



Asymptotic behaviour of solutions to the anisotropic doubly critical equation

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Abstract

The aim of this paper is to deal with the anisotropic doubly critical equation

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = u^{p^*-1} \quad \text{in } \mathbb{R}^N,$$

where H is in some cases called Finsler norm, H° is the dual norm, $1 < p < N$, $0 \leq \gamma < ((N-p)/p)^p$ and $p^* = Np/(N-p)$. In particular, we provide a complete asymptotic analysis of $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ near the origin and at infinity, showing that this solution has the same features of its euclidean counterpart. Some of the techniques used in the proofs are new even in the Euclidean framework.

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1 Introduction and main results

This work is devoted to the study of the following anisotropic doubly critical problem

$$\begin{cases} -\Delta_p^H u - \frac{\gamma}{H^\circ(x)^p} u^{p-1} = u^{p^*-1} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \quad (\mathcal{P}_H)$$

where $1 < p < N$, $p^* := Np/(N - p)$ is the Sobolev critical exponent, $0 \leq \gamma < C_H := ((N - p)/p)^p$ is the Hardy constant and

$$\Delta_p^H u := \operatorname{div}(H^{p-1}(\nabla u)\nabla H(\nabla u)), \tag{1.1}$$

where Δ_p^H is the so-called anisotropic p-Laplacian or Finsler p-Laplacian. We point out that H is a Finsler type norm and H° is its dual norm (H satisfies assumptions (h_H) , see Sect. 2 for further details). In particular, when $H(\xi) = |\xi| = H^\circ(\xi)$ the Finsler type p-Laplacian coincides with the classical p-Laplacian, and, hence it is singular when $1 < p < 2$ and degenerate when $p > 2$. Then, according with standard regularity theory [13, 27] and the regularity results in the anisotropic framework [2, 6], we say that any solution of (\mathcal{P}_H) has to be understood in the weak distributional meaning, i.e. $u \in D^{1,p}(\mathbb{R}^N)$ satisfies the following integral equality

$$\begin{aligned} & \int_{\mathbb{R}^N} \left(H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \varphi \rangle - \frac{\gamma}{H^\circ(x)^p} u^{p-1} \varphi \right) dx \\ & = \int_{\mathbb{R}^N} u^{p^*-1} \varphi dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N). \end{aligned} \tag{1.2}$$

The literature about critical problems is really huge. Going back to the Euclidean framework, i.e. when we consider $H(\xi) = |\xi| = H^\circ(\xi)$ in (\mathcal{P}_H) , we deal with

$$-\Delta_p u - \frac{\gamma}{|x|^p} u^{p-1} = u^{p^*-1} \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

In the seminal paper [5], Caffarelli, Gidas and Spruck classified any positive solution to (1.3) with $p = 2$, $N \geq 3$, and $\gamma = 0$. We point out that a first result, under stronger assumption on the decay of solutions, was obtained by Gidas, Ni and Nirenberg in [14]. Moreover, in this setting a complete answer in the subcritical case was done in the celebrated work of Gidas and Spruck [15], where the authors proved Liouville-type theorems.

In the quasilinear framework, the situation is much more involved due to the nonlinear nature of the operator. Recently, a classification result of positive solutions to (1.3) with $p > 2$, $\gamma = 0$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N) \mid \nabla u \in L^p(\mathbb{R}^N)\}$ was obtained in [24]. The proof of this result is based on a refined version of the well-known moving plane method of Alexandrov-Serrin [1, 23] and on some a priori estimates of the solutions and their gradients, proved in [29]. To be more precise, we note that the classification result of positive solution to the Sobolev critical quasilinear equation with finite energy started in [12] in the case, and then was extended in [29] for every $1 < p < 2$. Subsequently the full case was obtained in [24]. Recently, we refer to the papers [7, 21, 30] for new partial results on the classification of positive solutions without a priori assumption on the energy of solutions. In the anisotropic setting, Ciraolo, Figalli and Roncoroni [9], obtained a complete classification result for positive solution to (\mathcal{P}_H) with $\gamma = 0$ using different techniques that do not require the use of the moving plane method, which could not be used in the anisotropic context due to the lack of invariance.

When $\gamma \neq 0$ the situation is really different. In the seminal paper of Terracini [26], it was proved for the first time the classification result for positive solutions to (1.3) in the case $p = 2$. The author firstly showed the existence of solutions to this problem with a minimization argument based on the concentration and compactness principle. Subsequently, she proved that any solution to this problem is radial and radially decreasing about the origin combining the moving-plane technique and the use of the Kelvin transformation, in the same spirit of [5]. The case $p \neq 2$ and $\gamma \neq 0$ is much more involved and it is available in [20], where the techniques used are mainly based on a fine asymptotic analysis at infinity and refined versions of the moving plane procedure, and also on some asymptotic estimates proved in [31, 32].

Our aim is to prove some decay estimates for positive weak solutions to (\mathcal{P}_H) in the anisotropic framework $1 < p < N$ and $\gamma \neq 0$. More precisely, our first main result is the following:

Theorem 1.1 *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution of (\mathcal{P}_H) with $1 < p < N, 0 \leq \gamma < C_H$. Then there exist positive constants $0 < R_1 < 1 < R_2$ depending on N, p, γ and u , such that*

$$\frac{c_1}{[H^\circ(x)]^{\mu_1}} \leq u(x) \leq \frac{C_1}{[H^\circ(x)]^{\mu_1}} \quad x \in \mathcal{B}_{R_1}^{H^\circ}, \tag{1.4}$$

and

$$\frac{c_2}{[H^\circ(x)]^{\mu_2}} \leq u(x) \leq \frac{C_2}{[H^\circ(x)]^{\mu_2}} \quad x \in (\mathcal{B}_{R_2}^{H^\circ})^c, \tag{1.5}$$

where μ_1, μ_2 are the solutions of

$$\mu^{p-2}[(p-1)\mu^2 - (N-p)\mu] + \gamma = 0, \tag{1.6}$$

C_1, C_2 are positive constants depending on N, p, γ, H and u , c_1 is a positive constant depending on $N, p, \gamma, H, R_1, \mu_1$ and u , c_2 is a positive constant depending on $N, p, \gamma, H, R_2, \mu_2$ and u .

Remark 1.2 In the following we shall assume that $\mu_1 < \mu_2$ and it is easy to see that

$$0 \leq \mu_1 < \frac{N-p}{p} < \mu_2 \leq \frac{N-p}{p-1}.$$

furthermore $\mathcal{B}_R^{H^\circ}$ is the dual anisotropic ball also known as Frank diagram (see Sect. 2 for further details).

In the proof we will also exploit some clever ideas from [31] facing the difficulties of the anisotropic issue. A different approach is in fact needed for the study of the asymptotic behaviour of the gradient. In particular, the fact that the moving plane technique cannot be applied, a crucial point is given by the following classification result:

Theorem 1.3 *Let $v \in C_{loc}^{1,\alpha}(\mathbb{R}^N \setminus \{0\})$ be a positive weak solution of the equation*

$$-\Delta_p^H v - \frac{\gamma}{[H^\circ(x)]^p} v^{p-1} = 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \tag{1.7}$$

where $0 \leq \gamma < C_H$. Assume that there exist two positive constants C and c such that

$$\frac{c}{[H^\circ(x)]^{\mu_i}} \leq v(x) \leq \frac{C}{[H^\circ(x)]^{\mu_i}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{1.8}$$

where μ_i ($i = 1, 2$) are the roots of (1.6) and suppose that there exists a positive constant \hat{C} such that

$$|\nabla v(x)| \leq \frac{\hat{C}}{[H^\circ(x)]^{\mu_i+1}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{1.9}$$

then

$$v(x) = \frac{\bar{c}}{[H^\circ(x)]^{\mu_i}}, \tag{1.10}$$

for some $\bar{c} > 0$.

Theorem 1.3 is new and interesting in itself. The proof is very much different than the ones available in the euclidean case $H(\xi) = |\xi|$. Here we shall exploit it to deduce the precise asymptotic estimates for the gradient. more precisely we have the following:

Theorem 1.4 *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution of (\mathcal{P}_H) with $1 < p < N$, and $0 \leq \gamma < C_H$. Then there exist positive constants \tilde{c}, \tilde{C} depending on N, p, γ, H and u such that*

$$\frac{\tilde{c}}{[H^\circ(x)]^{\mu_1+1}} \leq |\nabla u(x)| \leq \frac{\tilde{C}}{[H^\circ(x)]^{\mu_1+1}} \quad x \in \mathcal{B}_{R_1}^{H^\circ}, \tag{1.11}$$

and

$$\frac{\tilde{c}}{[H^\circ(x)]^{\mu_2+1}} \leq |\nabla u(x)| \leq \frac{\tilde{C}}{[H^\circ(x)]^{\mu_2+1}} \quad x \in (\mathcal{B}_{R_2}^{H^\circ})^c, \tag{1.12}$$

where μ_1, μ_2 are roots of (1.6) as in Theorem 1.1, and $0 < R_1 < 1 < R_2$ are constants depending on N, p, γ and u .

The paper is structured as follows:

- In Sect. 2 we recall some notions about Finsler type anisotropic geometry, and we prove some technical lemmas that will be crucial in the proof of the main results.
- In Sect. 3 we prove some preliminary estimates, elliptic estimates and weak comparison principles in bounded and exterior domains that will be essential in the proof of Theorem 1.1.
- In Sect. 4 we give the proof of decay estimates of solutions to (\mathcal{P}_H) near the origin and at infinity, i.e. we prove Theorem 1.1. The, using this result we also prove decay estimates for the gradient of positive weak solutions to (\mathcal{P}_H) near the origin and at infinity, i.e. we prove Theorem 1.4.
- Although the existence of solutions can be easily deduced in the radial-anisotropic setting, in the ‘‘Appendix A’’ we show that problem (\mathcal{P}_H) admits at least a positive solution $u \in D^{1,p}(\mathbb{R}^N)$ that minimizes the Hardy-Sobolev anisotropic inequality. This result follows using classical arguments (see also [26]) that we decide to add for the readers’ convenience.

2 Preliminaries

Notation. Generic fixed and numerical constants will be denoted by C (with subscript in some case) and they will be allowed to vary within a single line or formula. By $|A|$ we will denote the Lebesgue measure of a measurable set A .

The aim of this section is to recall some properties and geometrical tools about the anisotropic elliptic operator defined above. For $a, b \in \mathbb{R}^N$ we denote by $a \otimes b$ the matrix whose entries are $(a \otimes b)_{ij} = a_i b_j$. We remark that for any $v, w \in \mathbb{R}^N$ it holds that:

$$\langle a \otimes b v, w \rangle = \langle b, v \rangle \langle a, w \rangle.$$

Now, we recall the definition of anisotropic norm.

(h_H) Let $H \in C^2(\mathbb{R}^N \setminus \{0\})$. In all the paper we assume that H is a anisotropic norm if it satisfies the following set of assumptions:

- (i) $H(\xi) > 0 \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}$;
- (ii) $H(s\xi) = |s|H(\xi) \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \forall s \in \mathbb{R}$;
- (iii) H is *uniformly elliptic*, that means the set $\mathcal{B}_1^H := \{\xi \in \mathbb{R}^N : H(\xi) < 1\}$ is *uniformly convex*

$$\exists \Lambda > 0 : \quad \langle D^2 H(\xi)v, v \rangle \geq \Lambda |v|^2 \quad \forall \xi \in \partial \mathcal{B}_1^H, \forall v \in \nabla H(\xi)^\perp. \tag{2.1}$$

A set is said uniformly convex if the principal curvatures of its boundary are all strictly positive. Moreover, assumption (iii) is equivalent to assume that $D^2(H^2)$ is definite positive.

The *dual norm* $H^\circ : \mathbb{R}^N \rightarrow [0, +\infty)$ is defined as:

$$H^\circ(x) = \sup_{H(\xi) \leq 1} \langle \xi, x \rangle.$$

It is possible to show that H° is also a Finsler norm and it has the same regularity properties of H . Moreover, it holds $(H^\circ)^\circ = H$. For $R > 0$ and $\bar{x} \in \mathbb{R}^N$ we define:

$$\mathcal{B}_R^H(\bar{x}) = \{x \in \mathbb{R}^N : H(x - \bar{x}) < R\}$$

and

$$\mathcal{B}_R^{H^\circ}(\bar{x}) = \{x \in \mathbb{R}^N : H^\circ(x - \bar{x}) < R\}.$$

For simplicity of exposition, when $\bar{x} = 0$, we set: $\mathcal{B}_R^H = \mathcal{B}_R^H(0)$, $\mathcal{B}_R^{H^\circ} = \mathcal{B}_R^{H^\circ}(0)$. In literature \mathcal{B}_R^H and $\mathcal{B}_R^{H^\circ}$ are also called ‘‘Wulff shape’’ and ‘‘Frank diagram’’ respectively. We remark that there holds the following identities:

$$H(\nabla H^\circ(x)) = 1 = H^\circ(\nabla H(x)); \tag{2.2}$$

and

$$H(x)\nabla H^\circ(\nabla H(x)) = x = H^\circ(x)\nabla H(\nabla H^\circ(x)). \tag{2.3}$$

We refer the reader to [3, 8] for further details. Observe also that H is a norm equivalent to the euclidean one, i.e. there exist $\alpha_1, \alpha_2 > 0$ such that:

$$\alpha_1 |\xi| \leq H(\xi) \leq \alpha_2 |\xi| \quad \forall \xi \in \mathbb{R}^N. \tag{2.4}$$

Moreover, recalling that H is 1-homogeneous, by the Euler’s Theorem it follows

$$\langle \nabla H(\xi), \xi \rangle = H(\xi) \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{2.5}$$

Since H is 1-homogeneous, we have that ∇H is 0-homogeneous and it satisfies

$$\nabla H(\xi) = \nabla H\left(\frac{|\xi| \xi}{|\xi|}\right) = \nabla H\left(\frac{\xi}{|\xi|}\right) \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

Hence, by the previous equality, we infer that there exists $M > 0$ such that

$$|\nabla H(\xi)| \leq M \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{2.6}$$

For the same reasons there exists a constant $\overline{M} > 0$ such that:

$$|D^2 H(\xi)| \leq \frac{\overline{M}}{|\xi|} \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \tag{2.7}$$

where $|\cdot|$ denotes the usual Euclidean norm of a matrix, and

$$D^2 H(\xi)\xi = 0 \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{2.8}$$

We start with some elliptic estimates that can be proved in the same spirit of the Euclidean framework.

Proposition 2.1 *For any $p > 1$ and $\eta, \eta' \in \mathbb{R}^N$ such that $|\eta| + |\eta'| > 0$, it holds*

$$|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \leq \tilde{C}_p(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|. \tag{2.9}$$

Moreover, any $p \geq 2$ it holds the following inequality

$$H^p(\eta) \geq H^p(\eta') + pH^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta' \rangle + \hat{C}(p)H^p(\eta - \eta'), \tag{2.10}$$

for any $\eta, \eta' \in \mathbb{R}^N$. Furthermore, if $1 < p < 2$ we have that

$$H^p(\eta) \geq H^p(\eta') + pH^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta' \rangle + C_p[H(\eta) + H(\eta')]^{p-2}H^2(\eta - \eta'), \tag{2.11}$$

for any $\eta, \eta' \in \mathbb{R}^N$ such that $|\eta| + |\eta'| > 0$.

Proof We start the proof showing (2.9). First of all we note that (2.9) is symmetric in η, η' . Hence, without loss of generality, we can assume that $|\eta'| \geq |\eta| > 0$. We note that for $j = 1, \dots, N$:

$$\begin{aligned} & H^{p-1}(\eta)\frac{\partial H}{\partial \eta_j}(\eta) - H^{p-1}(\eta')\frac{\partial H}{\partial \eta_j}(\eta') \\ &= \int_0^1 \sum_{i=1}^N \left[(p-1)H^{p-2}(\eta' + t(\eta - \eta')) \left(\frac{\partial H}{\partial \eta_i} \cdot \frac{\partial H}{\partial \eta_j} \right) (\eta' + t(\eta - \eta')) \right. \\ & \quad \left. + H^{p-1}(\eta' + t(\eta - \eta')) \frac{\partial^2 H}{\partial \eta_i \partial \eta_j}(\eta' + t(\eta - \eta')) \right] \cdot (\eta_i - \eta'_i) dt. \end{aligned} \tag{2.12}$$

By (2.12), using (2.4), (2.6) and (2.7) we have

$$\begin{aligned} & |H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \\ & \leq \int_0^1 |(p-1)H^{p-2}(\eta' + t(\eta - \eta'))\nabla H(\eta' + t(\eta - \eta')) \otimes \nabla H(\eta' + t(\eta - \eta')) \\ & \quad + H^{p-1}(\eta' + t(\eta - \eta'))D^2 H(\eta' + t(\eta - \eta'))| \cdot |\eta - \eta'| dt \\ & \leq ((p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 (|\eta' + t(\eta - \eta')|)^{p-2} dt, \end{aligned} \tag{2.13}$$

where $|\cdot|$ denotes the standard matrix Euclidean norm.

Now, we observe that

$$|\eta' + t(\eta - \eta')| \leq |\eta| + |\eta'|, \tag{2.14}$$

and, since $|\eta'| \geq |\eta|$, we have either

$$|\eta - \eta'| \leq \frac{|\eta'|}{2} \implies |\eta' + t(\eta - \eta')| \geq |\eta'| - |\eta - \eta'| \geq \frac{|\eta'|}{2} \geq \frac{|\eta'| + |\eta|}{4} \tag{2.15}$$

or, putting $t_0 := \frac{|\eta'|}{|\eta - \eta'|} \in (0, 2)$,

$$\begin{aligned} |\eta - \eta'| > \frac{|\eta'|}{2} &\implies |\eta' + t(\eta - \eta')| \geq \left| |\eta'| - t|\eta - \eta'| \right| = |t_0 - t| \cdot |\eta - \eta'| \\ &\geq |t_0 - t| \cdot \frac{|\eta'|}{2} = |t_0 - t| \frac{|\eta'|}{2} \\ &\geq |t_0 - t| \frac{|\eta| + |\eta'|}{4}. \end{aligned} \tag{2.16}$$

If $p > 2$, using (2.14) in (2.13) we have

$$|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \leq \tilde{C}_p(|\eta'| + |\eta|)^{p-2}|\eta - \eta'| \tag{2.17}$$

where $\tilde{C}_p = (p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M}$. Hence (2.9) holds.

If $p \leq 2$ and (2.15) holds, by (2.13) we obtain

$$\begin{aligned} &|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \\ &\leq ((p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 (|\eta' + t(\eta - \eta')|)^{p-2} dt. \\ &\leq ((p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 \left(\frac{|\eta'| + |\eta|}{4}\right)^{p-2} dt \\ &\leq \tilde{C}_p(|\eta'| + |\eta|)^{p-2}|\eta - \eta'|, \end{aligned} \tag{2.18}$$

where $\tilde{C}_p = ((p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})/4^{p-2}$. Hence (2.9) holds

If $p \leq 2$ and (2.16) holds, by (2.13) we obtain

$$\begin{aligned} &|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \\ &\leq ((p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 (|\eta' + t(\eta - \eta')|)^{p-2} dt. \\ &\leq ((p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 |t_0 - t|^{p-2} \left(\frac{|\eta'| + |\eta|}{4}\right)^{p-2} dt \\ &= \frac{(p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M}}{4^{p-2}}(|\eta'| + |\eta|)^{p-2}|\eta - \eta'| \int_0^1 |t_0 - t|^{p-2} dt \\ &\leq \frac{(p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M}}{4^{p-2}}(|\eta'| + |\eta|)^{p-2}|\eta - \eta'| 2 \int_0^1 z^{p-2} dz \\ &= \tilde{C}_p(|\eta'| + |\eta|)^{p-2}|\eta - \eta'|, \end{aligned} \tag{2.19}$$

where $\tilde{C}_p = (2(p - 1)\alpha_2^{p-2}M^2 + 2\alpha_2^{p-1}\overline{M})/4^{p-2}$.

Collecting the estimates above, we deduce that inequality (2.9) holds for every $p > 1$ and for $\tilde{C}_p = ((p - 1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M}) \cdot \max\{1, 4^{2-p}, 2 \cdot 4^{2-p}\}$.

Now we will show (2.10) and (2.11). For $\eta, \eta' \in \mathbb{R}^N$, we define

$$f(t) = H^p(\eta' + t(\eta - \eta'))$$

and Taylor’s formula yields

$$H^p(\eta) = H^p(\eta') + p H^{p-1}(\eta') \langle \nabla H(\eta'), \eta - \eta' \rangle + \int_0^1 (1-t) f''(t) dt, \tag{2.20}$$

provided $|\eta' + t(\eta - \eta')| \neq 0$, for $0 \leq t \leq 1$. But the case when $\eta' + t(\eta - \eta') = 0$ can be easily verified. By [10, Theorem 1.5] we obtain that

$$\begin{aligned} f''(t) &= \langle D^2(H^p(\eta' + t(\eta - \eta')))(\eta - \eta'), \eta - \eta' \rangle \\ &\geq C(p) |\eta' + t(\eta - \eta')|^{p-2} |\eta - \eta'|^2, \end{aligned} \tag{2.21}$$

where $C(p)$ is a constant depending on p . We remark that

$$\int_0^1 (1-t) f''(t) dt \geq \frac{3}{4} \int_0^{\frac{1}{4}} f''(t) dt. \tag{2.22}$$

If $1 < p < 2$, by (2.14) we have

$$(H(\eta) + H(\eta'))^{p-2} \leq \alpha_2^{p-2} |\eta' + t(\eta - \eta')|^{p-2}$$

and using (2.22) we arrive at (2.11).

If $p \geq 2$, using a similar argument as in the proof of the inequality (2.9), we obtain

$$\int_0^{\frac{1}{4}} f''(t) \geq \bar{C}(p) (|\eta| + |\eta'|)^{p-2} |\eta - \eta'|^2, \tag{2.23}$$

with $\bar{C}(p)$ constant depending on p . Since $|\eta - \eta'| \leq |\eta| + |\eta'|$ and using (2.4), we get (2.10). □

We state now the Hardy inequality for the anisotropic operator $\Delta_p^H u$, defined in (1.1). We refer to [28, Proposition 7.5].

Theorem 2.2 (Hardy inequality) *For any H satisfying the assumption (h_H) and any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $1 < p < N$,*

$$C_H \int_{\mathbb{R}^N} \frac{|u|^p}{H^c(x)^p} dx \leq \int_{\mathbb{R}^N} H^p(\nabla u) dx, \tag{2.24}$$

where $C_H = ((N - p)/p)^p$ is optimal.

Now we prove a technical lemma that will be very important in the proof of the asymptotic estimates.

Lemma 2.3 *Let $p > 1$ and $a, b \geq 0$. Then, for all $\delta > 0$ there exist $C_\delta > 0$ such that*

$$a^p \geq \frac{1}{1 + 2^{p+1}\delta} (a + b)^p - C_\delta b^p. \tag{2.25}$$

Proof Let us consider $p > 1$ as follows:

$$p = \lfloor p \rfloor + \{p\},$$

where $\lfloor \cdot \rfloor$ is the floor function and $\{\cdot\}$ is the mantissa function. Without loss of generality we assume that $\{p\} \neq 0$ and, moreover, we set $m := \lfloor p \rfloor$. Hence, we have

$$(a + b)^p = (a + b)^{\lfloor p \rfloor} (a + b)^m = (a + b)^{\lfloor p \rfloor} \sum_{k=0}^m \binom{m}{k} a^{m-k} b^k = (\star) \tag{2.26}$$

Noticing that $0 \leq \{p\} < 1$ it follows that

$$(a + b)^{\{p\}} \leq a^{\{p\}} + b^{\{p\}}.$$

Using this inequality in (2.26), we deduce

$$\begin{aligned} (\star) &\leq a^{\{p\}} \sum_{k=0}^m \binom{m}{k} a^{m-k} b^k + b^{\{p\}} \sum_{k=0}^m \binom{m}{k} a^{m-k} b^k \\ &= a^p + b^p + \sum_{k=1}^m \binom{m}{k} a^{p-k} b^k + \sum_{k=0}^{m-1} \binom{m}{k} a^{m-k} b^{k+\{p\}} = (*), \end{aligned} \tag{2.27}$$

where we used the fact that $p = m + \{p\}$. Now, we can apply the weighted Young’s inequality to each member of the first sum with conjugate exponents $(p/(p - k), p/k)$ and to each member of the second sum with conjugate exponents $(p/(m - k), p/(k + \{p\}))$ as follows

$$\begin{aligned} a^{p-k} b^k &\leq \frac{p-k}{p} \delta a^p + \frac{k}{p} C_\delta b^p \leq \delta a^p + C_\delta b^p, \\ a^{m-k} b^{k+\{p\}} &\leq \delta a^p + C_\delta b^p. \end{aligned} \tag{2.28}$$

Hence, using this estimate we deduce

$$\begin{aligned} (*) &\leq a^p + b^p + (\delta a^p + C_\delta b^p) \sum_{k=1}^m \binom{m}{k} + (\delta a^p + C_\delta b^p) \sum_{k=0}^{m-1} \binom{m}{k} \\ &\leq (1 + 2^{p+1} \delta) a^p + C_\delta b^p, \end{aligned} \tag{2.29}$$

where we renamed $C_\delta := (1 + 2^{p+1} \delta)$. Collecting (2.26), (2.27) and (2.29), we deduce that

$$a^p \geq \frac{1}{1 + 2^{p+1} \delta} (a + b)^p - C_\delta b^p, \tag{2.30}$$

with $C_\delta := C_\delta / (1 + 2^{p+1} \delta)$, and hence the thesis (2.25). □

Finally, we recall a lemma (see Lemma 4.19 in [16]) that will be very useful in the proofs of our results.

Lemma 2.4 [16] *Let \mathcal{L} and g be two nondecreasing functions on the interval $(0, \bar{R}]$, for some $\bar{R} > 0$. Suppose that it holds*

$$\mathcal{L}(\tau R) \leq \sigma \mathcal{L}(R) + g(R) \text{ for all } R \leq \bar{R},$$

for some $0 < \sigma, \tau < 1$. Then, for any $\mu \in (0, 1)$ and $R \leq \bar{R}$ we have

$$\mathcal{L}(R) \leq \frac{1}{\sigma} \left(\frac{R}{\bar{R}}\right)^\alpha \mathcal{L}(\bar{R}) + \frac{1}{1-\sigma} g(\bar{R}^\mu R^{1-\mu})$$

where $\alpha = \alpha(\sigma, \tau, \mu) = (1 - \mu) \log \sigma / \log \tau$.

3 Preliminary asymptotic estimates and comparison principles

The aim of this section is to prove some preliminary estimates that will be crucial in the proofs of the main results.

Lemma 3.1 *There exists a positive constant τ depending only on N, p and γ such that for any $\bar{R} > 0$ and for any solution u to problem (\mathcal{P}_H) satisfying*

$$\|u\|_{L^{p^*}(\mathcal{B}_{\bar{R}}^{H^\circ})} + \|u\|_{L^{p^*}(\mathbb{R}^N \setminus \mathcal{B}_{1/\bar{R}}^{H^\circ})} \leq \tau, \tag{3.1}$$

there exists a positive constant C depending only on N, p, γ and \bar{R} such that

$$\|u\|_{L^{p^*}(\mathcal{B}_R^{H^\circ})} \leq CR^{\sigma_1} \text{ for } R \leq \bar{R}, \tag{3.2}$$

and that

$$\|u\|_{L^{p^*}((\mathcal{B}_R^{H^\circ})^c)} \leq \frac{C}{R^{\sigma_2}} \text{ for } R \geq \frac{1}{\bar{R}}, \tag{3.3}$$

where σ_1, σ_2 are two positive constants depending on N, p and γ .

Proof We start proving (3.2). To this aim let us consider $R > 0$ and a cut-off function $\eta \in C_c^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} 0 \leq \eta \leq 1 & \text{in } \mathbb{R}^N \\ \eta \equiv 0 & \text{in } (\mathcal{B}_R^{H^\circ})^c \\ \eta \equiv 1 & \text{in } \mathcal{B}_{R/2}^{H^\circ} \\ |\nabla \eta| \leq \frac{4}{R} & \text{in } \mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}. \end{cases} \tag{3.4}$$

By density argument it is possible to put $\varphi = \eta^p u$ as test function in (1.2), so that we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} p\eta^{p-1}uH^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \eta \rangle dx + \int_{\mathbb{R}^N} \eta^p H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla u \rangle dx \\ &= \int_{\mathbb{R}^N} \frac{\gamma}{H^\circ(x)^p} u^{p-1} \eta^p u dx + \int_{\mathbb{R}^N} u^{p^*-1} \eta^p u dx. \end{aligned} \tag{3.5}$$

First of all, using Euler’s Theorem (2.5), the 0-homogeneity of ∇H (2.6) and Schwarz’s inequality, equation (3.5) becomes

$$\begin{aligned} & \int_{\mathbb{R}^N} H^p(\nabla u)\eta^p dx = \int_{\mathbb{R}^N} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla u \rangle \eta^p dx \\ & \leq p M \int_{\mathbb{R}^N} H^{p-1}(\nabla u)|\nabla \eta| \eta^{p-1} u dx + \int_{\mathbb{R}^N} \frac{\gamma}{H^\circ(x)^p} u^p \eta^p dx + \int_{\mathbb{R}^N} u^{p^*} \eta^p dx. \end{aligned} \tag{3.6}$$

Recalling that H is 1-homogeneous function, using the weighted Young’s inequality $ab \leq \varepsilon a^{\frac{p}{p-1}} + C_\varepsilon b^p$ on the first term of the right hand side of (3.6), for any $0 < \varepsilon < 1$ we have

$$\begin{aligned} & \int_{\mathbb{R}^N} H^p(\eta \nabla u) dx \leq \varepsilon \int_{\mathbb{R}^N} H^p(\eta \nabla u) dx + C(p, M, \varepsilon) \int_{\mathbb{R}^N} |\nabla \eta|^p u^p dx \\ & \quad + \int_{\mathbb{R}^N} \frac{\gamma}{H^\circ(x)^p} u^p \eta^p dx + \int_{\mathbb{R}^N} u^{p^*} \eta^p dx, \end{aligned} \tag{3.7}$$

where $C(p, M, \varepsilon) := (pM)^p C_\varepsilon$. Now, noticing that $\nabla(\eta u) = u \nabla \eta + \eta \nabla u$, by the triangular inequality, we deduce that for every $p > 1$ it holds

$$H^p(\nabla(\eta u)) = H^p(u \nabla \eta + \eta \nabla u) \leq [H(u \nabla \eta) + H(\eta \nabla u)]^p. \tag{3.8}$$

Thanks to (3.8) and applying Lemma 2.3 with $a = H(\eta \nabla u)$ and $b = H(u \nabla \eta)$, we deduce that

$$\int_{\mathbb{R}^N} H^p(\eta \nabla u) \, dx \geq \frac{1}{1 + 2^{p+1}\delta} \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx - C_\delta \int_{\mathbb{R}^N} H^p(u \nabla \eta) \, dx. \tag{3.9}$$

Using (3.9) in (3.7) we obtain

$$\begin{aligned} \frac{1 - \varepsilon}{1 + 2^{p+1}\delta} \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx &\leq C_\delta \int_{\mathbb{R}^N} H^p(u \nabla \eta) \, dx + C(p, M, \varepsilon) \int_{\mathbb{R}^N} |\nabla \eta|^p u^p \, dx \\ &\quad + \gamma \int_{\mathbb{R}^N} \frac{(u\eta)^p}{H^\circ(x)^p} \, dx + \int_{\mathbb{R}^N} u^{p^*} \eta^p \, dx. \end{aligned} \tag{3.10}$$

Now, applying the anisotropic Hardy inequality (see Theorem 2.2 or [28]) and (2.4) we have

$$\begin{aligned} \frac{1 - \varepsilon}{1 + 2^{p+1}\delta} \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx &\leq (\alpha_2^p C_\delta + C(p, M, \varepsilon)) \int_{\mathbb{R}^N} |\nabla \eta|^p u^p \, dx \\ &\quad + \frac{\gamma}{C_H} \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx + \int_{\mathbb{R}^N} u^{p^*} \eta^p \, dx. \end{aligned} \tag{3.11}$$

Let us fix $\varepsilon, \delta > 0$ sufficiently small such that $C_1 := (1 - \varepsilon)/(1 + 2^{p+1}\delta) - \gamma/C_H > 0$, so that

$$C_1 \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx \leq C_2 \int_{\mathbb{R}^N} |\nabla \eta|^p u^p \, dx + \int_{\mathbb{R}^N} u^{p^*} \eta^p \, dx, \tag{3.12}$$

where $C_2 := C(p, M, \varepsilon) + \alpha_2^p C_\delta$. By (2.4) we have

$$\alpha_1^p C_1 \int_{\mathbb{R}^N} |\nabla(\eta u)|^p \, dx \leq C_2 \int_{\mathbb{R}^N} |\nabla \eta|^p u^p \, dx + \int_{\mathbb{R}^N} u^{p^*} \eta^p \, dx. \tag{3.13}$$

Now, using the Sobolev inequality in the left hand side of (3.13), the Hölder inequality and (3.4) in the right hand side, we obtain

$$\begin{aligned} \alpha_1^p C_1 C_5^p \left(\int_{\mathbb{R}^N} |\eta u|^{p^*} \, dx \right)^{\frac{p}{p^*}} &\leq \alpha_1^p C_1 \int_{\mathbb{R}^N} |\nabla(\eta u)|^p \, dx \\ &\leq C_2 \int_{\mathbb{R}^N} |\nabla \eta|^p u^p \, dx + \int_{\mathbb{R}^N} u^{p^*-p} (\eta u)^p \, dx \\ &\leq C_2 \left(\int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} |\nabla \eta|^N \, dx \right)^{\frac{p}{N}} \left(\int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} u^{p^*} \, dx \right)^{\frac{p}{p^*}} \\ &\quad + \left(\int_{\mathcal{B}_R^{H^\circ}} u^{p^*} \, dx \right)^{\frac{p}{N}} \left(\int_{\mathbb{R}^N} (\eta u)^{p^*} \, dx \right)^{\frac{p}{p^*}} \\ &\leq C_2 C(p, N) \left(\int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} u^{p^*} \, dx \right)^{\frac{p}{p^*}} \\ &\quad + \|u\|_{p^*}^{p^*-p} \left(\int_{\mathbb{R}^N} (\eta u)^{p^*} \, dx \right)^{\frac{p}{p^*}}, \end{aligned} \tag{3.14}$$

hence we deduce

$$\begin{aligned} \left(\int_{\mathbb{R}^N} |\eta u|^{p^*} dx\right)^{\frac{p}{p^*}} &\leq \frac{C_2 C(p, N)}{\alpha_1^p C_1 C_S^p} \left(\int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} u^{p^*} dx\right)^{\frac{p}{p^*}} \\ &\quad + \frac{\|u\|_{L^{p^*}(\mathcal{B}_R^{H^\circ})}^{p^*-p}}{\alpha_1^p C_1 C_S^p} \left(\int_{\mathbb{R}^N} (\eta u)^{p^*} dx\right)^{\frac{p}{p^*}}, \end{aligned} \tag{3.15}$$

where $C(p, N)$ is a positive constant depending on p and N .

Setting

$$\tau = \left(\frac{\alpha_1^p C_1 C_S^p}{2}\right)^{\frac{1}{p^*-p}},$$

and choosing $\bar{R} > 0$ sufficiently small such that (3.1) holds, then $\|u\|_{L^{p^*}(\mathcal{B}_R^{H^\circ})}^{p^*-p} / (\alpha_1^p C_1 C_S^p) \leq 1/2$ for all $0 < R \leq \bar{R}$. Hence we obtain that

$$\int_{\mathcal{B}_{R/2}^{H^\circ}} u^{p^*} dx \leq \int_{\mathbb{R}^N} |\eta u|^{p^*} dx \leq \bar{C} \int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} u^{p^*} dx \quad \forall 0 < R \leq \bar{R},$$

where $\bar{C} := ((2C_2 C(p, N)) / (\alpha_1^p C_1 C_S^p))^{\frac{p}{p^*}}$ and it depends only on N, p and γ . Denoting with $\mathcal{L}(R) := \int_{\mathcal{B}_R^{H^\circ}} u^{p^*} dx$ for $0 < R \leq \bar{R}$, we get that

$$\mathcal{L}(R/2) \leq \vartheta \mathcal{L}(R) \quad \forall 0 < R \leq \bar{R},$$

where $\vartheta = \bar{C} / (\bar{C} + 1) \in (0, 1)$, depends only on N, p and γ . Now, by Lemma 2.4 it follows that

$$\mathcal{L}(R) \leq \frac{1}{\vartheta} \mathcal{L}(\bar{R}) \left(\frac{R}{\bar{R}}\right)^{\sigma'_1} \quad \forall 0 < R \leq \bar{R},$$

where $\sigma'_1 = \frac{1}{2} \log(1/\vartheta) / \log 2$ depends only on ϑ , Now (3.2) follows by setting $\sigma_1 = \sigma'_1 / p^*$ and $C = (\vartheta^{-1} \bar{R}^{-\sigma'_1} \mathcal{L}(\bar{R}))^{1/p^*}$. In a similar way, we can deduce (3.3). □

Now, we denote by $\mathcal{A}_R = \mathcal{B}_{8R}^{H^\circ} \setminus \mathcal{B}_{R/8}^{H^\circ}$ and $\mathcal{D}_R = \mathcal{B}_{4R}^{H^\circ} \setminus \mathcal{B}_{R/4}^{H^\circ}$ for $R > 0$.

Lemma 3.2 *Let $t \in (p^*, N/\mu_1)$. There exists a positive constant $\sigma = \sigma(N, p, \gamma, t)$ such that for any solution u to problem (\mathcal{P}_H) and for any $\bar{R} > 0$ satisfying the following inequality*

$$\|u\|_{L^{p^*}(\mathcal{B}_R)} + \|u\|_{L^{p^*}(\mathbb{R}^N \setminus B_{1/\bar{R}})} \leq \sigma, \tag{3.16}$$

then

$$\left(\int_{\mathcal{D}_R} u^t dx\right)^{\frac{1}{t}} \leq C \left(\int_{\mathcal{A}_R} u^{p^*} dx\right)^{\frac{1}{p^*}} \quad \forall R < \bar{R}/8 \text{ or } R > 8/\bar{R}, \tag{3.17}$$

where $\int_{\mathcal{D}_R} u^t dx = \frac{1}{|\mathcal{D}_R|} \int_{\mathcal{D}_R} u^t dx$ and C is a positive constant depending only on N, p, γ, \bar{R} and t .

Proof It is easy to see that, setting $\hat{u}(x) = u(Rx)$, for $R > 0$,

$$-\Delta_p^H \hat{u} - \gamma \frac{\hat{u}^{p-1}}{H^\circ(x)^p} = R^p \hat{u}^{p^*-1} \quad \text{in } \mathcal{A}_1.$$

Let $m \geq 1$ and set

$$\mathcal{A} := \mathcal{A}_1 \cap \{\hat{u} < m\} \quad \text{and} \quad \mathcal{B} := \mathcal{A}_1 \cap \{\hat{u} \geq m\}. \tag{3.18}$$

Hence we can consider the weak formulation of the last equation as follows

$$\begin{aligned} I_1 + I_2 &:= \int_{\mathcal{A}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), \nabla \varphi \rangle dx + \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), \nabla \varphi \rangle dx \\ &= \int_{\mathcal{A}_1} \gamma \frac{\hat{u}^{p-1}}{H^\circ(x)^p} \varphi dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \varphi dx \quad \forall \varphi \in C_c^\infty(\mathcal{A}_1). \end{aligned} \tag{3.19}$$

Let define $\hat{u}_m := \min(\hat{u}, m)$ for $m \geq 1$. By density argument, for any $\eta \in C_c^\infty(\mathcal{A}_1)$ it is possible to choose $\varphi = \eta^p \hat{u}_m^{p(s-1)} \hat{u}$, with $s \geq 1$, as test function in (3.19), so that, using (3.18) and (2.5), we can compute I_1 and I_2

$$\begin{aligned} I_1 &= \int_{\mathcal{A}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), p\eta^{p-1} \hat{u}^{p(s-1)+1} \nabla \eta \rangle dx \\ &\quad + [p(s-1) + 1] \int_{\mathcal{A}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), \eta^p \hat{u}^{p(s-1)} \nabla \hat{u} \rangle dx \\ &= p \int_{\mathcal{A}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), \nabla \eta \rangle \eta^{p-1} \hat{u}^{p(s-1)+1} dx \\ &\quad + [p(s-1) + 1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} dx. \end{aligned} \tag{3.20}$$

In the same way, we obtain

$$\begin{aligned} I_2 &= \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), p\eta^{p-1} m^{p(s-1)} \hat{u} \nabla \eta \rangle dx \\ &\quad + \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), \eta^p m^{p(s-1)} \nabla \hat{u} \rangle dx \\ &= p \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), \nabla \eta \rangle \eta^{p-1} m^{p(s-1)} \hat{u} dx + \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} dx. \end{aligned} \tag{3.21}$$

Collecting both (3.20), (3.21), using the Scwarz’s inequality and recalling (2.6) we obtain

$$\begin{aligned}
 & [p(s - 1) + 1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} dx + \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} dx \\
 & \leq pM \left[\int_{\mathcal{A}} H^{p-1}(\nabla \hat{u}) \eta^{p-1} \hat{u}^{p(s-1)+1} |\nabla \eta| dx + \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \eta^{p-1} m^{p(s-1)} |\nabla \eta| \hat{u} dx \right] \\
 & + \int_{\mathcal{A}_1} \gamma \frac{\hat{u}^{p-1}}{H^\circ(x)^p} \varphi dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \varphi dx \\
 & = pM \left[\int_{\mathcal{A}} H^{p-1}(\nabla \hat{u}) \eta^{p-1} \hat{u}^{(p-1)(s-1)} \hat{u}^s |\nabla \eta| dx \right. \\
 & \left. + \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \eta^{p-1} m^{(p-1)(s-1)} m^{s-1} |\nabla \eta| \hat{u} dx \right] \\
 & + \int_{\mathcal{A}_1} \gamma \frac{\hat{u}^{p-1}}{H^\circ(x)^p} \varphi dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \varphi dx.
 \end{aligned} \tag{3.22}$$

Now we can apply the weighted Young’s inequality to the first two terms in the right hand side of (3.22) with conjugate exponent $(p/(p - 1), p)$ in order to obtain

$$\begin{aligned}
 & [p(s - 1) + 1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} dx + \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} dx \\
 & \leq \varepsilon_1 [p(s - 1) + 1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} dx + C_{\varepsilon_1}(p, s, M) \int_{\mathcal{A}} \hat{u}^{ps} |\nabla \eta|^p dx \\
 & + \varepsilon_2 \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} dx + C_{\varepsilon_2}(p, M) \int_{\mathcal{B}} |\nabla \eta|^p m^{p(s-1)} \hat{u}^p dx \\
 & + \int_{\mathcal{A}_1} \gamma \frac{\hat{u}^{p-1}}{H^\circ(x)^p} \varphi dx \\
 & + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \varphi dx,
 \end{aligned} \tag{3.23}$$

where $C_{\varepsilon_1}(p, s, M)$ and $C_{\varepsilon_2}(p, M)$ are two positive constants. Hence we obtain

$$\begin{aligned}
 & (1 - \varepsilon_1)[p(s - 1) + 1] \int_{\mathcal{A}} H^p(\eta \hat{u}_m^{s-1} \nabla \hat{u}) dx + (1 - \varepsilon_2) \int_{\mathcal{B}} H^p(\eta \hat{u}_m^{s-1} \nabla \hat{u}) dx \\
 & = (1 - \varepsilon_1)[p(s - 1) + 1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} dx + (1 - \varepsilon_2) \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} dx \\
 & \leq C_{\varepsilon_1}(p, s, M) \int_{\mathcal{A}} |\nabla \eta|^p \hat{u}^{p(s-1)} \hat{u}^p dx + C_{\varepsilon_2}(p, M) \int_{\mathcal{B}} |\nabla \eta|^p m^{p(s-1)} \hat{u}^p dx \\
 & + \int_{\mathcal{A}_1} \frac{\gamma}{H^\circ(x)^p} \hat{u}^{p-1} \eta^p \hat{u}_m^{p(s-1)} \hat{u} dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \eta^p \hat{u}_m^{p(s-1)} \hat{u} dx.
 \end{aligned} \tag{3.24}$$

Thanks to (3.8) and applying Lemma 2.3, with $a = H(\eta \hat{u}_m^{s-1} \nabla \hat{u})$ and $b = H(\hat{u}^s \nabla \eta)$, we deduce that there hold the following inequalities in the sets \mathcal{A} and \mathcal{B} respectively:

$$\begin{aligned}
 & \int_{\mathcal{A}} H^p(\eta \hat{u}_m^{s-1} \nabla \hat{u}) dx = \int_{\mathcal{A}} H^p(\eta \hat{u}^{s-1} \nabla \hat{u}) dx \\
 & \geq \frac{1}{1 + 2^{p+1} \delta_1} \cdot \frac{1}{s^p} \int_{\mathcal{A}} H^p(\nabla(\eta \hat{u}^s)) dx - C_{\delta_1} \int_{\mathcal{A}} H^p(\hat{u}^s \nabla \eta) dx \\
 & = \frac{1}{1 + 2^{p+1} \delta_1} \cdot \frac{1}{s^p} \int_{\mathcal{A}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) dx - C_{\delta_1} \int_{\mathcal{A}} H^p(\hat{u}_m^{s-1} \hat{u} \nabla \eta) dx
 \end{aligned} \tag{3.25}$$

$$\begin{aligned}
 \int_{\mathcal{B}} H^p(\eta \hat{u}_m^{s-1} \nabla \hat{u}) \, dx &= \int_{\mathcal{B}} H^p(\eta m^{s-1} \nabla \hat{u}) \, dx \\
 &\geq \frac{1}{1 + 2^{p+1} \delta_2} \int_{\mathcal{B}} H^p(\nabla(\eta m^{s-1} \hat{u})) \, dx \\
 &\quad - C_{\delta_2} \int_{\mathcal{B}} H^p(m^{s-1} \hat{u} \nabla \eta) \, dx \tag{3.26} \\
 &= \frac{1}{1 + 2^{p+1} \delta_2} \int_{\mathcal{B}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx \\
 &\quad - C_{\delta_2} \int_{\mathcal{B}} H^p(\hat{u}_m^{s-1} \hat{u} \nabla \eta) \, dx
 \end{aligned}$$

By (3.25) and (3.26), we obtain

$$\begin{aligned}
 &\frac{1 - \varepsilon_1}{1 + 2^{p+1} \delta_1} \cdot \frac{p(s - 1) + 1}{s^p} \int_{\mathcal{A}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx + \frac{1 - \varepsilon_2}{1 + 2^{p+1} \delta_2} \int_{\mathcal{B}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx \\
 &\leq C_{\delta_1}(\varepsilon_1, p, s) \int_{\mathcal{A}} H^p(\hat{u}_m^{s-1} \hat{u} \nabla \eta) \, dx + C_{\delta_2}(\varepsilon_2) \int_{\mathcal{B}} H^p(\hat{u}_m^{s-1} \hat{u} \nabla \eta) \, dx \\
 &\quad + C_{\varepsilon_1}(p, s, M) \int_{\mathcal{A}} |\nabla \eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p \, dx + C_{\varepsilon_2}(p, M) \int_{\mathcal{B}} |\nabla \eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p \, dx \\
 &\quad + \int_{\mathcal{A}_1} \frac{\gamma}{H^\circ(x)^p} \hat{u}^{p-1} \eta^p \hat{u}_m^{p(s-1)} \hat{u} \, dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-p} \eta^p \hat{u}_m^{p(s-1)} \hat{u}^p \, dx. \tag{3.27}
 \end{aligned}$$

Using (2.4), we deduce that

$$\begin{aligned}
 &\frac{1 - \varepsilon_1}{1 + 2^{p+1} \delta_1} \cdot \frac{p(s - 1) + 1}{s^p} \int_{\mathcal{A}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx \\
 &\quad + \frac{1 - \varepsilon_2}{1 + 2^{p+1} \delta_2} \int_{\mathcal{B}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx \\
 &\leq \int_{\mathcal{A}_1} \frac{\gamma}{H^\circ(x)^p} \hat{u}^{p-1} \eta^p \hat{u}_m^{p(s-1)} \hat{u} \, dx + \hat{C} \int_{\mathcal{A}_1} |\nabla \eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p \, dx \\
 &\quad + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-p} (\eta \hat{u}_m^{s-1} \hat{u})^p \, dx, \tag{3.28}
 \end{aligned}$$

where \hat{C} depends on $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2, p, s, M$. Using Hardy’s and Hölder’s inequality in the right hand side of (3.28), and the definition of the sets \mathcal{A} and \mathcal{B} , we obtain

$$\begin{aligned}
 &\frac{1 - \varepsilon_1}{1 + 2^{p+1} \delta_1} \cdot \frac{p(s - 1) + 1}{s^p} \int_{\mathcal{A}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx + \frac{1 - \varepsilon_2}{1 + 2^{p+1} \delta_2} \int_{\mathcal{B}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx \\
 &\leq \frac{\gamma}{C_H} \int_{\mathcal{A}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx + \frac{\gamma}{C_H} \int_{\mathcal{B}} H^p(\nabla(\eta \hat{u}_m^{s-1} \hat{u})) \, dx \\
 &\quad + \hat{C} \int_{\mathcal{A}_1} |\nabla \eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p \, dx + \left(\int_{\mathcal{A}_1} \hat{u}^{p^*} R^N \, dx \right)^{\frac{p^*-p}{p^*}} \left(\int_{\mathcal{A}_1} (\eta \hat{u}_m^{s-1} \hat{u})^{p\chi} \, dx \right)^{\frac{1}{\chi}}, \tag{3.29}
 \end{aligned}$$

where $\chi = p^*/p$. Finally, we deduce

$$\begin{aligned} & \left(\frac{1 - \varepsilon_1}{1 + 2^{p+1}\delta_1} \cdot \frac{p(s-1) + 1}{s^p} - \frac{\gamma}{C_H} \right) \int_{\mathcal{A}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u})) dx \\ & + \left(\frac{1 - \varepsilon_2}{1 + 2^{p+1}\delta_2} - \frac{\gamma}{C_H} \right) \int_{\mathcal{B}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u})) dx \\ & \leq \hat{C} \int_{\mathcal{A}_1} |\nabla\eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p dx + \|u\|_{L^{p^*}(\mathcal{A}_R)}^{p^*-p} \left(\int_{\mathcal{A}_1} (\eta\hat{u}_m^{s-1}\hat{u})^{p\chi} dx \right)^{\frac{1}{\chi}}. \end{aligned} \tag{3.30}$$

Noticing that $(p(s-1) + 1)/s^p > \gamma/C_H$ for all $s \in ((N-p)/(p\mu_2), (N-p)/(p\mu_1))$, we can fix $\delta_1, \varepsilon_1 > 0$ sufficiently small such that

$$\frac{1 - \varepsilon_1}{1 + 2^{p+1}\delta_1} \cdot \frac{p(s-1) + 1}{s^p} - \frac{\gamma}{C_H} > 0,$$

and $\delta_2, \varepsilon_2 > 0$ sufficiently small such that

$$\frac{1 - \varepsilon_2}{1 + 2^{p+1}\delta_2} - \frac{\gamma}{C_H} > 0.$$

Hence by (3.30) we get

$$\begin{aligned} & \bar{C} \int_{\mathcal{A}_1} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u})) dx \leq \hat{C} \int_{\mathcal{A}_1} |\nabla\eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p dx + \|u\|_{L^{p^*}(\mathcal{A}_R)}^{p^*-p} \\ & \left(\int_{\mathcal{A}_1} (\eta\hat{u}_m^{s-1}\hat{u})^{p\chi} dx \right)^{\frac{1}{\chi}}, \end{aligned} \tag{3.31}$$

where \bar{C} is a positive constant depending on $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2, p, s, \gamma, C_H$. In conclusion, by (2.4) and Sobolev inequality we have

$$\begin{aligned} \alpha_1^p \bar{C} C_S^p \left(\int_{\mathcal{A}_1} (\eta\hat{u}_m^{s-1}\hat{u})^{p\chi} dx \right)^{\frac{1}{\chi}} & \leq \alpha_1^p \bar{C} \int_{\mathcal{A}_1} |\nabla(\eta\hat{u}_m^{s-1}\hat{u})|^p dx \leq \bar{C} \int_{\mathcal{A}_1} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u})) dx \\ & \leq \hat{C} \int_{\mathcal{A}_1} |\nabla\eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p dx \\ & + \|u\|_{L^{p^*}(\mathcal{A}_R)}^{p^*-p} \left(\int_{\mathcal{A}_1} (\eta\hat{u}_m^{s-1}\hat{u})^{p\chi} dx \right)^{\frac{1}{\chi}}. \end{aligned} \tag{3.32}$$

In order to apply the Moser’s iteration method we need to rewrite the last inequality as follows

$$\begin{aligned} & \left(\int_{\mathcal{A}_1} (\eta\hat{u}_m^{s-1}\hat{u})^{p\chi} dx \right)^{\frac{1}{\chi}} \\ & \leq C_1 \int_{\mathcal{A}_1} |\nabla\eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p dx + C_2 \|u\|_{L^{p^*}(\mathcal{A}_R)}^{p^*-p} \left(\int_{\mathcal{A}_1} (\eta\hat{u}_m^{s-1}\hat{u})^{p\chi} dx \right)^{\frac{1}{\chi}}, \end{aligned} \tag{3.33}$$

where $C_1 := \hat{C}/(\alpha_1^p \bar{C} C_S^p), C_2 := 1/(\alpha_1^p \bar{C} C_S^p)$ and $\chi = p^*/p$.

Fix $t \in (p^*, N/\mu_1)$ and $k \in \mathbb{N}$ so that $p\chi^k \leq t \leq p\chi^{k+1}$. Then there exist positive constants C_1 and C_2 such that (3.33) holds for all $1 \leq s \leq \min\{(N-p)/(p\mu_1), \chi^k\}$. Now, we set $\sigma = (1/(2C_2))^{\frac{1}{p^*-p}}$ and choosing \bar{R} sufficiently small such that (3.16) holds, we get

$$\left(\int_{\mathcal{A}_1} (\eta\hat{u}_m^{s-1}\hat{u})^{p\chi} dx \right)^{\frac{1}{\chi}} \leq C \int_{\mathcal{A}_1} |\nabla\eta|^p \hat{u}_m^{p(s-1)} \hat{u}^p dx \tag{3.34}$$

for all $1 \leq s \leq \min \{(N - p)/(p\mu_1), \chi^k\}$ and $C = C_1/2$. Applying Moser’s iteration method (see e.g. [16, 31] for further details), we conclude, after finitely many times of iteration,

$$\left(\int_{\mathcal{D}_1} |\hat{u}|^t dx \right)^{\frac{1}{t}} \leq C \left(\int_{\mathcal{A}_1} |\hat{u}|^{p^*} dx \right)^{\frac{1}{p^*}}. \tag{3.35}$$

for any $t \in (p^*, N/\mu_1)$. This proves (3.17). □

Let us now prove the following:

Theorem 3.3 *Let u be a weak solution of (\mathcal{P}_H) . Then there exists a positive constant $C = C(N, p, \gamma, u)$ such that*

$$|u(x)| \leq \frac{C}{[H^\circ(x)]^{\frac{N-p}{p}-\sigma_1}} \quad \text{in } \mathcal{B}_{R_1}^{H^\circ},$$

and that

$$|u(x)| \leq \frac{C}{[H^\circ(x)]^{\frac{N-p}{p}+\sigma_2}} \quad \text{in } (\mathcal{B}_{R_2}^{H^\circ})^c,$$

where σ_1, σ_2 are given in Lemma 3.1 and $R_1, R_2 > 0$ are constants depending on N, p, γ and u .

Proof Let us fix $t := (p^* + N/\mu_1)/2 \in (p^*, N/\mu_1)$ as in Lemma 3.2 and $\kappa := \min\{\tau, \sigma\}$, where τ and σ are respectively as in Lemma 3.1 and Lemma 3.2. Let $\bar{R} > 0$ such that (3.1) holds for κ and let us consider $\hat{u}(x) = u(Rx)$, for $R > 0$ fixed. We note that \hat{u} satisfies the equation

$$-\Delta_p^H \hat{u} + c(x)\hat{u}^{p-1} = 0 \quad \text{in } \mathcal{D}_1,$$

where

$$c(x) = -\frac{\gamma}{H^\circ(x)^p} - R^p \hat{u}^{p^*-p}(x).$$

We note that $H^\circ(x)^{-p}$ is bounded in \mathcal{D}_1 and $V_R(x) := R^p \hat{u}^{p^*-p}(x) \in L^q(\mathcal{D}_1)$ with $q = t/(p^* - p) > N/p$ due to Lemma 3.2. Hence, as in the proof of [22, Theorem 1] a classical Moser iteration argument yields

$$\sup_{\mathcal{B}_r^{H^\circ}(x)} \hat{u} \leq \mathbf{C} \left(\int_{\mathcal{B}_{2r}^{H^\circ}(x)} \hat{u}^p dx \right)^{\frac{1}{p}} \tag{3.36}$$

for any ball $\mathcal{B}_{2r}^{H^\circ}(x) \subset \mathcal{D}_1$, where $\mathbf{C} = \mathbf{C}(N, p, \gamma, \|V_R\|_{L^q(\mathcal{D}_1)})$. We claim that $\|V_R\|_{L^q(\mathcal{D}_1)}$ is uniformly bounded with respect to R . Indeed from Lemma 3.2, since

$$p - \frac{N}{q} + N \frac{p^* - p}{t} - N \frac{p^* - p}{p^*} = 0,$$

we get

$$\begin{aligned}
 \left(\int_{\mathcal{D}_1} V_R^q dx \right)^{\frac{1}{q}} &= R^{p-\frac{N}{q}} \left(\int_{\mathcal{D}_R} u^{(p^*-p)q} dx \right)^{\frac{1}{q}} \\
 &\leq CR^{p-\frac{N}{q}+N\frac{p^*-p}{t}} \left(\int_{\mathcal{D}_R} u^t dx \right)^{\frac{p^*-p}{t}} \\
 &\leq CR^{p-\frac{N}{q}+N\frac{p^*-p}{t}-N\frac{p^*-p}{p^*}} \left(\int_{\mathcal{A}_R} u^{p^*} dx \right)^{\frac{p^*-p}{p^*}} \\
 &\leq C \left(\int_{\mathbb{R}^N} u^{p^*} dx \right)^{\frac{p^*-p}{p^*}} \quad \forall 0 < R \leq \frac{\bar{R}}{8} \text{ or } R \geq \frac{8}{\bar{R}},
 \end{aligned}
 \tag{3.37}$$

where C is a positive constant depending on N, p, γ, q and \bar{R} .

Using a covering argument we deduce that

$$\sup_{\mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_1^{H^\circ}} \hat{u} \leq C \left(\int_{\mathcal{D}_1} \hat{u}^p dx \right)^{\frac{1}{p}}
 \tag{3.38}$$

Noticing that $\hat{u}(x) = u(Rx)$, by (3.38) we obtain that

$$\sup_{\mathcal{B}_{2R}^{H^\circ} \setminus \mathcal{B}_R^{H^\circ}} u \leq C \left(\int_{\mathcal{D}_R} u^p dx \right)^{\frac{1}{p}}
 \tag{3.39}$$

for each $0 < R \leq \bar{R}/8$ or $R \geq 8/\bar{R}$. By applying the Hölder’s inequality in (3.39), we get

$$\sup_{\mathcal{B}_{2R}^{H^\circ} \setminus \mathcal{B}_R^{H^\circ}} u \leq C \left(\int_{\mathcal{D}_R} u^p dx \right)^{\frac{1}{p}} \leq C \left(\int_{\mathcal{D}_R} u^{p^*} dx \right)^{\frac{1}{p^*}} = C \|u\|_{L^{p^*}(\mathcal{D}_R)} R^{\frac{p-N}{p}},
 \tag{3.40}$$

for each $0 < R \leq \bar{R}/8$ or $R \geq 8/\bar{R}$ and C depends only on N, p, γ, q, \bar{R} and $\|u\|_{L^{p^*}(\mathbb{R}^N)}$.

Now we note that, since $\mathcal{A}_R \subset \mathcal{B}_R^{H^\circ}$ for any $0 < R \leq \bar{R}/8$ and $\mathcal{A}_R \subset (\mathcal{B}_{1/\bar{R}}^{H^\circ})^c$ for any $R \geq 8/\bar{R}$, there exist, by Lemma 3.1, $\sigma_1, \sigma_2 > 0$ depending only on N, p, γ such that

$$\|u\|_{L^{p^*}(\mathcal{B}_R^{H^\circ})} \leq CR^{\sigma_1} \quad \text{for } 0 < R \leq \frac{\bar{R}}{8}$$

and that

$$\|u\|_{L^{p^*}((\mathcal{B}_R^{H^\circ})^c)} \leq \frac{C}{R^{\sigma_2}} \quad \text{for } R \geq 8/\bar{R}.$$

Now, if we set $R_1 = \bar{R}/8$ and $R_2 = 8/\bar{R}$, by (3.40) we get the thesis. \square

The next result is devoted to show the existence of some special supersolutions of our problem, in order to perform a comparison between them and the solutions of the doubly critical equation (\mathcal{P}_H) .

Proposition 3.4 *Given two constants $A > 0$ and $\alpha < p$, there exist constants $0 < \varepsilon, \delta < 1$, depending on N, p, γ, A, α , such that*

$$v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^\varepsilon}{[H^\circ(x)]^{\mu_1}} \in \mathcal{D}^{1,p}(\mathcal{B}_{R_1}^{H^\circ})
 \tag{3.41}$$

is a positive supersolution to equation

$$-\Delta_p^H v - \frac{\gamma}{H^\circ(x)^p} v^{p-1} = g(x)v^{p-1} \text{ in } \mathcal{B}_{R_1}^{H^\circ}, \tag{3.42}$$

for some positive constant $0 < R_1 < 1$ depending only on N, p, γ, A and α , where $g(x)$ is a positive function that belongs to $L^{\frac{N}{p}}(\mathcal{B}_{R_1}^{H^\circ})$ such that

$$g(x) \geq A[H^\circ(x)]^{-\alpha} \text{ in } \mathcal{B}_{R_1}^{H^\circ}. \tag{3.43}$$

In a similar way, given $A > 0$ and $\alpha > p$, there exist $0 < \varepsilon, \delta < 1$ such that

$$v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^{-\varepsilon}}{[H^\circ(x)]^{\mu_2}} \in \mathcal{D}^{1,p} \left((\mathcal{B}_{R_2}^{H^\circ})^c \right) \tag{3.44}$$

is a positive supersolution to equation

$$-\Delta_p^H v - \frac{\gamma}{H^\circ(x)^p} v^{p-1} = g(x)v^{p-1} \text{ in } (\mathcal{B}_{R_2}^{H^\circ})^c, \tag{3.45}$$

for some positive constant $R_2 > 1$ depending only on N, p, γ and α , where $g(x)$ is a positive function that belongs to $L^{\frac{N}{p}}((\mathcal{B}_{R_2}^{H^\circ})^c)$ such that

$$g(x) \geq A[H^\circ(x)]^{-\alpha} \text{ in } (\mathcal{B}_{R_2}^{H^\circ})^c. \tag{3.46}$$

Proof Let us consider $\mu, \delta, \varepsilon > 0$ and let us define the function

$$u(x) = v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^\varepsilon}{[H^\circ(x)]^\mu}.$$

It is easy to deduce that

$$\nabla u(x) = s(H^\circ(x))\nabla H^\circ(x), \tag{3.47}$$

where $s(t) := t^{-\mu-1}[-\mu + \delta(\mu - \varepsilon)t^\varepsilon]$. Using (2.2), we now compute

$$\begin{aligned} -\Delta_p^H u &= -\operatorname{div} (H^{p-1}(\nabla u)\nabla H(\nabla u)) = -\operatorname{div} (|s(H^\circ(x))|^{p-1} \operatorname{sign}(s)\nabla H(\nabla H^\circ(x))) \\ &= -(p-1)|s(H^\circ(x))|^{p-2}s'(H^\circ(x))\langle \nabla H^\circ(x), \nabla H(\nabla H^\circ(x)) \rangle \\ &\quad - |s(H^\circ(x))|^{p-1} \operatorname{sign}(s) \operatorname{div} (\nabla H(\nabla H^\circ(x))) \\ &= -(p-1)|s(H^\circ(x))|^{p-2}s'(H^\circ(x)) - (N-1)\frac{|s(H^\circ(x))|^{p-2}s(H^\circ(x))}{H^\circ(x)}, \end{aligned} \tag{3.48}$$

where in the last line we used the fact that $\langle \nabla H^\circ(x), \nabla H(\nabla H^\circ(x)) \rangle = 1$ and

$$\operatorname{div} (\nabla H(\nabla H^\circ(x))) = \frac{N-1}{H^\circ(x)}$$

due to (2.2) and (2.3).

Making standard computations on the right hand side of (3.48), one can deduce

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} |u|^{p-2}u = g(H^\circ(x))|u|^{p-2}u, \tag{3.49}$$

where

$$g(t) = \frac{h(t)}{|1 - \delta t^\varepsilon|^{p-2}(1 - \delta t^\varepsilon)t^p},$$

and

$$\begin{aligned}
 h(t) := & -|\mu - \delta(\mu - \varepsilon)t^\varepsilon|^{p-2} \{ \mu^2(p-1) - (N-p)\mu - (p-1)(\mu - \varepsilon)^2\delta t^\varepsilon \\
 & + (N-p)(\mu - \varepsilon)\delta t^\varepsilon \} \\
 & - \gamma|1 - \delta t^\varepsilon|^{p-2}(1 - \delta t^\varepsilon), \quad \text{for } t \in [0, +\infty).
 \end{aligned}
 \tag{3.50}$$

We note that $h(0) = -|\mu|^{p-2}[\mu^2(p-1) - \mu(N-p)] - \gamma$. Hence, using the definition of μ_1 and μ_2 , we deduce that $h(0) = 0$ when $\mu = \mu_1$ and $\mu = \mu_2$. Also we have $h'(0) > 0$ if $\mu = \mu_1, \varepsilon > 0$ or $\mu = \mu_2, \varepsilon < 0$. This implies there exist $0 < \delta_h < 1$ such that

$$2h'(0)t \geq h(t) \geq \frac{1}{2}h'(0)t > 0 \quad \forall t \in (0, \delta_h).
 \tag{3.51}$$

We set $\delta = \min\{\delta_h, 1/2\}$, $\varepsilon = (p - \alpha)/2$ and

$$R_0 = \left\{ 1, \left(\frac{\delta}{2A} h'(0) \right)^{1/(p-\alpha-\varepsilon)} \right\}.$$

It is easy to check that $v(H^\circ(x)) = (1 - \delta[H^\circ(x)]^\varepsilon)[H^\circ(x)]^{-\mu_1} \in \mathcal{D}^{1,p}(\mathcal{B}_{R_1}^{H^\circ})$ is positive, which thanks to (3.51), $g(x) \in L^{\frac{N}{p}}(\mathcal{B}_{R_1}^{H^\circ})$ and it satisfies (3.43). The other case is similar. \square

Now, we consider the following equation

$$-\Delta_p^H w - \frac{\gamma}{H^\circ(x)^p} w^{p-1} = f(x)w^{p-1} \quad \text{in } \Omega,
 \tag{3.52}$$

where Ω is an open subset of \mathbb{R}^N , $w > 0$ and $w \in \mathcal{D}^{1,p}(\Omega)$. Let us start with a comparison principle in bounded domains.

The first result is given by the following pointwise estimate, in the same spirit of [20, 31].

Proposition 3.5 *Let u, v two weakly differentiable strictly positive functions on a domain Ω .¹ We have that:*

(i) *if $p \geq 2$, then*

$$\begin{aligned}
 & H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \left(u - \frac{v^p}{u^p} u \right) \rangle + H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \left(v - \frac{u^p}{v^p} v \right) \rangle \\
 & \geq C_p(u^p + v^p)H^p(\nabla(\ln u - \ln v)),
 \end{aligned}
 \tag{3.53}$$

for some positive constant C_p depending only on p ;

(ii) *if $1 < p < 2$, then*

$$\begin{aligned}
 & H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \left(u - \frac{v^p}{u^p} u \right) \rangle + H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \left(v - \frac{u^p}{v^p} v \right) \rangle \\
 & \geq C_p(u^p + v^p) [H(\nabla \ln u) + H(\nabla \ln v)]^{p-2} H^2(\nabla(\ln u - \ln v)),
 \end{aligned}
 \tag{3.54}$$

for some positive constant C_p depending only on p .

¹ We mean that $u, v \geq C > 0$ a.e. in Ω .

Proof Let u, v two weakly differentiable positive functions and consider the following

$$\psi_1 := u - \frac{v^p}{u^p}u \quad \text{and} \quad \psi_2 := v - \frac{u^p}{v^p}v.$$

Then, thanks to the Euler’s theorem for 1-homogeneous functions, and since ∇H is 0-homogeneous, we deduce that

$$\begin{aligned} &H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1 \rangle \\ &= H^p(\nabla u) - p \frac{v^{p-1}}{u^{p-1}}H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla v \rangle + (p-1) \frac{v^p}{u^p}H^p(\nabla u) \\ &= H^p(\nabla u) - v^p \left[(1-p)H^p\left(\frac{\nabla u}{u}\right) + pH^{p-1}\left(\frac{\nabla u}{u}\right)\langle \nabla H(\nabla u), \frac{\nabla v}{v} \rangle \right] \tag{3.55} \\ &= H^p(\nabla u) - v^p \left[H^p(\nabla \ln u) + pH^{p-1}(\nabla \ln u)\langle \nabla H(\nabla \ln u), \nabla \ln v - \nabla \ln u \rangle \right] \end{aligned}$$

and

$$\begin{aligned} &H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2 \rangle \\ &= H^p(\nabla v) - p \frac{u^{p-1}}{v^{p-1}}H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla u \rangle + (p-1) \frac{u^p}{v^p}H^p(\nabla v) \\ &= H^p(\nabla v) - u^p \left[(1-p)H^p\left(\frac{\nabla v}{v}\right) + pH^{p-1}\left(\frac{\nabla v}{v}\right)\langle \nabla H(\nabla v), \frac{\nabla u}{u} \rangle \right] \tag{3.56} \\ &= H^p(\nabla v) - u^p \left[H^p(\nabla \ln v) + pH^{p-1}(\nabla \ln v)\langle \nabla H(\nabla \ln v), \nabla \ln u - \nabla \ln v \rangle \right]. \end{aligned}$$

(i) $p \geq 2$. We recall that when $p \geq 2$, it holds (2.10), i.e.

$$\begin{aligned} &H^p(\eta) \geq H^p(\eta') + pH^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta' \rangle \\ &+ \frac{1}{2^{p-1}-1}H^p(\eta - \eta'), \quad \forall \eta, \eta' \in \mathbb{R}^N. \end{aligned} \tag{3.57}$$

Hence we can apply this inequality, in order to give an estimate from below for (3.55) and (3.56):

$$\begin{aligned} &H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1 \rangle = H^p(\nabla u) - v^p \left[H^p(\nabla \ln u) + pH^{p-1}(\nabla \ln u) \right. \\ &\quad \left. \langle \nabla H(\nabla \ln u), \nabla \ln v - \nabla \ln u \rangle \right] \\ &\geq H^p(\nabla u) - v^p \\ &\quad \left[H^p(\nabla \ln v) - C_p H^p(\nabla(\ln v - \ln u)) \right] \\ &= H^p(\nabla u) - H^p(\nabla v) + C_p v^p H^p(\nabla(\ln v - \ln u)), \end{aligned} \tag{3.58}$$

$$\begin{aligned} &H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2 \rangle = H^p(\nabla v) - u^p \left[H^p(\nabla \ln v) + pH^{p-1}(\nabla \ln v) \right. \\ &\quad \left. \langle \nabla H(\nabla \ln v), \nabla \ln u - \nabla \ln v \rangle \right] \\ &\geq H^p(\nabla v) - u^p \left[H^p(\nabla \ln u) - C_p H^p(\nabla(\ln u - \ln v)) \right] \\ &= H^p(\nabla v) - H^p(\nabla u) + C_p u^p H^p(\nabla(\ln u - \ln v)). \end{aligned} \tag{3.59}$$

Adding both these two inequalities, we obtain

$$\begin{aligned}
 &H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1 \rangle + H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2 \rangle \\
 &\geq C_p(u^p + v^p)H^p(\nabla(\ln u - \ln v)).
 \end{aligned}
 \tag{3.60}$$

(ii) $1 < p < 2$. We recall that when $1 < p < 2$, it holds (2.11), i.e.

$$\begin{aligned}
 &H^p(\eta) \geq H^p(\eta') + p H^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta' \rangle \\
 &+ C_p[H(\eta) + H(\eta')]^{p-2}H^2(\eta - \eta'), \quad \forall \eta, \eta' \in \mathbb{R}^N,
 \end{aligned}
 \tag{3.61}$$

where C_p is a positive constant depending only on p .

Now we proceed exactly as in the previous case to get an estimate from below for (3.55) and (3.56):

$$\begin{aligned}
 &H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1 \rangle \\
 &= H^p(\nabla u) - v^p \left[H^p(\nabla \ln u) + p H^{p-1}(\nabla \ln u)\langle \nabla H(\nabla \ln u), \nabla \ln v - \nabla \ln u \rangle \right] \\
 &\geq H^p(\nabla u) - v^p \left[H^p(\nabla \ln v) - C_p[H(\nabla \ln u) + H(\nabla \ln v)]^{p-2}H^2(\nabla(\ln v - \ln u)) \right] \\
 &= H^p(\nabla u) - H^p(\nabla v) + C_p v^p \left[[H(\nabla \ln u) + H(\nabla \ln v)]^{p-2}H^2(\nabla(\ln v - \ln u)) \right],
 \end{aligned}
 \tag{3.62}$$

$$\begin{aligned}
 &H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2 \rangle \\
 &= H^p(\nabla v) - u^p \left[H^p(\nabla \ln v) + p H^{p-1}(\nabla \ln v)\langle \nabla H(\nabla \ln v), \nabla \ln u - \nabla \ln v \rangle \right] \\
 &\geq H^p(\nabla v) - u^p \left[H^p(\nabla \ln u) - C_p[H(\nabla \ln u) + H(\nabla \ln v)]^{p-2}H^2(\nabla(\ln v - \ln u)) \right] \\
 &= H^p(\nabla v) - H^p(\nabla u) + C_p u^p [H(\nabla \ln u) + H(\nabla \ln v)]^{p-2}H^2(\nabla(\ln v - \ln u)).
 \end{aligned}
 \tag{3.63}$$

Adding both these two inequalities, we obtain

$$\begin{aligned}
 &H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1 \rangle + H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2 \rangle \\
 &\geq C_p(u^p + v^p)[H(\nabla \ln u) + H(\nabla \ln v)]^{p-2}H^2(\nabla(\ln v - \ln u)).
 \end{aligned}
 \tag{3.64}$$

□

Now, we are ready to prove the comparison principles in bounded and exteriors domains.

Proposition 3.6 *Let Ω be an open bounded smooth domain of \mathbb{R}^N and $f \in L^{\frac{N}{p}}(\Omega)$. Let $u \in \mathcal{D}^{1,p}(\Omega)$ be a weak positive subsolution to (3.52) and $v \in \mathcal{D}^{1,p}(\Omega)$ be a weak positive supersolution of*

$$-\Delta^H v - \frac{\gamma}{H^\circ(x)^p} v^{p-1} = g(x)v^{p-1} \quad \text{in } \Omega,
 \tag{3.65}$$

with $g \in L^{\frac{N}{p}}(\Omega)$. Assume that $\inf_{\Omega} v > 0$ and $f \leq g$ in Ω . If $u \leq v$ on $\partial\Omega$, then

$$u \leq v \quad \text{in } \Omega.$$

Proof We will give the proof of this result in the case $p \geq 2$. The case $1 < p < 2$ is similar.

Let us define

$$\eta_1 := \frac{\min\{(u^p - v^p)^+, m\}}{u^{p-1}} \quad \text{and} \quad \eta_2 := \frac{\min\{(u^p - v^p)^+, m\}}{v^{p-1}}$$

where $m > 1$. It is quite standard to show that η_1 and η_2 are good test function that we can use in the weak formulations of (3.52) and (3.65). Taking both these test function and subtracting the two equations, we obtain

$$\begin{aligned} & \int_{\Omega} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \eta_1 \rangle dx - \int_{\Omega} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \eta_2 \rangle dx \\ & \leq \int_{\Omega} \frac{\gamma}{H^\circ(x)^p} u^{p-1} \eta_1 dx - \int_{\Omega} \frac{\gamma}{H^\circ(x)^p} v^{p-1} \eta_2 dx \\ & \quad + \int_{\Omega} f(x) u^{p-1} \eta_1 dx - \int_{\Omega} g(x) v^{p-1} \eta_2 dx \quad (3.66) \\ & = \int_{\Omega} \frac{\gamma}{H^\circ(x)^p} (u^{p-1} \eta_1 - v^{p-1} \eta_2) dx \\ & \quad + \int_{\Omega} f(x) u^{p-1} \eta_1 - g(x) v^{p-1} \eta_2 dx \leq 0, \end{aligned}$$

since $f \leq g$ in Ω . Hence, setting $\Omega_1 := \{x \in \Omega \mid 0 \leq u^p - v^p \leq m\}$ and $\Omega_2 := \{x \in \Omega \mid u^p - v^p \geq m\}$, we deduce that

$$\begin{aligned} & \int_{\Omega_1} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \left(u - \frac{v^p}{u^p} u\right) \rangle dx \\ & \quad + \int_{\Omega_1} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \left(v - \frac{u^p}{v^p} v\right) \rangle dx \\ & \quad + m(1-p) \left[\int_{\Omega_2} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), u^{-p} \nabla u \rangle dx \right. \\ & \quad \left. - \int_{\Omega_2} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), v^{-p} \nabla v \rangle dx \right] \leq 0. \end{aligned} \quad (3.67)$$

Applying (3.53) in (3.67) and making some computations we obtain

$$\begin{aligned} & C_p \int_{\Omega_1} (u^p + v^p) H^p(\nabla(\ln u - \ln v)) dx \\ & \quad + m(p-1) \left[\int_{\Omega_2} H^p(\nabla \ln v) dx - \int_{\Omega_2} H^p(\nabla \ln u) dx \right] \leq 0, \end{aligned} \quad (3.68)$$

but this implies that

$$C_p \int_{\Omega_1} (u^p + v^p) H^p(\nabla(\ln u - \ln v)) dx \leq m(p-1) \int_{\Omega_2} H^p(\nabla \ln u) dx. \quad (3.69)$$

For the right hand side of (3.69), we have

$$\begin{aligned} & m(p-1) \int_{\Omega_2} H^p(\nabla \ln u) dx = (p-1) \int_{\Omega_2} m u^{-p} H^p(\nabla u) dx \\ & \leq (p-1) \int_{\Omega'_2} H^p(\nabla u) dx, \end{aligned} \quad (3.70)$$

where $\Omega'_2 := \{x \in \Omega \mid u^p \geq m\}$ and it holds that

$$\int_{\Omega'_2} H^p(\nabla u) dx \rightarrow 0 \quad \text{for } m \rightarrow +\infty.$$

Hence, passing to the limit for $m \rightarrow +\infty$ in (3.69) we obtain that

$$\int_{\{u^p - v^p \geq 0\}} (u^p + v^p) H^p(\nabla(\ln u - \ln v)) \, dx = 0, \tag{3.71}$$

which clearly implies that

$$u = Kv \quad \text{in the set } \{x \in \Omega \mid u(x) \geq v(x)\},$$

for some positive constant K . By our assumptions $\inf_{x \in \Omega} v > 0$ and $u \leq v$ on $\partial\Omega$, hence it follows that $K = 1$. But this implies that

$$u \leq v \quad \text{in } \Omega$$

and this complete the proof of this result in the case $p \geq 2$. The case $1 < p < 2$ follows repeating verbatim the proof of the case $p \geq 2$, but applying inequality (3.54) instead of (3.53). \square

Now we want to prove the corresponding result of Proposition 3.6 in exterior domains.

Proposition 3.7 *Let Ω be an exterior domain such that $\mathbb{R}^N \setminus \Omega$ is bounded and $f \in L^{\frac{N}{p}}(\Omega)$. Let $u \in \mathcal{D}^{1,p}(\Omega)$ be a weak positive subsolutions to (3.52) and $v \in \mathcal{D}^{1,p}(\Omega)$ be a positive supersolution of*

$$-\Delta_p^H v - \frac{\gamma}{H^\circ(x)^p} v^{p-1} = g(x)v^{p-1} \quad \text{in } \Omega, \tag{3.72}$$

with $g \in L^{\frac{N}{p}}(\Omega)$. Assume that $\inf_{\Omega} v > 0$ and $f \leq g$ in Ω . If $u \leq v$ on $\partial\Omega$ and

$$\limsup_{R \rightarrow +\infty} \frac{1}{R} \int_{B_{2R} \setminus B_R} u^p |\nabla \log v|^{p-1} = 0, \tag{3.73}$$

then

$$u \leq v \quad \text{in } \Omega.$$

Proof In the same spirit of Proposition 3.6 we prove our result in the case $p \geq 2$. The other case is similar and it can be shown using the same arguments. To this aim, let $\varphi_R \in C_c^\infty(B_{2R})$ be a standard cut-off function such that

$$\begin{cases} 0 \leq \varphi_R \leq 1 \\ \varphi_R \equiv 1 & \text{on } B_R \\ |\nabla \varphi_R| \leq \frac{2}{R} & \text{on } B_{2R} \setminus B_R. \end{cases}$$

Let us define

$$\eta_1 := \varphi_R \frac{\min\{(u^p - v^p)^+, m\}}{u^{p-1}} \quad \text{and} \quad \eta_2 := \varphi_R \frac{\min\{(u^p - v^p)^+, m\}}{v^{p-1}},$$

where $m > 1$. As pointed out in the proof of previous proposition, it is possible to show, by standard arguments, that η_1 and η_2 are good test functions for the weak formulations (3.52) and (3.72). Hence, we obtain

$$\begin{aligned} & \int_{\Omega} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \eta_1 \rangle \, dx - \int_{\Omega} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \eta_2 \rangle \, dx \\ & \leq \int_{\Omega} \frac{\gamma}{H^\circ(x)^p} (u^{p-1} \eta_1 - v^{p-1} \eta_2) \, dx \tag{3.74} \\ & + \int_{\Omega} f(x) u^{p-1} \eta_1 - g(x) v^{p-1} \eta_2 \, dx \leq 0, \end{aligned}$$

since $f \leq g$ in Ω . Now we explicitly compute the left hand side of (3.74).

$$\begin{aligned}
 0 &\geq \int_{\Omega} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \eta_1 \rangle dx - \int_{\Omega} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \eta_2 \rangle dx \\
 &= \int_{\Omega_1} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \left(u - \frac{v^p}{u^p} u \right) \rangle \varphi_R \\
 &\quad + H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \left(v - \frac{u^p}{v^p} v \right) \rangle \varphi_R dx \\
 &\quad + \int_{\Omega_1} \left(u - \frac{v^p}{u^p} u \right) H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \varphi_R \rangle \\
 &\quad + \left(v - \frac{u^p}{v^p} v \right) H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \varphi_R \rangle dx \\
 &\quad + \int_{\Omega_2} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla (u^{1-p}) \rangle m \varphi_R dx \\
 &\quad + \int_{\Omega_2} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \varphi_R \rangle m u^{1-p} dx \\
 &\quad - \int_{\Omega_2} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla (v^{1-p}) \rangle m \varphi_R dx \\
 &\quad - \int_{\Omega_2} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \varphi_R \rangle m v^{1-p} dx \\
 &=: I_1 + I_2 + I_3 + I_4,
 \end{aligned} \tag{3.75}$$

where $\Omega_1 := \{x \in \Omega \mid 0 \leq u^p - v^p \leq m\}$ and $\Omega_2 := \{x \in \Omega \mid u^p - v^p \geq m\}$. By Proposition 3.5 and using the definition of φ_R , it follows that there exists a positive constant depending only on p such that

$$I_1 \geq C_p \int_{\Omega_1 \cap B_R} (u^p + v^p) H^p(\nabla(\ln u - \ln v)) dx. \tag{3.76}$$

Now we are going to give estimates for I_2, I_3 and I_4 . We start with I_2 . Using (2.6) and the Cauchy–Schwarz inequality, setting $\tilde{\Omega}_1 := \{x \in \Omega : v^p \leq u^p\}$, we have

$$\begin{aligned}
 |I_2| &\leq M \int_{\Omega_1} \left| u - \frac{v^p}{u^p} u \right| H^{p-1}(\nabla u) |\nabla \varphi_R| dx + M \int_{\Omega_1} \left| v - \frac{u^p}{v^p} v \right| H^{p-1}(\nabla v) |\nabla \varphi_R| dx \\
 &\leq M \int_{\tilde{\Omega}_1} (2u H^{p-1}(\nabla u) + v H^{p-1}(\nabla v)) |\nabla \varphi_R| dx + M \int_{\tilde{\Omega}_1} u^p H^{p-1}(\nabla \ln v) |\nabla \varphi_R| dx \\
 &\leq \frac{C}{R} \int_{B_{2R} \setminus B_R} u \cdot H^{p-1}(\nabla u) dx + \frac{C}{R} \int_{B_{2R} \setminus B_R} v \cdot H^{p-1}(\nabla v) dx \\
 &\quad + \frac{C}{R} \int_{B_{2R} \setminus B_R} u^p H^{p-1}(\nabla \ln v) dx \\
 &\leq C \left(\int_{B_{2R} \setminus B_R} H^p(\nabla u) dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} u^{p^*} dx \right)^{\frac{1}{p^*}} \\
 &\quad + C \left(\int_{B_{2R} \setminus B_R} H^p(\nabla v) dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} v^{p^*} dx \right)^{\frac{1}{p^*}} \\
 &\quad + \frac{C}{R} \int_{B_{2R} \setminus B_R} u^p H^{p-1}(\nabla \ln v) dx,
 \end{aligned} \tag{3.77}$$

where in the last line we applied the Hölder inequality with conjugate exponent $(N, p/(p - 1), p^*)$ and $C := 2M$. Passing to the limit for R that goes to $+\infty$ in the right hand side of (3.77), using also assumption (3.73) and (2.4), we deduce that I_2 goes to zero.

Now we proceed with the estimate of the term I_3 . By (2.5), we have

$$I_3 = m \int_{\Omega_2} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \varphi_R \rangle u^{1-p} dx + m(1 - p) \int_{\Omega_2} H^p(\nabla u) \varphi_R u^{-p} dx$$

Therefore

$$|I_3| \leq mM \int_{\tilde{\Omega}_2} H^{p-1}(\nabla u) |\nabla \varphi_R| u^{1-p} dx + m(p - 1) \int_{\tilde{\Omega}_2} H^p(\nabla u) \varphi_R u^{-p} dx,$$

where $\tilde{\Omega}_2 := \{x \in \Omega \mid u^p \geq m\}$. Using this definition and also the properties of φ_R we deduce that

$$\begin{aligned} |I_3| &\leq \frac{2M}{R} \int_{\tilde{\Omega}_2} u \cdot H^{p-1}(\nabla u) dx + (p - 1) \int_{\tilde{\Omega}_2} H^p(\nabla u) dx \\ &\leq C \left(\int_{B_{2R} \setminus B_R} H^p(\nabla u) dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} u^{p^*} dx \right)^{\frac{1}{p^*}} + (p - 1) \int_{\tilde{\Omega}_2} H^p(\nabla u) dx, \end{aligned} \tag{3.78}$$

where $C := 2M$. Passing to the limit for m, R that go to $+\infty$, we deduce that

$$\lim_{m, R \rightarrow +\infty} I_3 = 0. \tag{3.79}$$

For the last term I_4 , by (2.5), recalling $\tilde{\Omega}_2 := \{x \in \Omega : u^p \geq m\}$, we have

$$\begin{aligned} I_4 &= m(p - 1) \int_{\Omega_2} H^p(\nabla v) \varphi_R v^{-p} dx - \int_{\Omega_2} m H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \varphi_R \rangle v^{1-p} dx \\ &\geq -M \int_{\Omega_2} m H^{p-1}(\nabla \ln v) |\nabla \varphi_R| dx \geq M \int_{\tilde{\Omega}_2} u^p H^{p-1}(\nabla \ln v) |\nabla \varphi_R| dx \\ &\geq -\frac{2M}{R} \int_{B_{2R} \setminus B_R} u^p H^{p-1}(\nabla \ln v) dx \end{aligned} \tag{3.80}$$

Hence, passing to the limit the right hand side of (3.80), by (3.73) we have that I_4 goes to zero when R tends to $+\infty$.

Finally, if we combine all the estimates (3.76), (3.77), (3.79), (3.80) and we pass to the limit for $m, R \rightarrow +\infty$ we deduce that

$$\begin{aligned} 0 &\geq \int_{\Omega} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \eta_1 \rangle dx - \int_{\Omega} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \eta_2 \rangle dx \\ &= \limsup_{R \rightarrow +\infty} (I_1 + I_2 + I_3 + I_4) \geq \liminf_{R \rightarrow +\infty} (I_1 + I_2 + I_3 + I_4) \\ &\geq \int_{\{u \geq v\}} (u^p + v^p) H^p(\nabla(\ln u - \ln v)) dx \geq 0, \end{aligned}$$

which implies $u \leq v$ in Ω as we concluded in the proof of Proposition 3.6. □

4 Proof of the main results

This section is dedicated to the proof of our main results: Theorems 1.1, 1.3 and 1.4.

Proof of Theorem 1.1 We start by proving (1.4). To this aim, let us consider u a solution of

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = u^{p^*-p} u^{p-1} \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

In particular, we have that u is a subsolution of (4.1) in any bounded domain $\Omega = \mathcal{B}_{R_1}^{H^\circ}$. We note that $f(x) := u^{p^*-p} \in L^{\frac{N}{p}}(\mathcal{B}_{R_1}^{H^\circ})$ and satisfies $|f(x)| \leq A[H^\circ(x)]^{-\alpha}$ with $\alpha = ((N - p)/p - \sigma_1)(p^* - p) < p$ for $x \in \mathcal{B}_{R_1}^{H^\circ}$ due to Theorem 3.3. By Proposition 3.4 we have that the function

$$v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^\varepsilon}{[H^\circ(x)]^{\mu_1}} \in \mathcal{D}^{1,p}(\mathcal{B}_{R_1}^{H^\circ})$$

is a positive supersolution of (3.65) in $\mathcal{B}_{R_1}^{H^\circ} \subset \Omega$, with $g \in L^{\frac{N}{p}}(\mathcal{B}_{R_1}^{H^\circ})$ satisfying $g(x) \geq A[H^\circ(x)]^{-\alpha}$ and where $0 < \delta, \varepsilon, R_1 < 1$ are positive constants depending only on N, p, γ, A and α .

Let us consider $\Gamma > 0, \mathcal{M} = \sup_{\partial \mathcal{B}_{R_1}^{H^\circ}} u, \mathcal{N} = \sup_{\partial \mathcal{B}_{R_1}^{H^\circ}} 1/v$ and define

$$w(x) := \mathcal{N}(\mathcal{M} + \Gamma)v(H^\circ(x)).$$

It is easy to check that w is a positive supersolution of (3.65) in $\mathcal{B}_{R_1}^{H^\circ} \subset \Omega$ and $\inf_{\mathcal{B}_{R_1}^{H^\circ}} w = \mathcal{M} + \Gamma > 0$ and $u \leq w$ on $\partial \mathcal{B}_{R_1}^{H^\circ}$. Hence, by Proposition 3.6 we deduce that

$$u \leq w \quad \text{in } \mathcal{B}_{R_1}^{H^\circ}.$$

Passing to the limit for $\Gamma \rightarrow 0$ we obtain that

$$u(x) \leq \frac{C}{[H^\circ(x)]^{\mu_1}} \quad \text{in } \mathcal{B}_{R_1}^{H^\circ},$$

where $C = \mathcal{M} \cdot \mathcal{N}$.

Now we have to show the estimate from below. Let u be a weak solution of (\mathcal{P}_H) , then u is a supersolution of

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = 0 \quad \text{in } \mathcal{B}_{R_1}^{H^\circ}. \tag{4.2}$$

We set $c_1 := \inf_{\mathcal{B}_{R_1}^{H^\circ}} u > 0$.

Now, we define

$$\tilde{w}(x) := c \frac{c_1}{[H^\circ(x)]^{\mu_1}} \quad \text{in } \mathcal{B}_{R_1}^{H^\circ},$$

where $c = \inf_{\partial \mathcal{B}_{R_1}^{H^\circ}} [H^\circ(x)]^{\mu_1} = R_1^{\mu_1}$. The function \tilde{w} is a subsolution to (4.2) in $\mathcal{B}_{R_1}^{H^\circ}$. Since, it holds that $u \geq \tilde{w}$ in $\partial \mathcal{B}_{R_1}^{H^\circ}$, we conclude by using Proposition 3.6 to obtain

$$u \geq \tilde{w} \quad \text{in } \mathcal{B}_{R_1}^{H^\circ},$$

and hence combining the estimates from above and below we deduce that (1.4) is proved.

Now, our aim is to prove (1.5). Let us consider u a subsolution of

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = u^{p^*-p} u^{p-1} \quad \text{in } (\mathcal{B}_{R_2}^{H^\circ})^c. \tag{4.3}$$

We note that $f(x) := u^{p^*-p} \in L^{\frac{N}{p}}((\mathcal{B}_{R_2}^{H^\circ})^c)$ and by Theorem 3.3 we have $|f(x)| \leq A[H^\circ(x)]^{-\alpha}$ with $\alpha = ((N - p)/p + \sigma_2)(p^* - p) > p$ for $x \in (\mathcal{B}_{R_2}^{H^\circ})^c$. By Proposition 3.4 the function

$$v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^{-\varepsilon}}{[H^\circ(x)]^{\mu_2}} \in \mathcal{D}^{1,p}((\mathcal{B}_{R_2}^{H^\circ})^c)$$

is a positive supersolution of (3.72) in $(\mathcal{B}_{R_2}^{H^\circ})^c \subset \Omega$, with $g \in L^{\frac{N}{p}}((\mathcal{B}_{R_2}^{H^\circ})^c)$ satisfying $g(x) \geq A[H^\circ(x)]^{-\alpha}$ and where $0 < \delta, \varepsilon < 1$ and $R_2 > 1$ are positive constants depending only on N, p, γ, A and α .

Let us consider $\Gamma > 0, \mathcal{M} = \sup_{(\partial\mathcal{B}_{R_2}^{H^\circ})^c} u, \mathcal{N} = \sup_{(\partial\mathcal{B}_{R_2}^{H^\circ})^c} 1/v$ and define

$$w(x) := \mathcal{N}(\mathcal{M} + \Gamma)v(H^\circ(x)).$$

We note that w is a positive supersolution of (3.72) in $(\mathcal{B}_{R_2}^{H^\circ})^c \subset \Omega$ and $\inf_{(\mathcal{B}_{R_2}^{H^\circ})^c} w = \mathcal{M} + \Gamma > 0$ and $u \leq w$ on $(\partial\mathcal{B}_{R_2}^{H^\circ})^c$. We verify the condition (3.73). Since $|\nabla \log v(x)| \leq C|x|^{-1}$, by Hölder inequality we have

$$\limsup_{R \rightarrow +\infty} \frac{1}{R} \int_{B_{2R} \setminus B_R} u^p |\nabla \log v|^{p-1} dx \leq \limsup_{R \rightarrow +\infty} \frac{C}{R} \left(\int_{B_{2R} \setminus B_R} |u|^{p^*} dx \right)^{\frac{p}{p^*}} = 0, \tag{4.4}$$

where C is a constant independent of R .

Hence, by Proposition 3.7 we deduce that

$$u \leq w \quad \text{in } (\mathcal{B}_{R_2}^{H^\circ})^c.$$

Passing to the limit for $\Gamma \rightarrow 0$ we obtain that

$$u(x) \leq \frac{C}{[H^\circ(x)]^{\mu_2}} \quad \text{in } (\mathcal{B}_{R_2}^{H^\circ})^c,$$

where $C = \mathcal{M} \cdot \mathcal{N}$

We conclude with the estimate from below. Let u be a weak solution of (\mathcal{P}_H) . Then u is a supersolution of

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = 0 \quad \text{in } \mathbb{R}^N. \tag{4.5}$$

We claim that

$$\int_{B_{2R} \setminus B_R} |\nabla \log u|^p \leq CR^{N-p} \tag{4.6}$$

for R sufficiently large and constant C independent of R . Indeed, by (4.5), we have

$$-\Delta_p^H u \geq 0 \quad \text{in } \mathbb{R}^N. \tag{4.7}$$

Therefore, considering the test function $\eta = \zeta^p u^{1-p}$, with $\zeta \in C_c^1(\mathbb{R}^N)$ nonnegative function, and taking in (4.7) we have

$$-(p-1) \int_{\mathbb{R}^N} H^p(\nabla u) \zeta^p u^{-p} dx + p \int_{\mathbb{R}^N} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla \zeta \rangle \zeta^{p-1} u^{1-p} dx \geq 0. \tag{4.8}$$

By Hölder inequality and by 0-homogeneity of ∇H we get

$$\int_{\mathbb{R}^N} H^p(\nabla \log u)\zeta^p dx \leq M \frac{p}{p-1} \left(\int_{\mathbb{R}^N} H^p(\nabla \log u)\zeta^p dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |\nabla \zeta|^p dx \right)^{\frac{1}{p}}. \tag{4.9}$$

Taking a standard cutoff function ζ in (4.9) we get the claim (4.6).

Now we set $c_2 = \inf_{(\partial B_{R_2}^{H^\circ})^c} u > 0$ and $c = \inf_{(\partial B_{R_2}^{H^\circ})^c} [H^\circ(x)]^{\mu_2}$. We note that $v = c_2 c [H^\circ(x)]^{-\mu_2}$ is a weak solution of (4.5). Moreover the condition (3.73) is verified. Indeed by Hölder inequality and (4.6) we have

$$\begin{aligned} \frac{1}{R} \int_{B_{2R} \setminus B_R} v^p |\nabla \log u|^{p-1} dx &\leq C R^{-1-p\mu_2+\frac{N}{p}} \left(\int_{B_{2R} \setminus B_R} |\nabla \log u|^p dx \right)^{\frac{p-1}{p}} \\ &\leq C R^{-1-p\mu_2+\frac{p-1}{p}(N-p)+\frac{N}{p}} \rightarrow 0 \end{aligned} \tag{4.10}$$

since $\mu_2 > (N - p)/p$. Applying the Proposition 3.7 we conclude that

$$u(x) \geq v(x) = c \frac{c_2}{[H^\circ(x)]^{\mu_2}} \quad \text{in } (B_{R_2}^{H^\circ})^c$$

and therefore the thesis. □

Now we prove Theorem 1.3 that will be essential to prove the asymptotic behavior of the gradient of solutions to (\mathcal{P}_H) . For the reader convenience we state a more detailed statement contained in the following:

Theorem 4.1 *Let $v \in C_{loc}^{1,\alpha}(\mathbb{R}^N \setminus \{0\})$ be a positive weak solution of the equation*

$$-\Delta_p^H v - \frac{\gamma}{[H^\circ(x)]^p} v^{p-1} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{4.11}$$

where $0 \leq \gamma < C_H$. Assume that there exist two positive constants C and c such that

$$\frac{c}{[H^\circ(x)]^{\mu_2}} \leq v(x) \leq \frac{C}{[H^\circ(x)]^{\mu_2}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.12}$$

and suppose that there exists a positive constant \hat{C} such that

$$|\nabla v(x)| \leq \frac{\hat{C}}{[H^\circ(x)]^{\mu_2+1}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.13}$$

then

$$v(x) = \frac{\bar{c}_1}{[H^\circ(x)]^{\mu_2}}, \tag{4.14}$$

with

$$\bar{c}_1 := \limsup_{|x| \rightarrow 0} [H^\circ(x)]^{\mu_2} v(x).$$

On the other hand, suppose that there exist two positive constants \tilde{C} and \tilde{c} such that

$$\frac{\tilde{c}}{[H^\circ(x)]^{\mu_1}} \leq v(x) \leq \frac{\tilde{C}}{[H^\circ(x)]^{\mu_1}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.15}$$

and suppose that there exists a positive constant \bar{C} such that

$$|\nabla v(x)| \leq \frac{\bar{C}}{[H^\circ(x)]^{\mu_1+1}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.16}$$

then

$$v(x) = \frac{\bar{c}_2}{[H^\circ(x)]^{\mu_1}}, \tag{4.17}$$

with

$$\bar{c}_2 := \limsup_{|x| \rightarrow +\infty} [H^\circ(x)]^{\mu_1} v(x).$$

Proof We prove this result only for $p \geq 2$. The other case is similar. Suppose that (4.12) holds. From definition of \bar{c}_1 , there exist r (sufficiently small) such that

$$v(x) \leq \frac{\bar{c}_1 + a_n}{[H^\circ(x)]^{\mu_2}} \quad \text{in } B_r(0) \setminus \{0\}, \tag{4.18}$$

given $a_n \rightarrow 0$.

On the other hand, there exist sequences of radii R_n and points x_n with R_n tending to zero and $H^\circ(x_n) = R_n$, such that

$$v(x_n) \geq \frac{\bar{c}_1 - a_n}{[H^\circ(x_n)]^{\mu_2}}. \tag{4.19}$$

Now we set

$$w_n(x) := R_n^{\mu_2} v(R_n x) \quad \forall x \in \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}. \tag{4.20}$$

By (4.12), it follows that w_n is uniformly bounded in $L^\infty(\mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ})$ and, since w_n satisfies the equation (4.11), by [2, 17, 18], it is also uniformly bounded in $C^{1,\alpha}(K)$, for $0 < \alpha < 1$ and for any compact set $K \subset \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}$. Moreover, from Ascoli-Arzelà's Theorem, we deduce that $w_n \rightarrow w_\infty$ in the norm $\|\cdot\|_{C^{1,\alpha}(K)}$, for any compact set $K \subset \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}$. By (4.18), we have

$$w_\infty \leq \frac{\bar{c}_1}{[H^\circ(x)]^{\mu_2}}$$

and by (4.19), there exist a point $\bar{x} \in \partial \mathcal{B}_1^{H^\circ}$ such that $w_\infty(\bar{x}) = \bar{c}_1$. By the strong comparison principle [11], that holds under our assumption on H (see Sect. 2), we have

$$w_\infty \equiv \frac{\bar{c}_1}{[H^\circ(x)]^{\mu_2}} \quad \text{in } \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}. \tag{4.21}$$

Now we set

$$\hat{v} := \frac{\bar{c}_1}{[H^\circ(x)]^{\mu_2}} \tag{4.22}$$

and we note that it solves (4.11). Fix $R > 0$ sufficiently large and let $\varphi_R \in C_c^\infty(B_{2R})$ be a standard cut-off function such that

$$\begin{cases} 0 \leq \varphi_R \leq 1 \\ \varphi_R \equiv 1 & \text{on } B_R \\ |\nabla \varphi_R| \leq \frac{2}{R} & \text{on } B_{2R} \setminus B_R. \end{cases} \tag{4.23}$$

Let us define

$$\varphi_1 := \varphi_R \frac{\min\{(v^p - \hat{v}^p)^+, m\}}{v^{p-1}} \quad \text{and} \quad \varphi_2 := \varphi_R \frac{\min\{(v^p - \hat{v}^p)^+, m\}}{\hat{v}^{p-1}}, \quad (4.24)$$

where $m > 1$.

We remark that φ_1 and φ_2 are good test function in any domain $0 \notin \bar{\Omega}$. Testing (4.24) in (4.11) on the domain $\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}$, and, since the stress field $H(\nabla u)^{p-1} \nabla H(\nabla u) \in W_{loc}^{1,2}(\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ})$ (see [19]), exploiting the divergence theorem, we have

$$\begin{aligned} & \int_{\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \varphi_1 \rangle dx - \int_{\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \nabla \varphi_2 \rangle dx \\ & - \int_{\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}} \frac{\gamma}{H^\circ(x)^p} (v^{p-1} \varphi_1 - \hat{v}^{p-1} \varphi_2) dx \\ & = \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \eta_n \rangle \varphi_1 - H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle \varphi_2 dx, \end{aligned} \quad (4.25)$$

where η_n is the inner unite normal vector at $\partial \mathcal{B}_{R_n}^{H^\circ}$. Now we set $\mathcal{A} := \{0 \leq v^p - \hat{v}^p \leq m\}$ and $\mathcal{B} := \{v^p - \hat{v}^p \geq m\}$. Then (4.25) becomes:

$$\begin{aligned} & \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} -H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \eta_n \rangle \varphi_1 + H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle \varphi_2 dx \\ & = \int_{(\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}) \cap \mathcal{A}} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \left(v - \frac{\hat{v}^p}{v^p} v \right) \rangle \varphi_R dx \\ & + \int_{(\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}) \cap \mathcal{A}} H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \nabla \left(\hat{v} - \frac{v^p}{\hat{v}^p} \hat{v} \right) \rangle \varphi_R dx \\ & + \int_{(\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}) \cap \mathcal{A}} \left(v - \frac{\hat{v}^p}{v^p} v \right) H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \varphi_R \rangle dx \\ & + \int_{(\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}) \cap \mathcal{A}} \left(\hat{v} - \frac{v^p}{\hat{v}^p} \hat{v} \right) H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \nabla \varphi_R \rangle dx \\ & + \int_{(\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}) \cap \mathcal{B}} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla (v^{1-p}) \rangle m \varphi_R \\ & + H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \varphi_R \rangle m v^{1-p} dx \\ & - \int_{(\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}) \cap \mathcal{B}} (H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \nabla (\hat{v}^{1-p}) \rangle) m \varphi_R \\ & + H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \nabla \varphi_R \rangle m \hat{v}^{1-p} dx \\ & =: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \quad (4.26)$$

Now we set $\Omega_1 := (\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}) \cap \mathcal{A}$ and $\Omega_2 := (\mathbb{R}^N \setminus \mathcal{B}_{R_n}^{H^\circ}) \cap \mathcal{B}$. By Proposition 3.5, it follows that there exists a positive constant depending only on p such that

$$I_1 + I_2 \geq C_p \int_{\Omega_1 \cap B_{2R}} (v^p + \hat{v}^p) H^p(\nabla(\ln v - \ln \hat{v})) \varphi_R dx. \quad (4.27)$$

Now we estimate $I_3 + I_4$. Using (2.6) and the Hölder inequality with conjugate exponent $(N, p/(p - 1), p^*)$ we have

$$\begin{aligned}
 |I_3 + I_4| &\leq M \int_{\Omega_1} \left| v - \frac{\hat{v}^p}{v^p} v \right| H^{p-1}(\nabla v) |\nabla \varphi_R| dx \\
 &\quad + M \int_{\Omega_1} \left| \hat{v} - \frac{v^p}{\hat{v}^p} \hat{v} \right| H^{p-1}(\nabla \hat{v}) |\nabla \varphi_R| dx \\
 &\leq M \int_{\Omega_1} (2v H^{p-1}(\nabla v) + \hat{v} H^{p-1}(\nabla \hat{v})) |\nabla \varphi_R| dx \\
 &\quad + M \int_{\Omega_1} v^p H^{p-1}(\nabla \ln \hat{v}) |\nabla \varphi_R| dx \\
 &\leq \frac{2M}{R} \int_{B_{2R} \setminus B_R} v H^{p-1}(\nabla v) dx + \frac{2M}{R} \int_{B_{2R} \setminus B_R} \hat{v} H^{p-1}(\nabla \hat{v}) dx \tag{4.28} \\
 &\quad + \frac{2M}{R} \int_{B_{2R} \setminus B_R} v^p H^{p-1}(\nabla \ln \hat{v}) dx \\
 &\leq 2M \left(\int_{B_{2R} \setminus B_R} H^p(\nabla v) dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} v^{p^*} dx \right)^{\frac{1}{p^*}} \\
 &\quad + 2M \left(\int_{B_{2R} \setminus B_R} H^p(\nabla \hat{v}) dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} \hat{v}^{p^*} dx \right)^{\frac{1}{p^*}} \\
 &\quad + \frac{2M}{R} \int_{B_{2R} \setminus B_R} v^p H^{p-1}(\nabla \ln \hat{v}) dx.
 \end{aligned}$$

By (4.22) we have $H(\nabla \ln \hat{v}(x)) \leq C[H^\circ(x)]^{-1}$, where $C = \mu_2$, we have that

$$\begin{aligned}
 \frac{1}{R} \int_{B_{2R} \setminus B_R} v^p H^{p-1}(\nabla \ln \hat{v}) &\leq \frac{C}{R^p} \int_{B_{2R} \setminus B_R} |v|^p \\
 &\leq C \left(\int_{B_{2R} \setminus B_R} |v|^{p^*} \right)^{\frac{p}{p^*}}, \tag{4.29}
 \end{aligned}$$

where we used the Hölder inequality. Since v has the right summability at the infinity, for R that goes to infinity in the right hand side of (4.28), we obtain that $I_3 + I_4$ goes to zero.

Now we proceed with the estimate of the term I_5 . Recalling that $v^p \geq m$ in Ω_2 , we get

$$|I_5| \leq mM \int_{\Omega_2} H^{p-1}(\nabla v) |\nabla \varphi_R| v^{1-p} dx + m(p - 1) \int_{\Omega_2} H^p(\nabla v) \varphi_R v^{-p} dx,$$

Using the properties of φ_R we obtain that

$$\begin{aligned}
 |I_5| &\leq \frac{M}{R} \int_{B_{2R} \setminus B_R} v H^{p-1}(\nabla v) dx + (p - 1) \int_{\Omega_2} H^p(\nabla v) dx \\
 &\leq M \left(\int_{B_{2R} \setminus B_R} H^p(\nabla v) dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} v^{p^*} dx \right)^{\frac{1}{p^*}} + (p - 1) \int_{\Omega_2} H^p(\nabla v) dx, \tag{4.30}
 \end{aligned}$$

Passing to the limit for m, R that go to $+\infty$, we deduce that I_5 goes to zero since the set Ω_2 vanishes as $m \rightarrow +\infty$.

For the last term I_6 we obtain

$$\begin{aligned}
 I_6 &= m(p-1) \int_{\Omega_2} H^p(\nabla \hat{v}) \varphi_R \hat{v}^{-p} dx - \int_{\Omega_2} m H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \nabla \varphi_R \rangle \hat{v}^{1-p} dx \\
 &\geq -M \int_{\Omega_2} m H^{p-1}(\nabla \ln \hat{v}) |\nabla \varphi_R| dx \\
 &\geq -\frac{2M}{R} \int_{B_{2R} \setminus B_R} v^p H^{p-1}(\nabla \ln \hat{v}) dx
 \end{aligned} \tag{4.31}$$

Hence, passing to the limit in the right hand side of (4.31) and using (4.29) we have that the right hand of (4.31) tends to zero when R tends to the infinity.

Now we estimate the left hand of (4.26). By (4.24) and Proposition 2.1 (in particular see (2.11)), we get

$$\begin{aligned}
 &\int_{\partial \mathcal{B}_{R_n}^{H^\circ}} -H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle \varphi_2 + H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \eta_n \rangle \varphi_1 dx \\
 &= - \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} \varphi_1 (H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle - H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \eta_n \rangle) dx \\
 &\quad - \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} (\varphi_2 - \varphi_1) H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle dx \\
 &\leq \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} \frac{m}{v^{p-1}} \tilde{C}_p (|\nabla \hat{v}|^{p-2} + |\nabla v|^{p-2}) |\nabla \hat{v} - \nabla v| + \frac{2m}{\hat{v}^{p-1}} M \alpha_2 |\nabla \hat{v}|^{p-1} dx, \\
 &:= J_1 + J_2.
 \end{aligned} \tag{4.32}$$

where, in the last line, we used (2.4) and (2.6).

We set $x = R_n y$, with $y \in \partial \mathcal{B}_1^{H^\circ}$. Using this change of variables and recalling (4.12), (4.20) and (4.21), J_1 can be estimated as

$$\begin{aligned}
 J_1 &\leq \tilde{C}_1 \int_{\partial \mathcal{B}_1^{H^\circ}} m R_n^{N-1+\mu_2(p-1)-(\mu_2+1)(p-1)} |-\mu_2 \bar{c}_1 \nabla H^\circ(y) - \nabla w_n(y)| dy \\
 &= \tilde{C}_1 R_n^{N-p-\epsilon}
 \end{aligned} \tag{4.33}$$

where we choose $m = R_n^{-\epsilon}$, for $\epsilon > 0$ fixed sufficiently small, and \tilde{C}_1 is a positive constant.

In a similar way J_2 is estimated as

$$J_2 \leq \tilde{C}_2 R_n^{N-p-\epsilon}, \tag{4.34}$$

where \tilde{C}_2 is a positive constant.

Finally, if we combine all the estimates (4.27), (4.28), (4.30), (4.31), (4.33), (4.34) and passing to the limit for $R_n \rightarrow 0$, and then, exploiting the Fatou's lemma, for $R \rightarrow +\infty$, we deduce that

$$\int_{\{v \geq \hat{v}\}} (v^p + \hat{v}^p) H^p(\nabla(\ln v - \ln \hat{v})) dx = 0,$$

which implies $v \leq \hat{v}$ as we concluded in the proof of Proposition 3.6.

To prove that $\hat{v} \leq v$, let us consider

$$\varphi_1 := \varphi_R \frac{\min\{(\hat{v}^p - v^p)^+, m\}}{\hat{v}^{p-1}} \quad \text{and} \quad \varphi_2 := \varphi_R \frac{\min\{(\hat{v}^p - v^p)^+, m\}}{v^{p-1}}, \tag{4.35}$$

where $m > 1$ and φ_R is the standard cutoff function defined in (4.23). Using (4.35) in the weak formulation of (4.11), proceeding in a similar way as above we obtain (4.14). The only thing to check is that

$$\frac{1}{R} \int_{B_{2R} \setminus B_R} \hat{v}^p H^{p-1}(\nabla \ln v) dx \rightarrow 0. \tag{4.36}$$

Assumption (4.13) ensures (4.36), and therefore we are done.

Now, assume that (4.15) holds and suppose that $p \geq 2$, the other case is similar. The proof is similar to the previous one and, for this reason, we omit some details. From definition of \bar{c}_2 , there exist r (sufficiently large) such that

$$v(x) \leq \frac{\bar{c}_2 + a_n}{[H^\circ(x)]^{\mu_1}} \text{ in } \mathbb{R}^N \setminus B_r(0), \tag{4.37}$$

given $a_n \rightarrow 0$.

On the other hand, there exist sequences of radii R_n and points x_n with R_n tending to infinity and $H^\circ(x_n) = R_n$, such that

$$v(x_n) \geq \frac{\bar{c}_2 - a_n}{[H^\circ(x_n)]^{\mu_1}}. \tag{4.38}$$

Now we set

$$w_n(x) := R_n^{-\mu_1} v\left(\frac{x}{R_n}\right) \quad \forall x \in B_2^{H^\circ} \setminus B_{1/2}^{H^\circ}. \tag{4.39}$$

As in the previous case we obtain

$$w_\infty \equiv \frac{\bar{c}_2}{[H^\circ(x)]^{\mu_1}} \text{ in } B_2^{H^\circ} \setminus B_{1/2}^{H^\circ}. \tag{4.40}$$

Now we set

$$\hat{v} := \frac{\bar{c}_2}{[H^\circ(x)]^{\mu_1}}$$

and we remark that it solves (4.11). We show that $v = \hat{v}$ in \mathbb{R}^N . To this aim, fix $\varepsilon > 0$ sufficiently small and let $\varphi_\varepsilon \in C^\infty(\mathbb{R}^N)$ be a function such that

$$\begin{cases} 0 \leq \varphi_\varepsilon \leq 1 \\ \varphi_\varepsilon \equiv 0 & \text{on } B_\varepsilon \\ \varphi_\varepsilon \equiv 1 & \text{on } (B_{2\varepsilon})^c \\ |\nabla \varphi_\varepsilon| \leq \frac{2}{\varepsilon} & \text{on } B_{2\varepsilon} \setminus B_\varepsilon. \end{cases} \tag{4.41}$$

Let us define

$$\varphi_1 := \varphi_\varepsilon \frac{\min\{(v^p - \hat{v}^p)^+, m\}}{v^{p-1}} \quad \text{and} \quad \varphi_2 := \varphi_\varepsilon \frac{\min\{(v^p - \hat{v}^p)^+, m\}}{\hat{v}^{p-1}}, \tag{4.42}$$

where $m > 1$.

We note that φ_1 and φ_2 are good test function in any bounded domain $0 \in \Omega$. Testing (4.42) in (4.11) on the domain $\mathcal{B}_{R_n}^{H^\circ}$ we get

$$\begin{aligned} & \int_{\mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \nabla \varphi_1 \rangle dx - \int_{\mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \nabla \varphi_2 \rangle dx \\ & - \int_{\mathcal{B}_{R_n}^{H^\circ}} \frac{\gamma}{H^\circ(x)^p} (v^{p-1} \varphi_1 - v^{p-1} \varphi_2) dx \\ & = \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \eta_n \rangle \varphi_1 - H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle \varphi_2 dx, \end{aligned} \tag{4.43}$$

where η_n is the outward unite normal vector at $\partial \mathcal{B}_{R_n}^{H^\circ}$. By (4.42) and Proposition 2.1, the right hand of (4.43) becomes

$$\begin{aligned} & \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} -H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle \varphi_2 + H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \eta_n \rangle \varphi_1 dx \\ & = \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} -\varphi_1 (H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle - H^{p-1}(\nabla v) \langle \nabla H(\nabla v), \eta_n \rangle) dx \\ & - \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} (\varphi_2 - \varphi_1) H^{p-1}(\nabla \hat{v}) \langle \nabla H(\nabla \hat{v}), \eta_n \rangle dx \\ & \leq \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} \tilde{C}_p v (|\nabla \hat{v}|^{p-2} + |\nabla v|^{p-2}) (\nabla \hat{v} - \nabla v) + \frac{v^p}{\hat{v}^{p-1}} M \alpha_2 |\nabla \hat{v}|^{p-1} dx, \\ & := \tilde{J}_1 + \tilde{J}_2. \end{aligned} \tag{4.44}$$

where, in the last line, we used (2.4) and (2.6).

We consider the following change of variables $x = y/R_n$, with $y \in \partial \mathcal{B}_1^{H^\circ}$. Using this fact, by (4.39) and (4.40), we have

$$\begin{aligned} \tilde{J}_1 & \leq \bar{C}_1 \int_{\partial \mathcal{B}_1^{H^\circ}} R_n^{1-N-\mu_1+(\mu_1+1)(p-1)} | -\mu_1 \bar{c}_2 y - \nabla w_n(y) | dy \\ & = \bar{C}_1 R_n^{-N+\mu_1 p+p} \end{aligned} \tag{4.45}$$

where \bar{C}_1 is a positive constant.

In a similar way \tilde{J}_2 is estimated as

$$\tilde{J}_2 \leq \bar{C}_2 R_n^{-N+\mu_1 p+p}, \tag{4.46}$$

where \bar{C}_2 is a positive constant. We note that $-N+\mu_1 p+p < 0$ since $0 \leq \mu_1 < (N-p)/p$. Proceeding as in the case (4.12), we prove that

$$\int_{\{v \geq \hat{v}\}} (v^p + \hat{v}^p) H^p(\nabla(\ln v - \ln \hat{v})) dx = 0, \tag{4.47}$$

which implies $v \leq \hat{v}$ in Ω as we concluded in the proof of Proposition 3.6.

To prove that $\hat{v} \leq v$, let us consider

$$\varphi_1 := \varphi_\varepsilon \frac{\min\{(\hat{v}^p - v^p)^+, m\}}{\hat{v}^{p-1}} \quad \text{and} \quad \varphi_2 := \varphi_\varepsilon \frac{\min\{(\hat{v}^p - v^p)^+, m\}}{v^{p-1}}, \tag{4.48}$$

where $m > 1$ and φ_ε is defined as (4.41). Testing (4.48) in (4.11), using the assumption (4.16) and proceeding in a similar way as above we get (4.17). \square

At this point we are ready to prove

Proof of Theorem 1.4 We prove (1.12), the other case is similar and it can be proved in a similar way. We start by showing the estimate from above. For R_n tending to infinity, let us consider

$$w_n(x) := R_n^{\mu_2} u(R_n x) \quad \forall x \in \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}. \tag{4.49}$$

We remark that w_n is uniformly bounded in $L^\infty(\mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ})$ and it weakly solves

$$-\Delta_p^H w_n - \frac{\gamma}{H^\circ(x)^p} w_n^{p-1} = R_n^{\mu_2(p-1)+p-\mu_2(p^*-1)} w_n^{p^*-1} \quad \text{in } \mathbb{R}^N. \tag{4.50}$$

Since $\mu_2(p-1) + p - \mu_2(p^*-1) < 0$, by [2, 17, 18], w_n is also uniformly bounded in $C^{1,\alpha}(K)$, for $0 < \alpha < 1$ and for any compact set $K \subset \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}$. For R_n sufficiently large we get the estimate from above in (1.12).

Now we prove the estimate from below. Suppose by contradiction that there exist sequences of points x_n such that

$$[H^\circ(x_n)]^{\mu_2+1} |\nabla u(x_n)| \rightarrow 0 \quad \text{for } |x_n| \rightarrow \infty. \tag{4.51}$$

For $0 < a < A$ fixed, let us consider

$$w_n(x) := R_n^{\mu_2} u(R_n x).$$

For n sufficiently large, from Theorem 1.1, relabeling the constants, we have

$$\frac{c}{A^{\mu_2}} \leq w_n(x) \leq \frac{C}{a^{\mu_2}} \quad \text{in } \overline{\mathcal{B}_A^{H^\circ} \setminus \mathcal{B}_a^{H^\circ}}.$$

Furthermore, recalling the estimate from above of the gradients of the weak solution u , proved previously, we get

$$|\nabla w_n(x)| \leq \frac{\tilde{C}}{a^{\mu_2+1}} \quad \text{in } \overline{\mathcal{B}_A^{H^\circ} \setminus \mathcal{B}_a^{H^\circ}}. \tag{4.52}$$

For a, A fixed, by [2, 17, 18], w_n is also uniformly bounded in $C^{1,\alpha}(K)$, for $0 < \alpha < 1$ and for any compact set $K \subset \mathcal{B}_A^{H^\circ} \setminus \mathcal{B}_a^{H^\circ}$. Moreover

$$w_n(x) \rightarrow w_{a,A} \quad \text{in } B_A \setminus B_a,$$

in the norm $C^{1,\alpha'}$, for $0 < \alpha' < \alpha$. Moreover, since w_n weakly solves

$$-\Delta_p^H w_n - \frac{\gamma}{H^\circ(x)^p} w_n^{p-1} = R_n^{\mu_2(p-1)+p-\mu_2(p^*-1)} w_n^{p^*-1} \quad \text{in } \mathcal{B}_A^{H^\circ} \setminus \mathcal{B}_a^{H^\circ}, \tag{4.53}$$

we deduce that

$$-\Delta_p^H w_{a,A} - \frac{\gamma}{H^\circ(x)^p} w_{a,A}^{p-1} = 0 \quad \text{in } \mathcal{B}_A^{H^\circ} \setminus \mathcal{B}_a^{H^\circ}. \tag{4.54}$$

Now we take $a_j = 1/j$ and $A_j = j$, for $j \in \mathbb{N}$ and we construct w_{a_j, A_j} as above. For j goes to infinity, using a standard diagonal process, we construct a limiting profile w_∞ so that

$$-\Delta_p^H w_\infty - \frac{\gamma}{H^\circ(x)^p} w_\infty^{p-1} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{4.55}$$

with $w_\infty \equiv w_{a_j, A_j}$ in $\mathcal{B}_{A_j}^{H^\circ} \setminus \mathcal{B}_{a_j}^{H^\circ}$.

Since w_∞ satisfies the assumptions (4.12), (4.13) of the Theorem 4.1, we get

$$w_\infty(x) = \frac{\bar{c}_2}{[H^\circ(x)]^{\mu_2}}, \tag{4.56}$$

where we set $\bar{c}_2 := \limsup_{|x| \rightarrow 0} [H^\circ(x)]^{\mu_2} w_\infty(x)$.

Now we set $y_n = x_n/R_n$, and by (4.51), we deduce that $|\nabla w_n(y_n)|$ tends to zero as R_n tends to infinity. This fact and the uniform convergence of the gradients imply that there exist $\bar{y} \in \partial B_1^{H^\circ}$ such that

$$|\nabla w_\infty(\bar{y})| = 0.$$

This is an absurd since the solution w_∞ has no critical points. □

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Data Availability All data generated or analyzed during this study are included in this published article.

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Appendix A

In this section we prove the existence of a weak solution to problem (\mathcal{P}_H) by means of a minimization problem. For this reason we define the following minimization problem: Let $S(\gamma)$ defined as

$$S(\gamma) := \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}} \frac{\mathcal{L}(u)}{\left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{\frac{p}{p^*}}}, \tag{A.1}$$

where

$$\mathcal{L}(u) = \int_{\mathbb{R}^N} H^p(\nabla u) - \gamma \frac{|u|^p}{H^\circ(x)^p} dx. \tag{A.2}$$

Thanks to the Hardy inequality (2.24) we deduce that $S(\gamma) \geq 0$.

Theorem 5.1 *Let $0 \leq \gamma < C_H$. The problem (\mathcal{P}_H) has a positive weak solution that it minimizes the quotient (A.1).*

Proof Let $\{u_n\}$ be a minimizing sequence to (A.1). Without loss of generality, because of the homogeneity of the quotient in (A.1), we assume that

$$\mathcal{L}(u_n) = 1. \tag{A.3}$$

By Hardy inequality (observe that H is a norm equivalent to the euclidean one), $\mathcal{L}(\cdot)^{1/p}$ is an equivalent norm to the standard one $\|\cdot\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}$, we have that

$$\|u_n\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \leq C,$$

with C that does not depend on n . Hence up to a subsequence $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Moreover let us also assume that (we will prove it later)

$$u_0 \neq 0. \tag{A.4}$$

Recalling (A.1), using the weak lower semicontinuity of the norm and (A.3), we deduce

$$\begin{aligned} S(\gamma) &\leq \frac{\mathcal{L}(u_0)}{\left(\int_{\mathbb{R}^N} u_0^{p^*} dx\right)^{\frac{p}{p^*}}} \leq \liminf_n \frac{\mathcal{L}(u_n)}{\left(\int_{\mathbb{R}^N} u_n^{p^*} dx\right)^{\frac{p}{p^*}}} \\ &= \frac{1}{\left(\int_{\mathbb{R}^N} u_0^{p^*} dx\right)^{\frac{p}{p^*}}} \end{aligned} \tag{A.5}$$

Using Sobolev inequality we have that $u_n \rightarrow u_0$ a.e. in \mathbb{R}^N . Therefore by the Breis–Lieb result [4], it follows that

$$\int_{\mathbb{R}^N} u_0^{p^*} dx = \int_{\mathbb{R}^N} u_n^{p^*} dx - \int_{\mathbb{R}^N} (u_n - u_0)^{p^*} dx + o(1).$$

Then from (A.5) we obtain

$$S(\gamma) \leq \frac{1}{\left(\int_{\mathbb{R}^N} u_n^{p^*} dx - \int_{\mathbb{R}^N} (u_n - u_0)^{p^*} dx + o(1)\right)^{\frac{p}{p^*}}} \tag{A.6}$$

From (A.1), we deduce

$$\mathcal{L}(u_n) = S(\gamma) \left(\int_{\mathbb{R}^N} u_n^{p^*} dx\right)^{\frac{p}{p^*}} + o(1)$$

and

$$\mathcal{L}(u_0 - u_n) \geq S(\gamma) \left(\int_{\mathbb{R}^N} |u_0 - u_n|^{p^*} dx\right)^{\frac{p}{p^*}}.$$

Using these last inequalities in (A.6) we get

$$\begin{aligned} S(\gamma) &\leq S(\gamma) \frac{1}{\left(\mathcal{L}^{\frac{p^*}{p}}(u_n) - \mathcal{L}^{\frac{p^*}{p}}(u_0 - u_n) + o(1)\right)^{\frac{p}{p^*}}} \\ &\leq S(\gamma) \frac{1}{\left(1 - \mathcal{L}^{\frac{p^*}{p}}(u_0 - u_n) + o(1)\right)^{\frac{p}{p^*}}}, \end{aligned} \tag{A.7}$$

where in the last line we used (A.3). Taking the limit superior of (A.7) we get a contradiction, i.e. $S(\gamma) < S(\gamma)$ unless the sequence $\mathcal{L}(u_0 - u_n) \rightarrow 0$. Hence, since $\mathcal{L}(\cdot)^{1/p}$ is an equivalent norm to $\|\cdot\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}$, we finally get that $u_n \rightarrow u_0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Therefore passing to the limit in (A.1), we obtain

$$S(\gamma) = \frac{\mathcal{L}(u_0)}{\left(\int_{\mathbb{R}^N} |u_0|^{p^*} dx\right)^{\frac{p}{p^*}}},$$

namely u_0 (eventually redefining it as Cu_0 , for some positive constant) is a weak solution to (\mathcal{P}_H) .

Now we prove that actually (A.4) holds, concluding indeed the proof. Let $\{u_n\}$ the minimizing sequence such that (A.3) holds. For every n let us take a sequence of radii R_n such that

$$\int_{B_{R_n}} H^p(\nabla u_n) - \gamma \frac{|u_n|^p}{H^\circ(x)^p} dx = \int_{\mathbb{R}^N \setminus B_{R_n}} H^p(\nabla u_n) - \gamma \frac{|u_n|^p}{H^\circ(x)^p} dx = \frac{\mathcal{L}(u_n)}{2} \tag{A.8}$$

and let us define the rescaled sequence

$$w_n = R_n^{\frac{p-N}{p}} u_n \left(\frac{x}{R_n} \right). \tag{A.9}$$

Using assumptions (h_H) , (A.8) and (A.9) we deduce that

$$\int_{B_1} H^p(\nabla w_n) - \gamma \frac{|w_n|^p}{H^\circ(x)^p} dx = \int_{\mathbb{R}^N \setminus B_1} H^p(\nabla w_n) - \gamma \frac{|w_n|^p}{H^\circ(x)^p} dx = \frac{\mathcal{L}(u_n)}{2} \tag{A.10}$$

Now we prove the following

Lemma 5.2 *Let $\{u_n\}$ be a minimizing sequence, weakly converging to zero. Then, for every ball B_r and for every $\varepsilon \in (-r, r)$ there exists $\rho \in (0, \varepsilon) \cup (\varepsilon, 0)$ such that for a subsequence*

$$\text{either } \int_{B_{r+\rho}} H^p(\nabla u_n) dx \rightarrow 0, \quad \text{or } \int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n) dx \rightarrow 0. \tag{A.11}$$

Proof By the homogeneity of the quotient in (A.1) we can assume that the minimizing sequence $\{u_n\}$ is such that $\|u_n\|_{L^{p^*}(\mathbb{R}^N)} = 1$, so that $\mathcal{L}(u_n) \rightarrow S(\gamma)$. By Ekeland’s ε -principle, we can suppose that the minimizing sequence has the Palais-Smale property, that is

$$\begin{aligned} & \int_{\mathbb{R}^N} H^{p-1}(\nabla u_n) \langle \nabla H(\nabla u_n), \nabla \varphi \rangle - \frac{\gamma}{H^\circ(x)^p} u_n^{p-1} \varphi dx \\ & = S(\gamma) \int_{\mathbb{R}^N} u_n^{p^*-1} \varphi dx + o(1) \|\varphi\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}, \end{aligned} \tag{A.12}$$

for all $\varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ We have that

$$\int_r^{r+\varepsilon} d\rho \int_{\rho S^{N-1}} H^p(\nabla u_n) = \int_{B_{r+\varepsilon} \setminus B_r} H^p(\nabla u_n)$$

is bounded. Then we can find $\rho \in (0, \varepsilon)$ such that for infinitely many n ’s it holds

$$\int_{(r+\rho)S^{N-1}} H^p(\nabla u_n) \leq C \int_{B_{r+\varepsilon} \setminus B_r} H^p(\nabla u_n),$$

for some positive constant C and hence up to redefining the constant

$$\int_{(r+\rho)S^{N-1}} |\nabla u_n|^p \leq C \int_{B_{r+\varepsilon} \setminus B_r} H^p(\nabla u_n).$$

Therefore (see [25, Theorem A.8]) since

$$W^{1,p}((r+\rho)S^{N-1}) \hookrightarrow W^{1-\frac{1}{p},p}((r+\rho)S^{N-1}) \hookrightarrow L^p((r+\rho)S^{N-1}) \tag{A.13}$$

with both embedding compact, we can assume that a subsequence converges strongly to some limit, say u in the trace space $W^{1-\frac{1}{p},p}((r + \rho)S^{N-1})$. Using the fact that the trace operator has a continuous embedding from $W^{1,p}(B_{r+\rho})$ into $W^{1-\frac{1}{p},p}((r + \rho)S^{N-1})$, by the weak convergence to zero of $\{u_n\}$, we deduce that indeed $u \equiv 0$.

Now we show the following

CLAIM: Let $\Omega \subset \mathbb{R}^N$ a generic smooth bounded domain. The inverse operator

$$(-\Delta_p^H)^{-1} : W^{1-\frac{1}{p},p}(\partial\Omega) \rightarrow W^{1,p}(\Omega),$$

is continuous. Indeed we consider a succession $g_n \rightarrow g$ in $W^{1-\frac{1}{p},p}(\Omega)$, and let $u_n, u \in W^{1,p}(\Omega)$ be the solutions to

$$\begin{cases} -\Delta_p^H u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} -\Delta_p^H u_n = 0 & \text{in } \Omega \\ u_n = g_n & \text{on } \partial\Omega. \end{cases} \tag{A.14}$$

The solution to (A.14) can be obtained minimizing the functional

$$J(u) = \frac{1}{p} \int_{\Omega} H^p(\nabla u) dx$$

on the set $\{g\} + W_0^{1,p}(\Omega)$, $\{g_n\} + W_0^{1,p}(\Omega)$ respectively. Since $(u - g), (u_n - g_n) \in W_0^{1,p}(\Omega)$, integrating by parts (A.14) and subtracting the equations, we obtain

$$\begin{aligned} 0 &= \int_{\Omega} H^{p-1}(\nabla u) \langle \nabla H(\nabla u), \nabla(u - g) \rangle dx \\ &\quad - \int_{\Omega} H^{p-1}(\nabla u_n) \langle \nabla H(\nabla u_n), \nabla(u_n - g_n) \rangle dx \\ &= \int_{\Omega} \langle H^{p-1}(\nabla u) \nabla H(\nabla u) - H^{p-1}(\nabla u_n) \nabla H(\nabla u_n), \nabla u - \nabla u_n \rangle dx \\ &\quad - \int_{\Omega} \langle H^{p-1}(\nabla u) \nabla H(\nabla u) - H^{p-1}(\nabla u_n) \nabla H(\nabla u_n), \nabla g - \nabla g_n \rangle dx. \end{aligned}$$

We recall (see [6, Lemma 4.1]) that for $x \in \mathbb{R}^N \setminus \{0\}$, $y \in \mathbb{R}^N$ there exist a constant $C > 0$ such that

$$(H^{p-1}(x) \nabla H(x) - H^{p-1}(y) \nabla H(y), x - y) \geq C(|x| + |y|)^{p-2} |x - y|^2, \tag{A.15}$$

Therefore, by (A.15) and (2.9) we get

$$\begin{aligned} &C \int_{\Omega} (|\nabla u| + |\nabla u_n|)^{p-2} (|\nabla u - \nabla u_n|)^2 dx \\ &\leq \tilde{C}_p \int_{\Omega} (|\nabla u| + |\nabla u_n|)^{p-2} |\nabla u - \nabla u_n| |\nabla g - \nabla g_n| dx \\ &\leq \tilde{C}_p \left(\int_{\Omega} (|\nabla u| + |\nabla u_n|)^p dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |\nabla g - \nabla g_n|^p dx \right)^{\frac{1}{p}}, \end{aligned} \tag{A.16}$$

where in the last inequality we have used the Hölder inequality.

We recall that by $W^{1-\frac{1}{p},p}(\partial\Omega)$ we denote the space of traces $u|_{\partial\Omega}$, namely the set (of equivalence classes) $\{u\} + W_0^{1,p}(\Omega)$, $u \in W^{1,p}(\Omega)$, endowed with the trace norm

$$\|u|_{\partial\Omega}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = \inf\{\|v\|_{W^{1,p}(\Omega)} : u - v \in W_0^{1,p}(\Omega)\}. \tag{A.17}$$

Hence using (A.16) letting the boundary data $g_n \rightarrow g$ in the sense of (A.17) we obtain the claim.

Let us define the two auxiliary sequences of $\mathcal{D}^{1,p}(\mathbb{R}^N)$ as follows:

$$u_{1,n}(x) = \begin{cases} u_n(x) & \text{if } x \in B_{r+\rho} \\ w_{1,n}(x) & \text{if } x \in B_{r+\varepsilon} \setminus B_{r+\rho} \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$u_{2,n}(x) = \begin{cases} 0 & \text{if } x \in B_{r-\varepsilon} \\ w_{2,n}(x) & \text{if } x \in B_{r+\rho} \setminus B_{r-\varepsilon} \\ u_n(x) & \text{elsewhere,} \end{cases} \tag{A.18}$$

where $w_{1,n}$ respectively $w_{2,n}$ denote the solutions to

$$\begin{cases} -\Delta_p^H w_{1,n} = 0 & \text{in } B_{r+\varepsilon} \setminus B_{r+\rho} \\ w_{1,n} = 0 & \text{on } \partial B_{r+\varepsilon} \\ w_{1,n} = u_n & \text{on } \partial B_{r+\rho} \end{cases} \tag{A.19}$$

respectively

$$\begin{cases} -\Delta_p^H w_{2,n} = 0 & \text{in } B_{r+\rho} \setminus B_{r-\varepsilon} \\ w_{2,n} = u_n & \text{on } \partial B_{r+\rho} \\ w_{2,n} = 0 & \text{on } \partial B_{r-\varepsilon}. \end{cases} \tag{A.20}$$

Since $u_n \rightarrow 0$ on $\partial B_{r+\rho}$ in the $W^{1-\frac{1}{p},p}$ norm, see (A.13), by the above claim we immediately get that both

$$w_{1,n} \xrightarrow{W^{1,p}} 0 \text{ in } B_{r+\varepsilon} \setminus B_{r+\rho} \quad \text{and} \quad w_{2,n} \xrightarrow{W^{1,p}} 0 \text{ in } B_{r+\rho} \setminus B_{r-\varepsilon}. \tag{A.21}$$

Using $u_{1,n}$ as test function in (A.12) we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} H^{p-1}(\nabla u_n) \langle \nabla H(\nabla u_n), \nabla u_{1,n} \rangle - \frac{\gamma}{H^\circ(x)^p} u_n^{p-1} u_{1,n} \, dx \\ & = S(\gamma) \int_{\mathbb{R}^N} u_n^{p^*-1} u_{1,n} \, dx + o(1) \|u_{1,n}\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \end{aligned}$$

and recalling the definition of $u_{1,n}$ and by (A.21), we obtain

$$\int_{B_{r+\rho}} H^p(\nabla u_n) - \frac{\gamma}{H^\circ(x)^p} u_n^p \, dx = S(\gamma) \int_{B_{r+\rho}} u_n^{p^*} \, dx + o(1) = S(\gamma) \int_{B_{r+\rho}} u_{1,n}^{p^*} \, dx + o(1)$$

In the same way, using $u_{2,n}$ as test function in (A.12) we obtain

$$\int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n) - \frac{\gamma}{H^\circ(x)^p} u_n^p \, dx = S(\gamma) \int_{\mathbb{R}^N \setminus B_{r+\rho}} u_n^{p^*} \, dx + o(1)$$

$$= S(\gamma) \int_{\mathbb{R}^N \setminus B_{r+\rho}} u_{2,n}^{p^*} dx + o(1).$$

Moreover, by definition (A.2), using the two sequences $\{u_{1,n}\}$ and $\{u_{2,n}\}$ we infer that actually

$$\mathcal{L}(u_{1,n}) = \int_{B_{r+\rho}} H^p(\nabla u_n) - \frac{\gamma}{H^\circ(x)^p} u_n^p dx + o(1)$$

and

$$\mathcal{L}(u_{2,n}) = \int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n) - \frac{\gamma}{H^\circ(x)^p} u_n^p dx + o(1)$$

so that

$$\mathcal{L}(u_n) = \mathcal{L}(u_{1,n}) + \mathcal{L}(u_{2,n}) + o(1) \tag{A.22}$$

and $\|u_n\|_{p^*}^{p^*} = \|u_{1,n}\|_{p^*}^{p^*} + \|u_{2,n}\|_{p^*}^{p^*} + o(1)$. Let us assume for example, that $\{u_{1,n}\}$ does not converges to zero. Since $\{u_n\}$ is a minimizing sequence we have that (see (A.1))

$$\mathcal{L}(u_n) = S(\gamma) \|u_n\|_{p^*}^p + o(1)$$

and also that $\mathcal{L}(u_{2,n}) \geq S(\gamma) \|u_{2,n}\|_{p^*}^p$ and

$$\begin{aligned} \frac{\mathcal{L}(u_{1,n})}{\|u_{1,n}\|_{p^*}^p} &= \frac{\mathcal{L}(u_n) - \mathcal{L}(u_{2,n}) + o(1)}{(\|u_n\|_{p^*}^{p^*} - \|u_{2,n}\|_{p^*}^{p^*} + o(1))^{\frac{p}{p^*}}} \\ &\leq S(\gamma) \frac{\mathcal{L}(u_n) - \mathcal{L}(u_{2,n}) + o(1)}{(\mathcal{L}(u_n)^{\frac{p}{p^*}} - \mathcal{L}(u_{2,n})^{\frac{p}{p^*}} + o(1))^{\frac{p}{p^*}}}. \end{aligned}$$

By (A.22) we deduce that

$$\limsup_n \mathcal{L}(u_{2,n}) = \limsup_n (\mathcal{L}(u_n) - \mathcal{L}(u_{1,n}) + o(1)) < \limsup_n \mathcal{L}(u_n),$$

by some computations we deduce that actually

$$\limsup_n \frac{\mathcal{L}(u_{1,n})}{\|u_{1,n}\|_{p^*}^p} < S(\gamma),$$

a contradiction with (A.1) unless $\mathcal{L}(u_{2,n})$ tends to zero. Using Hardy inequality in (A.2), recalling (A.18), we obtain

$$\begin{aligned} 0 < \int_{\mathbb{R}^N} H^p(\nabla u_{2,n}) dx &= \int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n) dx + \int_{B_{r+\rho} \setminus B_{r-\varepsilon}} H^p(\nabla w_{2,n}) dx \\ &= \int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n) dx + o(1). \end{aligned}$$

The other case of (A.11) can be proved arguing in the same as we have done above, assuming that $\{u_{2,n}\}$ does not converge to zero. This concludes the proof of the lemma. \square

Using the invariance of the problem under the scaling $R_n^{(p-N)/p} u(x/R_n)$, the sequence $\{w_n\}$ (see (A.9) and (A.10)) is still a minimizing sequence bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$: hence it admits (up to a subsequence) a weakly convergence sequence. We want to show that the weak limit cannot be zero. We argue by contradiction and we apply Lemma 5.2 twice choosing $r = 1$

and $\varepsilon = \pm 1/4$ respectively. We find the existence of $\rho^+ \in (0, 1/4)$ and $\rho^- \in (-1/4, 0)$ such that (A.11) holds. Using the alternative (A.11) together with (A.10), we obtain that

$$\int_{B_{1+\rho^-}} H^p(\nabla w_n) \, dx \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_{1+\rho^+}} H^p(\nabla w_n) \, dx \rightarrow 0. \tag{A.23}$$

Since $w_n \rightarrow 0$ using the strong convergence on compacts K of $w_n \rightarrow 0$ in $L^p(K)$ we deduce that

$$\int_{B(0)_{\frac{3}{2}} \setminus B(0)_{\frac{1}{2}}} \frac{|w_n|^p}{H^\circ(x)^p} \, dx \rightarrow 0. \tag{A.24}$$

Let us take a smooth cut-off function η , with $0 \leq \eta \leq 1$, such that $\eta \equiv 1$ in $B(0)_{5/4} \setminus B(0)_{3/4}$ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus (B(0)_{3/2} \setminus B(0)_{1/2})$. Take in to account Hardy inequality, (A.23) and (A.24) we deduce that

$$\int_{\mathbb{R}^N} |H^p(\eta \nabla w_n) - H^p(\nabla w_n)| \, dx \rightarrow 0,$$

for $n \rightarrow +\infty$ and therefore we infer that

$$\int_{\mathbb{R}^N} H^p(\eta \nabla w_n) \, dx = \int_{\mathbb{R}^N} H^p(\nabla w_n) \, dx + o(1).$$

Then, recalling also (A.1) we get

$$S(0) \leq \frac{\int_{\mathbb{R}^N} H^p(\eta \nabla w_n) \, dx}{\left(\int_{\mathbb{R}^N} |\eta w_n|^{p^*} \, dx\right)^{\frac{p}{p^*}}} \leq \frac{\mathcal{L}(w_n) + o(1)}{\left(\int_{\mathbb{R}^N} |w_n|^{p^*} \, dx\right)^{\frac{p}{p^*}} + o(1)}.$$

Passing to the limit we obtain

$$S(0) \leq S(\gamma). \tag{A.25}$$

We claim that (A.25) is not possible. Indeed let u_0 be a minimizer of (A.1) for $\gamma = 0$ ([9]). Therefore we clearly deduce the following

$$S(\gamma) \leq \frac{\int_{\mathbb{R}^N} H^p(\nabla u_0) - \gamma \frac{|u_0|^p}{H^\circ(x)^p} \, dx}{\left(\int_{\mathbb{R}^N} |u_0|^{p^*} \, dx\right)^{\frac{p}{p^*}}} < \frac{\int_{\mathbb{R}^N} H^p(\nabla u_0) \, dx}{\left(\int_{\mathbb{R}^N} |u_0|^{p^*} \, dx\right)^{\frac{p}{p^*}}} = S(0). \tag{A.26}$$

Inequality (A.26) gives the desired contradiction, concluding the proof. □

References

1. Alexandrov, A.D.: A characteristic property of the spheres. *Ann. Mat. Pura Appl.* (4) **58**, 303–315 (1962)
2. Antonini, C.A., Ciraolo, G., Farina, A.: Interior regularity results for inhomogeneous anisotropic quasilinear equations. *Math. Ann.* (2022). <https://doi.org/10.1007/s00208-022-02500-x>
3. Bellettini, G., Paolini, M.: Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J.* **25**(3), 537–566 (1996)
4. Brezis, H., Lieb, E.: A relation between pointwise convergence of functions and convergence of functionals. *Proc. Am. Math. Soc.* **88**(3), 486–490 (1983)
5. Caffarelli, L.A., Gidas, B., Spruck, J.: Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth. *Commun. Pure Appl. Math.* **42**(3), 271–297 (1989)

6. Castorina, D., Riey, G., Sciunzi, B.: Hopf Lemma and regularity results for quasilinear anisotropic elliptic equations. *Calc. Var. Partial Differ. Equ.* **58**(3), 95, 18 (2019)
7. Catino, G., Monticelli, D.D., Roncoroni, A.: On the critical p -Laplace equation. Preprint [arXiv:2204.06940](https://arxiv.org/abs/2204.06940)
8. Cianchi, A., Salani, P.: Overdetermined anisotropic elliptic problems. *Math. Ann.* **345**(4), 859–881 (2009)
9. Ciraolo, G., Figalli, A., Roncoroni, A.: Symmetry results for critical anisotropic p -Laplacian equations in convex cones. *Geom. Funct. Anal.* **30**(3), 770–803 (2020)
10. Cozzi, M., Farina, A., Valdinoci, E.: Monotonicity formulae and classification results for singular, degenerate, anisotropic PDEs. *Adv. Math.* **293**, 343–381 (2016)
11. Damascelli, L.: Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15**(4), 493–516 (1998)
12. Damascelli, L., Merchán, S., Montoro, L., Sciunzi, B.: Radial symmetry and applications for a problem involving the operator $-\Delta_p(\cdot)$ and critical nonlinearity in \mathbb{R}^N . *Adv. Math.* **265**, 313–335 (2014)
13. Di Benedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.* **7**(8), 827–850 (1983)
14. Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry of positive solutions of nonlinear elliptic equations in \mathbb{R}^N . *Mathematical analysis and applications, Part A*, pp. 369–402, *Adv. in Math. Suppl. Stud.*, 7a. Academic Press, New York (1981)
15. Gidas, B., Spruck, J.: Global and local behavior of positive solutions of nonlinear elliptic equations. *Commun. Pure Appl. Math.* **34**(4), 525–598 (1981)
16. Han, Q., Lin, F.: *Elliptic partial differential equations*. Courant Lect. Notes Math., 1 New York University, Courant Institute of Mathematical Sciences, New York American Mathematical Society, Providence, RI, xx+144 pp (1997)
17. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.* **12**(11), 1203–1219 (1988)
18. Lieberman, G.M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. *Commun. Partial Differ. Equ.* **16**(2–3), 311–361 (1991)
19. Montoro, L., Sciunzi, B.: Pohozaev identity for Finsler anisotropic problems. *NoDEA Nonlinear Differ. Equ. Appl.* **30**(3), 33, 15 (2023)
20. Oliva, F., Sciunzi, B., Vaira, G.: Radial symmetry for a quasilinear elliptic equation with a critical Sobolev growth and Hardy potential. *J. Math. Pures Appl.* (9) **140**, 89–109 (2020)
21. Ou, Q.: On the classification of entire solutions to the critical p -Laplace equation. Preprint [arXiv:2210.05141](https://arxiv.org/abs/2210.05141)
22. Serrin, J.: Local behaviour of solutions of quasilinear equations. *Acta Math.* **111**, 247–302 (1964)
23. Serrin, J.: A symmetry problem in potential theory. *Arch. Ration. Mech. Anal.* **43**, 304–318 (1971)
24. Sciunzi, B.: Classification of positive $D^{1,p}(\mathbb{R}^N)$ -solutions to the critical p -Laplace equation in \mathbb{R}^N . *Adv. Math.* **291**, 12–23 (2016)
25. Struwe, M.: *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*. Fourth edition, *Ergeb. Math. Grenzgeb.* (3), 34 [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics] Springer, Berlin, xx+302 pp (2008)
26. Terracini, S.: On positive entire solutions to a class of equations with a singular coefficient and critical exponent. *Adv. Differ. Equ.* **1**(2), 241–264 (1996)
27. Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. *J. Differ. Equ.* **51**(1), 126–150 (1984)
28. Van Schaftingen, J.: Anisotropic symmetrization. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23**(4), 539–565 (2006)
29. Vétois, J.: A priori estimates and application to the symmetry of solutions for critical p -Laplace equations. *J. Differ. Equ.* **260**(1), 149–161 (2016)
30. Vétois, J.: A note on the classification of positive solutions to the critical p -Laplace equation in \mathbb{R}^n . Preprint [arXiv:2304.02600](https://arxiv.org/abs/2304.02600)
31. Xiang, C.L.: Asymptotic behaviors of solutions to quasilinear elliptic equations with critical Sobolev growth and Hardy potential. *J. Differ. Equ.* **259**(8), 3929–3954 (2015)
32. Xiang, C.L.: Gradient estimates for solutions to quasilinear elliptic equations with critical Sobolev growth and Hardy potential. *Acta Math. Sci. Ser. B* **37**(1), 58–68 (2017)