher integrability follows from Theorem 1.1. So, as in [55], we shall rely on a technique introduced by Caffarelli and Peral [16, 17], and based on Calderón and Zygmund type covering arguments and iteration of level sets, combined with a clever use of Harmonic Analysis tools such as weighted versions of Maximal function inequalities. Finally, in Sect. 7 we present global versions of Theorem 1.1 and Theorem 1.2. We study the Dirichlet problem with zero boundary condition on a regular C^2 domain. Since mollifiers and quasiconformal homeomorphisms preserve the BMO norm [5] and the distance $\mathscr{D}_K$ [10] respectively, the proof of these results follows in a standard way (see Theorems 7.1, 7.2). When Ω is not regular the problem is more delicate [25, 29, 54].

We point out that through all the paper we consider $p \geq 2$. As known, in the subquadratic case the assumptions and the results change, according to the properties of the p-Laplacean operator and the degeneracy of the problem. In the case of systems with right-hand side affected by weak integrability properties, the existence of solutions to boundary value problems obtained as the limit of smooth solutions to approximating problems is only known under the assumption that $p > 2 - \frac{1}{n}$ [28]. Theorem 1.1 extends for homogeneous regular systems [1, 9, 25, 29]. For regular systems also CZ type estimates are available when $p > 2-\frac{1}{n}$ [60]. For nonhomogeneous p-Laplacean systems Theorem 1.1 holds when $p > \frac{3}{2}$ for a datum not in divergence form and lying in L^2 [25]. An improvement of the range of p is given in [7]. Some techniques presented here are suitable to be extended, but since they are already delicate, at this stage we prefer to confine ourselves to the superquadratic case in order to highlight the main ideas and novelties.

2 Preliminaries

This section is devoted to notation and preliminary results useful for our aims. Regarding definitions, notations and main properties of Lorentz spaces and difference quotients, we refer to sections 2.3 and 2.4 in [65].

2.1 BMO spaces

Definition 2.1 ([12, 50]) Let Ω be a cube or the entire space $\mathbb{R}^n$. The $BMO(\Omega)$ space consists of all functions b which are integrable on every cube $Q \subset \Omega$ with sides parallel to those of Ω and satisfy:

$$\|b\|_* = \sup_Q \left\{ \frac{1}{|Q|} \int_Q |b - b_Q|\, dx \right\} < \infty,$$

where $b_Q = \frac{1}{|Q|} \int_Q b(y)\, dy$ and $|Q|$ denotes the Lebesgue measure of Q.

It is clear that the functional $\|\cdot\|_*$ does not define a norm since it vanishes on constant functions. However BMO becomes a Banach space provided we identify functions which differ a. e. by a constant.

Bounded functions are in BMO. On the other hand, BMO contains unbounded functions and is contained in L^p_{loc} spaces. The standard example of BMO function is

$$f(x) = \log|x|, \qquad x \in B_1(0) \setminus 0,$$

where, for $R > 0$ and $x_0 \in \mathbb{R}^n$, we define

$$B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\},$$

but in the case no ambiguity arises, we shall use the short notation B_R. We also recall the following property

Theorem 2.1 ([12]) *For any cube $Q \subset \mathbb{R}^n$ the following inclusion holds with continuous embedding:*

$$W^{1,n}(Q) \hookrightarrow BMO(Q).$$

2.2 Muckenhoupt weights

Definition 2.2 ([33]) Given a weight w, i.e. a nonnegative function locally integrable in $\mathbb{R}^n$, we say that w belongs to the A_p class of Muckenhoupt, with $1 < p < \infty$, if

$$A_p(w) := \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty$$

where the supremum is taken all over cubes Q of $\mathbb{R}^n$. We say w belongs to the A_1 class of Muckenhoupt if

$$A_1(w) := \sup_Q \left(\fint_Q w \right) \left(\operatorname{ess\,sup}_Q \left(w^{-1} \right) \right) < \infty,$$

where the supremum is taken all over cubes Q of $\mathbb{R}^n$. The number $A_p(w)$ is called the A_p constant of w.

Note that, if $1 \le p < q$, then $A_p \subset A_q$. In fact, if $p > 1$, by Hölder's inequality, we have

$$\begin{aligned} \left(\fint_Q w \right) \left(\fint_Q w^{-\frac{1}{q-1}} \right)^{q-1} &\le \left(\fint_Q w \right) \frac{\left(\int_Q 1 \right)^{q-p} \left(\int_Q w^{-\frac{1}{p-1}} \right)^{p-1}}{|Q|^{q-1}} \\ &= \left(\fint_Q w \right) \left(\fint_Q w^{-\frac{1}{p-1}} \right)^{p-1} \le A_p(w). \end{aligned}$$

If $p = 1$ then

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{q-1}} \right)^{q-1} \le \operatorname{ess\,sup}_{x \in Q} \left(w^{-1}(x) \right) &= \operatorname{ess\,sup}_{x \in Q} \left(w^{-1}(x) \right) \left(\fint_Q w \right) \left(\fint_Q w \right)^{-1} \\ &\le A_1(w) \left(\frac{|Q|}{w(Q)} \right), \end{aligned}$$

where we have set, for every measurable set $E \subset \mathbb{R}^n$,

$$w(E) := \int_E w \, dx. \tag{2.1}$$

Note that the measure defined in (2.1) is doubling (see [70, Chapter IX, Theorem 2.1]).

Definition 2.3 ([68]) Let $k(x) : \mathbb{R}^n \to \mathbb{R}^n$. We will call k a Calderon–Zygmund kernel (CZ kernel) if k satisfies the following properties

- $k(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$,
- $k(x)$ is homogeneous of degree $-n$,
- $\int_\Sigma k(x) \, d\sigma_x = 0$ where Σ is the unit sphere of $\mathbb{R}^n$.

Given such a kernel, one can define a bounded operator in L^p, $1 < p < \infty$, called Calderon–Zygmund singular operator, as follows

$$Kf(x) = P.V.(k\star f)(x) := P.V.\int_{\mathbb{R}^n} k(x-y)f(y)\,dy.$$

Given a measurable subset E of $\mathbb{R}^n$, we will denote by $L^p(w, E; \mathbb{R}^N)$, $1 < p < \infty$, the Banach space of all measurable functions f defined on E for which

$$\|f\|_{L^p_w(E)} = \left(\int_E |f(x)|^p w(x)\,dx\right)^{\frac{1}{p}} < \infty.$$

It is well known that the singular integral operators are bounded on weighted L^p spaces for weights belonging to the A_p class. A theorem due to Buckley explicitly gives the dependence of the $L^p(w, \mathbb{R}^n)$ - norm of a singular integral operator on the A_p constant of w. Namely

Theorem 2.2 ([13]) *Let w be an A_p weight and K a singular integral operator. Then, for every $f \in L^p(w, E; \mathbb{R}^N)$, there exists a constant $c = c(n, p)$ such that*

$$\|Kf\|^p_{L^p_w(E)} \leq c A_p(w)^{p+p'} \|f\|^p_{L^p_w(E)}$$

where $p' = \frac{p}{p-1}$.

Since both A_p condition and the definition of BMO deal with the averaging of functions it is natural to consider the connections between these two classes. Among a lot of results in this direction, we point out the following

Lemma 2.1 ([51]) *Let $b(x)$ be a function such that b, $\frac{1}{b}$ both belong to $BMO(\mathbb{R}^n)$. Then*

$$b \in \bigcap_{p>1} A_p$$

and

$$A_p(b) \leq c + c\|b\|_*$$

where c is a constant depending only on p.

We can state the following weighted versions of Imbedding Theorem and Sobolev–Poincaré inequality:

Theorem 2.3 ([30]) *Given $1 < p < \infty$ and $w \in A_p$, there exist constants c, depending on n, p and the A_p constant of w, and $\zeta > 0$, depending on n and p, such that for all balls B_R, all $u \in C^\infty_0(B_R)$ and all numbers k satisfying $1 \leq k \leq \frac{n}{n-1} + \zeta$,*

$$\left(\frac{1}{w(B_R)}\int_{B_R} |u|^{kp} w\,dx\right)^{\frac{1}{kp}} \leq cR\left(\frac{1}{w(B_R)}\int_{B_R} |\nabla u|^p w\,dx\right)^{\frac{1}{p}}.$$

Theorem 2.4 ([30]) *Let $1 < p < \infty$ and $w \in A_p$. Then there are constants c, depending on n, p and the A_p constant of w, and $\zeta > 0$, depending on n and p, such that for all Lipschitz continuous functions u defined on $\overline{B_R}$ and for all $1 \leq k \leq \frac{n}{n-1} + \zeta$,*

$$\left(\frac{1}{w(B_R)}\int_{B_R} \left|u(x) - A_{B_R}\right|^{kp} w\,dx\right)^{\frac{1}{kp}} \leq cR\left(\frac{1}{w(B_R)}\int_{B_R} |\nabla u|^p\, w\,dx\right)^{\frac{1}{p}},$$

where either $A_{B_R} = \frac{1}{w(B_R)}\int_{B_R} u(x)w(x)\,dx$ or $A_{B_R} = \frac{1}{|B_R|}\int_{B_R} u(x)\,dx$.

2.3 Hodge decomposition

We shall now discuss briefly the Hodge decomposition of vector fields; for a more complete treatment see [46, 49]. For a given vector field $L = (l^1, \ldots, l^n) \in L^p(\mathbb{R}^n; \mathbb{R}^n)$, $1 < p < \infty$, the Poisson equation $\Delta u = \operatorname{div} L$ can be solved by using the Riesz transforms in $\mathbb{R}^n$, $\mathcal{R} = (R_1, \ldots, R_n)$,

$$\nabla u = -(\mathcal{R} \otimes \mathcal{R})(L) =: \mathscr{K}(L).$$

Here the tensor product operator $\mathscr{K} = -\mathcal{R} \otimes \mathcal{R} = -[R_{ij}]$ is the $n \times n$ matrix of the second order Riesz transforms $R_{ij} = R_i \circ R_j$, $i, j = 1, \ldots, n$. Notice that the range of the operator

$$\mathscr{H} := \mathrm{Id} - \mathscr{K} : L^p(\mathbb{R}^n; \mathbb{R}^n) \to L^p(\mathbb{R}^n; \mathbb{R}^n)$$

consists of the divergence free vector fields. We then arrive at the familiar Hodge decomposition of L

$$L = \nabla u + H, \quad \operatorname{div} H = 0.$$

Hence, L^p-estimates for Riesz transform yield an uniform estimate

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} + \|H\|_{L^p(\mathbb{R}^n)} \le c(p)\|L\|_{L^p(\mathbb{R}^n)}.$$

Let $\Omega \subset \mathbb{R}^n$ be a domain and $G = G(x, y)$ the Green's function. For $h \in C_0^\infty(\Omega)$ the integral

$$u(x) = \int_\Omega G(x, y)h(y)\,dy$$

defines a solution of the Poisson equation $\Delta u = h$ with u vanishing on the boundary of Ω. If h has a divergence form, say $h = \operatorname{div} L$ with $L = (l^1, \ldots, l^n) \in C_0^\infty(\Omega; \mathbb{R}^n)$, then integration by parts yields

$$u(x) = -\int_\Omega \nabla_y G(x, y)L(y)\,dy.$$

Hence the gradient of u is expressed by a singular integral

$$\nabla u(x) = -\int_\Omega \nabla_x \nabla_y G(x, y)L(y)\,dy =: (\mathscr{K}_\Omega L)(x).$$

By Theorem 2.2 and Lemma 2.1, if $b \in BMO$ and $\frac{1}{b} \in BMO$, we have

$$\|\mathscr{K}_\Omega L\|^p_{L^p_b} \le c(1 + \|b\|_*)^{p+p'}\|L\|^p_{L^p_b}$$

If $1 < p < \infty$, let $\mathcal{D}^p(b, \Omega; \mathbb{R}^n)$ denote the closure of the range of the gradient operator $\nabla : C_0^\infty(\Omega) \to L^p(b, \Omega; \mathbb{R}^n)$, i.e.

$$\mathcal{D}^p(b, \Omega; \mathbb{R}^n) := \overline{\left\{\nabla v \,:\, v \in C_0^\infty(\Omega)\right\}}^{L^p_b}.$$

If Ω is smooth, then $\mathscr{K}_\Omega$ extends continuously to all $L^p(b, \Omega; \mathbb{R}^n)$ spaces. Consequently the formula $\nabla u = \mathscr{K}_\Omega L$ extends to all $L \in L^p(b, \Omega; \mathbb{R}^n)$ giving a solution with $\nabla u \in \mathcal{D}^p(b, \Omega; \mathbb{R}^n)$, $1 < p < \infty$.

Definition 2.4 ([49]) A domain $\Omega \subset \mathbb{R}^n$ will be called regular if the operator $\mathscr{K}_\Omega$ acts boundedly in all $L^p(b, \Omega; \mathbb{R}^n)$-spaces, for $1 < p < \infty$.

For Ω a regular domain we introduce, as before, the operator

$$\mathscr{H}_\Omega := \mathrm{Id} - \mathscr{K}_\Omega : L^p(b, \Omega; \mathbb{R}^n) \to L^p(b, \Omega; \mathbb{R}^n).$$

Obviously, the range of $\mathscr{H}_\Omega$ consists of the divergence free vector fields on Ω. We have the Hodge decomposition of $L \in L^p(b, \Omega; \mathbb{R}^n)$,

$$L = \nabla u + H, \quad \operatorname{div} H = 0, \quad \nabla u \in \mathcal{D}^p(b, \Omega; \mathbb{R}^n).$$

We deduce the following stability property in our decomposition

Lemma 2.2 ([22]) *Let Ω be a regular domain of $\mathbb{R}^n$ and consider $b \in BMO$ such that $\frac{1}{b} \in BMO$. If $u \in W_0^{1,r-\varepsilon}(b, \Omega; \mathbb{R}^N)$, $1 < r < \infty$, $-1 < 2\varepsilon < r - 1$, there exist $\phi \in W_0^{1,\frac{r-\varepsilon}{1-\varepsilon}}(b, \Omega; \mathbb{R}^N)$ and a divergence free vector field $H \in L^{\frac{r-\varepsilon}{1-\varepsilon}}(b, \Omega; \mathbb{R}^{N\times n})$ such that*

$$|Du|^{-\varepsilon} Du = D\phi + H.$$

Moreover

$$\begin{aligned} \|D\phi\|_{L_b^{\frac{r-\varepsilon}{1-\varepsilon}}(\Omega)} &\le c(n,r)(1 + \|b\|_*)^{\gamma(r)} \|Du\|^{1-\varepsilon}_{L_b^{r-\varepsilon}(\Omega)} \\ \|H\|_{L_b^{\frac{r-\varepsilon}{1-\varepsilon}}(\Omega)} &\le c(n,r)(1 + \|b\|_*)^{\gamma(r)} |\varepsilon|\, \|Du\|^{1-\varepsilon}_{L_b^{r-\varepsilon}(\Omega)} \end{aligned}$$

where $\gamma(r)$ is an exponent depending only on r.

2.4 Maximal operators

Let $Q_0 \subset \mathbb{R}^n$ be a cube. We shall consider, in the following, the Restricted Maximal Function Operator relative to Q_0. This is defined as

$$M^*_{Q_0}(f)(x) := \sup_{\substack{Q \subseteq Q_0 \\ x \in Q}} \fint_Q |f(y)|\, dy, \qquad x \in Q_0,$$

whenever $f \in L^1(Q_0)$, where Q denotes any cube contained in Q_0 with sides parallel to those of Q_0, as long as $x \in Q$. We recall the following weak type (p, p) estimate for $M^*_{Q_0}$, valid for any $p \in [1, \infty)$:

$$\left|\left\{x \in Q_0 : M^*_{Q_0}(f)(x) \ge t\right\}\right| \le \frac{c(n,p)}{t^p} \int_{Q_0} |f(y)|^p\, dy \qquad t > 0 \tag{2.2}$$

which is valid for any $f \in L^p(Q_0)$. For this and related issues we refer to [67].
If w is a weight and $Q_0 \subset \mathbb{R}^n$ is a cube, we define the weighted Restricted Maximal Function Operator relative to Q_0 as

$$M^*_{w,Q_0}(f)(x) := \sup_{\substack{Q \subseteq Q_0 \\ x \in Q}} \frac{\int_Q |f(y)| w(y)\, dy}{w(Q)}, \qquad x \in Q_0,$$

whenever $f \in L^1(w, Q_0)$, where Q denotes any cube contained in Q_0 with sides parallel to those of Q_0, as long as $x \in Q$. We have the following weighted generalization of (2.2):

Theorem 2.5 ([70]) *Suppose $w \in A_p$, $1 < p < \infty$. Then M^*_{w,Q_0} maps $L^p(w, Q_0)$ into weak-$L^p(w, Q_0)$, with norm independent in A_p.*

Finally we recall the following useful lemmas:

Lemma 2.3 ([36]) *For $R_0 < R_1$, consider a bounded function $f : [R_0, R_1] \to [0, \infty)$ with*

$$f(s) \le \theta f(t) + \frac{A}{(s-t)^\delta} + B \quad \text{for all } R_0 < s < t < R_1,$$

where A, B and δ denote non - negative constants and $\theta \in (0, 1)$. Then we have

$$f(R_0) \le c(\delta, \theta)\left(\frac{A}{(R_1 - R_0)^\delta} + B\right),$$

where $c(\delta, \theta)$ is increasing with respect to δ.

Lemma 2.4 ([37]) *For any $p \ge 2$, we have*

$$c^{-1}(\mu^2 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}}|\eta - \xi|^2 \le |V_\mu(\eta) - V_\mu(\xi)|^2 \le c(\mu^2 + |\eta|^2 + |\xi|^2)^{\frac{p-2}{2}}|\eta - \xi|^2$$

for any η, $\xi \in \mathbb{R}^k$, $\mu \in [0, 1]$ and a constant $c = c(p) > 0$.

Lemma 2.5 ([55]) *Let $p \in [2, \infty)$ and $\mu \in [0, 1]$, then*

$$(\mu^2 + |z|^2)^{\frac{p-2}{2}}|z|\,|\zeta| \le \varepsilon(\mu^2 + |z|^2)^{\frac{p-2}{2}}|z|^2 + \varepsilon^{1-p}(\mu^2 + |\zeta|^2)^{\frac{p-2}{2}}|\zeta|^2$$

for every z, $\zeta \in \mathbb{R}^{N\times n}$ and $\varepsilon \in (0, 1]$.

Lemma 2.6 ([17]) *Let $Q_0 \subset \mathbb{R}^n$ be a cube and $\mathcal{D}(Q_0)$ be the class of all dyadic cubes obtained from Q_0. Let $a \in (0, 1)$. Assume that $X \subset Y \subset Q_0$ are measurable sets satisfying the following conditions:*

- $|X| < a|Q_0|$
- *if* $Q \in \mathcal{D}(Q_0)$ *then* $|X \cap Q| > a|Q| \implies \tilde{Q} \subset Y$,

where $\tilde{Q}$ denotes the predecessor of Q. Then

$$|X| < a|Y|.$$

By means of an analogous proof of [38, Proposition 5.1] we get

Lemma 2.7 *Let $f \in L^r(w, \Omega)$ and $g \in L^s(w, \Omega)$ be non-negative functions, where $1 < r < s$, Ω is an open set, w is a weight. If the following*

$$\frac{\int_{B_R} f^r w\,dx}{w(B_R)} \le c\left(\left(\frac{\int_{B_{2R}} f w\,dx}{w(B_{2R})}\right)^r + \frac{\int_{B_{2R}} g^r w\,dx}{w(B_{2R})}\right), \qquad c > 1,$$

holds for every pair of concentric balls $B_R \subset B_{2R} \subset\subset \Omega$, then there exists $\varepsilon > 0$ such that $f \in L^{r+\varepsilon}_{loc}(w, \Omega)$.

3 Higher integrability

In this section we prove a higher integrability result useful to our aims. We consider the nonlinear elliptic system

$$\operatorname{div} A(x, Du(x)) = \operatorname{div} |G|^{p-2}G \quad \text{in} \quad \Omega. \tag{3.1}$$

We begin with a reverse Hölder inequality.

Lemma 3.1 *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions (1.2), (1.3) and (1.5), $\mu\in[0,1]$, and let $G\in L^{p+\delta}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$, for $\delta\geq 0$. Then there exist positive constants ε_1 and c, depending on n, λ_0, $\|b\|_*$, k, p, with $0<\varepsilon_1<\frac{1}{2}$, such that if $u\in W^{1,p-\varepsilon}_{loc}$ verifies (3.1) and $-\min\{\varepsilon_1,\delta\}<\varepsilon\leq\varepsilon_1$, then*

$$\frac{1}{b(B_R)}\int_{B_R}\left(\mu^2+|Du|^2+|u|^2\right)^{\frac{p-\varepsilon}{2}}b\,dx\leq c\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^2+|Du|^2+|u|^2)^{\frac{\sigma}{2}}b\,dx\right)^{\frac{p-\varepsilon}{\sigma}}+\frac{c}{b(B_{2R})}\int_{B_{2R}}\left(\mu^2+|G|^2\right)^{\frac{p-\varepsilon}{2}}b\,dx \tag{3.2}$$

for every σ with $\max\left\{1,\frac{(n-1)(p-\varepsilon)}{n}\right\}\leq\sigma<p-\varepsilon$ and for every pair of concentric balls $B_R\subset B_{2R}\subset\subset\Omega$ with $R<1$.

Proof Fix a ball $B_R(x_0)$ with $R<1$ such that $B_{2R}(x_0)\subset\subset\Omega$. For $R\leq s<t\leq 2R$, we consider the balls centered at x_0 with radii $R,s,t,2R$. Let $\xi:\mathbb{R}^n\to\mathbb{R}$ be the usual cut-off function, that is $\xi\in C^\infty_0(B_t)$ with $0\leq\xi\leq 1$, $\xi=1$ on B_s and $|\nabla\xi|\leq\frac{1}{t-s}$. Let us assume that $u\in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$ is a local solution of (3.1), with $-1<2\varepsilon<p-1$. By Lemma 2.2 applied to $\xi(u-\lambda)$, $\lambda\in\mathbb{R}^N$, there exist $\phi\in W_0^{1,\frac{p-\varepsilon}{1-\varepsilon}}(b,\Omega;\mathbb{R}^N)$ and $H\in L^{\frac{p-\varepsilon}{1-\varepsilon}}(b,\Omega;\mathbb{R}^{N\times n})$ such that

$$\begin{aligned}&|D[\xi(u-\lambda)]|^{-\varepsilon}D[\xi(u-\lambda)]=D\phi+H\\&\|D\phi\|_{L_b^{\frac{p-\varepsilon}{1-\varepsilon}}(\Omega)}\leq c(n,p)(1+\|b\|_*)^\gamma\|D[\xi(u-\lambda)]\|^{1-\varepsilon}_{L_b^{p-\varepsilon}(\Omega)}\\&\|H\|_{L_b^{\frac{p-\varepsilon}{1-\varepsilon}}(\Omega)}\leq c(n,p)(1+\|b\|_*)^\gamma|\varepsilon|\,\|D[\xi(u-\lambda)]\|^{1-\varepsilon}_{L_b^{p-\varepsilon}(\Omega)},\end{aligned}$$

where γ is an exponent depending only on p. We use ϕ as a test function in (1.8); this yields

$$\begin{aligned}\int_{B_t}\langle|G|^{p-2}G,D\phi\rangle\,dx&=\int_{B_t}\langle A(x,Du),D\phi\rangle\,dx\\&=\int_{B_t}\langle A(x,D[\xi(u-\lambda)]),D\phi\rangle\,dx+\int_{B_t}\langle[A(x,Du)-A(x,D[\xi(u-\lambda)])],D\phi\rangle\,dx\\&=\int_{B_t}\langle A(x,D[\xi(u-\lambda)]),|D[\xi(u-\lambda)]|^{-\varepsilon}D[\xi(u-\lambda)]\rangle\,dx\\&\quad-\int_{B_t}\langle A(x,D[\xi(u-\lambda)]),H\rangle\,dx+\int_{B_t}\langle[A(x,Du)-A(x,D[\xi(u-\lambda)])],D\phi\rangle\,dx.\end{aligned}$$

Hence

$$\begin{aligned}\int_{B_t}\langle A(x,D[\xi(u-\lambda)]),|D[\xi(u-\lambda)]|^{-\varepsilon}D[\xi(u-\lambda)]\rangle&=\int_{B_t}\langle A(x,D[\xi(u-\lambda)]),H\rangle\\&-\int_{B_t}\langle[A(x,Du)-A(x,D[\xi(u-\lambda)])],D\phi\rangle+\int_{B_t}\langle|G|^{p-2}G,D\phi\rangle.\end{aligned}$$

Now we use (1.3) and (1.5) to obtain

$$\frac{1}{k}\int_{B_t}|D[\xi(u-\lambda)]|^{p-\varepsilon}b\,dx\leq\int_{B_t}\langle A(x,D[\xi(u-\lambda)]),|D[\xi(u-\lambda)]|^{-\varepsilon}D[\xi(u-\lambda)]\rangle\,dx. \tag{3.3}$$

We then apply (1.2), (1.5), together with Hölder's and Young's inequalities with exponents $\frac{p-\varepsilon}{p-1}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, to get

$$\begin{aligned}
&\int_{B_t} |\langle A(x, D[\xi(u-\lambda)]), H\rangle|\,dx \le k \int_{B_t} (\mu^2 + |D[\xi(u-\lambda)]|^2)^{\frac{p-1}{2}} |H| b\,dx \\
&\le c(k,p) \int_{B_t} (\mu^{p-1} + |D[\xi(u-\lambda)])|^{p-1})|H| b\,dx \\
&\le c(n, \|b\|_*, k, p)\,|\varepsilon| \Bigg(\left(\int_{B_t} \mu^{p-\varepsilon} b\,dx \right)^{\frac{p-1}{p-\varepsilon}} \left(\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx \right)^{\frac{1-\varepsilon}{p-\varepsilon}} \qquad (3.4) \\
&\quad + \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx \Bigg) \\
&\le c(n, \|b\|_*, k, p)|\varepsilon| \left(\int_{B_t} \mu^{p-\varepsilon} b\,dx + \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx \right).
\end{aligned}$$

On the other hand,

$$|Du| \le |Du - D[\xi(u-\lambda)]| + |D[\xi(u-\lambda)]|,$$

thus

$$\begin{aligned}
(\mu^2 + |Du|^2 + |D[\xi(u-\lambda)]|^2)^{\frac{p-2}{2}} &\le c(p)(\mu^{p-2} + |Du - D[\xi(u-\lambda)]|^{p-2} \\
&\quad + |D[\xi(u-\lambda)]|^{p-2})
\end{aligned}$$

and, by (1.2), we have

$$\begin{aligned}
&\int_{B_t} |\langle A(x, Du) - A(x, D[\xi(u-\lambda)]), D\phi\rangle|\,dx \\
&\le c \int_{B_t} |Du - D[\xi(u-\lambda)]|\,|D\phi|(\mu^{p-2} + |Du - D[\xi(u-\lambda)]|^{p-2} \\
&+ |D[\xi(u-\lambda)]|^{p-2}) b\,dx \\
&= c\Bigg(\int_{B_t} \mu^{p-2} |Du - D[\xi(u-\lambda)]|\,|D\phi| b\,dx + \int_{B_t} |Du - D[\xi(u-\lambda)]|^{p-1} |D\phi| b\,dx \\
&\quad + \int_{B_t} |Du - D[\xi(u-\lambda)]|\,|D[\xi(u-\lambda)]|^{p-2}\,|D\phi| b\,dx \Bigg),
\end{aligned}$$

with $c = c(k, p)$. Next, we apply the straightforward equality $Du - D[\xi(u-\lambda)] = (1 - \xi)Du - (u-\lambda)\nabla\xi$:

$$\begin{aligned}
&\int_{B_t} |\langle A(x, Du) - A(x, D[\xi(u-\lambda)]), D\phi\rangle|\,dx \le c(k,p) \Bigg(\int_{B_t} \mu^{p-2}(1-\xi)\,|Du|\,|D\phi| b\,dx \\
&+ \int_{B_t} \mu^{p-2} |\nabla\xi|\,|u-\lambda||D\phi| b\,dx + \int_{B_t} |(1-\xi)Du|^{p-1} |D\phi| b\,dx \\
&+ \int_{B_t} |(u-\lambda)\nabla\xi|^{p-1} |D\phi| b\,dx + \int_{B_t} (1-\xi)|Du|\,|D[\xi(u-\lambda)]|^{p-2} |D\phi| b\,dx \\
&+ \int_{B_t} |\nabla\xi|\,|u-\lambda|\,|D[\xi(u-\lambda)[|^{p-2} |D\phi| b\,dx \Bigg) =: I + II + III + IV + V + VI.
\end{aligned}$$

In order to estimate I, if $p > 2$, using Hölder's and Young's inequalities with exponents $p-\varepsilon$, $\frac{p-\varepsilon}{p-2}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, and for $\dot{\varepsilon} > 0$, we get:

$$\begin{aligned}
I &\le \left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx\right)^{\frac{1}{p-\varepsilon}} \left(\int_{B_t} \mu^{p-\varepsilon} b\,dx\right)^{\frac{p-2}{p-\varepsilon}} \left(\int_{B_t} |D\phi|^{\frac{p-\varepsilon}{1-\varepsilon}} b\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le c(n, \|b\|_*, p) \left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx\right)^{\frac{1}{p-\varepsilon}} \left(\int_{B_t} \mu^{p-\varepsilon} b\,dx\right)^{\frac{p-2}{p-\varepsilon}} \\
&\quad \times \left(\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le c(n, \|b\|_*, p)^{p-\varepsilon} \left(\frac{1}{\dot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx + \int_{B_t} \mu^{p-\varepsilon} b\,dx \\
&\quad + \dot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx.
\end{aligned} \tag{3.5}$$

Now, since c may be assumed greater than 1, $c^{p-\varepsilon}$ is less than c^p. Therefore, we can assume that the constant c is independent of ε. If $p = 2$, since $\mu \le 1$, we argue as before by applying Hölder's and Young's inequalities with exponents $p-\varepsilon$ and $\frac{p-\varepsilon}{p-\varepsilon-1}$:

$$\begin{aligned}
I &\le \left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx\right)^{\frac{1}{p-\varepsilon}} \left(\int_{B_t} |D\phi|^{\frac{p-\varepsilon}{p-\varepsilon-1}} b\,dx\right)^{\frac{p-\varepsilon-1}{p-\varepsilon}} \\
&\le c(n, \|b\|_*, p) \left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx\right)^{\frac{1}{p-\varepsilon}} \left(\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le c(n, \|b\|_*, p) \left(\frac{1}{\dot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx + \dot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx.
\end{aligned} \tag{3.6}$$

Replacing $|(1-\xi)Du|$ by $|(u-\lambda)\nabla\xi|$, we get the desired estimate for II, if $p > 2$:

$$\begin{aligned}
II &\le c \left(\frac{1}{\dot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\,dx + \int_{B_t} \mu^{p-\varepsilon} b\,dx + \dot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \\
&\quad \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx
\end{aligned} \tag{3.7}$$

and $p = 2$:

$$II \le c \left(\frac{1}{\dot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\,dx + \dot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx, \tag{3.8}$$

with $c = c(n, \|b\|_*, p)$. Using Hölder's and Young's inequalities with exponents $\frac{p-\varepsilon}{p-1}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, and if $\ddot{\varepsilon} > 0$, we obtain:

$$\begin{aligned}
III &\le c(n, \|b\|_*, p) \left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx\right)^{\frac{p-1}{p-\varepsilon}} \left(\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx\right)^{\frac{1-\varepsilon}{p-\varepsilon}} \\
&\le c(n, \|b\|_*, p) \left(\frac{1}{\ddot{\varepsilon}}\right)^{\frac{p-\varepsilon}{p-1}} \int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx + \ddot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx.
\end{aligned} \tag{3.9}$$

In the same way, we estimate IV, namely

$$
\begin{aligned}
IV &= \int_{B_t} |(u-\lambda)\nabla\xi|^{p-1}|D\phi| b\, dx \\
&\le c(n, \|b\|_*, p)\left(\frac{1}{\ddot{\varepsilon}}\right)^{\frac{p-\varepsilon}{p-1}} \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\, dx + \ddot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\, dx.
\end{aligned}
\tag{3.10}
$$

Arguing as for I and II, with $\dddot{\varepsilon} > 0$, we estimate V

$$
\begin{aligned}
V &= \int_{B_t} |(1-\xi)Du|\,|D[\xi(u-\lambda)]|^{p-2}|D\phi| b\, dx \\
&\le c(n, \|b\|_*, p)\left(\frac{1}{\dddot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\, dx + \dddot{\varepsilon}^{\frac{p-\varepsilon}{p-\varepsilon-1}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\, dx
\end{aligned}
\tag{3.11}
$$

and VI

$$
\begin{aligned}
VI &= \int_{B_t} |(u-\lambda)\nabla\xi|\,|D[\xi(u-\lambda)]|^{p-2}|D\phi| b\, dx \\
&\le c(n, \|b\|_*, p)\left(\frac{1}{\dddot{\varepsilon}}\right)^{p-\varepsilon} \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\, dx + \dddot{\varepsilon}^{\frac{p-\varepsilon}{p-\varepsilon-1}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\, dx.
\end{aligned}
\tag{3.12}
$$

Finally, by Hölder's and Young's inequalities with exponents $\frac{p-\varepsilon}{p-1}$ and $\frac{p-\varepsilon}{1-\varepsilon}$, supposing $|\varepsilon| \le \delta$, and if $\ddddot{\varepsilon} > 0$, we have

$$
\begin{aligned}
\int_{B_t} \left|\langle |G|^{p-2}G, D\phi\rangle\right| dx &\le \frac{1}{\lambda_0}\int_{B_t} |G|^{p-1}|D\phi| b\, dx \le \frac{1}{\lambda_0}\|G\|^{p-1}_{L_b^{p-\varepsilon}(B_t)}\|D\phi\|_{L_b^{\frac{p-\varepsilon}{1-\varepsilon}}(B_t)} \\
&\le c(n, \|b\|_*, p, \lambda_0)\left(\frac{1}{\ddddot{\varepsilon}}\right)^{\frac{p-\varepsilon}{p-1}} \int_{B_t} |G|^{p-\varepsilon} b\, dx + \ddddot{\varepsilon}^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\, dx.
\end{aligned}
\tag{3.13}
$$

Combining estimates (3.3)–(3.13) yields

$$
\begin{aligned}
&\frac{1}{k}\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\, dx \\
&\le c(n, \|b\|_*, k, p, \lambda_0)\Bigg(\left(|\varepsilon| + (\dot{\varepsilon} + \ddot{\varepsilon} + \ddddot{\varepsilon})^{\frac{p-\varepsilon}{1-\varepsilon}} + \dddot{\varepsilon}^{\frac{p-\varepsilon}{p-\varepsilon-1}}\right) \int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\, dx \\
&\quad + \left(\left(\frac{1}{\dot{\varepsilon}} + \frac{1}{\dddot{\varepsilon}}\right)^{p-\varepsilon} + \left(\frac{1}{\ddot{\varepsilon}}\right)^{\frac{p-\varepsilon}{p-1}}\right)\left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\, dx + \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\, dx\right) \\
&\quad + (|\varepsilon| + 1)\int_{B_t} \mu^{p-\varepsilon} b\, dx + \left(\frac{1}{\ddddot{\varepsilon}}\right)^{\frac{p-\varepsilon}{1-\varepsilon}} \int_{B_t} |G|^{p-\varepsilon} b\, dx\Bigg)
\end{aligned}
$$

for arbitrary positive numbers $\dot{\varepsilon}$, $\ddot{\varepsilon}$, $\dddot{\varepsilon}$, $\ddddot{\varepsilon}$. We now choose ε, $\dot{\varepsilon}$, $\ddot{\varepsilon}$, $\dddot{\varepsilon}$ and $\ddddot{\varepsilon}$ to be such that $c(n, \|b\|_*, k, p, \lambda_0) \cdot \left(|\varepsilon| + (\dot{\varepsilon} + \ddot{\varepsilon} + \dddot{\varepsilon})^{\frac{p-\varepsilon}{1-\varepsilon}} + \ddddot{\varepsilon}^{\frac{p-\varepsilon}{p-\varepsilon-1}}\right) < \frac{1}{2k}$. To this effect, we fix $\varepsilon_1 > 0$ sufficiently small. Accordingly,

$$\int_{B_t} |D[\xi(u-\lambda)]|^{p-\varepsilon} b\,dx \le c(n, \|b\|_*, k, p, \lambda_0)\left(\int_{B_t} |(1-\xi)Du|^{p-\varepsilon} b\,dx \right.$$
$$\left. + \int_{B_t} |(u-\lambda)\nabla\xi|^{p-\varepsilon} b\,dx + \int_{B_t} \mu^{p-\varepsilon} b\,dx + \int_{B_t} |G|^{p-\varepsilon} b\,dx\right),$$

for $-\min\{\varepsilon_1, \delta\} < \varepsilon \le \varepsilon_1$ and c independent of ε. The properties of the cut-off function ξ and the previous inequality yield

$$\int_{B_s} |Du|^{p-\varepsilon} b\,dx \le c\Big(\int_{B_t\setminus B_s} |Du|^{p-\varepsilon} b\,dx + \frac{1}{(t-s)^{p-\varepsilon}} \int_{B_t} |u-\lambda|^{p-\varepsilon} b\,dx$$
$$+ \int_{B_t} (\mu^{p-\varepsilon} + |G|^{p-\varepsilon}) b\,dx\Big).$$

Adding $c\int_{B_s} |Du|^{p-\varepsilon} b\,dx$ to both sides yields

$$\int_{B_s} |Du|^{p-\varepsilon} b\,dx \le \frac{c}{c+1} \int_{B_t} |Du|^{p-\varepsilon} b\,dx + \frac{c}{(c+1)(t-s)^{p-\varepsilon}} \int_{B_{2R}} |u-\lambda|^{p-\varepsilon} b\,dx$$
$$+ \frac{c}{c+1} \int_{B_{2R}} (\mu^{p-\varepsilon} + |G|^{p-\varepsilon}) b\,dx.$$

With the notation (2.1), we set

$$\lambda = u_R := \frac{\int_{B_{2R}} u(x) b\,dx}{b(B_{2R})}$$

and we apply Lemma 2.3 to get

$$\int_{B_R} |Du|^{p-\varepsilon} b\,dx \le c\left(R^{-(p-\varepsilon)} \int_{B_{2R}} |u-u_R|^{p-\varepsilon} b\,dx + \int_{B_{2R}} (\mu^{p-\varepsilon} + |G|^{p-\varepsilon}) b\,dx\right).$$

We add $\int_{B_R} \left(|u|^{p-\varepsilon} + \mu^{p-\varepsilon}\right) b\,dx$ to both sides. Since $|u|^{p-\varepsilon} \le c(|u-u_R|^{p-\varepsilon} + |u_R|^{p-\varepsilon})$ and $R < 1$, we obtain

$$\int_{B_R} (\mu^{p-\varepsilon} + |Du|^{p-\varepsilon} + |u|^{p-\varepsilon}) b\,dx \le c\Bigg(R^{-(p-\varepsilon)} \int_{B_{2R}} |u-u_R|^{p-\varepsilon} b\,dx$$
$$+ \int_{B_{2R}} |u_R|^{p-\varepsilon} b\,dx + \int_{B_{2R}} \left(\mu^2 + |G|^2\right)^{\frac{p-\varepsilon}{2}} b\,dx\Bigg), \tag{3.14}$$

with $c = c(n, \|b\|_*, k, p, \lambda_0)$. Note that by Jensen's inequality

$$\int_{B_{2R}} |u_R|^{p-\varepsilon} b\,dx = b(B_{2R}) \left(\frac{\int_{B_{2R}} |u(x)|\, b\,dx}{b(B_{2R})}\right)^{p-\varepsilon}$$
$$\le \int_{B_{2R}} |u(x)|^{p-\varepsilon} b\,dx \le R^{-(p-\varepsilon)} \int_{B_{2R}} |u(x)|^{p-\varepsilon} b\,dx. \tag{3.15}$$

Theorem 2.3 and Lemma 2.1 yield

$$\frac{R^{-(p-\varepsilon)}}{b(B_{2R})} \int_{B_{2R}} |u|^{p-\varepsilon} b\,dx \le c\left(\frac{1}{b(B_{2R})} \int_{B_{2R}} |\nabla u|^{\sigma} b\,dx\right)^{\frac{p-\varepsilon}{\sigma}} \tag{3.16}$$

for all σ such that $\max\left\{1, \frac{(n-1)p-\varepsilon}{n}\right\} \leq \sigma < p-\varepsilon$ and with $c = c(n, \|b\|_*, k, p, \lambda_0)$. Theorem 2.4 and Lemma 2.1 yield

$$\frac{R^{-(p-\varepsilon)}}{b(B_{2R})}\int_{B_{2R}}|u-u_R|^{p-\varepsilon}b\,dx \leq c\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}|\nabla u|^\sigma b\,dx\right)^{\frac{p-\varepsilon}{\sigma}} \tag{3.17}$$

for all σ such that $\max\left\{1, \frac{(n-1)p-\varepsilon}{n}\right\} \leq \sigma < p-\varepsilon$ and with $c = c(n, \|b\|_*, k, p, \lambda_0)$. Putting together (3.14)–(3.17), since the measure $b\,dx$ is doubling and by means of Lemma 2.1, we have

$$\begin{aligned}
&\frac{c(n,p,\|b\|_*)}{b(B_R)}\int_{B_R}(\mu^2+|Du|^2+|u|^2)^{\frac{p-\varepsilon}{2}}b\,dx\\
&\leq \frac{c(n,p,\|b\|_*)}{b(B_R)}\int_{B_R}(\mu^{p-\varepsilon}+|Du|^{p-\varepsilon}+|u|^{p-\varepsilon})b\,dx\\
&\leq \frac{1}{b(B_{2R})}\int_{B_R}(\mu^{p-\varepsilon}+|Du|^{p-\varepsilon}+|u|^{p-\varepsilon})b\,dx\\
&\leq \frac{c(n,\|b\|_*,k,p,\lambda_0)}{b(B_{2R})}\Bigg(R^{-(p-\varepsilon)}\int_{B_{2R}}|u-u_R|^{p-\varepsilon}b\,dx\\
&\quad + R^{-(p-\varepsilon)}\int_{B_{2R}}|u(x)|^{p-\varepsilon}b\,dx+\int_{B_{2R}}\left(\mu^2+|G|^2\right)^{\frac{p-\varepsilon}{2}}b\,dx\Bigg)\\
&\leq c(n,\|b\|_*,k,p,\lambda_0)\Bigg[\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^\sigma+|Du|^\sigma+|u|^\sigma)b\,dx\right)^{\frac{p-\varepsilon}{\sigma}}\\
&\quad+\frac{1}{b(B_{2R})}\int_{B_{2R}}\left(\mu^2+|G|^2\right)^{\frac{p-\varepsilon}{2}}b\,dx\Bigg],
\end{aligned} \tag{3.18}$$

for all σ such that $\max\left\{1, \frac{(n-1)p-\varepsilon}{n}\right\} \leq \sigma < p-\varepsilon$ and we can conclude the proof of the reverse Hölder's inequality (3.2). □

Now we are ready to prove the main result of this section.

Theorem 3.1 *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions (1.2), (1.3) and (1.5), $\mu \in [0,1]$, and let $G \in L^{p+\delta}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$, for $\delta \geq 0$. Then there exists $0 < \varepsilon_1 < \frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$, with $0 \leq \varepsilon < \varepsilon_1$, is a local solution of (3.1), then $u \in W^{1,p-\tilde{\varepsilon}}_{loc}(b,\Omega,\mathbb{R}^N)$, for any $0 \leq |\tilde{\varepsilon}| \leq \min\{\delta,\varepsilon_1\}$.*

Proof Let $u \in W^{1,p-\bar{\varepsilon}}_{loc}(b,\Omega;\mathbb{R}^N)$ verify the Eq. (1.8), with $0 \leq \bar{\varepsilon} \leq \varepsilon_1$. For $\Omega' \subset\subset \Omega$, set $A = \{\varepsilon \in [-\min\{\delta,\varepsilon_1\},\varepsilon_1] : u \in W^{1,p-\varepsilon}_{loc}(b,\Omega';\mathbb{R}^N)\}$. We claim that $A = [-\min\{\delta,\varepsilon_1\},\varepsilon_1]$. In order to see this, we first note that A is not empty, since $\varepsilon_1 \in A$. Our goal is to show that A is open and closed in $[-\min\{\delta,\varepsilon_1\},\varepsilon_1]$. First we prove that A is open. Indeed, if $\varepsilon_2 \in A$, by the reverse Hölder inequality in Lemma 3.1 and by the higher integrability result stated in Lemma 2.7, there exists $\varepsilon > 0$ such that $\max\{\varepsilon_2-\varepsilon, -\min\{\delta,\varepsilon_1\}\} \in A$. Therefore A is open. Now we prove that A is closed too. Let $\{\rho_k\} \subset A$ be such that $\rho_k \to \rho$. We want to prove that $\rho \in A$. Obviously, $u \in W^{1,p-\rho_k}_{loc}(b,\Omega';\mathbb{R}^N)$ and

$$
\begin{aligned}
&\frac{1}{b(B_R)}\int_{B_R}\left(\mu^2+|Du|^2+|u|^2\right)^{\frac{p-\rho_k}{2}} b\,dx \le \\
&\le c\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^2+|Du|^2+|u|^2)^{\frac{\sigma_k}{2}} b\,dx\right)^{\frac{p-\rho_k}{\sigma_k}} \\
&+\frac{c}{b(B_{2R})}\int_{B_{2R}}\left(\left(\mu^2+|G|^2\right)^{\frac{p-\rho_k}{2}} b\,dx\right)
\end{aligned} \tag{3.19}
$$

with $\sigma_k = \max\{1, (n-1)(p-\rho_k)/n\}$. Let us observe that $(\mu^2+|Du|^2+|u|^2)^{\frac{p-\rho_k}{2}} \to (\mu^2+|Du|^2+|u|^2)^{\frac{p-\rho}{2}}$, almost everywhere in B_{2R}. Therefore, we can apply Fatou's Lemma in order to estimate the left-hand side of (3.19). We obtain

$$
\begin{aligned}
&\frac{1}{b(B_R)}\int_{B_R}(\mu^2+|Du|^2+|u|^2)^{\frac{p-\rho}{2}} b\,dx \le \liminf_k \frac{1}{b(B_R)} \\
&\int_{B_R}(\mu^2+|Du|^2+|u|^2)^{\frac{p-\rho_k}{2}} b\,dx.
\end{aligned} \tag{3.20}
$$

In order to pass to the limit on the right-hand side of (3.19), we assume that $\rho_k > \rho$ for every k, since otherwise $u \in W^{1,p-\rho}_{loc}(b,\Omega';\mathbb{R}^N)$ and the conclusion is obvious. With this assumption we have

$$
\sigma_k \le \max\{1, (n-1)(p-\rho)/n\} < p-\rho,
$$

since $1 < 2-\varepsilon_1 \le p-\rho_k < p-\rho$. Set $\sigma = \max\{1, (n-1)(p-\rho)/n\}$; the following inequality holds:

$$
(\mu^2+|Du|^2+|u|^2)^{\frac{\sigma_k}{2}} \le 1+(\mu^2+|Du|^2+|u|^2)^{\frac{\sigma}{2}}. \tag{3.21}
$$

For k large enough, we have $\sigma < p-\rho_k < p-\rho$ and, since $u \in W^{1,\sigma_k}_{loc}(b,\Omega';\mathbb{R}^N)$, the right-hand side of (3.21) is in L^1. Therefore, we can apply Lebesgue's Convergence Theorem in order to pass to the limit on the right-hand side of (3.19). Taking also into account that we may assume $|\rho_k| \le \delta$ for every k and $G \in L^{p+\delta}$, we get, recalling (3.20),

$$
\begin{aligned}
&\frac{1}{b(B_R)}\int_{B_R}\left(\mu^2+|Du|^2+|u|^2\right)^{\frac{p-\rho}{2}} b\,dx \\
&\le c\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^2+|Du|^2+|u|^2)^{\frac{\sigma}{2}} b\,dx\right)^{\frac{p-\rho}{\sigma}} \\
&+\frac{c}{b(B_{2R})}\int_{B_{2R}}\left(\left(\mu^2+|G|^2\right)^{\frac{p-\rho}{2}} b\,dx\right)
\end{aligned}
$$

Hence $\rho \in A$, therefore A is closed. □

Remark 3.1 By virtue of Theorem 3.1, in Sects. 4 and 5 we can assume $u \in W^{1,p+\varepsilon_1}_{loc}(b,\Omega;\mathbb{R}^N)$, for $\varepsilon_1 > 0$ sufficiently small. In fact, in Theorem 1.1 and in a priori estimate 4.1 we assume $F \in W^{1,2}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$. Therefore, with the notation $F = |G|^{p-2}G$, we have $|G|^{p-1} \in L^{2^*}_b$, with $2^* := \frac{2n}{n-2}$, i.e. equivalently $|G| \in L_b^{\frac{2np-2n}{n-2}}$. Finally, we note that $\frac{2np-2n}{n-2} > p$ for $p > 2_* := \frac{2n}{n+2}$.

4 A priori estimate

In this section we assume the weak differentiability of $V_\mu(Du)$ in order prove an a priori estimate.

Theorem 4.1 *Let Ω be a regular domain and $A(x,\xi)$ a mapping verifying assumptions (1.2)–(1.5). If $D(V_\mu(Du)) \in L^2_{loc}(b,\Omega)$, $\mu \in (0,1]$ and $F \in W^{1,2}_{loc}(b,\,\Omega;\mathbb{R}^{N\times n})$, there exists $\alpha > 0$, depending on p, n, λ_0, μ and k, such that, if*

$$\mathscr{D}_K := dist_{L^{n,\infty}}(K(x), L^\infty) < \alpha, \tag{4.1}$$

the following estimate holds:

$$\int_{B_R} |D(V_\mu(Du))|^2 b\,dx \le c\int_{B_{2R}} \left(\left(1+\frac{1}{R^2}\right)(\mu^2+|Du|^2)^{\frac{p}{2}} + (\mu^2+|DF|^2)\right) b\,dx \tag{4.2}$$

for every ball $B_{2R} \subset\subset \Omega$ and for a constant c depending on p, k, λ_0, n, μ and $\mathscr{D}_K$.

Proof For a fixed ball $B_{2R} \subset\subset \Omega$ and radii $R < s < t < 2R$ with R small enough, consider a function $\xi \in C_0^\infty(B_t)$, $0 \le \xi \le 1$, $\xi = 1$ on B_s, $|\nabla\xi| \le \frac{1}{t-s}$ and set $\psi = \xi^2\tau_h u$ for sufficiently small $h > 0$. Since u is a local solution of (1.1), we can choose $\phi = \tau_{-h}\psi$ as a test function. By virtue of the properties of the difference quotients, we have

$$\int_{B_t} \langle \tau_h A(x,Du), D\psi\rangle\,dx = \int_{B_t} \langle \tau_h F(x), D\psi\rangle\,dx$$

that is

$$\int_{B_t} \langle \tau_h A(x,Du), D(\xi^2\tau_h u)\rangle\,dx = \int_{B_t} \langle \tau_h F(x), D(\xi^2\tau_h u)\rangle\,dx.$$

It follows that

$$\begin{aligned}&\int_{B_t} \xi^2\langle \tau_h A(x,Du), \tau_h Du\rangle\,dx + 2\int_{B_t} \xi\langle \tau_h A(x,Du), \nabla\xi\otimes\tau_h u\rangle\,dx\\ &\quad= \int_{B_t} \langle \tau_h F(x), D(\xi^2\tau_h u)\rangle\,dx,\end{aligned} \tag{4.3}$$

and observing that

$$\begin{aligned}\tau_h A(x,Du) &= [A(x+he_i, Du(x+he_i)) - A(x+he_i, Du(x))]\\ &\qquad + [A(x+he_i, Du(x)) - A(x, Du(x))] =: \mathcal{A}_h + \mathcal{A}'_h\end{aligned}$$

the equality (4.3) can be rewritten as

$$\begin{aligned}\int_{B_t} \xi^2\langle \mathcal{A}_h, \tau_h Du\rangle\,dx &= -\int_{B_t} \xi^2\langle \mathcal{A}'_h, \tau_h Du\rangle\,dx - 2\int_{B_t} \xi\langle \mathcal{A}_h, \nabla\xi\otimes\tau_h u\rangle\,dx\\ &\quad - 2\int_{B_t} \xi\langle \mathcal{A}'_h, \nabla\xi\otimes\tau_h u\rangle\,dx + \int_{B_t} \langle \tau_h F, D(\xi^2\tau_h u)\rangle\,dx =: I_1+I_2+I_3+I_4.\end{aligned}$$

By Assumption (1.3), we immediately obtain for the left hand side that

$$\int_{B_t} \xi^2\langle \mathcal{A}_h, \tau_h Du\rangle\,dx \ge \frac{1}{k}\int_{B_t} \xi^2(\mu^2+|Du(x)|^2+|Du(x+he_i)|^2)^{\frac{p-2}{2}}|\tau_h Du|^2 b\,dx$$

and hence

$$\frac{1}{k}\int_{B_t}\xi^2(\mu^2+|Du(x)|^2+|Du(x+he_i)|^2)^{\frac{p-2}{2}}\frac{|\tau_h Du|^2}{|h|^2}b\,dx\le\frac{1}{|h|^2}\sum_{i=1}^4|I_i|.$$

Now let $K_0\in L^\infty(\Omega)$. In order to estimate $|I_j|$, $j=1,\dots,4$, we introduce the notation

$$\begin{aligned}\mathcal{K}(h)&:=K(x+he_i)+K(x),\\ \mathcal{D}(h)&:=(\mu^2+|Du(x)|^2+|Du(x+he_i)|^2)^{\frac{1}{2}}.\end{aligned}$$

By Assumption (1.4), we immediately have

$$\begin{aligned}|I_1|&\le\int_{B_t}\xi^2|\mathcal{A}_h'|\,|\tau_h Du|\,dx\\ &\le\int_{B_t}\xi^2|h|\mathcal{K}(h)(\mu^2+|Du|^2)^{\frac{p-1}{2}}|\tau_h Du|\,dx.\end{aligned}$$

Then, defining

$$\mathcal{K}_0(h):=K_0(x+he_i)+K_0(x),$$

the use of Young's inequality with a constant $\nu\in(0,1)$ that will be chosen later yields

$$\begin{aligned}|I_1|\le&\int_{B_t}\xi^2|h|\,\mathcal{K}(h)\,(\mu^2+|Du|^2)^{\frac{p-1}{2}}\,|D\tau_h u|\,dx\\ \le&\int_{B_t}\xi^2|h|\,|\mathcal{K}(h)-\mathcal{K}_0(h)|\,(\mu^2+|Du|^2)^{\frac{p-1}{2}}\,|D\tau_h u|\,dx\\ &+\int_{B_t}\xi^2|h|\,\|\mathcal{K}_0(h)\|_{L^\infty}\,(\mu^2+|Du|^2)^{\frac{p-1}{2}}\,|D\tau_h u|\,dx\\ \le&\frac{\nu}{2}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}\,|D\tau_h u|^2\,dx+\frac{|h|^2}{2\nu}\int_{B_t}|\mathcal{K}(h)-\mathcal{K}_0(h)|^2(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})^2\,dx\\ &+\frac{\nu}{2}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}|D\tau_h u|^2\,dx+\frac{|h|^2}{2\nu}\int_{B_t}\xi^2\|\mathcal{K}_0(h)\|^2_{L^\infty}(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx\\ \le&\,\nu\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}\,|D\tau_h u|^2\,dx+\frac{|h|^2}{\nu}\int_{B_t}|K(x)-K_0(x)|^2(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})^2\,dx\\ &+\frac{|h|^2}{\nu}\int_{B_t}|K(x+he_i)-K_0(x+he_i)|^2(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})^2\,dx\\ &+\frac{|h|^2}{2\nu}\int_{B_t}\xi^2\|\mathcal{K}_0(h)\|^2_{L^\infty}(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx\end{aligned}$$

Now note that, thanks to Lemma 2.4, the assumption $V_\mu(Du)\in W^{1,2}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$ guarantees that $(\mu^2+|Du|^2)^{\frac{p}{2}}\in W^{1,2}_{loc}(b,\Omega)$, and therefore it is in $W^{1,2}(\Omega)$, and in particular, by Sobolev embedding Theorem in Lorentz spaces (see [4]), that $\xi(\mu^2+|Du|^2)^{\frac{p}{2}}\in L^{\frac{n}{n-2},1}$ Consequently, by Hölder's inequality in Lorentz spaces (see [66]), set $2^*:=\frac{2n}{n-2}$, we can estimate the second integral in the right hand side of previous inequality as follows

$$|I_1| \le \nu \int_{B_t} \xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}|D\tau_h u|^2\,dx + \frac{2|h|^2}{\nu}\|K(x)-K_0\|^2_{L^{n,\infty}(B_t)}$$
$$\|\xi(\mu^2+|Du|^2)^{\frac{p}{4}}\|^2_{L^{2^*,2}(B_t)} + \frac{|h|^2}{2\nu}\int_{B_t}\xi^2\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx$$

Finally by Sobolev embedding Theorem in Lorentz spaces, and taking into account that $b(x)\ge\lambda_0$ and $(\mu^2+|Du(x)|^2)^{\frac{p-2}{2}} \le \mathcal{D}(h)^{p-2}$, we have that

$$\begin{aligned}|I_1| \le{}& \frac{\nu}{\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}|D\tau_h u|^2 b\,dx\\&+2\frac{|h|^2}{\nu}S^2_{2,n}\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)}\\&+\frac{|h|^2}{2\nu\lambda_0}\int_{B_t}\xi^2\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx \le\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\le \frac{\nu}{\lambda_0}\int_{B_t}\xi^2\,\mathcal{D}(h)^{p-2}|D\tau_h u|^2 b\,dx\\&+2\frac{|h|^2}{\nu}S^2_{2,n}\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)}\\&+\frac{|h|^2}{2\nu\lambda_0}\int_{B_t}\xi^2\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx.\end{aligned}$$

Next we estimate I_2. Observe that assumption (1.2) yields

$$\begin{aligned}|\mathcal{A}_h| &\le k\,b(x+he_i)\,|\tau_h Du|\,\left(\mu^2+|Du(x)|^2+|Du(x+he_i)|^2\right)^{\frac{p-2}{2}}\\&= k\,b(x+he_i)\,|\tau_h Du|\,\mathcal{D}(h)^{p-2}\end{aligned}$$

and hence, by the aid of Young's inequality, we obtain

$$\begin{aligned}|I_2| &\le 2\int_{B_t\setminus B_s}\xi\,|\mathcal{A}_h|\,|\nabla\xi|\,|\tau_h u|\,dx\\&\le 2k\int_{B_t\setminus B_s}\xi\,b(x+he_i)\,|\tau_h Du|\,\mathcal{D}(h)^{p-2}\,|\nabla\xi|\,|\tau_h u|\,dx\\&\le \nu\int_{B_t\setminus B_s}\xi^2\mathcal{D}(h)^{p-2}|\tau_h Du|^2 b(x+he_i)\,dx\\&\quad+\frac{k^2}{\nu}\int_{B_t\setminus B_s}|\nabla\xi|^2\,\mathcal{D}(h)^{p-2}|\tau_h u|^2 b(x+he_i)\,dx\\&\le \nu\int_{B_t\setminus B_s}\xi^2\mathcal{D}(h)^{p-2}|\tau_h Du|^2 b(x+he_i)\,dx\\&\quad+\frac{k^2}{\nu\,(t-s)^2}\int_{B_t\setminus B_s}\mathcal{D}(h)^{p-2}|\tau_h u|^2 b(x+he_i)\,dx.\end{aligned} \tag{4.5}$$

For I_3 we proceed as follows. The Assumption (1.4) yields

$$|I_3| \le 2\int_{B_t\setminus B_s}\xi\,|\mathcal{A}'_h|\,|\nabla\xi|\,|\tau_h u|\,dx \le 2|h|\int_{B_t\setminus B_s}\xi\,\mathcal{K}(h)\,|\nabla\xi|\,(\mu^2+|Du|^2)^{\frac{p-1}{2}}|\tau_h u|\,dx.$$

Arguing similarly as we have done for I_1, we have

$$\begin{aligned}
|I_3| &\le 2|h| \int_{B_t\setminus B_s} \xi |\mathcal{K}(h)-\mathcal{K}_0(h)|\,|\nabla\xi|(\mu^2+|Du|^2)^{\frac{p-1}{2}}|\tau_h u|\,dx \\
&\quad +2|h|\,\|\mathcal{K}_0(h)\|_{L^\infty(\Omega)} \int_{B_t\setminus B_s} \xi\,|\nabla\xi|(\mu^2+|Du|^2)^{\frac{p-1}{2}}|\tau_h u|\,dx \\
&\le \frac{|h|^2}{\nu}\int_{B_t}\xi^2|\mathcal{K}(h)-\mathcal{K}_0(h)|^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx \\
&\quad +\frac{2\nu}{(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|\tau_h u|^2\,dx \\
&\quad +\frac{|h|^2}{\nu}\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx \\
&\le \frac{2|h|^2}{\nu}\int_{B_t}\xi^2|K(x)-K_0(x)|^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx \\
&\quad +\frac{2|h|^2}{\nu}\int_{B_t}\xi^2|K(x+he_i)-K_0(x+he_i)|^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx \\
&\quad +\frac{2\nu}{(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|\tau_h u|^2dx+\frac{|h|^2}{\nu}\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,dx \\
&\le 4\frac{|h|^2}{\nu}S^2_{2,n}\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)} \\
&\quad +\frac{2\nu}{\lambda_0(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|\tau_h u|^2 b\,dx \\
&\quad +\frac{|h|^2}{\nu\lambda_0}\|K_0\|^2_{L^\infty(\Omega)}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx.
\end{aligned} \tag{4.6}$$

Finally we estimate I_4. By using Young's inequality, together with $b(x)\ge\lambda_0$, we get

$$\begin{aligned}
|I_4| &= \left|\int_{B_t}\langle\tau_h F, D(\xi^2\tau_h u)\rangle\,dx\right| \\
&\le \int_{B_t}\xi^2|\tau_h F|\,|D\tau_h u|\,dx+2\int_{B_t}\xi\,|\nabla\xi|\,|\tau_h F|\,|\tau_h u|\,dx \\
&\le \frac{\nu}{2}\int_{B_t}\xi^2|D\tau_h u|^2\,dx+\frac{1}{2\nu}\int_{B_t}\xi^2|\tau_h F|^2\,dx+\int_{B_t}\xi^2|\tau_h F|^2\,dx+\int_{B_t}|\nabla\xi|^2|\tau_h u|^2\,dx \\
&\le \frac{\nu}{2\lambda_0}\int_{B_t}\xi^2|D\tau_h u|^2 b\,dx+\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2|\tau_h F|^2\,dx+\frac{1}{(t-s)^2}\int_{B_t\setminus B_s}|\tau_h u|^2\,dx \\
&\le \frac{\nu}{2\mu^2\lambda_0}\int_{B_t}\xi^2\,(\mu^2+|Du|^2)^{\frac{p-2}{2}}|D\tau_h u|^2 b\,dx+\frac{1}{\lambda_0}\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2|\tau_h F|^2 b\,dx \\
&\quad +\frac{1}{\mu^2\lambda_0(t-s)^2}\int_{B_t\setminus B_s}(\mu^2+|Du|^2)^{\frac{p-2}{2}}|\tau_h u|^2 b\,dx.
\end{aligned} \tag{4.7}$$

Combining estimates (4.4)–(4.7), we get

$$\frac{1}{k}\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{\tau_h Du}{h}\right|^2 b\,dx \le \frac{\nu}{\lambda_0}\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{D\tau_h u}{h}\right|^2 b\,dx$$

$$+\frac{6}{\nu}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)}$$
$$+\frac{\nu}{2\mu^2\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}\left|\frac{D\tau_h u}{h}\right|^2 b\,dx+\nu\int_{B_t\setminus B_s}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{\tau_h Du}{h}\right|^2 b(x+he_i)\,dx$$
$$+\frac{3\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}}{2\nu\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx+\frac{2\mu^2\nu+1}{\mu^2\lambda_0(t-s)^2}\int_{B_t}(1+|Du|^2)^{\frac{p-2}{2}}\left|\frac{\tau_h u}{h}\right|^2 b\,dx$$
$$+\frac{k^2}{\nu(t-s)^2}\int_{B_t\setminus B_s}\mathcal{D}(h)^{p-2}\left|\frac{\tau_h u}{h}\right|^2 b(x+he_i)\,dx$$
$$+\frac{1}{\lambda_0}\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2\left|\frac{\tau_h F}{h}\right|^2 b\,dx.$$

Assuming $\nu<\frac{\lambda_0}{k}$ and reabsorbing the first integral in the right hand side by the left hand side, we get

$$\left(\frac{1}{k}-\frac{\nu}{\lambda_0}\right)\int_{B_t}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{\tau_h Du}{h}\right|^2 b\,dx$$
$$\le\frac{6}{\nu}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)}$$
$$+\frac{\nu}{2\mu^2\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p-2}{2}}\left|\frac{D\tau_h u}{h}\right|^2 b\,dx$$
$$+\nu\int_{B_t\setminus B_s}\xi^2\mathcal{D}(h)^{p-2}\left|\frac{\tau_h Du}{h}\right|^2 b(x+he_i)\,dx$$
$$+\frac{3\|\mathcal{K}_0(h)\|^2_{L^\infty(\Omega)}}{2\nu\lambda_0}\int_{B_t}\xi^2(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx$$
$$+\frac{2\mu^2\nu+1}{\mu^2\lambda_0(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p-2}{2}}\left|\frac{\tau_h u}{h}\right|^2 b\,dx$$
$$+\frac{k^2}{\nu(t-s)^2}\int_{B_t\setminus B_s}\mathcal{D}(h)^{p-2}\left|\frac{\tau_h u}{h}\right|^2 b(x+he_i)\,dx$$
$$+\frac{1}{\lambda_0}\left(\frac{1}{2\nu}+1\right)\int_{B_t}\xi^2\left|\frac{\tau_h F}{h}\right|^2 b\,dx.$$

Let us note that, by the properties of ξ and using the fact that $b(x)\ge\lambda_0$,

$$\begin{aligned}\|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})\|^2_{L^2(B_t)}&\le\frac{1}{\lambda_0}\int_{B_t}|D(\xi(\mu^2+|Du|^2)^{\frac{p}{4}})|^2 b\,dx\\&\le\frac{2}{\lambda_0}\int_{B_t}|\nabla\xi|^2(\mu^2+|Du|^2)^{\frac{p}{2}}\,b\,dx\\&\quad+\frac{2}{\lambda_0}\int_{B_t}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p}{4}}]|^2 b\,dx\\&\le\frac{2}{\lambda_0(t-s)^2}\int_{B_t}(\mu^2+|Du|^2)^{\frac{p}{2}}\,b\,dx\\&\quad+\frac{2}{\lambda_0}\int_{B_t}\xi^2|D[(\mu^2+|Du|^2)^{\frac{p}{4}}]|^2 b\,dx,\end{aligned}$$

and, by (1.6) and since by Lemma 8.1 in [39] we are legitimate to pass to the limit for $h \to 0$,

$$\int_{B_t \setminus B_s} \lim_{h \to 0} \left(\mathcal{D}(h)^{p-2} \left| \frac{\tau_h u}{h} \right|^2 b(x + he_i) \right) dx = \int_{B_t \setminus B_s} (\mu^2 + 2|Du|^2)^{\frac{p-2}{2}} |Du|^2 b \, dx$$
$$\leq \int_{B_t \setminus B_s} (2\mu^2 + 2|Du|^2)^{\frac{p-2}{2}} (\mu^2 + |Du|^2) b \, dx = 2^{\frac{p-2}{2}} \int_{B_t \setminus B_s} (\mu^2 + |Du|^2)^{\frac{p}{2}} b \, dx.$$

Applying now Lemma 2.4 and again Lemma 8.1 in [39], we have

$$\frac{1}{c(p)} \left(\frac{1}{k} - \frac{\nu}{\lambda_0} \right) \int_{B_t} \xi^2 |D[(\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du]|^2 b \, dx$$
$$\leq \frac{12}{\nu \lambda_0} S_{2,n}^2 \|K(x) - K_0\|_{L^{n,\infty}(\Omega)}^2 \int_{B_t} \xi^2 (D[(\mu^2 + |Du|^2)^{\frac{p}{4}}])^2 b \, dx$$
$$+ \frac{\nu}{2\mu^2 \lambda_0} \int_{B_t} \xi^2 (\mu^2 + |Du|^2)^{\frac{p-2}{2}} |D^2 u|^2 b \, dx$$
$$+ c(p)\, \nu \int_{B_t \setminus B_s} \xi^2 |D[(\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du]|^2 b \, dx$$
$$+ \frac{3\|K_0\|_{L^\infty(\Omega)}^2}{\nu \lambda_0} \int_{B_t} \xi^2 (\mu^2 + |Du|^2)^{\frac{p}{2}} b \, dx + \frac{c(p, k, \lambda_0, \nu, \mu)}{(t-s)^2} \int_{B_t} (\mu^2 + |Du|^2)^{\frac{p}{2}} b \, dx$$
$$+ \frac{12}{\nu \lambda_0 (t-s)^2} S_{2,n}^2 \|K(x) - K_0\|_{L^{n,\infty}(\Omega)}^2 \int_{B_t} (\mu^2 + |Du|^2)^{\frac{p}{2}} b \, dx$$
$$+ \frac{1}{\lambda_0} \left(\frac{1}{2\nu} + 1 \right) \int_{B_t} \xi^2 \left| \frac{\tau_h F}{h} \right|^2 b \, dx.$$

Therefore

$$\frac{1}{c(p)} \left(\frac{1}{k} - \frac{\nu}{\lambda_0} \right) \int_{B_t} \xi^2 |D[(\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du]|^2 b \, dx$$
$$\leq \left(\frac{12}{\nu \lambda_0} S_{2,n}^2 \|K(x) - K_0\|_{L^{n,\infty}(\Omega)}^2 + \frac{\nu}{2\mu^2 \lambda_0} \right) \int_{B_t} \xi^2 |D[(\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du]|^2 b \, dx$$
$$+ c(p) \nu \int_{B_t \setminus B_s} \xi^2 |D[(\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du]|^2 b \, dx$$
$$+ \frac{3\|K_0\|_{L^\infty(\Omega)}^2}{\nu \lambda_0} \int_{B_t} \xi^2 (\mu^2 + |Du|^2)^{\frac{p}{2}} b \, dx + \frac{c(p, k, \lambda_0, \nu, \mu)}{(t-s)^2} \int_{B_t} (\mu^2 + |Du|^2)^{\frac{p}{2}} b \, dx \tag{4.8}$$
$$+ \frac{12}{\nu \lambda_0 (t-s)^2} S_{2,n}^2 \|K(x) - K_0\|_{L^{n,\infty}(\Omega)}^2 \int_{B_t} (\mu^2 + |Du|^2)^{\frac{p}{2}} b \, dx$$
$$+ \frac{1}{\lambda_0} \left(\frac{1}{2\nu} + 1 \right) \int_{B_t} \xi^2 (\mu^2 + |DF|^2) b \, dx.$$

Let us fix $\nu := \nu_0$ such that

$$\frac{1}{c(p)} \left(\frac{1}{k} - \frac{\nu_0}{\lambda_0} \right) - \frac{\nu_0}{2\mu^2 \lambda_0} > 0,$$

for example $\nu_0 := \frac{\mu^2 \lambda_0}{k(2\mu^2 + c(p))}$. Set $\eta := \frac{\sqrt{\nu_0 \lambda_0}}{2\sqrt{3} S_{2,n}} \sqrt{\frac{1}{c(p)} \left(\frac{1}{k} - \frac{\nu_0}{\lambda_0} \right) - \frac{\nu}{2\mu^2 \lambda_0}}$, let α be a number such that $0 < \alpha < \eta$. If

$$\mathscr{D}_K < \alpha,$$

then we can choose $K_0 \in L^\infty(\Omega)$ such that

$$\left(\frac{1}{c(p)}\left(\frac{1}{k}-\frac{\nu_0}{\lambda_0}\right)-\frac{12}{\nu_0\lambda_0}S_{2,n}^2\|K(x)-K_0\|^2_{L^{n,\infty}(\Omega)}-\frac{\nu_0}{2\mu^2\lambda_0}\right)>0.$$

Then, by reabsorbing the first term of the right hand side of (4.8) in the left hand side, since $\xi = 1$ on B_s and $0 \le \xi \le 1$, we get

$$\begin{aligned}
C\int_{B_s}|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,dx &\le c(p,k,\lambda_0,\mu)\int_{B_t\setminus B_s}|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,dx\\
&+c(p,k,\lambda_0,n,\mathscr{D}_K,\mu)\left(\int_{B_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx+\frac{1}{(t-s)^2}\int_{B_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx\right.\\
&\left.+\int_{B_{2R}}(\mu^2+|DF|^2)b\,dx\right),
\end{aligned}$$

where $C=\frac{1}{c(p)}\left(\frac{1}{k}-\frac{\nu_0}{\lambda_0}\right)-\frac{3}{\nu_0\lambda_0}S_{2,n}^2\alpha^2-\frac{\nu_0}{2\mu^2\lambda_0}$.

Now we fill the hole, having

$$\begin{aligned}
\int_{B_s}|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,dx &\le \frac{c(p,k,\lambda_0,\mu)}{C+c(p,k,\lambda_0,\mu)}\int_{B_t}|D[(\mu^2+|Du|^2)^{\frac{p-2}{4}}Du]|^2 b\,dx\\
&+\frac{c(p,k,\lambda_0,n,\mathscr{D}_K,\mu)}{C+c(p,k,\lambda_0,\mu)}\left(\int_{B_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx+\frac{1}{(t-s)^2}\int_{B_{2R}}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx\right.\\
&\left.+\int_{B_{2R}}(\mu^2+|DF|^2)b\,dx\right).
\end{aligned}$$

Then by Lemma 2.3

$$\begin{aligned}
\int_{B_R}|D(V_\mu(Du))|^2 b\,dx &\le c\int_{B_{2R}}\left(1+\frac{1}{R^2}\right)(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx\\
&+c\int_{B_{2R}}(\mu^2+|DF|^2)b\,dx,
\end{aligned}\tag{4.9}$$

where $c=c(p,k,\lambda_0,n,\mathscr{D}_K,\mu)$, and therefore we have the result. □

Remark 4.1 Note that, even if we do not provide the exact value of the constant α in (4.1), a bound on it is given at the end of the proof of Theorem 4.1.

Remark 4.2 We point out that the dependence of the constant c in (4.9) on $\mathscr{D}_K$ occurs only through the norm of K_0 in L^∞.

Remark 4.3 By a careful analysis of the proof, it is evident that the degenerate case, that is for $\mu = 0$, causes further difficulties only when dealing with the integral involving the datum F. More specifically, in the estimate of $|I_4|$, an integral which can blow up appears. Consequently, if $F \equiv 0$, the proof proceeds in the same way even if $\mu = 0$.

5 Regularity

In this section we prove Theorem 1.1.

Proof of Theorem 1.1 Fix a ball $B_{2R} \subset\subset \Omega$, let u be a local solution of the system (1.1) and let us consider, for $x \in \Omega$, $\xi \in \mathbb{R}^{N\times n}$ and $j \in \mathbb{N}$ sufficiently large,

$$A_j(x,\xi) := \begin{cases} A(x,\xi) & \text{if} \quad b(x) < j \\ j\frac{A(x,\xi)}{b(x)} & \text{if} \quad b(x) \geq j. \end{cases}$$

Let b_j be the truncated of b at level j, i.e. for $x \in \Omega$ and $j \in \mathbb{N}$ sufficiently large

$$b_j(x) := \begin{cases} b(x) & \text{if} \quad b(x) < j \\ j & \text{if} \quad b(x) \geq j. \end{cases}$$

Since

$$A_j(x,\xi) = \frac{b_j(x)}{b(x)} A(x,\xi),$$

it is easy to check that for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^{N\times n}$ we have

$$|A_j(x,\xi) - A_j(x,\eta)| \leq k b_j(x)|\xi - \eta|\,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}, \tag{5.1}$$

$$\frac{1}{k} b_j(x)|\xi - \eta|^2\,(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \langle A_j(x,\xi) - A_j(x,\eta), \xi - \eta\rangle, \tag{5.2}$$

$$A_j(x,0) = 0.$$

For a.e. $x, y \in \Omega$ one easily gets

$$|b_j(x) - b_j(y)| \leq |b(x) - b(y)| \leq |x - y|\,[K(x) + K(y)].$$

In particular

$$|b_j(x) - b_j(y)| \leq (k+1)\,|x-y|\,[K(x) + K(y)].$$

In order to simplify the proof, we will prove

$$|A_j(x,\eta) - A_j(y,\eta)| \leq (k+1)\,|x-y|\,[K(x) + K(y)]\,(\mu^2 + |\eta|^2)^{\frac{p-1}{2}}, \tag{5.3}$$

at the end of the paper, see the Appendix.

Let us consider the following Dirichlet problem

$$\begin{cases} \operatorname{div} A_j(x, Dv) = \operatorname{div} F & \text{in} \quad B_{2R} \\ v = u & \text{on} \quad \partial B_{2R}. \end{cases}$$

If we denote by $u_j \in W^{1,p}(B_{2R})$ the solution of this problem, then $D(V_\mu(Du_j)) \in L^2_{loc}(b_j, \Omega)$ (see [35]) and, if

$$\mathscr{D}_K < \alpha_1 := \frac{1}{k+1}\alpha,$$

we can use estimate (4.2) to obtain

$$\int_{B_R} |D(V_\mu(Du_j))|^2 b_j\,dx \leq c \int_{B_{2R}} \left(\left(1 + \frac{1}{R^2}\right)(\mu^2 + |Du_j|^2)^{\frac{p}{2}} + (\mu^2 + |DF|^2)\right) b_j\,dx. \tag{5.4}$$

Let us remark that by Lemma 3.1 there exists $\delta > 0$ such that $|Du| \in L^{p+\delta}(b, B_{2R})$. Now we prove the strong convergence of $\{|Du_j|\}_j$ to $|Du|$ in $L^p(b, B_{2R})$. Using $(u - u_j)$ as test function, we easily get, thanks to (5.2),

$$\begin{aligned}\int_{B_{2R}} |Du - Du_j|^p b_j(x)\,dx &\le c(k) \int_{B_{2R}} \langle A_j(x, Du_j) - A_j(x, Du), Du_j - Du\rangle\,dx\\ &= c(k) \int_{B_{2R}} \langle F, Du_j - Du\rangle\,dx - \int_{B_{2R}} \langle A_j(x, Du), Du_j - Du\rangle\,dx\\ &= c(k) \int_{B_{2R}} \langle A(x, Du) - A_j(x, Du), Du_j - Du\rangle\,dx\\ &= c(k) \int_{B_{2R}} \left(1 - \frac{b_j}{b}\right) \langle A(x, Du), Du_j - Du\rangle\,dx.\end{aligned}$$

Then from (1.2) and (1.5) we derive

$$\int_{B_{2R}} |Du - Du_j|^p b_j(x)\,dx \le c(k) \int_{B_{2R}} (b - b_j)|Du - Du_j|(\mu^2 + |Du|^2)^{\frac{p-1}{2}}\,dx.$$

Finally by Hölder's inequality, since $b_j(x) \le b(x)$, we obtain

$$\begin{aligned}\int_{B_{2R}} |Du - Du_j|^p b_j(x)\,dx \le c(k, \lambda_0) &\left(\int_{B_{2R}} (\mu^2 + |Du|^2)^{\frac{p+\delta}{2}} b(x)\,dx\right)^{\frac{p}{p+\delta}}\\ &\cdot \left(\int_{B_{2R}} b(x)|b - b_j|^{p'\left(\frac{p+\delta}{\delta}\right)}\,dx\right)^{\frac{\delta}{p+\delta}}\end{aligned} \tag{5.5}$$

and the last term goes to zero. Previous relation easily implies the conclusion.

At this point, estimate (5.4) and (5.5) yield $\|D(V_\mu(Du_j))\|_{L^2_b(B_R)} \le C$, so that we deduce that, up to a subsequence, $D(V_\mu(Du_j))$ is weakly converging to $D(V_\mu(Du))$ in $L^2(b, B_R)$ and $V_\mu(Du_j)$ is strongly converging in $L^2(b, B_R)$. Therefore, we can pass to the limit in the estimate (5.4) having the validity of the desired inequality for the function u. □

In account of Remark 4.3, we can state the following

Proposition 5.1 *Let Ω be a regular domain and $A(x, \xi)$ a mapping verifying assumptions (1.2)–(1.5). There exist $0 < \varepsilon_1 < \frac{1}{2}$, depending on k, n, λ_0, p and the BMO -norm of $b(x)$, and $\alpha_2 > 0$, depending on p, n, λ_0 and k, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b, \Omega; \mathbb{R}^N)$, with $0 \le \varepsilon < \varepsilon_1$, is a local solution of*

$$\operatorname{div} A(x, Du(x)) = 0$$

and

$$\mathcal{D}_K := \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_2,$$

then $D(V_\mu(Du)) \in L^2_{loc}(b, \Omega)$ and the following estimate holds:

$$\int_{B_R} |D(V_\mu(Du))|^2 b\,dx \le c\left(1 + \frac{1}{R^2}\right) \int_{B_{2R}} (\mu^2 + |Du|^2)^{\frac{p}{2}} b\,dx,$$

for every ball $B_{2R} \subset\subset \Omega$ and for a constant c depending on p, k, λ_0, n and $\mathcal{D}_K$.

For $2 < p < n$, the following corollaries of fractional higher integrability easily derive from Theorem 1.1.

Corollary 5.1 *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions (1.2)–(1.5), and $F \in W^{1,2}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$. There exist $0<\varepsilon_1<\frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, and $\alpha_1>0$, depending on p, n, λ_0, μ and k, such that, if $u \in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$, with $0\le\varepsilon<\varepsilon_1$, is a local solution of (1.1) and*

$$\mathscr{D}_K := \mathrm{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1,$$

then $Du \in W^{\beta,p}_{loc}(b,\Omega;\mathbb{R}^N)$ for every $\beta \in \left(0,\frac{2}{p}\right)$.

Proof Since we can estimate for every $i\in\{1,\dots,n\}$

$$|\tau_{h,i}Du|^p \le c(n,p)|\tau_{h,i}V_\mu(Du)|^2, \tag{5.6}$$

summing up on $i\in\{1,\dots,n\}$ and taking into account the estimate given by Theorem 1.1, we get for $\rho\in(0,R)$ and h sufficiently small

$$\int_{B_\rho}\sum_{i=1}^n |\tau_{h,i}Du|^p b\,dx \le c\cdot\left(|h|^{\frac{2}{p}}\right)^p \int_{B_{2R}}\left(\left(1+\frac{1}{R^2}\right)(\mu^2+|Du|^2)^{\frac{p}{2}}+(\mu^2+|DF|^2)\right)b\,dx.$$

It follows that Du belongs to the Nikolskii space $\mathcal{H}^{\frac{2}{p},p}$ and hence the conclusion by embedding (see [3], 7.73 and also [62]). □

In the next corollary we show that a higher integrability of the function F improves the integrability of the fractional derivatives.

Corollary 5.2 *Let Ω be a regular domain, $A(x,\xi)$ a mapping verifying assumptions (1.2)–(1.5), and $F \in W^{1,r}_{loc}(b,\Omega;\mathbb{R}^{N\times n})$, for some $r>2$. There exist $0<\varepsilon_1<\frac{1}{2}$, depending on k, n, λ_0, p and the BMO - norm of $b(x)$, and $\alpha_1>0$, depending on p, n, λ_0, μ and k, such that, if $u\in W^{1,p-\varepsilon}_{loc}(b,\Omega;\mathbb{R}^N)$, with $0\le\varepsilon<\varepsilon_1$, is a local solution of (1.1) and*

$$\mathscr{D}_K := \mathrm{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1,$$

then $Du\in W^{\beta,q}_{loc}(b,\Omega;\mathbb{R}^N)$ for some $q>p$ and for every $\beta\in\left(0,\frac{2}{p}\right)$.

Proof Without loss of generality, we assume $0<R<1$. The estimate given by Theorem 1.1 and the use of Lemma 2.4 yield

$$\frac{1}{b(B_R)}\int_{B_R}|DV_\mu(Du)|^2 b\,dx \le c\left(\left(1+\frac{1}{R^2}\right)\frac{1}{b(B_{2R})}\int_{B_{2R}}|V_\mu(Du)-(V(Du))_{B_{2R}}|^2+(\mu^2+|DF|^2)b\,dx\right).$$

Hence, applying Sobolev–Poincaré inequality, we have the following reverse Hölder's inequality

$$\frac{1}{B_R}\int_{B_R}|DV_\mu(Du)|^2 b\,dx \le c\left[\left(\frac{1}{b(B_{2R})}\int_{B_{2R}}(|DV_\mu(Du)|^2)^{\frac{n-1}{n}}b\,dx\right)^{\frac{n}{n-1}}+\frac{1}{b(B_{2R})}\int_{B_{2R}}(\mu^2+|DF|^2)b\,dx\right]$$

getting the existence of an exponent $s > 2$ such that $|DV_\mu(Du)| \in L^s_{loc}$ and

$$\frac{1}{b\left(B_{\frac{R}{2}}\right)} \int_{B_{\frac{R}{2}}} |DV_\mu(Du)|^s b\, dx$$
$$\leq c\left[\left(\frac{1}{b\left(B_{\frac{R}{2}}\right)} \int_{B_{\frac{R}{2}}} |DV_\mu(Du)|^2 b\, dx\right)^{\frac{s}{2}} + \frac{1}{b(B_{2R})} \int_{B_{2R}} (\mu^2 + |DF|^2)^{\frac{s}{2}} b\, dx\right].$$

Then, using the pointwise inequality in (5.6), we easily obtain that

$$\frac{\|\tau_h Du\|_{L_b^{\frac{ps}{2}}}}{|h|^2} \leq c\|DV_\mu(Du)\|^{\frac{2}{p}}_{L^2_b}$$

which allows to conclude that Du belongs to the Nikolskii space $\mathcal{H}^{\frac{2}{p},\frac{ps}{2}}$ and hence, setting $q := \frac{ps}{2}$, by embedding $Du \in W^{\beta,q}_{loc}(b, \Omega; \mathbb{R}^N)$ for every $\beta \in \left(0, \frac{2}{p}\right)$. □

6 Calderón–Zygmund estimates

In this section we prove Theorem 1.2. Here the cubes considered will always have sides parallel to the coordinate axes.
We recall a few basic facts concerning the interior regularity of solutions of elliptic systems in divergence form of the type

$$\operatorname{div} A(x, Dw) = 0 \quad \text{in} \quad 3Q \subset \Omega,$$

where Q is a generic cube. The same proof of Theorem 1.1 works for balls of the type B_{2R}, B_{3R} and, by a covering argument by means of countable disjoint balls, we have the estimate also over cubes instead of balls. Thanks to Theorem 1.1, Remark 4.3 and Sobolev embedding Theorem 2.4, arguing as in (3.18) and if $\mathscr{D}_K < \alpha_2$, with α_2 as in Proposition 5.1, we get the following reverse Hölder inequality:

$$\left(\frac{1}{b(2Q)} \int_{2Q} (\mu^2 + |Dw|^2)^{\frac{r}{2}}\, b\, dx\right)^{\frac{1}{r}} \leq c\left(\frac{1}{b(3Q)} \int_{3Q} (\mu^2 + |Dw|^2)^{\frac{p}{2}}\, b\, dx\right)^{\frac{1}{p}},$$

with $c = c(p, k, \lambda_0, n, \|b\|_*, \mathscr{D}_K)$ and $r = \frac{np}{n-1}$. Then, by Gehring's Lemma, there exists an exponent

$$s := r + \delta, \tag{6.1}$$

with $\delta = \delta(p, k, \lambda_0, n, \|b\|_*, \mathscr{D}_K)$, such that

$$\left(\frac{1}{b(2Q)} \int_{2Q} (\mu^2 + |Dw|^2)^{\frac{s}{2}} b\, dx\right)^{\frac{1}{s}} \leq c\left(\frac{1}{b(3Q)} \int_{3Q} (\mu^2 + |Dw|^2)^{\frac{p}{2}} b\, dx\right)^{\frac{1}{p}}$$

with $c = c(p, k, \lambda_0, n, \|b\|_*, \mathscr{D}_K)$. Then

$$
\begin{aligned}
c\left(\int_{3Q}(\mu^2+|Dw|^2)^{\frac{p}{2}}b\,dx\right)^{\frac{1}{p}} &\geq \frac{[b(3Q)]^{\frac{1}{p}}}{[b(2Q)]^{\frac{1}{s}}}\left(\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}\\
&\geq [b(2Q)]^{\frac{1}{p}-\frac{1}{s}}\left(\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}\\
&\geq \lambda_0|2Q|^{\frac{1}{p}-\frac{1}{s}}\left(\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}} \geq \lambda_0 c'(n)\frac{|3Q|^{\frac{1}{p}}}{|2Q|^{\frac{1}{s}}}\left(\int_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}.
\end{aligned}
$$

Equivalently we have

$$
\left(\fint_{2Q}(\mu^2+|Dw|^2)^{\frac{s}{2}}\,b\,dx\right)^{\frac{1}{s}} \leq c\left(\fint_{3Q}(\mu^2+|Dw|^2)^{\frac{p}{2}}\,b\,dx\right)^{\frac{1}{p}}, \tag{6.2}
$$

with $c = c(p, k, \lambda_0, n, \|b\|_*, \mathscr{D}_K)$. We wish to emphasize the fact that the above constants and exponents are independent of the number $\mu \in [0, 1]$.
The following Lemma is fundamental in order to prove Theorem 1.2.

Lemma 6.1 *Let $u \in W^{1,p}(b, Q_{2R}; \mathbb{R}^N)$ be a solution to (1.11). Let $B > 1$; there exists a number $\varepsilon = \varepsilon(p, k, \lambda_0, n, \mathscr{D}_K, \|b\|_*, B)$ such that the following is true:*
If $\lambda > 0$ and $Q \subset Q_R$ is a dyadic subcube of Q_R such that

$$
\begin{aligned}
\Bigg|Q\cap\Bigg\{x\in Q_R : M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) > \frac{AB\lambda}{\lambda_0},\\
M^*\left[\left(\mu^2+|G|^2\right)^{\frac{p}{2}}\right](x) < \varepsilon\lambda\Bigg\}\Bigg| > B^{-\frac{s}{p}}|Q|,
\end{aligned}
\tag{6.3}
$$

then its predecessor $\tilde{Q}$ satisfies

$$
\tilde{Q} \subseteq \left\{x \in Q_R : M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}b\right](x) > \lambda\right\}. \tag{6.4}
$$

Here $M^*_{(b)} \equiv M^*_{(b,)Q_{2R}}$ denotes the (weighted) restricted Maximal Function relative to Q_{2R}, i.e. if $f_1 \in L^1(Q_{2R})$, $f_2 \in L^1(b, Q_{2R})$ and $x \in Q_{2R}$

$$
M^*(f_1)(x) := \sup_{\substack{Q\subseteq Q_{2R}\\ x\in Q}} \fint_Q |f_1(y)|\,dy, \qquad M_b^*(f_2)(x) := \sup_{\substack{Q\subseteq Q_{2R}\\ x\in Q}} \frac{\int_Q |f_2(y)|b(y)\,dy}{b(Q)}.
$$

Moreover here s is the number defined in (6.1), and $A = A(p, k, \lambda_0, n, \mathscr{D}_K, \|b\|_*) > 1$ is an absolute constant. All the constants and quantities are uniform with respect to $\mu \in [0, 1]$.

Proof We prove this lemma by contradiction. The constants A and ε will be chosen toward the end while all the arguments will be worked out for a general $\mu \in [0, 1]$. Suppose (6.4) is not satisfied although (6.3) holds. Then there exists $\tilde{x} \in \tilde{Q}$ such that

$$
\lambda_0 M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](\tilde{x}) \leq M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}b\right](\tilde{x}) \leq \lambda. \tag{6.5}
$$

Since $\tilde{Q} \subset 3Q$ because $\tilde{Q}$ is the predecessor of Q, we have

$$\fint_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b\, dx \le \lambda.$$

Note that $3Q \subset Q_{2R}$. Moreover by (6.3) we can find $\bar{x} \in Q$ such that

$$M^*\left[\left(\mu^2 + |G|^2\right)^{\frac{p}{2}}\right](\bar{x}) \le \varepsilon\lambda$$

and therefore

$$\fint_{3Q} (\mu^2 + |G|^2)^{\frac{p}{2}}\, dx \le \varepsilon\lambda. \tag{6.6}$$

Now we define $w \in W^{1,p}(b, 3Q; \mathbb{R}^N)$ as the unique solution of the following Dirichlet problem

$$\begin{cases} \operatorname{div} A(x, Dw) = 0 & \text{in } \; 3Q \\ w - u \in W_0^{1,p}(3Q, \mathbb{R}^N). \end{cases} \tag{6.7}$$

The existence and the uniqueness of such a solution follows from Minty–Browder Theorem. Let us first derive the following estimate

$$\int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b\, dx \le c \int_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b\, dx \tag{6.8}$$

where $c = c(k, p)$. By using $w - u$ as test function in (6.7) we get

$$\int_{3Q} \langle A(x, Dw), Dw\rangle\, dx = \int_{3Q} \langle A(x, Dw), Du\rangle\, dx.$$

By (1.2), (1.3) and (1.5) we get

$$\int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dw|^2 b\, dx \le k^2 \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dw|\, |Du| b\, dx$$

Using the Young's type inequality in Lemma 2.5, which holds uniformly in $\mu \in [0, 1]$, with $\varepsilon < \min\left\{\frac{1}{2}, \frac{1}{2k^2}\right\}$, we obtain

$$\int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p-2}{2}} |Dw|^2 b\, dx \le c(k, p) \int_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{2}} |Du|^2 b\, dx.$$

Now using again Young's inequality, from previous relation we have

$$\begin{aligned} \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b\, dx &= \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p-2}{2}} (\mu^2 + |Dw|^2) b\, dx \\ &\le c \int_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{2}} |Du|^2 b\, dx + \frac{1}{2} \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b\, dx + c \int_{3Q} \mu^p b\, dx \\ &\le \frac{1}{2} \int_{3Q} \left(\mu^2 + |Dw|^2\right)^{\frac{p}{2}} b\, dx + c \int_{3Q} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b\, dx, \end{aligned}$$

with $c = c(k, p)$ independent of μ. Then estimate (6.8) follows.

Now by (6.2) we find that

$$\left(\fint_{2Q}\left(\mu^2+|Dw|^2\right)^{\frac{s}{2}}b\,dx\right)^{\frac{1}{s}}\le c\left(\fint_{3Q}\left(\mu^2+|Dw|^2\right)^{\frac{p}{2}}b\,dx\right)^{\frac{1}{p}}$$
$$\le c\left(\fint_{3Q}\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}b\,dx\right)^{\frac{1}{p}}\le c\lambda^{\frac{1}{p}}, \tag{6.9}$$

where $c=c(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)$. Notice that since $p\ge 2$, by (1.3)

$$\fint_{3Q}|Du-Dw|^p b\,dx\le c(p,k)\fint_{3Q}\langle A(x,Du)-A(x,Dw),Du-Dw\rangle\,dx$$
$$=c(p,k)\fint_{3Q}\langle A(x,Du),Du-Dw\rangle\,dx=c(p,k)\fint_{3Q}\langle|G|^{p-2}G,Du-Dw\rangle\,dx$$
$$\le c(p,k,\lambda_0)\left(\fint_{3Q}(\mu^2+|G|^2)^{\frac{p}{2}}\,dx\right)+\frac{\lambda_0}{2}\fint_{3Q}|Du-Dw|^p\,dx$$
$$\le c(p,k,\lambda_0)\left(\fint_{3Q}(\mu^2+|G|^2)^{\frac{p}{2}}\,dx\right)+\frac{1}{2}\fint_{3Q}|Du-Dw|^p b\,dx.$$

Then from (6.6)

$$\fint_{3Q}|Du-Dw|^p b\,dx\le c\varepsilon\lambda \tag{6.10}$$

where $c=c(p,k,\lambda_0)$. In the following we shall denote by $M^{**}_{(b)}$ the (weighted) Restricted Maximal operator relative to the cube $2Q$, while $M^{*}_{(b)}$ keeps on denoting the (weighted) Restricted Maximal Operator relative to the cube Q_{2R}. Now, with the notation (2.1),

$$\begin{aligned}
&\left|\left\{x\in Q: M_b^{**}\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x)>\frac{AB\lambda}{\lambda_0}\right\}\right|\\
&\le\left|\left\{x\in Q: M_b^{**}\left[\left(\mu^2+|Dw|^2\right)^{\frac{p}{2}}\right](x)>\frac{AB\lambda}{\lambda_0 8^p}\right\}\right|\\
&\quad+\left|\left\{x\in Q: M_b^{**}\left[|Du-Dw|^p\right](x)>\frac{AB\lambda}{\lambda_0 8^p}\right\}\right|\\
&\le\frac{1}{\lambda_0}b\left(\left\{x\in Q: M_b^{**}\left[\left(\mu^2+|Dw|^2\right)^{\frac{p}{2}}\right](x)>\frac{AB\lambda}{\lambda_0 8^p}\right\}\right)\\
&\quad+\left|\left\{x\in Q: M^{**}\left[|Du-Dw|^p\,b\right](x)>\frac{AB\lambda}{\lambda_0^2 8^p}\right\}\right|\\
&\le\frac{c(s,p,\lambda_0,\|b\|_*)}{(AB\lambda)^{\frac{s}{p}}}\int_{2Q}\left(\mu^2+|Dw|^2\right)^{\frac{s}{2}}b\,dx+\frac{c(n,p,\lambda_0)}{AB\lambda}\int_{2Q}|Du-Dw|^p b\,dx,
\end{aligned} \tag{6.11}$$

where in the last inequality we used (2.2), Theorem 2.5 and Lemma 2.1. From (6.1) and (6.9) we get

$$\begin{aligned}\frac{c(s,p,\lambda_0,\|b\|_*)}{(AB\lambda)^{\frac{s}{p}}}\int_{2Q}\left(\mu^2+|Dw|^2\right)^{\frac{s}{2}}b\,dx &\le \tilde{c}(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)\frac{|Q|}{(AB)^{\frac{s}{p}}}\\ &\le \frac{1}{100^{n+2}B^{\frac{s}{p}}}|Q|,\end{aligned}\tag{6.12}$$

where the last inequality is true provided we choose, for instance, $A := \left(100^{n+2}(\tilde{c}+1)\right)^{\frac{p}{s}}$; this fixes the constant A and yields the absolute dependence mentioned in the statement. From (6.11) thanks to (6.10) and (6.12) we obtain

$$\left|\left\{x\in Q : M_b^{**}\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) > \frac{AB\lambda}{\lambda_0}\right\}\right| \le \frac{|Q|}{100^{n+2}B^{\frac{s}{p}}} + c\frac{\varepsilon}{AB}|Q|, \tag{6.13}$$

with $c = c(n,p,k,\lambda_0)$. Now we can choose ε such that

$$c(n,p,k,\lambda_0)\frac{\varepsilon}{A} \le \frac{1}{8B^{\frac{s}{p}-1}}$$

and so from (6.13)

$$\left|\left\{x\in Q : M_b^{**}\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) > \frac{AB\lambda}{\lambda_0}\right\}\right| \le \frac{|Q|}{8^nB^{\frac{s}{p}}}. \tag{6.14}$$

To conclude we remark that (6.5) implies

$$M_b^{*}\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) \le \max\left\{M_b^{**}\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x),\, 100^n\frac{\lambda}{\lambda_0}\right\} \tag{6.15}$$

for every $x\in Q$. Indeed, let $x_0\in Q$ and let $C\subset Q_{2R}$ be a cube such that $x_0\in C$. In the case when $C\subset 2Q$, by the definition of M^{**}, we trivially have

$$\frac{\int_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx}{b(C)} \le M_b^{**}\left[(\mu^2+|Du|^2)^{\frac{p}{2}}\right](x_0).$$

In the case when $C\not\subset 2Q$, we must have $2^n|C|\ge|Q|$, then $\tilde{Q}\subset 10C$ and in particular $\tilde{x}\in 10C$; at this point we further distinguish two cases. If $20C\subset Q_{2R}$ then, using (6.5), we obtain

$$\begin{aligned}\frac{\int_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx}{b(C)} &\le \frac{1}{\lambda_0}\fint_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx \le \frac{20^n}{\lambda_0}\fint_{20C}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx\\ &\le \frac{20^n}{\lambda_0}M^*\left[(\mu^2+|Du|^2)^{\frac{p}{2}}b\right](\tilde{x}) \le \frac{100^n}{\lambda_0}\lambda.\end{aligned}$$

If finally $20C\not\subset Q_{2R}$ then $Q_{2R}\subset 70C$ and therefore

$$\begin{aligned}\frac{\int_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx}{b(C)} &\le \frac{1}{\lambda_0}\fint_C(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx \le \frac{70^n}{\lambda_0}\fint_{20C}(\mu^2+|Du|^2)^{\frac{p}{2}}b\,dx\\ &\le \frac{70^n}{\lambda_0}M^*\left[(\mu^2+|Du|^2)^{\frac{p}{2}}b\right](\tilde{x}) \le \frac{100^n}{\lambda_0}\lambda,\end{aligned}$$

so that (6.15) is completely proved. Since $AB > A > 100^n$, by using (6.14) we get

$$\left|\left\{x \in Q : M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) > \frac{AB\lambda}{\lambda_0}\right\}\right| \leq \frac{|Q|}{2B^{\frac{s}{p}}},$$

which is a contradiction to (6.3). The proof is complete. □

When applying Lemma 6.1 we shall need to fix the constant B, depending on the choice of an integrability exponent $q \in (p, s)$. With q being fixed, we do this in the following canonical way:

$$\frac{1}{B^{\frac{s-q}{p}}} = \frac{1}{2A^{\frac{q}{p}}}, \tag{6.16}$$

where the constant $A \equiv A(p, k, \lambda_0, n, \|b\|_*, \mathscr{D}_K)$ is the absolute constant appearing in Lemma 6.1. This fixes in turn $B \equiv B(p, k, \lambda_0, n, s-q, \|b\|_*, \mathscr{D}_K)$. Note that $B \nearrow \infty$ when $q \nearrow s$; consequently, in Lemma 6.1 $\varepsilon \searrow 0$ when $q \nearrow s$. Once the choice of B has been made, this canonically fixes the choice of ε, with the following absolute dependence

$$\varepsilon_0 \equiv \varepsilon \equiv \varepsilon(p, k, \lambda_0, n, s-q, \|b\|_*, \mathscr{D}_K) > 0. \tag{6.17}$$

Proof of Theorem 1.2 Following the notation of Lemma 6.1, let us set

$$\mu_1(t) := \left|\left\{x \in Q_R : M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) > t\right\}\right|,$$
$$\mu_2(t) := \left|\left\{x \in Q_R : M^*\left[\left(\mu^2+|G|^2\right)^{\frac{p}{2}}\right](x) > t\right\}\right|.$$

Then, with $B > 1$ as in (6.16), we take

$$\tilde{\lambda} := \frac{10^n}{\lambda_0} c B^{\frac{s}{p}} \fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\, dx,$$

where $c \equiv c(n)$ is the constant appearing in the weak type inequality (2.2) when $p \equiv 1$; note that $\tilde{\lambda}$ is positive. Therefore,

$$\begin{aligned}\mu_1(\tilde{\lambda}) &\leq \left|\left\{x \in Q_R : M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}} b\right](x) > \tilde{\lambda}\lambda_0\right\}\right| \\ &\leq \frac{c}{\tilde{\lambda}\lambda_0}\int_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\, dx < \frac{|Q_R|}{2B^{\frac{s}{p}}},\end{aligned} \tag{6.18}$$

and consequently, since $AB > 1$,

$$\mu_1((AB)^h\tilde{\lambda}) < \frac{|Q_R|}{2B^{\frac{s}{p}}} \qquad \forall h \in \mathbb{N}, \tag{6.19}$$

where A is the constant appearing in Lemma 6.1. Next, we recall that the constant B has been chosen according to (6.16). With such a choice of B, and in view of (6.18)–(6.19), we can combine Lemmas 6.1 and 2.6 at the levels $\lambda \equiv (AB)^h\tilde{\lambda}$, $h \in \mathbb{N}$. To this end, note that

$$\left\{x \in Q_R : M_b^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}}\right](x) > (AB)^h\tilde{\lambda}\right\}$$
$$\subset \left\{x \in Q_R : M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}} b\right](x) > \tilde{\lambda}\lambda_0\right\}$$

Therefore, an elementary induction argument leads to

$$\mu_1((AB)^{h+1}\tilde{\lambda}) \le B^{-\frac{s}{p}(h+1)}\mu_1(\tilde{\lambda}) + \sum_{i=0}^{h} B^{-\frac{s}{p}(h-i)}\mu_2((AB)^i\varepsilon_0\tilde{\lambda})$$

for every $h \in \mathbb{N}$; the number ε_0 is defined in (6.17). Summing up over h, we have, for every $M \in \mathbb{N}$

$$\begin{aligned}\sum_{h=0}^{M}(AB)^{\frac{q}{p}(h+1)}\mu_1((AB)^{h+1}\tilde{\lambda}) \le \left(\sum_{h=0}^{M}\left[B^{-\frac{s}{p}}(AB)^{\frac{q}{p}}\right]^{h+1}\right)\mu_1(\tilde{\lambda})\\ + \sum_{h=0}^{M}\sum_{i=0}^{h}(AB)^{\frac{q}{p}(h+1)}B^{-\frac{s}{p}(h-i)}\mu_2((AB)^i\varepsilon_0\tilde{\lambda}).\end{aligned} \tag{6.20}$$

As for the first sum in the right-hand side, we notice that (6.16) leads to

$$\sum_{h=0}^{\infty}\left[B^{-\frac{s}{p}}(AB)^{\frac{q}{p}}\right]^{h+1} \le 1.$$

Concerning the second sum appearing in the right-hand side of (6.20), we have

$$\begin{aligned}&\sum_{h=0}^{M}\sum_{i=0}^{h}(AB)^{\frac{q}{p}(h+1)}B^{-\frac{s}{p}(h-i)}\mu_2((AB)^i\varepsilon_0\tilde{\lambda})\\ &= (AB)^{\frac{q}{p}}\sum_{i=0}^{M}(AB)^{\frac{q}{p}i}\mu_2((AB)^i\varepsilon_0\tilde{\lambda})\sum_{h=0}^{M-i}\left[B^{-\frac{s}{p}}(AB)^{\frac{q}{p}}\right]^h\\ &\le 2(AB)^{\frac{q}{p}}\sum_{k=0}^{M}(AB)^{\frac{q}{p}k}\mu_2((AB)^k\varepsilon_0\tilde{\lambda}).\end{aligned}$$

Combining the previous estimates with (6.20) we finally obtain

$$\sum_{k=1}^{\infty}(AB)^{\frac{q}{p}k}\mu_1((AB)^k\tilde{\lambda}) \le \mu_1(\tilde{\lambda}) + 2(AB)^{\frac{q}{p}}\sum_{k=0}^{\infty}(AB)^{\frac{q}{p}k}\mu_2((AB)^k\varepsilon_0\tilde{\lambda}).$$

Now we will do a straight, readable estimate, although it is justified only if read backwards: when the series in (6.22) will be shown to converge, we will have proved that the power q of the maximal function is integrable, which implies that also the first integral we are about to write is finite. We observe that

$$\begin{aligned}\int_{Q_R}(\mu^2+|Du|^2)^{\frac{q}{2}}b\,dx &\le \int_{Q_R}\left|M_b^*\left[(\mu^2+|Du|^2)^{\frac{p}{2}}\right](x)\right|^{\frac{q}{p}}dx\\ &= \int_0^{\infty}\frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda\\ &= \int_0^{\tilde{\lambda}}\frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda + \int_{\tilde{\lambda}}^{\infty}\frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda\end{aligned} \tag{6.21}$$

and

$$\int_0^{\tilde\lambda} \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda \le \tilde\lambda^{\frac{q}{p}}|Q_R| = c(n,\lambda_0)^{\frac{q}{p}} B^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,dx\right)^{\frac{q}{p}} |Q_R|$$

$$\le c(n,\lambda_0)^{\frac{s}{p}} B^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,dx\right)^{\frac{q}{p}} |Q_R|$$

$$= c(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K) B^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,dx\right)^{\frac{q}{p}} |Q_R|,$$

where we assumed $c(n,\lambda_0) > 1$. In a similar way we have

$$\int_{\tilde\lambda}^{\infty} \frac{q}{p}\lambda^{\frac{q}{p}-1}\mu_1(\lambda)\,d\lambda = \sum_{n=0}^{\infty}\int_{(AB)^{\frac{n}{p}}\tilde\lambda}^{(AB)^{\frac{n+1}{p}}\tilde\lambda} q\lambda^{q-1}\mu_1(\lambda)\,d\lambda$$

$$\le (AB\tilde\lambda)^{\frac{q}{p}}\sum_{n=0}^{\infty}(AB)^{\frac{nq}{p}}\mu_1((AB)^n\tilde\lambda).$$

Again,

$$(AB\tilde\lambda)^{\frac{q}{p}}\mu_1(\tilde\lambda) \le (AB\tilde\lambda)^{\frac{q}{p}}\left|\left\{x\in Q_R : M^*\left[\left(\mu^2+|Du|^2\right)^{\frac{p}{2}} b\right](x) > \tilde\lambda\lambda_0\right\}\right|$$

$$\le c(n,\lambda_0)(AB)^{\frac{q}{p}}\tilde\lambda^{\frac{q}{p}-1}\int_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,dx$$

$$\le c(p,k,\lambda_0,n,\|b\|_*,\mathscr{D}_K)(A)^{\frac{s}{p}} B^{-\frac{s-q}{p}} B^{\frac{s}{p^2}q}|Q_{2R}|\left(\fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,dx\right)^{\frac{q}{p}}.$$

Joining the last three estimates to (6.21) yields

$$\begin{aligned}
\int_{Q_R} (\mu^2+|Du|^2)^{\frac{q}{2}} b\,dx &\le cB^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,dx\right)^{\frac{q}{p}} |Q_R| \\
&\quad + (AB\tilde\lambda)^{\frac{q}{p}}\mu_1(\tilde\lambda) + (AB\tilde\lambda)^{\frac{q}{p}}\sum_{k=1}^{\infty}(AB)^{k\frac{q}{p}}\mu_1((AB)^k\tilde\lambda) \\
&\le cB^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,dx\right)^{\frac{q}{p}} |Q_R| \\
&\quad + 2(AB\tilde\lambda)^{\frac{q}{p}}\mu_1(\tilde\lambda) + 2(AB\tilde\lambda)^{\frac{q}{p}}\sum_{k=0}^{\infty}(AB)^{k\frac{q}{p}}\mu_2((AB)^k\varepsilon_0\tilde\lambda) \\
&\le cB^{\frac{s}{p^2}q}\left(\fint_{Q_{2R}} (\mu^2+|Du|^2)^{\frac{p}{2}} b\,dx\right)^{\frac{q}{p}} |Q_R| + cB^{\frac{2s}{p}}\tilde\lambda\sum_{k=0}^{\infty}(AB)^{k\frac{q}{p}}\mu_2((AB)^k\varepsilon_0\tilde\lambda),
\end{aligned} \tag{6.22}$$

where $c = c(p, k, \lambda_0, n, s-q, \|b\|_*, \mathscr{D}_K)$. It remains to estimate the last series. To this aim, observe that, as before,

$$\int_{Q_R} \left| M^*\left[\left(\mu^2 + |G|^2\right)^{\frac{p}{2}} \right](x) \right|^{\frac{q}{p}} dx = \int_0^\infty \frac{q}{p} \lambda^{\frac{q}{p}-1} \mu_2(\lambda)\, d\lambda$$
$$= \int_0^{\varepsilon_0 \tilde{\lambda}} \frac{q}{p} \lambda^{\frac{q}{p}-1} \mu_2(\lambda)\, d\lambda + \int_{\varepsilon_0 \tilde{\lambda}}^\infty \frac{q}{p} \lambda^{\frac{q}{p}-1} \mu_2(\lambda)\, d\lambda.$$

Then

$$\int_0^{\varepsilon_0 \tilde{\lambda}} \frac{q}{p} \lambda^{\frac{q}{p}-1} \mu_2(\lambda)\, d\lambda \ge (\varepsilon_0 \tilde{\lambda})^{\frac{q}{p}} \mu_2(\varepsilon_0 \tilde{\lambda}),$$

and

$$\int_{\varepsilon_0 \tilde{\lambda}}^\infty \frac{q}{p} \lambda^{\frac{q}{p}-1} \mu_2(\lambda)\, d\lambda = \sum_{k=0}^\infty \int_{(AB)^k \varepsilon_0 \tilde{\lambda}}^{(AB)^{k+1} \varepsilon_0 \tilde{\lambda}} \frac{q}{p} \lambda^{\frac{q}{p}-1} \mu_2(\lambda)\, d\lambda$$
$$\ge \sum_{k=0}^\infty \mu_2((AB)^{k+1} \varepsilon_0 \tilde{\lambda}) \left[((AB)^{k+1} \varepsilon_0 \tilde{\lambda})^{\frac{q}{p}} - ((AB)^k \varepsilon_0 \tilde{\lambda})^{\frac{q}{p}} \right]$$
$$= (\varepsilon_0 \tilde{\lambda})^{\frac{q}{p}} \sum_{k=0}^\infty (AB)^{(k+1)\frac{q}{p}} \mu_2((AB)^{k+1} \varepsilon_0 \tilde{\lambda}) \left[1 - (AB)^{-\frac{q}{p}} \right]$$
$$\ge \frac{1}{2} (\varepsilon_0 \tilde{\lambda})^{\frac{q}{p}} \sum_{k=1}^\infty (AB)^{k\frac{q}{p}} \mu_2((AB)^k \varepsilon_0 \tilde{\lambda}).$$

Combining the last estimates with the maximal inequality we finally get

$$\frac{p}{2q} (\varepsilon_0 \tilde{\lambda})^{\frac{q}{p}} \sum_{k=1}^\infty (AB)^{k\frac{q}{p}} \mu_2((AB)^k \varepsilon_0 \tilde{\lambda}) + \frac{p(\varepsilon_0 \tilde{\lambda})^{\frac{q}{p}}}{q} \mu_2(\varepsilon_0 \tilde{\lambda})$$
$$\le \frac{p}{q} (\varepsilon_0 \tilde{\lambda})^{\frac{q}{p}} \sum_{k=0}^\infty (AB)^{k\frac{q}{p}} \mu_2((AB)^k \varepsilon_0 \tilde{\lambda}) \le \frac{2p}{q} \int_{Q_R} \left| M^*\left[\left(\mu^2 + |G|^2\right)^{\frac{p}{2}} \right](x) \right|^{\frac{q}{p}} dx$$
$$\le c(n, p, s, \|b\|_*) \int_{Q_{2R}} \left(\mu^2 + |G|^2\right)^{\frac{q}{2}} dx \le c(n, p, s, \lambda_0, \|b\|_*) \int_{Q_{2R}} \left(\mu^2 + |G|^2\right)^{\frac{q}{2}} b\, dx.$$

Using this estimate in (6.22) and passing to averages we have

$$\left(\fint_{Q_R} \left(\mu^2 + |Du|^2\right)^{\frac{q}{2}} b\, dx \right)^{\frac{1}{q}} \le c \left(\fint_{Q_{2R}} \left(\mu^2 + |Du|^2\right)^{\frac{p}{2}} b\, dx \right)^{\frac{1}{p}}$$
$$+ c \left(\fint_{Q_{2R}} \left(\mu^2 + |G|^2\right)^{\frac{q}{2}} b\, dx \right)^{\frac{1}{q}},$$

□

7 Globality

The goal of this section is to prove global versions of the estimates in Theorems 1.1 and 1.2 when we consider solutions of the corresponding Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary. In the following we assume that

$$\Omega \subset\subset B_{2R} \subset Q_0,$$

where, without loss of generality, we suppose that the ball B_{2R} and the cube Q_0 are centered in the origin.
Let conditions (1.2)–(1.6) hold in $\overline{B}_{2R}$. Set $A(x,\xi) = 0$ for any $x \in \mathbb{R}^n \setminus Q_0$, we consider a standard mollifier $\rho : \mathbb{R}^n \to [0,\infty)$ with compact support contained in $B_1 \subset \mathbb{R}^n$. If $0 < \varepsilon < \min\{R, 1\}$, for any $x \in B_{2R-\varepsilon}$ and $\xi \in \mathbb{R}^{N\times n}$ we consider

$$A_\varepsilon(x,\xi) := \int_{B_1} A(x+\varepsilon y, \xi)\, \rho(y)\, dy,$$

$$K_\varepsilon(x) := \int_{B_1} K(x+\varepsilon y)\, \rho(y)\, dy,$$

$$b_\varepsilon(x) := \int_{B_1} b(x+\varepsilon y)\, \rho(y)\, dy,$$

$$F_\varepsilon(x) := \int_{B_1} F(x+\varepsilon y) \rho(y)\, dy.$$

It is easy to verify that for a.e. $x, y \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^{N\times n}$

$$|A_\varepsilon(x,\xi) - A_\varepsilon(x,\eta)| \le k b_\varepsilon(x)|\xi-\eta|\, (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}, \tag{7.1}$$

$$\frac{1}{k} b_\varepsilon(x)|\xi-\eta|^2\, (\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \le \langle A_\varepsilon(x,\xi) - A_\varepsilon(x,\eta), \xi - \eta\rangle, \tag{7.2}$$

$$|A_\varepsilon(x,\eta) - A_\varepsilon(y,\eta)| \le |x-y|\, [K_\varepsilon(x) + K_\varepsilon(y)]\, (\mu^2 + |\eta|^2)^{\frac{p-1}{2}}, \tag{7.3}$$

$$A_\varepsilon(x,0) = 0, \tag{7.4}$$

$$|b_\varepsilon(x) - b_\varepsilon(y)| \le |x-y|\, [K_\varepsilon(x) + K_\varepsilon(y)]. \tag{7.5}$$

7.1 Global differentiability

The following lemma holds

Lemma 7.1 *Let U a bounded Lipschitz domain such that, if we denote by Q_r a cube centered in the origin and with side of length r and*

$$Q_r^+ = Q_r \cap \{x_n > 0\},$$

then

$$Q_{4d}^+ \subset U \subseteq Q_1^+,$$

with $d > 0$. Let $A_\varepsilon : U \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ satisfy Assumptions (7.1)–(7.5) for $x \in U$, $p \ge 2$ and $b_\varepsilon \in L^\infty(U)$. Let $F \in W^{1,2}(b, U; \mathbb{R}^{N\times n})$. Consider the problem

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Du_\varepsilon) = \operatorname{div} F_\varepsilon & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U \cap \{x_n = 0\}. \end{cases} \tag{7.6}$$

If $\alpha_1 > 0$ *is the constant, depending on* p, n, λ_0, μ *and* k, *in Theorem* 1.1 *and if*

$$\mathscr{D}_K \equiv \mathrm{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_1,$$

then

$$\int_{Q_{2d}^+} |D(V_\mu(Du_\varepsilon))|^2 b_\varepsilon\, dx \le c \int_{Q_{4d}^+} \left(\left(1+\frac{1}{d^2}\right)(\mu^2+|Du_\varepsilon|^2)^{\frac{p}{2}} + (\mu^2+|DF_\varepsilon|^2)\right) b_\varepsilon, \tag{7.7}$$

for a constant c *depending on* p, k, λ_0, n, μ *and* $\mathscr{D}_K$.

Regarding the proof of Lemma 7.1, we refer to [29, Theorem 2.3]. Actually, we can firstly repeat the proof of Theorem 1.1 by using the standard difference quotient method in the tangential directions. This allows to prove the existence of $D_s(V_\mu(Du_\varepsilon))$, $s = 1, \ldots, n-1$, in L^2. Secondly, we can use the definition of (7.6) to bound the L^2-norm of $D_n(V_\mu(Du_\varepsilon))$ by the L^2-norm of the tangential derivatives.
Now let $u \in W_0^{1,p}(b, \Omega; \mathbb{R}^{N\times n})$ be the unique solution of the problem

$$\begin{cases} \operatorname{div} A(x, Du) = \operatorname{div} F & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $F \in W^{1,2}(b, \Omega; \mathbb{R}^{N\times n})$. Then

Theorem 7.1 *There exists* $\alpha_3 > 0$, *depending on* p, n, λ_0, μ, k *and* Ω, *such that, if*

$$\mathscr{D}_K \equiv \mathrm{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_3,$$

then $D(V_\mu(Du)) \in L^2(b, \Omega; \mathbb{R}^{N\times n})$ *and*

$$\int_\Omega |D(V_\mu(Du))|^2 b\, dx \le c \int_\Omega \left((\mu^2+|Du|^2)^{\frac{p}{2}} + (\mu^2+|DF|^2)\right) b\, dx, \tag{7.8}$$

where $c = c(p, k, \lambda_0, n, \mu, \mathscr{D}_K, \Omega)$.

Proof Firstly we prove (7.8) when $b(x) \in L^\infty(\Omega)$ with a constant $c = c(p, k, \lambda_0, n, \mu, \mathscr{D}_K, \Omega)$. If $b(x) \in L^\infty(\Omega)$, let $A_\varepsilon(x, \xi)$ satisfy (7.1)–(7.5) and let u_ε be the unique solution of the system

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Du_\varepsilon) = \operatorname{div} F_\varepsilon & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

In a standard way (see for example [19, 42, 44, 59, 65]), we cover Ω by a family of open sets Ω', Ω'', $U_1, \ldots, U_m$, $V_1, \ldots, V_m$ such that

- $\Omega' \subset\subset \Omega'' \subset\subset \Omega$;
- U_l, V_l are cubes centered in $x_l \in \partial\Omega$, with $l = 1, \ldots m$;
- $V_l \subset\subset U_l$, with $l = 1, \ldots, m$;
- $\cup_{l=1}^m V_l \supsetneq \partial\Omega$;
- $\Omega \subsetneq \cup_{l=1}^m V_l \cup \Omega'$.

Covering $\bar{\Omega}'$ by a finite number of balls, by Theorem 1.1 we have that

$$\int_{\Omega'} |D(V_\mu(Du_\varepsilon))|^2 b_\varepsilon\, dx \le c \int_\Omega \left((\mu^2+|Du_\varepsilon|^2)^{\frac{p}{2}} + (\mu^2+|DF_\varepsilon|^2)\right) b_\varepsilon\, dx,$$

with $c = c(p, k, \lambda_0, n, \mu, \mathscr{D}_K, \Omega)$. Regarding the boundary regularity of the solution, on every U_l we can consider a diffeomorphism Φ which maps $\Omega_l \equiv U_l \cap \Omega$ to an open set of $\mathbb{R}^n$ and such that

$$\Phi(U_l \cap \Omega) \subset \{y \in \mathbb{R}^n : y_n > 0\}, \qquad \Phi\{U_l \cap \partial\Omega\} \subset \{y \in \mathbb{R}^n : y_n = 0\}.$$

If $\tilde{u}_\varepsilon$ is such that $u_\varepsilon(x) = (\tilde{u}_\varepsilon \circ \Phi)(x)$, with $x \in U_l \cap \bar{\Omega}$, then $\tilde{u}_\varepsilon$ solves in $\tilde{\Omega}_l \equiv \Phi(U_l \cap \Omega)$ a system

$$\begin{cases} \operatorname{div} \tilde{A}_\varepsilon(x, D\tilde{u}_\varepsilon) = \operatorname{div} \tilde{F}_\varepsilon & \text{in} \quad \tilde{\Omega}_l \\ \tilde{u}_\varepsilon = 0 & \text{on} \quad \{y_n = 0\} \cap \partial\tilde{\Omega}_l, \end{cases}$$

where $\tilde{A}_\varepsilon$ satisfies conditions similar to (7.1)–(7.5) with new constants $\tilde{\lambda}_0$ and $\tilde{k}$ depending on Φ. This diffeomorphism preserves the BMO norm and the distance (see [5, 10]), that is

$$\|b_\varepsilon \circ \Phi\|_* \le \|b_\varepsilon\|_* \le \|b\|_*,$$
$$\mathscr{D}_{K_\varepsilon \circ \Phi} \le \mathscr{D}_{K_\varepsilon} \le \mathscr{D}_K$$

and we apply Lemma 7.1 in $\tilde{\Omega}_l$, giving (7.7) with a new constant $c = c(\tilde{\lambda}_0, \tilde{k}, p, n, \mu, \mathscr{D}_K)$. Coming back to the original variables and summing on l, we get that, for any $0 < \varepsilon < 1$, $|DV_\mu(Du_\varepsilon)| \in L^2(\Omega)$ and

$$\int_\Omega |DV_\mu(Du_\varepsilon)|^2 b_\varepsilon \, dx \le c \left(\int_\Omega (\mu^2 + |Du_\varepsilon|^2)^{\frac{p}{2}} b_\varepsilon \, dx + \int_\Omega (\mu^2 + |DF_\varepsilon|^2) b_\varepsilon \, dx \right). \tag{7.9}$$

Now we prove that $Du_\varepsilon \to Du$ in $L^p(b, \Omega; \mathbb{R}^{N \times n})$. From (7.2)

$$\begin{aligned} &\frac{1}{k} \int_\Omega (\mu^2 + |Du_\varepsilon|^2 + |Du|^2)^{\frac{p-2}{2}} |Du - Du_\varepsilon|^2 b_\varepsilon \, dx \\ &\quad \le \int_\Omega \langle A_\varepsilon(x, Du) - A_\varepsilon(x, Du_\varepsilon), Du - Du_\varepsilon \rangle \, dx \\ &\quad = \int_\Omega \langle F - F_\varepsilon, Du - Du_\varepsilon \rangle \, dx + \int_\Omega \langle A_\varepsilon(x, Du) - A(x, Du), Du - Du_\varepsilon \rangle \, dx \\ &\quad \le \frac{\nu}{p} \|Du - Du_\varepsilon\|_p^p + \frac{c}{\nu} \|F - F_\varepsilon\|_{p'}^{p'} + \frac{c}{\nu} \int_\Omega |A_\varepsilon(x, Du) - A(x, Du)|^{p'} \, dx. \end{aligned} \tag{7.10}$$

We remark that from (7.3) we deduce that $A_\varepsilon(x, Du) \to A(x, Du)$ a.e. Moreover (7.1) and (7.4) give

$$|A_\varepsilon(x, Du)|^{\frac{p}{p-1}} \le k^{\frac{p}{p-1}} \|b\|_{L^\infty}^{\frac{p}{p-1}} (\mu^2 + |Du|^2)^{\frac{p}{2}},$$

and by dominated convergence theorem we obtain that $A_\varepsilon(x, Du) \to A(x, Du)$ in $L^{\frac{p}{p-1}}$. Then, from (7.10) with a suitable choice of ν, we get that $Du_\varepsilon \to Du$ in L^p. From (7.9) and the semicontinuity of the norm with respect to weak convergence, we get (7.8) for $b(x) \in L^\infty$. Now let $A_j(x, \xi)$, $j \in \mathbb{N}$, be the operators defined in Sect. 5. We consider the problem

$$\begin{cases} \operatorname{div} A_j(x, Du_j) = \operatorname{div} F & \text{in} \quad \Omega \\ u_j = u & \text{on} \quad \partial\Omega, \end{cases} \tag{7.11}$$

Since (7.8) holds for $b_j(x)$, we get that for any $j \in \mathbb{N}$

$$\int_\Omega |DV_\mu(Du_j)|^2 b_j \, dx \le c \left(\int_\Omega (\mu^2 + |Du_j|^2)^{\frac{p}{2}} b_j \, dx + \int_\Omega (\mu^2 + |DF|^2) b \, dx \right).$$

(7.12)

Now we prove that $Du_j \to Du$ in $L^p(b,\Omega;\mathbb{R}^{N\times n})$. From (1.3) we get

$$\begin{aligned}
&\frac{1}{k}\int_\Omega (\mu^2+|Du|^2+|Du_j|^2)^{\frac{p-2}{2}}|Du-Du_j|^2 b\,dx\\
&\le \int_\Omega \langle A(x,Du)-A(x,Du_j), Du-Du_j\rangle\,dx\\
&= \int_\Omega \langle A_j(x,Du_j)-A(x,Du_j), Du-Du_j\rangle\,dx\\
&\le k\int_\Omega \left(1-\frac{b_j}{b}\right)(\mu^2+|Du_j|^2)^{\frac{p-1}{2}}|Du-Du_j|b\,dx\\
&= k\int_\Omega (b-b_j)(\mu^2+|Du_j|^2)^{\frac{p-1}{2}}|Du-Du_j|\,dx.
\end{aligned}$$

Then from Young inequality we get

$$\begin{aligned}
\int_\Omega |Du-Du_j|b\,dx &\le c\int_\Omega (b-b_j)^{\frac{p}{p-1}}(\mu^2+|Du_j|^2)^{\frac{p}{2}}\,dx\\
&\le c\left(\int_\Omega (b-b_j)^r\,dx\right)^{\frac{1}{2}}\cdot\left(\int_\Omega (\mu^2+|Du_j|^2)^{\frac{np}{n-2}}\,dx\right)^{\frac{n-2}{2n}},
\end{aligned} \tag{7.13}$$

where $r=\frac{p}{p-1}\cdot\frac{2n}{n+1}$. The last term goes to zero as $j\to+\infty$ thanks to (7.12), the embedding Sobolev Theorem and the convergence of b_j to b in every Lebesgue space L^q with $1\le q<n$. Now, from (7.12), by using (7.11) and (5.1), we obtain that $\{|DV_\mu(Du_j)|\}$ is a bounded sequence in $L^2(b_j,\Omega)$. Then, by the semicontinuity of the norm with respect to the weak convergence, we get the result. □

7.2 Global integrability

For $G\in L^p(b,Q_{2R};\mathbb{R}^{N\times n})$, consider the problem

$$\begin{cases}
\operatorname{div} A_\varepsilon(x,Du_\varepsilon)=\operatorname{div}|G_\varepsilon|^{p-2}G_\varepsilon & \text{in } Q_{2R}^+\\
u_\varepsilon=0 & \text{on } Q_{2R}\cap\{x_n=0\}.
\end{cases}$$

Lemma 7.2 *If $\alpha_2>0$ is the constant, depending on p, n, λ_0 and k, in Theorem 1.2, if*

$$\mathscr{D}_K\equiv \operatorname{dist}_{L^{n,\infty}}(K(x),L^\infty)<\alpha_2$$

and if $G\in L^q(b,Q_{2R};\mathbb{R}^{N\times n})$, for $q\in(p,s)$, then

$$\begin{aligned}
\left(\fint_{Q_R^+}(\mu^2+|Du_\varepsilon|^2)^{\frac{q}{2}}b_\varepsilon\,dx\right)^{\frac{1}{q}} &\le c\left(\fint_{Q_{2R}^+}(\mu^2+|Du_\varepsilon|^2)^{\frac{p}{2}}b_\varepsilon\,dx\right)^{\frac{1}{p}}\\
&\quad+c\left(\fint_{Q_{2R}^+}(\mu^2+|G_\varepsilon|^2)^{\frac{q}{2}}b_\varepsilon\,dx\right)^{\frac{1}{q}},
\end{aligned}$$

where $c = c(p, n, \lambda_0, k, \mathscr{D}_K, \|b\|_*)$.

Proof We proceed as in the proof of Theorem 1.2. We consider in Lemma 6.1 the comparison map defined as the unique solution of the problem

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Dw_\varepsilon) = 0 & \text{in} \quad 3Q^+ \\ w_\varepsilon - u_\varepsilon \in W_0^{1,p}(3Q^+; \mathbb{R}^N). \end{cases}$$

Moreover $M^*_{b_\varepsilon} \equiv M^*_{b_\varepsilon, Q^+_{2R}}$ and we continue arguing as in [55, Lemma 7.5]. □

Now we consider the problem

$$\begin{cases} \operatorname{div} A(x, Du) = \operatorname{div} |G|^{p-2}G & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

we prove the following

Theorem 7.2 *Let* $u \in W^{1,p}(b, \Omega, \mathbb{R}^N)$ *be the solution of* (7.14). *There exists* $\alpha_4 > 0$, *depending on* p, n, λ_0, k *and* Ω, *such that, if*

$$\mathscr{D}_K \equiv \operatorname{dist}_{L^{n,\infty}}(K(x), L^\infty) < \alpha_4$$

and $G \in L^q(b, \Omega; \mathbb{R}^{N\times n})$, *for* $q \in (p, s)$, *then*

$$\left(\fint_\Omega (\mu^2 + |Du|^2)^{\frac{q}{2}} b\, dx \right)^{\frac{1}{q}} \le c \left(\fint_\Omega (\mu^2 + |G|^2)^{\frac{q}{2}} b\, dx \right)^{\frac{1}{q}},$$

where $c = c(p, n, \lambda_0, k, \mathscr{D}_K, \|b\|_*, \Omega)$.

Proof Firstly, assuming $b(x) \in L^\infty(\Omega)$, we consider for any $0 < \varepsilon < 1$ the system

$$\begin{cases} \operatorname{div} A_\varepsilon(x, Du_\varepsilon) = \operatorname{div} |G_\varepsilon|^{p-2}G_\varepsilon & \text{in} \quad \Omega \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{7.14}$$

Following the lines of Theorem 7.1, by using Lemma 7.2, since u_ε solve the system (7.14), we obtain that

$$\begin{aligned} \int_\Omega (\mu^2 + |Du_\varepsilon|^2)^{\frac{q}{2}} b_\varepsilon\, dx &\le c \int_\Omega (\mu^2 + |G_\varepsilon|^2)^{\frac{q}{2}} b_\varepsilon\, dx \\ &\le c \int_\Omega (\mu^2 + |G|^2)^{\frac{q}{2}} b\, dx, \end{aligned} \tag{7.15}$$

where $c = c(p, n, k, \lambda_0, \mathscr{D}_K, \|b\|_*)$. Arguing as in (7.10) we get that $Du_\varepsilon \to Du$ in $L^p(b, \Omega; \mathbb{R}^{N\times n})$. Then the result follows by (7.15). In order to study the case $b(x) \in BMO$, as in Theorem 7.1 we consider for any $j \in \mathbb{N}$ the problems

$$\begin{cases} \operatorname{div} A_j(x, Du_j) = \operatorname{div} |G|^{p-2}G & \text{in} \quad \Omega \\ u_j = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and we apply (7.15) to get

$$\int_\Omega (\mu^2 + |Du_j|^2)^{\frac{q}{2}} b_j\, dx \le c \int_\Omega (\mu^2 + |G|^2)^{\frac{q}{2}} b\, dx. \tag{7.16}$$

As in Theorem 7.1 we get that $Du_j \to Du$ in $L^p(b, \Omega; \mathbb{R}^{N\times n})$. Indeed in (7.13) we have

$$\int_\Omega |Du - Du_j|^p b\, dx \le c \int_\Omega (b - b_j)^{\frac{p}{p-1}} \cdot (\mu^2 + |Du_j|^2)^{\frac{p}{2}}\, dx$$
$$\le c \left(\int_\Omega (b - b_j)^{\frac{pr}{p-1}}\, dx \right)^{\frac{1}{r}} \cdot \left(\int_\Omega (\mu^2 + |Du_j|^2)^{\frac{q}{2}}\, dx \right)^{\frac{p}{q}},$$

where $r = \frac{p}{p-q}$, and from (7.16) $\|Du_j - Du\| \to 0$ as $j \to +\infty$. Now, from the semicontinuity of the norm with respect to weak convergence, (7.16) gives the conclusion. □

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

Appendix

In this section we prove (5.3). Assume for the moment that $A(x, \eta) \ge 0$ and $A(y, \eta) \ge 0$. We note that

- If $b(x) \le j$ and $b(y) \le j$, then

$$|A_j(x, \eta) - A_j(y, \eta)| = |A(x, \eta) - A(y, \eta)|$$
$$\le |x - y|\, [K(x) + K(y)]\, (\mu^2 + |\eta|^2)^{\frac{p-1}{2}}$$
$$\le (k + 1)\, |x - y|\, [K(x) + K(y)]\, (\mu^2 + |\eta|^2)^{\frac{p-1}{2}}.$$

- If $b(y) \geq b(x) > j$, then

$$
\begin{aligned}
&-(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq -\frac{j}{b(y)}\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq \frac{j}{b(y)}\,[A(x,\eta)-A(y,\eta)] \leq A_j(x,\eta)-A_j(y,\eta) \\
&= \frac{j}{b(x)}\,[A(x,\eta)-A(y,\eta)] + \left(\frac{j}{b(x)}-\frac{j}{b(y)}\right)A(y,\eta) \\
&\leq \frac{j}{b(x)}\,|x-y|\,[K(x)+K(y)](\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\quad + \frac{j}{b(x)}k\,|x-y|\,[K(x)+K(y)](\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq (k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

- If $b(x) > b(y) > j$, then

$$
\begin{aligned}
&-(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq -\frac{j}{b(x)}(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&= -\frac{j}{b(x)}|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\quad - \frac{j}{b(x)}k\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq \frac{j}{b(x)}\,[A(x,\eta)-A(y,\eta)] - j\left(\frac{b(x)-b(y)}{b(y)b(x)}\right)A(y,\eta) \\
&= A_j(x,\eta)-A_j(y,\eta) \leq \frac{j}{b(y)}\,[A(x,\eta)-A(y,\eta)] \\
&\leq (k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

- If $b(y) > j \geq b(x)$ then

$$
\begin{aligned}
&-(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq -\frac{j}{b(y)}|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq \frac{j}{b(y)}\,[A(x,\eta)-A(y,\eta)] \leq A_j(x,\eta)-A_j(y,\eta) \\
&= A(x,\eta)-A(y,\eta) + \left(1-\frac{j}{b(y)}\right)A(y,\eta) \\
&\leq |x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} + \left(\frac{b(y)-b(x)}{b(y)}\right)kb(y)(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
&\leq (k+1)|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

- If $b(x) > j \geq b(y)$ then

$$
\begin{aligned}
& -(k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
& \leq -\frac{j}{b(x)}\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
& \quad -\frac{j}{b(x)}k\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
& \leq -\frac{j}{b(x)}\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
& \quad -\frac{b(y)}{b(x)}k|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}} \\
& \leq \frac{j}{b(x)}\,[A(x,\eta)-A(y,\eta)] - \left(\frac{b(x)-b(y)}{b(x)}\right) A(y,\eta) \\
& = A_j(x,\eta) - A_j(y,\eta) \leq [A(x,\eta)-A(y,\eta)] \\
& \leq (k+1)\,|x-y|\,[K(x)+K(y)]\,(\mu^2+|\eta|^2)^{\frac{p-1}{2}}.
\end{aligned}
$$

The proof of the remaining cases is analogous, therefore (5.3) is proved.

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