Calculus of Variations

Energy scaling laws for microstructures: from helimagnets to martensites

Janusz Ginster[1] · **Barbara Zwicknagl**[1]

Abstract

We consider scalar-valued variational models for pattern formation in helimagnetic compounds and in shape memory alloys. Precisely, we consider a non-convex multi-well bulk energy term on the unit square, which favors four gradients $(\pm \alpha, \pm \beta)$, regularized by a singular perturbation in terms of the total variation of the second derivative. We derive scaling laws for the minimal energy in the case of incompatible affine boundary conditions in terms of the singular perturbation parameter as well as the ratio α/β and the incompatibility of the boundary condition. We discuss how well-studied models for martensitic microstructures in shape-memory alloys arise as a limiting case, and relations between the different models in terms of scaling laws. In particular, we show that scaling regimes arise in which an interpolation between the rather different branching-type constructions in the spirit of Kohn and Müller (Commun Pure Appl Math 47(4):405–435, 1994) and Ginster and Zwicknagl (J Nonlinear Sci 33:20, 2023), respectively, occurs. A particular technical difficulty in the lower bounds arises from the fact that the energy scalings involve various logarithmic terms that we capture in matching upper and lower scaling bounds.

Mathematics Subject Classification 49J40 · 74N15 · 74G65 · 82B21 · 82B24

1 Introduction

We explore relations between certain scalar-valued variational models for microstructures in shape-memory alloys and in helimagnetic compounds, respectively. In these models, the formation of microstructures is induced by a lack of convexity. More precisely, the energy functionals favor functions whose gradients lie in discrete sets consisting of two (for martensites) or four (for helimagnets) vectors, respectively. To introduce a length scale to the problem, these non-convex terms are typically complemented by a higher-order regular-

✉ Janusz Ginster
ginsterj@hu-berlin.de

Barbara Zwicknagl
barbara.zwicknagl@hu-berlin.de

[1] Insitut für Mathematik, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

🙂 Springer

ization term which is interpreted as a surface energy term and in particular penalizes changes between regions of preferred gradients (see e.g. [1] and the discussion in [2]). There are only very few special cases in which minimizers for the resulting variational problems are known (see [3] and the references given there), or qualitative properties of them are proven (see [4, 5] and the references given there). Therefore, starting with the work by Kohn and Müller on martensitic microstructures (see [6, 7] and also below for a more detailed description), it has often proven useful to understand the scaling behaviour of the minimal energy in terms of the problem parameters in order to explain pattern formation in materials. More precisely, such results often indicate that in certain parameter regimes, optimal configurations are rather uniform while in other regimes, complex branching patterns are predicted. Similar results have been obtained for a variety of models for very different phenomena, including among many others, various models for the classes of materials considered here, that is, martensitic microstructures (see e.g. [6–18] and the references therein) and microstructures in micromagnetics (see e.g. [19–27] and the references therein).

1.1 The model and related results

Throughout the text, we consider a generic square domain $(0, 1)^2$. For recent related results addressing the domain dependence we refer to [28]. Pattern formation in helimagnets is often described in terms of (discrete) frustrated spin systems (see e.g. [29]). We continue here a study of two-dimensional $J_1 - J_3$-type models on a square lattices. It has been found (see [30–32]) that zooming into the helimagnetic/ferromagnetic transition point and simultaneously taking the continuum limit as the lattice spacing vanishes, such discrete models from statistical mechanics (if properly rescaled) converge in the sense of $\Gamma-$convergence to singularly perturbed continuum functionals. More precisely, the latter are closely related to functionals of the form

$$J(u) := \int_{\Omega} \text{dist}^2(\nabla u, \mathcal{P}_{\alpha,\beta}) \, dx \, dy + \sigma \mid D^2 u \mid (\Omega), \tag{1}$$

see [30–32] for precise results and [27] for a discussion of the simplifications we make here. In (1), $\Omega = (0, 1)^2 \subset \mathbb{R}^2$ denotes the domain occupied by the magnetic body, and the set of preferred gradients $\mathcal{P}_{\alpha,\beta} \subset \mathbb{R}^2$ contains four vectors. The latter can be seen as order parameters and correspond to chiralities of the discrete spin fields. More precisely, it has been found that in the parameter regime we consider here, optimal spin field configurations correspond to helical spins field, where the spin vectors rotate between nearest neighbors by a fixed angle (clockwise or counterclockwise) horizontally and by a fixed angle vertically. The preferred gradients $\mathcal{P}_{\alpha,\beta} = \{(\pm\alpha, \pm\beta)\}$ with $\alpha, \beta > 0$ correspond to appropriately rescaled versions of these optimal rotation angles, horizontally (α) and vertically (β). The actual angles are determined by the parameters of the $J_1 - J_3-$model (see [30, 31]). We note that in [27, 31] the case $\alpha = \beta = 1$ has been considered. The more general case considered here corresponds to the additional freedom in the discrete spin systems that the ratio between nearest neighbor and next to nearest neighbor interactions may be different (details will be discussed in [32]).

We consider the case of incompatible affine Dirichlet boundary conditions, i.e.,

$$u(0, y) = (1 - 2\theta)\beta y \tag{2}$$

with a parameter $\theta \in (0, 1/2)$ that measures the incompatibility between the rigid field on the boundary and the preferred gradients inside the sample: If $\theta = 0$, the boundary condition

is compatible with the preferred gradients in Ω and the minimal energy vanishes, but the larger $\theta \in (0, 1/2)$ gets, the more incompatible are the boundary conditions, and we expect a larger minimal energy.

Let us briefly explain the relation to variational problems from the literature: The case $\alpha = \beta = 1$ has been studied in [27]. The main result there is a scaling law for the minimal energy which takes the form

$$\min J(u) \sim \min \left\{ \theta^2, \ \sigma \left(\frac{|\log \sigma|}{|\log \theta|} + 1 \right) \right\}. \tag{3}$$

Here and in the following, we use the notation $G \sim H$ for functions G, H depending on the problem parameters, to indicate that there are constants $c, C > 0$ such that for all admissible choices of the problem parameters there holds $cH \leq G \leq CH$. The first scaling in (3) is achieved by an affine function, while the second scaling corresponds to a branching-type construction, which (up to an interpolation layer close to the interface $\{x = 0\}$) uses only the preferred gradients in $\mathcal{P}_{\alpha,\beta}$.

On the other hand, if we set $\alpha = 0$ and $\beta = 1$, then the functional given in (1) reduces to a variant of the Kohn-Müller model for martensitic microstructures. It is well-known that the scaling law for the minimal energy in this case reads (see e.g. [33, 34])

$$\min J(u) \sim \min \left\{ \theta^2, \ \sigma^{2/3} \theta^{2/3} \right\}. \tag{4}$$

Again, the first regime corresponds to an affine function, while minimizers in the second regime are expected to show almost self-similar behaviour (see [4, 5] for rigorous results for a simplified model). However, in contrast to the situation above, (almost) minimizers in this regime only satisfy (up to an interpolation layer) $\partial_2 u \in \{\pm\beta\}$ but not $\partial_1 u \in \{\pm\alpha\}$.

1.2 Main results

We shall prove scaling laws for the minimal energies (1) under the boundary conditions (2). As discussed above, this can be seen as generalizations of results on Kohn-Müller-type models (see e.g. [2, 7, 33, 34]) and of the analysis in [27]. In particular, we show how this "transition" from two-gradient models to four-gradient models takes place in terms of "interpolating" scaling regimes of the minimal energy. A particular technical difficulty in the analysis lies in the fact that the scaling law contains various logarithmic terms that we capture precisely in the upper and lower bounds.

We restrict ourselves to a generic domain $(0, 1)^2$ to keep notation simple, but we expect that more general rectangles can be treated along the lines of [27, Section 4.4]. We treat the cases $\alpha \leq \beta$ and $\alpha \geq \beta$ separately in Sects. 2 and 3, respectively. As the absolute values α and β can be adjusted by the rescaling chosen for the spin model (see [31, 32]), we set the larger of the two values to one and call the smaller one $\gamma \in (0, 1]$, see Remarks 2 und 3 for details. Our main results concern the complete characterization of the scaling laws of the minimal energies, which we will outline in the sequel. We treat the cases of small preferred y- und small preferred x-derivatives separately.

1.2.1 The case of small y-derivative

For $\sigma > 0$, $\theta \in (0, 1/2]$, and $\gamma \in (0, 1]$ we set

$$\mathcal{A}_{\theta,\gamma} := \left\{ u \in W^{1,2}((0,1)^2) : \ \nabla u \in BV((0,1)^2), \ u(0, y) = (1 - 2\theta)\gamma y \right\}.$$

We note that any function in $\mathcal{A}_{\theta,\gamma}$ has a continuous representative up to the boundary (see e.g. [35, Lemma 9]). In the following, we always refer to this representative without further mentioning. We consider the functional $E_{\sigma,\gamma,\theta} : \mathcal{A}_{\theta,\gamma} \to [0,\infty)$ defined by

$$E_{\sigma,\gamma,\theta}(u) := \int_{(0,1)^2} \operatorname{dist}^2(\nabla u, \mathcal{K}_\gamma)\, dx\, dy + \sigma \mid D^2 u \mid (\Omega)$$

where

$$\mathcal{K}_\gamma := \{(\pm 1, \pm\gamma)\}.$$

While for $\gamma \to 1$, we end up with the four-gradient problem studied in [27], we shall also exploit the above-mentioned relation to two-gradient problems for martensitic microstructure formation. Precisely, we make use of the following relation.

Remark 1 For $u \in \mathcal{A}_{\theta,\gamma}$ consider $v(x,y) := \frac{u(x,y)-x}{\gamma}$. Then

$$\begin{aligned}
&E_{\sigma,\gamma,\theta}(u) \\
&\leq \int_{(0,1)^2} |\partial_1 u - 1|^2 + \min\{|\partial_2 u - \gamma|, |\partial_2 u + \gamma|\}^2 \, dx\, dy + \sigma|D^2 u|((0,1)^2) \\
&= \gamma^2 \left(\int_{(0,1)^2} |\partial_1 v|^2 + \min\{|\partial_2 v - 1|, |\partial_2 v + 1|\}^2 \, dx\, dy + \sigma\gamma^{-1}|D^2 v|((0,1)^2) \right).
\end{aligned}$$

In this way, the upper bound for the Kohn-Müller-type functional (see (4)) immediately yield bounds for our problem which in some parameter regimes turn out to be sharp, in others not. This will be explored more specifically in the proof of Proposition 3.

It turns out that the scaling law for the minimal energy $E_{\sigma,\gamma,\theta}$ shows a transition between the scaling behaviour of the Kohn-Müller-type two-gradient energies and the ones derived in [27] for the four preferred gradients $(\pm 1, \pm 1)$. Precisely, compared to the setting of [27], besides uniform phases and branching-type structures involving all four preferred gradients, also a Kohn-Müller type branching construction is relevant for the scaling behaviour of the minimal energy. We note that in the Kohn-Müller model there are only two preferred gradients, and the construction uses (up to a boundary layer) only the preferred values for the $y-$derivative. However, in contrast to the branching-type construction using four gradients, the refinement is done in an anisotropic way so that the x-derivatives become very large (and therefore get far away from the preferred value) after only a few refinement steps. Our main result is the following scaling law of the minimal energy. Note that we prove matching upper and lower bounds.

Theorem 1 *There are constants* $c, C > 0$ *such that for all* $\theta \in (0, 1/2]$, *all* $\gamma \in (0, 1]$, *and all* $\sigma > 0$, *there holds*

$$c\, s(\gamma, \theta, \sigma) \leq \min_{\mathcal{A}_{\theta,\gamma}} E_{\sigma,\gamma,\theta} \leq C\, s(\gamma, \theta, \sigma),$$

where $s(\gamma, \theta, \sigma) = \min\left\{ \gamma^2\theta^2,\ \sigma^{2/3}\gamma^{4/3}\theta^{2/3},\ \sigma\left(\frac{|\log\sigma|}{|\log(\gamma^2\theta)|} + 1\right) \right\}$.

Proof The upper bound is proven in Proposition 3, the lower bound in Proposition 4. □

Remark 2 The case of arbitrary preferred gradients $(\pm\alpha, \pm\beta)$ with $0 < \beta \leq \alpha$ can be reduced to the setting considered here by rescaling. Precisely, for a function $u \in \mathcal{A}_{\theta, \beta/\alpha}$, the function $u_\alpha := \alpha u$ satisfies $u_\alpha(0, y) = \alpha(1 - 2\theta)y$, and

$$\int\limits_{(0,1)^2} \text{dist}^2\left(\nabla u_\alpha, \mathcal{P}_{\alpha,\beta}\right) dx\, dy + \sigma \left|D^2 u_\alpha\right| ((0,1)^2) = \alpha^2 E_{\sigma/\alpha, \beta/\alpha, \theta}(u).$$

1.2.2 The case of small x-derivative

For $\sigma > 0$, $\theta \in (0, 1/2]$, and $\gamma \in (0, 1]$ we also consider on

$$\mathcal{B}_\theta := \left\{u \in W^{1,2}((0,1)^2) : \nabla u \in BV((0,1)^2), \ u(0, y) = (1 - 2\theta)y\right\}$$

the functional

$$F_{\sigma,\gamma,\theta}(u) = \int\limits_{(0,1)^2} \text{dist}^2(\nabla u, \mathcal{M}_\gamma)\, dx\, dy + \sigma \mid D^2 u \mid (\Omega)$$

where

$$\mathcal{M}_\gamma := \{(\pm\gamma, \pm 1)\}.$$

Also in this case, we consider for any function in $\mathcal{B}_\theta$ always its continuous representative (see e.g. [35, Lemma 9]). Again, for $\gamma \to 1$, the problem turns into the four-gradient problem studied in [27], while for $\gamma \to 0$, this problem turns into a Kohn–Müller-type model. Our main result is the following scaling law for the minimal energy. Note that we a prove matching upper and lower bound in all parameter regimes.

Theorem 2 *There are constants c_i, $C_i > 0$, $i = 1, 2, 3$ such that for all $\theta \in (0, 1/2]$, all $\gamma \in (0, 1]$, and all $\sigma > 0$, the following statements hold:*

1. *If $0 < \gamma \leq \theta/8$ then*

$$c_1 s_1(\gamma, \theta, \sigma) \leq \min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \leq C_1 s_1(\gamma, \theta, \sigma),$$

 where $s_1(\gamma, \theta, \sigma) = \min\left\{\theta^2, \sigma^{2/3}\theta^{2/3}, \frac{\sigma\theta}{\gamma}\left(\left|\log\frac{\sigma\theta}{\gamma^3}\right| + 1\right)\right\}$.
2. *If $0 < \gamma^2/2 \leq \theta/8 < \gamma$ then*

$$c_2 s_2(\gamma, \theta, \sigma) \leq \min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \leq C_2 s_2(\gamma, \theta, \sigma),$$

 where $s_2(\gamma, \theta, \sigma) = \min\left\{\theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \frac{\sigma\theta}{\gamma}\left(\left|\log\frac{\sigma}{\theta^2}\right| + 1\right)\right\}$.
3. *If $0 < \theta/8 < \gamma^2/2$ then*

$$c_3 s_3(\gamma, \theta, \sigma) \leq \min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \leq C_3 s_3(\gamma, \theta, \sigma),$$

 where $s_3(\gamma, \theta, \sigma) = \min\left\{\theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \sigma\gamma\left(\frac{|\log\sigma\gamma^2/\theta^3|}{|\log\gamma^2/\theta|} + 1\right)\right\}$.

Proof The upper bounds follow from Corollary 1, the lower bounds from Proposition 6. □

Remark 3 By rescaling, one can obtain similar results also for the case of general preferred gradients of the form $(\pm\alpha, \pm\beta)$ with $0 < \alpha \leq \beta$. For a function $u \in \mathcal{B}_\theta$ consider $u_\beta := \beta u$. Then $u_\beta(0, y) = \beta(1 - 2\theta)y$ and

$$\int_{(0,1)^2} \text{dist}^2(\nabla u_\beta, \mathcal{P}_{\alpha,\beta}) \, dx \, dy + \sigma \left| D^2 u_\beta \right| ((0, 1)^2) = \beta^2 F_{\sigma/\beta,\theta,\alpha/\beta}(u).$$

Let us briefly discuss how the above mentioned relations to the well-known scaling laws are reflected in this result.

Remark 4 Suppose that $\sigma > 0$ and $\theta \in (0, 1/2]$ are fixed.

1. If $\gamma \to 0$ for $\theta > 0$ fixed, we are in case (1) and we recover the well-known scaling law for Kohn-Müller type models (see (4)).
2. Consider now the case $\gamma \to 1$. Note that then we are necessarily in case (3) since for (2) there holds $\gamma^2/2 \leq \theta/8 \leq 1/16$. If we are in case (3) then we have

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \sim \min\left\{\theta^2, \sigma + \theta^3, \sigma\left(\frac{|\log \sigma/\theta^3|}{|\log \theta|} + 1\right)\right\}$$

$$\sim \min\left\{\theta^2, \sigma + \theta^3, \sigma\left(\frac{|\log \sigma|}{|\log \theta|} + 1\right)\right\}$$

$$\sim \min\left\{\theta^2, \sigma\left(\frac{|\log \sigma|}{|\log \theta|} + 1\right)\right\},$$

which is the scaling law from [27, Theorem 1] (see (3)).

Throughout the text, we denote by c and C generic constants that may change from expression to expression and do not depend on the problem parameters. In the absence of ambiguities, we will not distinguish between row and column vectors. For $B \subseteq \mathbb{R}^2$ open and $u \in W^{1,2}(B)$ with $\nabla u \in BV(B)$, we use the notation $E_{\sigma,\gamma,\theta}(u; B)$ and $F_{\sigma,\gamma,\theta}(u; B)$ for the energy on B, i.e.,

$$E_{\sigma,\gamma,\theta}(u; B) := \int_B \text{dist}^2(\nabla u, \mathcal{K}_\gamma) \, dx \, dy + \sigma |D^2 u|(B)$$

$$\text{and } F_{\sigma,\gamma,\theta}(u; B) := \int_B \text{dist}^2(\nabla u, \mathcal{M}_\gamma) \, dx \, dy + \sigma |D^2 u|(B). \tag{5}$$

In addition for $x \in (0, 1)$ and $I \subseteq (0, 1)$ Lebesgue-measurable, we write for $u \in \mathcal{A}_{\theta,\gamma}$

$$E_{\sigma,\gamma,\theta}(u; \{x\} \times I) := \int_I \text{dist}^2(\nabla u(x, y), \mathcal{K}_\gamma) \, dy + |\partial_1 \nabla u(x, \cdot)|(I).$$

Note that since $\nabla u \in BV((0, 1)^2)$ this formula makes sense for almost every $x \in (0, 1)$ in the sense of slicing of BV-functions, see [36]. Similarly, we write for $y \in (0, 1)$ and $u \in \mathcal{A}_{\theta,\gamma}$

$$E_{\sigma,\gamma,\theta}(u; I \times \{y\}) := \int_I \text{dist}^2(\nabla u(x, y), \mathcal{K}_\gamma) \, dx + |\partial_2 \nabla u(\cdot, y)|(I).$$

We use analogous notation for $F_{\sigma,\gamma,\theta}$.

2 Proof of Theorem 1

2.1 Upper bound

We start with the proof of the upper bound in Theorem 1. Before we present the precise constructions, let us start with a brief heuristic explanation. Essentially, the boundary conditions can be met in two ways, namely the y-derivative being approximately $(1 - 2\theta)\gamma$ and fast oscillations close to $x = 0$ between y-derivatives $+\gamma$ and $-\gamma$ with volume fraction $1 - \theta$ and θ, respectively. The first case is penalized by the first term in $E_{\sigma,\gamma,\theta}$, whereas the second case introduces a certain energy through the second term of $E_{\sigma,\gamma,\theta}$. If $\sigma > 0$ is large, uniform structures should be energetically favorable. For small $\sigma > 0$, we present two constructions in which oscillations in the y-derivative refine in a self-similar way towards $x = 0$. If $\gamma > 0$ is rather small then the set $\mathcal{K}_\gamma = \{(1, \pm\gamma)\} \cup \{(-1, \pm\gamma)\}$ is the disjoint union of two rather far apart sets of two preferred gradients with a small distance. Hence, it is energetically favorable to use only two of the four vectors in $\mathcal{K}_\gamma$, i.e. we restrict ourselves to the setting of only two preferred gradients. By a simple change of variables, in this scenario the construction from [7] can be invoked. The second construction uses all four preferred gradients and isotropically rescaled building blocks, cf. Fig. 4 and [27]. Assuming that $\nabla u \in \mathcal{K}_\gamma$, we find that $|u(x, y) - (1 - 2\theta)\gamma y| \leq x$. It then follows, again assuming that $\nabla u \in \mathcal{K}_\gamma$, that the number of jumps of the y-derivative is of order $\frac{\theta\gamma}{x}$. The presented construction realizes this as the number of jumps of the y-derivatives grows from approximately $(\theta\gamma^2)^{-i}$ to approximately $(\theta\gamma^2)^{-i-1}$ between $x_i \approx \theta\gamma(\theta\gamma^2)^i$ and $x_{i+1} \approx \theta\gamma(\theta\gamma^2)^{i+1}$. Moreover, $|\partial_2\partial_2 u|((x_{i+1}, x_i) \times (0, 1)) \approx \gamma(x_i - x_{i+1})(\theta\gamma^2)^{-i} \approx 1$ which balances the energy contributions from $|\partial_1\partial_1 u|$ per refinement step. The lower bound indicates that this is necessary. Eventually, we note that instead of the described refinement step (purely creating surface energy), we could alternatively interpolate to the boundary conditions using gradients that are not in $\mathcal{K}_\gamma$, see Fig. 4 (middle). This creates an energy contribution through the first part of the energy. By scaling this contribution becomes smaller as the number of refinement steps taken before increases. The use of refinement steps in the construction then terminates once this interpolation is energetically favorable.

Precisely, we have the following result.

Proposition 3 *There is a constant $C > 0$ such that for all $\theta \in (0, 1/2]$, all $\gamma \in (0, 1]$ and all $\sigma > 0$, there is a function $u \in \mathcal{A}_{\theta,\gamma}$ with*

$$E_{\sigma,\gamma,\theta}(u) \leq C \min\left\{\gamma^2\theta^2, \ \sigma^{2/3}\gamma^{4/3}\theta^{2/3}, \ \sigma\left(\frac{|\log\sigma|}{|\log(\gamma^2\theta)|} + 1\right)\right\}.$$

Proof We use the labels from Fig. 1 to indicate in which parameter regime which test function is used. The first two scaling regimes arise from the scaling regimes of the Kohn–Müller-type models (see Remark 1, with constructions along the lines of [2, 7, 33, 34]) (Fig. 2).

1. *Uniform configuration (UC).* The first scaling, $\gamma^2\theta^2$, corresponds to a *uniform* test function $u_{\mathrm{UC}}(x, y) = (1 - 2\theta)\gamma y + x$, which has energy $E(u_{\mathrm{UC}}) = 4\gamma^2\theta^2$.

2. *Kohn–Müller-type two-gradient branching (TG).* The second energy scaling, $\sigma^{2/3}\gamma^{4/3}\theta^{2/3}$, can be achieved by a *Kohn–Müller type branching* construction, which involves only two of the four preferred gradients, see Fig. 3. A similar construction is also given in detail in Proposition 5 (b) below, cf. also the upper bound constructions in [34, 37]. Precisely, it is shown in [34] that there is a constant $C > 0$ such that for all $\varepsilon \in (0, \infty)$ and all $\theta \in (0, 1/2]$ with $\varepsilon \leq \theta^2$, there is a function $v := v_{\varepsilon,\theta} : (0, 1)^2 \to \mathbb{R}$ with $v(0, y) = 0$ for all $y \in (0, 1)$
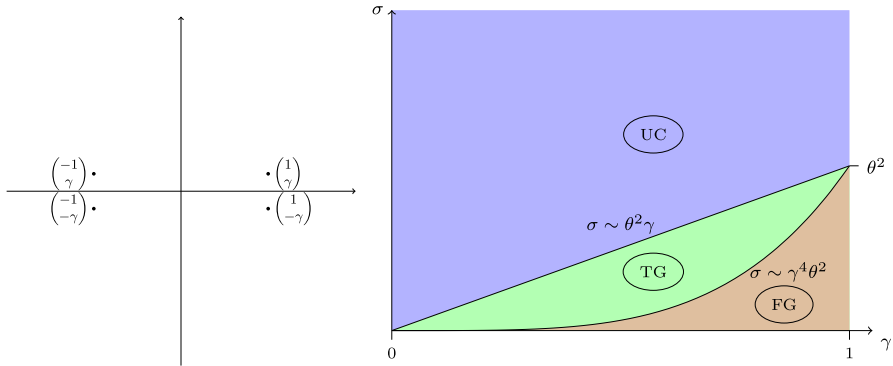
Fig. 1 Left: Sketch of the set of preferred gradients with small y-derivative $\mathcal{K}_\gamma$. Right: Sketch of the appearing regimes for small y-derivatives in Theorem 1 for fixed θ in the σ-γ-plane: the uniform configuration (UC), the Kohn–Müller type branching using two gradients (TG), and the isotropically rescaled branching using all four preferred gradients (FG). For the corresponding upper bound constructions see Sect. 2.1
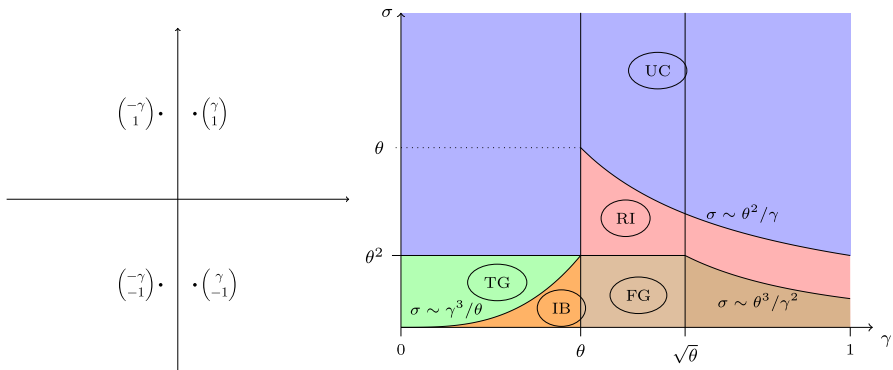

Fig. 2 Left: Sketch of the set of preferred gradients with small x-derivative $\mathcal{M}_\gamma$. Right: Sketch of the appearing regimes for small x-derivatives in Theorem 2 for fixed θ in the σ-γ-plane: the uniform configuration (UC), the Kohn–Müller type branching using two gradients (TG), the interpolation between the Kohn–Müller type branching and the isotropically rescaled branching using four gradients (IB), the isotropically rescaled branching using all four preferred gradients (FG), and the rotated interphase configuration (RI). For the corresponding upper bound constructions see Sect. 3.1

such that

$$
\int_{(0,1)^2} \left((\partial_1 v)^2 + \min \left\{ (\partial_2 v + 1 - \theta)^2, \ (\partial_2 v - \theta)^2 \right\} \right) dx\, dy + \varepsilon |D^2 v|((0,1)^2) \leq C \varepsilon^{2/3} \theta^{2/3}.
$$

Suppose now $\sigma \leq \gamma \theta^2$ (otherwise $\gamma^2 \theta^2 \leq \sigma^{2/3} \gamma^{4/3} \theta^{2/3}$, and the second energy scaling is not relevant). We then set $\varepsilon := \sigma \gamma^{-1} \leq \theta^2$, and define $u_{\mathrm{TG}} : (0,1)^2 \to \mathbb{R}$ by

$$
u_{\mathrm{TG}}(x, y) := \gamma \left(2v(x, y) + (1 - 2\theta)y \right) + x.
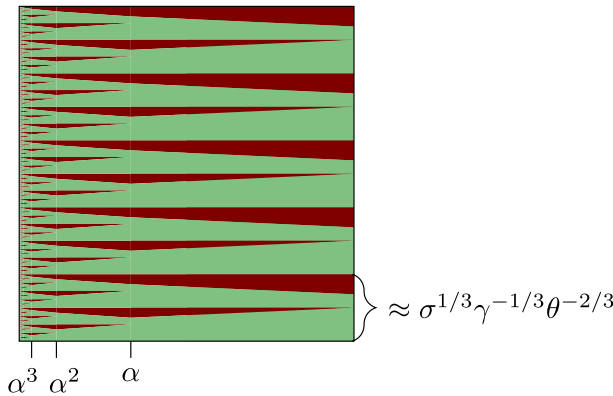$$

Fig. 3 Sketch of the Kohn-Müller-type branching construction. The areas of $\partial_2 u = 1$ and $\partial_2 u = -1$ are colored in green and red, respectively. The parameter α can be chosen in the interval $(1/4, 1/2)$

This yields $u_{\text{TG}} \in \mathcal{A}_{\theta,\gamma}$, and by (4)

$$E_{\sigma,\gamma,\theta}(u_{\text{TG}})$$

$$\leq \int\limits_{(0,1)^2} |\partial_1 u_{\text{TG}} - 1|^2 + \min\{|\partial_2 u_{\text{TG}} - \gamma|, |\partial_2 u_{\text{TG}} + \gamma|\}^2 \, dx \, dy + \sigma |D^2 u_{\text{TG}}|((0,1)^2)$$

$$= \gamma^2 \left(\int\limits_{(0,1)^2} 4|\partial_1 v|^2 + 4\min\{|\partial_2 v - \theta|, |\partial_2 v + 1 - \theta|\}^2 \, dx \, dy + 2\sigma\gamma^{-1}|D^2 v|((0,1)^2) \right)$$

$$\leq 4C\gamma^2\varepsilon^{2/3}\theta^{2/3} = 4C\sigma^{2/3}\gamma^{4/3}\theta^{2/3}.$$

3. *Four-gradient branching (FG)*. We now turn to the third and last scaling regime, $\sigma \left(\frac{|\log \sigma|}{|\log(\gamma^2\theta)|} + 1 \right)$. We may assume that $\sigma \leq \gamma^4\theta^2$ (otherwise $\sigma \geq \sigma^{2/3}\gamma^{4/3}\theta^{2/3}$, and the last energy scaling is not relevant). Here, we use a branching-type construction (see Fig. 4), which is a variant of the construction that has been introduced in [27] for the special case $\gamma = 1$. We introduce some auxiliary notation and set

$$m := \left\lceil \frac{1}{\theta} \left\lceil \frac{1}{\gamma^2} \right\rceil \right\rceil, \quad \delta := \frac{1}{m}, \quad \text{and} \quad n := \left\lfloor \frac{\theta}{\delta} \right\rfloor,$$

where $\lfloor x \rfloor := \max\{\ell \in \mathbb{N} : \ell \leq x\}$ and $\lceil x \rceil := \min\{\ell \in \mathbb{N} : \ell \geq x\}$. Then

$$m < \frac{1}{\theta} \left\lceil \frac{1}{\gamma^2} \right\rceil + 1, \quad \frac{\theta}{\theta + \left\lceil \frac{1}{\gamma^2} \right\rceil} < \delta \leq \gamma^2\theta \leq \frac{1}{2}, \quad n \geq 1, \quad \text{and} \quad \frac{m}{n} \geq \frac{m\delta}{\theta} \geq 2.$$

Step 1: Construction of the building block.
The building block is sketched in Fig. 4, left panel. We define $V : [\gamma\theta^2, \gamma\theta] \times \mathbb{R} \to \mathbb{R}^2$ as the function which is 1-periodic in y-direction and satisfies the following:

(i) If $(x, y) \in [\gamma\theta^2, \gamma\theta] \times [1 - l\delta, 1 - (l-1)\delta)$ for $1 \le l \le n$ then

$$V(x, y) = \begin{cases} (1, -\gamma) & \text{if } x \le \gamma\theta - \delta\gamma(1-\theta)(n-l+1) \text{ and} \\ & y \ge 1 - (l-1)\delta - \theta\delta, \\ (1, \gamma) & \text{if } x \le \gamma\theta - \delta\gamma(1-\theta)(n-l), \\ & y \le 1 - (l-1)\delta - \theta\delta, \text{ and} \\ & y \le 1 - l\delta - \gamma^{-1}(x - \gamma\theta - \gamma\delta(1-\theta)(n-l)), \\ (-1, -\gamma) & \text{else.} \end{cases}$$

(ii) If $(x, y) \in [\gamma\theta^2, \gamma\theta] \times [1 - (n+1)\delta, 1 - n\delta)$ then

$$V(x, y) = \begin{cases} (1, -\gamma) & \text{if } y \ge \max\{1 - (n+\theta)\delta, 1 - \theta + \gamma^{-1}(x - \gamma\theta)\}, \\ (1, \gamma) & \text{if } y \le 1 - (n+\theta)\delta \text{ and } x \le 2\gamma\theta - (n+\theta)\gamma\delta, \\ (-1, \gamma) & \text{else.} \end{cases}$$

(iii) If $(x, y) \in [\gamma^2\theta, \gamma\theta] \times [1 - \ell\delta, 1 - (\ell-1)\delta)$ for $n + 1 < \ell \le m$ then

$$V(x, y) = \begin{cases} (1, -\gamma) & \text{if } y \ge 1 - \ell\delta + (1-\theta)\delta \text{ and} \\ & y \ge 1 - (\ell-1)\delta + \gamma^{-1}(x - 2\gamma\theta + (n+\theta)\gamma\delta + (\ell-n-2)\theta\gamma\delta), \\ (1, \gamma) & \text{if } y \le 1 - \ell\delta + (1-\theta)\delta \text{ and} \\ & x \le 2\gamma\theta - (n+\theta)\gamma\delta - (\ell-n-3)\gamma\theta\delta, \\ (-1, \gamma) & \text{else,} \end{cases}$$

It can be seen that V is curl-free as it is piecewise constant and $\nu \parallel (V^- - V^+)$ on its jump set J_V, where ν is the measure-theoretic normal to J_V. Consequently, V is a gradient field on $(\gamma^2\theta, \gamma\theta) \times \mathbb{R}$, and additionally, $V(x, y) \in K$ for almost all (x, y), and

$$|DV|\left((\gamma^2\theta, \gamma\theta) \times (0, 1)\right) \le C(\delta\gamma(1-\gamma)\gamma m + 1) \le C.$$

We will use in the next step that for the second component V_2 of V, we have

$$V_2(\theta\gamma, y) = V_2(\theta^2\gamma, \delta y) \qquad \text{for all } y \in \mathbb{R}. \tag{6}$$

Additionally, consider the function $V_{bd} : (0, \gamma\theta(1-\theta)) \times \mathbb{R} \to \mathbb{R}^2$ which is 1-periodic in the y-component and for $(x, y) \in (0, \gamma\theta(1-\theta)) \times (0, 1)$ defined as (see Fig. 4)

$$V_{bd}(x, y) = \begin{cases} (-1, -\gamma) & \text{if } y \ge 1 - \frac{1}{\gamma(1-\theta)}x, \\ (-1, \gamma) & \text{if } y \le \frac{x}{\gamma\theta} \\ (1, \gamma(1-2\theta)) & \text{else.} \end{cases}$$

Again, we note that also V_{bd} is curl-free and $|D^2 V_{bd}|((0, \theta\gamma(1-\theta)) \times (0, 1)) \le C$.

Step 2: Branching construction.

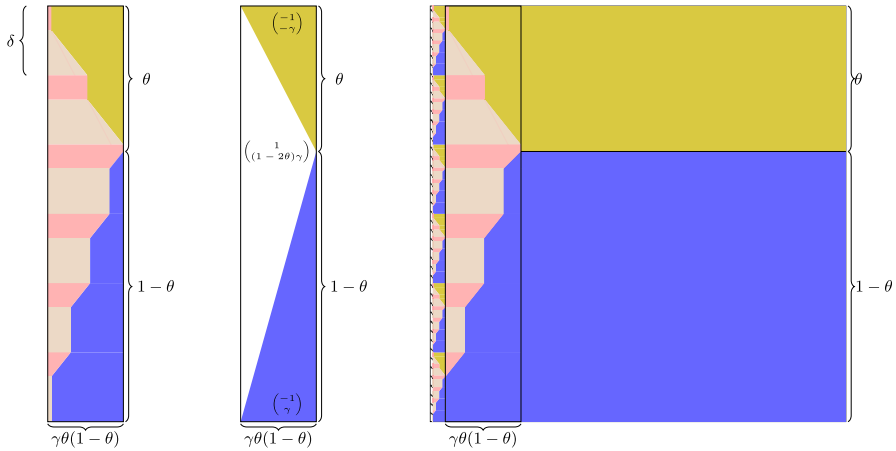The branching-type construction is sketched in Fig. 4, right panel.

Fig. 4 Left: Sketch of the building block. Middle: Sketch of the function V_{bd}. Right: Sketch of the branching construction for small y-derivatives

We now set $V_N : (0, 1)^2 \to \mathbb{R}^2$ for fixed $N \in \mathbb{N}$ as

$$V_N(x, y) = \begin{cases} (1, -\gamma) & \text{if } x \geq \gamma\theta\frac{1-\theta}{1-\delta} \text{ and } y \geq 1-\theta, \\ (1, \gamma) & \text{if } x \geq \gamma\theta\frac{1-\theta}{1-\delta} \text{ and } y \leq 1-\theta, \\ V(\delta^{-k+1}x + \frac{\theta-\delta}{1-\delta}, \delta^{-k+1}y) & \text{if } x \in [\delta^k\gamma\theta\frac{1-\theta}{1-\delta}, \delta^{k-1}\gamma\theta\frac{1-\theta}{1-\delta}) \\ & \quad \text{with } 1 \leq k \leq N, \\ V_{bd}(\delta^{-N}(x - \delta^{N+1}\gamma\theta\frac{1-\theta}{1-\delta}), \delta^{-N}y) & \text{if } x \in (\delta^{N+1}\gamma\theta\frac{1-\theta}{1-\delta}, \delta^N\gamma\theta\frac{1-\theta}{1-\delta}), \\ (1, (1-2\theta)\gamma) & \text{if } x \in (0, \delta^{N+1}\gamma\theta\frac{1-\theta}{1-\delta}). \end{cases}$$

We note that V_N is curl-free as $\nu \parallel (V_N^- - V_N^+)$ on J_{V_N}, where ν is the measure theoretic normal to J_{V_N}, see also (6). Moreover, $V_N(x, y) \in \mathcal{K}_\gamma$ for almost all $(x, y) \in (\frac{\gamma\theta(1-\theta)}{1-\delta}\delta^N, 1) \times (0, 1)$, and

$$| DV_N | \big((0, 1)^2\big) \leq CN + 2\gamma \leq CN. \tag{7}$$

Let $u_N : (\frac{\gamma\theta(1-\theta)}{1-\delta}\delta^N, 1) \times (0, 1) \to \mathbb{R}$ be a corresponding primitive, i.e., $\nabla u_N = V_N$, such that $u_N(0, 0) = 0$.

Step 3: Estimate for the energy.
We have

$$\int_{(0,1)^2} \text{dist}(\nabla u_N, \mathcal{K}_\gamma)^2 \, dx \, dy \leq 4\gamma^2\theta^2\gamma\theta\frac{1-\theta}{1-\delta}\delta^N \leq C(\gamma^2\theta)^N. \tag{8}$$

Moreover, by (7)

$$| D^2 u_N | ((0, 1)^2) \leq CN. \tag{9}$$

Now fix

$$N_0 := \left\lceil \frac{\log \sigma}{\log(\gamma^2\theta)} \right\rceil$$

and set $u_{FG} := u_{N_0}$. Then combining (8) and (9), we obtain

$$E_{\sigma,\gamma,\theta}(u_{FG}) \leq C(\gamma^2\theta)^{N_0} + C\sigma N_0 \leq C\sigma\left(1 + \frac{\log\sigma}{\log(\gamma^2\theta)}\right) = C\sigma\left(\frac{|\log\sigma|}{|\log(\gamma^2\theta)|} + 1\right).$$

$\square$

2.2 Lower bound

We now turn to the proof of the lower bound in Theorem 1. Precisely, we show the following result.

Proposition 4 *There is a constant $c > 0$ such that for all $\theta \in (0, 1/2]$, all $\gamma \in (0, 1]$, and all $\sigma > 0$, there holds*

$$\min_{\mathcal{A}_{\theta,\gamma}} E_{\sigma,\gamma,\theta} \geq c \min\left\{\gamma^2\theta^2, \sigma^{2/3}\gamma^{4/3}\theta^{2/3}, \sigma\left(\frac{|\log\sigma|}{|\log(\gamma^2\theta)|} + 1\right)\right\}.$$

Proof The proof is structured in a similar way as the proof of [27, Theorem 1] and is split in several parts that we prove as separate lemmas below. We briefly outline the main steps and how they yield the claimed lower bound.

First, in Lemma 1, we prove a (weaker) lower bound without the logarithmic term in the third regime, i.e.,

$$\min_{\mathcal{A}_{\theta,\gamma}} E_{\sigma,\gamma,\theta} \geq c \min\{\gamma^2\theta^2, \sigma^{2/3}\gamma^{4/3}\theta^{2/3}, \sigma\}.$$

This concludes the proof of the lower bound in the first two regimes, in particular if $\sigma \geq \gamma^4\theta^2$ (since in this case, $\gamma^{4/3}\sigma^{2/3}\theta^{2/3} \leq \sigma$, and the last regime is not relevant).

Let us now consider the remaining case $\sigma \leq \gamma^4\theta^2$.

- If $(\gamma^2\theta)^{34} \leq \sigma \leq (\gamma^2\theta)^2$ then $|\log\sigma|/|\log(\gamma^2\theta)| \sim 1$, and the lower bound is also concluded by Lemma 1 described above.

- If $\sigma \leq (\gamma^2\theta)^{34}$ and $\alpha_0 \leq \gamma^2\theta$ with some fixed constant $\alpha \in (0, 1)$, we have $|\log(\gamma^2\theta)| \sim 1$ (recall that always $\gamma^2\theta \leq 1/2$), and we prove in Lemma 3 below that

$$\min_{\mathcal{A}_{\theta,\gamma}} E_{\sigma,\gamma,\theta} \geq c\sigma(|\log\sigma| + 1),$$

which concludes the proof of the lower bound in this case.

- Finally, consider the case $\sigma < (\gamma^2\theta)^{34}$ and $\gamma^2\theta \leq \alpha_0$ with some fixed constant $\alpha_0 \in (0, 1)$. Then there exists some $k \geq 32$ such that $\gamma^4\theta^2(\gamma^2\theta)^{k+1} \leq \sigma < \gamma^4\theta^2(\gamma^2\theta)^k$, and we prove in Lemma 2 below that

$$\min_{\mathcal{A}_{\theta,\gamma}} E_{\sigma,\gamma,\theta} \geq ck\sigma \geq c\sigma \frac{|\log\sigma|}{|\log(\gamma^2\theta)|} \geq c\sigma\left(\frac{|\log\sigma|}{|\log(\gamma^2\theta)|} + 1\right)$$

where in the last estimate we used that $\sigma < (\gamma^2\theta)^2$. This concludes the proof of the lower bound in this parameter regime.

$\square$

We first prove a weaker lower bound without the logarithmic term.

Lemma 1 *There exists $c > 0$ such that for all $u \in \mathcal{A}_{\theta,\gamma}$ there holds*

$$E_{\sigma,\gamma,\theta}(u) \geq c \min\{\gamma^2\theta^2, \sigma^{2/3}\gamma^{4/3}\theta^{2/3}, \sigma\}.$$

Proof Assume that there exists $u \in \mathcal{A}_{\theta,\gamma}$ such that $E_{\sigma,\gamma,\theta}(u) \leq \frac{1}{32^2} \min\{\gamma^2\theta^2, \sigma^{1/2}\gamma^{3/2}\theta, \sigma\}$.
We define

$$N := 8 \begin{cases} 1 & \text{if } \gamma^2\theta^2 = \min\{\gamma^2\theta^2, \sigma^{2/3}\gamma^{4/3}\theta^{2/3}, \sigma\}, \\ \sigma^{-1/3}\gamma^{1/3}\theta^{2/3} & \text{if } \sigma^{2/3}\gamma^{4/3}\theta^{2/3} = \min\{\gamma^2\theta^2, \sigma^{2/3}\gamma^{4/3}\theta^{2/3}, \sigma\}, \\ \gamma^{-1} & \text{if } \sigma = \min\{\gamma^2\theta^2, \sigma^{2/3}\gamma^{4/3}\theta^{2/3}, \sigma\}. \end{cases}$$

Note that $N \geq 8$. First find $\bar{y} \in (0, 1 - \frac{2}{N})$ such that

$$E_{\sigma,\gamma,\theta}\left(u; (0, 1) \times (\bar{y}, \bar{y} + \frac{2}{N})\right) \leq \frac{4}{N} E_{\sigma,\gamma,\theta}(u).$$

Then by a Fubini-type argument find $\bar{x} \in (0, 1)$ such that

$$E_{\sigma,\gamma,\theta}\left(u; \{\bar{x}\} \times (\bar{y}, \bar{y} + \frac{2}{N})\right) \leq \frac{4}{N} E_{\sigma,\gamma,\theta}(u).$$

Lastly, again by a Fubini-type argument, we find $y_1, y_2 \in (\bar{y}, \bar{y} + \frac{2}{N})$ such that $y_2 - y_1 = \frac{1}{N}$ and

$$E_{\sigma,\gamma,\theta}(u; (0, 1) \times \{y_1\}) + E_{\sigma,\gamma,\theta}(u; (0, 1) \times \{y_2\}) \leq 4E_{\sigma,\gamma,\theta}(u).$$

First note that $\int_0^1 \min\{|\partial_1 u(x, y_1) - 1|, |\partial_1 u(x, y_1) + 1|\}^2 \, dx \leq \frac{4}{32^2}\gamma^2\theta^2$. Hence, there exists $t \in (0, 1)$ such that $\min\{|\partial_1 u(t, y_1) - 1|, |\partial_1 u(t, y_1) + 1|\} \leq \frac{1}{16}\gamma\theta < \frac{1}{2}$. Without loss of generality we assume that $|\partial_1 u(t, y_1) - 1| < \frac{1}{2}$, i.e. $\partial_1 u(t, y_1) \geq \frac{1}{2}$. Moreover, it holds that

$$|\partial_1 \nabla u(\cdot, y_1)|((0, 1)) + |\partial_1 \nabla u(\cdot, y_2)|((0, 1)) + |\partial_2 \nabla u|(\{\bar{x}\} \times (y_1, y_2)) \leq \sigma^{-1} 8 E_{\sigma,\gamma,\theta}(u)$$
$$\leq \frac{8}{32^2} < \frac{1}{2}.$$

Hence, $\partial_1 u(s, y_i) \geq 0$ for $i = 1, 2$ and almost all $s \in (0, 1)$, i.e.,

$$|\partial_1 u(s, y_i) - 1| = \min\{|\partial_1 u(s, y_i) - 1|, |\partial_1 u(s, y_i) + 1|\}.$$

Then we estimate

$$|u(\bar{x}, y_2) - u(\bar{x}, y_1) - (1 - 2\theta)\gamma(y_2 - y_1)|$$
$$= \left| \int_0^{\bar{x}} \partial_1 u(t, y_2) - 1 - \partial_1 u(t, y_1) + 1 \, dt \right|$$
$$\leq \bar{x}^{\frac{1}{2}} \left(E(u; (0, 1) \times \{y_1\})^{1/2} + E_{\sigma,\gamma,\theta}(u; (0, 1) \times \{y_2\})^{1/2} \right)$$
$$\leq \frac{1}{8} \min\{\gamma\theta, \sigma^{1/3}\gamma^{2/3}\theta^{1/3}, \sigma^{1/2}\}$$
$$\leq \gamma\theta \frac{1}{N} = \gamma\theta(y_2 - y_1). \tag{10}$$

Here, we used that $\sigma \leq \gamma^4\theta^2$ if $\sigma = \min\{\gamma^2\theta^2, \sigma^{2/3}\gamma^{4/3}\theta^{2/3}, \sigma\}$. Next, we note that

$$|\partial_2 \partial_2 u(\bar{x}, \cdot)|((y_1, y_2)) \leq \sigma^{-1} \frac{4}{32^2 N} \min\{\gamma^2\theta^2, \gamma^{4/3}\sigma^{2/3}\theta^{2/3}, \sigma\} \leq \frac{1}{2 \cdot 32^2}\gamma < \gamma/2.$$

By a similar argument as before, we may assume that it holds for almost all $y \in (y_1, y_2)$ that $|\partial_2 u(\bar{x}, y) - \gamma| = \min\{|\partial_2 u(\bar{x}, y) - \gamma|, |\partial_2 u(\bar{x}, y) - \gamma|\}$. Then we estimate

$$\left| \int_{y_1}^{y_2} (\partial_2 u(\bar{x}, y) - \gamma) \, dy \right|^2 \leq (y_2 - y_1) \int_{y_1}^{y_2} |\partial_2 u(\bar{x}, y) - \gamma|^2 \, dy$$

$$\leq (y_2 - y_1) E_{\sigma, \gamma, \theta}(u; \{\bar{x}\} \times (y_1, y_2)) \leq \frac{4}{N^2} E_{\sigma, \gamma, \theta}(u). \quad (11)$$

On the other hand, we have by (10)

$$\left| \int_{y_1}^{y_2} (\partial_2 u(\bar{x}, y) - \gamma) \, dy \right|^2 = |u(\bar{x}, y_2) - u(\bar{x}, y_1) - \gamma (y_2 - y_1)|^2$$

$$\geq \left| 2\theta\gamma(y_2 - y_1) - |u(\bar{x}, y_2) - u(\bar{x}, y_1) - (1 - 2\theta)\gamma(y_2 - y_1)| \right|^2$$

$$\geq \gamma^2 \theta^2 (y_2 - y_1)^2 = \gamma^2 \theta^2 \frac{1}{N^2}. \quad (12)$$

Combining (11) and (12) yields

$$E_{\sigma, \gamma, \theta}(u) \geq \frac{1}{4} \gamma^2 \theta^2 \geq \frac{1}{4} \min\{\gamma^2 \theta^2, \sigma^{2/3} \gamma^{4/3} \theta^{2/3}, \sigma\}.$$

This concludes the proof. □

The next lemma is along the lines of [27, Lemma 6] and its proof, with the necessary careful amendments to deal with the additional parameter γ.

Lemma 2 *There exist $\alpha_0 > 0$ and $c > 0$ such that for all $k \geq k_0 = 32$, all $\gamma \in (0, 1]$, all $\theta \in (0, \alpha_0/\gamma^2]$, and all*

$$\sigma \in \left[\gamma^4 \theta^2 (\gamma^2 \theta)^{k+1}, \gamma^4 \theta^2 (\gamma^2 \theta)^k \right)$$

there holds

$$E_{\sigma, \gamma, \theta}(u) \geq ck\sigma. \quad (13)$$

Proof Set $k_0 := 32$ and $0 < \alpha_0 < 1/(63)^2$ such that $2 \cdot 64 \cdot 21^2 k \alpha_0^{k/4} \leq 1$ for all $k \geq k_0$. Let $k \geq k_0$, γ, θ and σ be as in the lemma and assume that $E_{\sigma, \gamma, \theta}(u) \leq k\sigma$.

For $i = 1, \ldots, k$, there are by a Fubini-type argument $x_i \in \left(\gamma\theta(\gamma^2\theta)^i, \frac{3}{2}\gamma\theta(\gamma^2\theta)^i \right)$ such that (cf. Fig. 5)

$$E_{\sigma, \gamma, \theta}(u; \{x_i\} \times (0, 1)) \leq 2(\gamma\theta)^{-1} (\gamma^2\theta)^{-i} E_{\sigma, \gamma, \theta} \left(u; \left(\gamma\theta(\gamma^2\theta)^i, \frac{3}{2}\gamma\theta(\gamma^2\theta)^i \right) \times (0, 1) \right).$$

Claim: There exists a constant $c > 0$ such that for all $i \leq k/2$ it holds

$$\gamma\theta(\gamma^2\theta)^i E_{\sigma, \gamma, \theta}(u, \{x_i\} \times (0, 1)) + E_{\sigma, \gamma, \theta}(u; (x_{i+1}, x_i) \times (0, 1)) \geq c\sigma.$$

Note that once we prove the claim, the lower bound (13) follows via

$$2E_{\sigma, \gamma, \theta}(u)$$

$$\geq \sum_{i=1}^{\lfloor k/2 \rfloor} \left[E_{\sigma, \gamma, \theta} \left(u; \left(\gamma\theta(\gamma^2\theta)^i, \frac{3}{2}\gamma\theta(\gamma^2\theta)^i \right) \times (0, 1) \right) + E_{\sigma, \gamma, \theta}(u; (x_{i+1}, x_i) \times (0, 1)) \right]$$

$$\geq \frac{1}{2} \sum_{i=1}^{\lfloor k/2 \rfloor} \left[\gamma\theta(\gamma^2\theta)^i E_{\sigma, \gamma, \theta}(u; \{x_i\} \times (0, 1)) + E_{\sigma, \gamma, \theta}(u; (x_{i+1}, x_i) \times (0, 1)) \right]$$

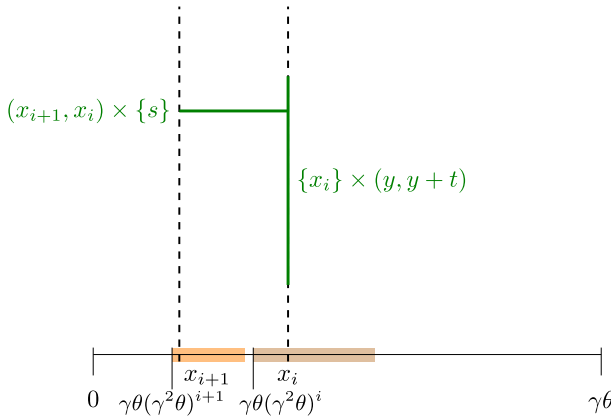$$\geq \lfloor k/2 \rfloor c\sigma \geq \frac{ck}{4}\sigma.$$

Fig. 5 Sketch of the important quantities in the proof of Lemma 2

Hence, it remains to prove the claim. From now on fix $i \leq k/2$. We define the set

$$N_i = \{s \in (0, 1) : |\partial_2 u(x_i, s) + \gamma| \leq 3\, |\partial_2 u(x_i, s) - \gamma|\}$$

and claim that $\mathcal{L}^1(N_i) \leq 2/3$, where we denote by $\mathcal{L}^1$ the 1-dimensional Lebesgue-measure. For a contradiction, assume that $\mathcal{L}^1(N_i) > 2/3$. There are $y_1, y_2 \in (0, 1)$ such that $y_2 - y_1 \geq 1 - \frac{1}{12}$ and

$$\int_0^1 \min\{|\,\partial_1 u(x, y_j) - 1\,|, |\,\partial_1 u(x, y_j) + 1\,|\}^2\, dx \leq 24 E_{\sigma,\gamma,\theta}(u) \text{ for } j = 1, 2.$$

This yields for $j = 1, 2$

$$|u(x_i, y_j) - (1 - 2\theta)\gamma y_j| \leq x_i + x_i^{1/2} \left(24 E_{\sigma,\gamma,\theta}(u)\right)^{1/2} \leq 2x_i,$$

and hence

$$u(x_i, y_2) - u(x_i, y_1) \geq (1 - 2\theta)\gamma(y_2 - y_1) - 4x_i.$$

This leads to

$$\int\limits_{(y_1,y_2) \cap \{\partial_2 u(x_i, \cdot) \geq \gamma/2\}} \partial_2 u(x_i, s)\, ds$$

$$= u(x_i, y_2) - u(x_i, y_1) - \int\limits_{(y_1,y_2) \cap \{\partial_2 u(x_i, \cdot) < \gamma/2\}} \partial_2 u(x_i, s) + \gamma - \gamma\, ds$$

$$\geq (1 - 2\theta)\gamma(1 - \frac{1}{12}) - 4x_i + (2/3 - 1/12)\gamma - 3 E_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0, 1))^{1/2}$$

$$\geq \frac{1}{2}\gamma, \tag{14}$$

where we used that $4x_i \leq 6\gamma\theta(\gamma^2\theta) \leq \frac{1}{24}\gamma$ and $E_{\sigma,\gamma,\theta}(u; \{x_i\}\times(0,1)) \leq 2\gamma^{-1}\theta^{-1}(\gamma^2\theta)^{-i}$ $k\sigma \leq \frac{\gamma^2}{9\cdot 24^2}$.

On the other hand, we estimate

$$\int\limits_{(y_1,y_2)\cap\{\partial_2 u(x_i,\cdot)\geq\gamma/2\}} \partial_2 u(x_i,s)\,ds \leq \frac{1}{3}\gamma + E_{\sigma,\gamma,\theta}(\{x_i\}\times(0,1))^{1/2} < \frac{1}{2}\gamma.$$

This contradicts (14). Hence, it holds $\mathcal{L}^1(N_i) \leq 2/3$.

Now, let $t := 120(\theta\gamma^2)^{i+1}$. Then there is a point $y \in (0,1)$ such that the interval $(y, y+t) \subseteq (0,1)$ is not completely contained in N_i,

$$E_{\sigma,\gamma,\theta}(u; (0,1)\times(y,y+t)) \leq 48t\,E_{\sigma,\gamma,\theta}(u),$$
$$E_{\sigma,\gamma,\theta}(u: (x_{i+1},x_i)\times(y,y+t)) \leq 48t\,E_{\sigma,\gamma,\theta}(u; (x_{i+1},x_i)\times(0,1)),$$
$$E_{\sigma,\gamma,\theta}(u; \{x_i\}\times(y,y+t)) \leq 48t\,E_{\sigma,\gamma,\theta}(u; \{x_i\}\times(0,1)), \text{ and}$$
$$E_{\sigma,\gamma,\theta}(u; \{x_{i+1}\}\times(y,y+t)) \leq 48t\,E_{\sigma,\gamma,\theta}(u; \{x_{i+1}\}\times(0,1)).$$

Then one of the following statements (1), (2), or (3) holds in $\{x_i\}\times(y,y+t)$, where

1. $|\partial_2\partial_2 u(x_i,\cdot)|((y,y+t)) \geq \frac{1}{2}\gamma$,
2. $|\partial_2 u(x_i,\cdot)-\gamma| \leq 3|\partial_2 u(x_i,\cdot)+\gamma|$, or
3. $|\partial_2 u(x_i,\cdot)+\gamma| \leq 3|\partial_2 u(x_i,\cdot)-\gamma|$.

As the interval $(y, y+t)$ is not a subset of N_i, assertion (3) cannot be true. Hence, it suffices to consider the cases (1) and (2).

Suppose that estimate (1) *holds* : Then

$$\frac{\sigma}{2} = \frac{\sigma\gamma}{2\gamma} \leq \gamma^{-1}E_{\sigma,\gamma,\theta}(u; \{x_i\}\times(y,y+t)) \leq 48t\gamma^{-1}E_{\sigma,\gamma,\theta}(u; \{x_i\}\times(0,1))$$

$$= 48\cdot 120(\gamma^2\theta)^i\gamma\theta\,E_{\sigma,\gamma,\theta}(u; \{x_i\}\times(0,1))$$

and thus the claim follows.

Suppose that estimate (2) *holds* : Note that by the triangle inequality

$$\frac{1}{2}\gamma\theta t^2 \leq \|u(x_i,s+t/2)-u(x_i,s)-\gamma t/2\|_{L^1(y,y+t/2)}$$
$$+ \|u(x_i,s+t/2)-u(x_i,s)-(1-2\theta)\gamma t/2\|_{L^1(y,y+t/2)}.$$

Next, we use that $E_{\sigma,\gamma,\theta}(u; \{x_i\}\times(0,1)) \leq 2\gamma^{-1}\theta^{-1}(\gamma^2\theta)^{-i}k\sigma \leq \frac{1}{2^{12}\cdot 64}\gamma^2\theta^2$ and estimate

$$\left|\int\limits_{y}^{y+t/2} u(x_i,s+t/2)-u(x_i,s)-\gamma t/2\,ds\right|$$

$$\leq \int\limits_{y}^{y+t/2}\int\limits_{s}^{s+t/2} |\partial_2 u(x_i,r)-\gamma|\,dr\,ds$$

$$\leq t^{1/2}\int\limits_{y}^{y+t/2}\left(\int\limits_{y}^{y+t} |\partial_2 u(x_i,r)-\gamma|^2\,dr\right)^{1/2}ds$$

$$\leq 21t^2 E_{\sigma,\gamma,\theta}(u; \{x_i\}\times(0,1))^{1/2} \leq \frac{1}{8}t^2\gamma\theta.$$

Then, by (2) and Poincaré's inequality we have using the notation $a := \frac{2}{t}\int\limits_{y}^{y+t/2}(u(x_i, s+t/2)$

$- u(x_i, s) - \gamma t/2)\,ds$

$$\|u(x_i, s + t/2) - u(x_i, s) - \gamma t/2 - a\|_{L^1(y, y+t/2)}$$

$$\leq t\|\partial_2 u(x_i, s+t/2) - \partial_2 u(x_i, s)\|_{L^1(y, y+t/2)}$$

$$\leq t\|\partial_2 u(x_i, s) - \gamma\|_{L^1(y, y+t)}$$

$$\leq 3t^{3/2}E_{\sigma,\gamma,\theta}(u; \{x_i\} \times (y, y+t))^{1/2}$$

$$\leq 21t^2 E_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0, 1))^{1/2}.$$

Consequently, if $\|u(x_i + t/2, s) - u(x_i, s) - \gamma t/2\|_{L^1(y, y+t/2)} \geq \frac{1}{4}\gamma\theta t^2$ it follows that

$$21^2 t^4 E_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0, 1)) \geq \left(\frac{1}{4}\theta\gamma t^2 - \frac{t}{2}|a|\right)^2 \geq \frac{t^4}{64}\gamma^2\theta^2 \geq \frac{t^4}{64}\gamma^3\theta(\gamma^2\theta)^{k-i}$$

$$\geq \frac{t^4}{64}\frac{\sigma}{\gamma\theta(\gamma^2\theta)^i}, \tag{15}$$

which yields the claim.

Hence, from now on we will assume that $\|u(x_i, s) - u(x_i, s+t/2) - (1-2\theta)\gamma t/2\|_{L^1(y, y+t/2)} \geq \frac{1}{4}\gamma\theta t^2$. First, observe that it holds for all $s \in (y, y+t/2)$

$$|u(x_i, s+t/2) - u(x_i, s) - (1-2\theta)\gamma t/2| \leq \gamma\theta t + 21t E_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0, 1))^{1/2} \leq 2\gamma\theta t. \tag{16}$$

Moreover, define

$$A_i = \left\{s \in (y, y+t) : E_{\sigma,\gamma,\theta}(u; (0, 1) \times \{s\}) \leq \frac{80}{\theta t}E_{\sigma,\gamma,\theta}(u; (0, 1) \times (y, y+t))\right\},$$

and note that $\mathcal{L}^1(A_i) \geq \left(1 - \frac{\theta}{80}\right)t$. For $s \in A_i$ we estimate

$$|u(x_{i+1}, s) - (1-2\theta)\gamma s| \leq x_{i+1} + x_{i+1}^{1/2} E_{\sigma,\gamma,\theta}(u; (0, 1) \times \{s\})^{1/2}$$

$$\leq x_{i+1} + x_{i+1}^{1/2}\left(\frac{80}{\theta t}E_{\sigma,\gamma,\theta}(u; (0, 1) \times (y, y+t))\right)^{1/2}$$

$$\leq x_{i+1} + 63 \cdot x_{i+1}^{1/2}\theta^{-1/2}E_{\sigma,\gamma,\theta}(u)^{1/2}$$

$$\leq 2x_{i+1}$$

$$\leq \frac{1}{40}\gamma\theta t,$$

where for the second to last inequality we used that for i small enough versus k (recall that $i \leq k/2$) we have $k\sigma \leq k\gamma^4\theta^2(\gamma^2\theta)^k \leq \gamma\theta(\gamma^2\theta)^{i+3} \leq \frac{1}{63^2}\theta x_{i+1}$. For $s \in (y, y+t)$ we find $\bar{s} \in A_i$ such that $|s - \bar{s}| \leq \frac{\theta}{80}t$. Then we obtain

$$|u(x_{i+1}, s) - (1-2\theta)\gamma s|$$

$$\leq |u(x_{i+1}, s) - u(x_{i+1}, \bar{s})| + |u(x_{i+1}, \bar{s}) - (1-2\theta)\gamma\bar{s}| + \gamma|s - \bar{s}|$$

$$\leq 2\gamma|s - \bar{s}| + |s - \bar{s}|^{1/2}E_{\sigma,\gamma,\theta}(u; \{x_{i+1}\} \times (y, y+t))^{1/2} + \frac{1}{40}\gamma\theta t$$

$$\leq \frac{1}{16}\gamma\theta t,$$

where we used similarly to above that for $i \leq k/2$ it holds $E_{\sigma,\gamma,\theta}(u; \{x_{i+1}\} \times (y, y+t)) \leq \frac{1}{80}\gamma^2\theta t$. In particular, using (16) we obtain for almost all $s \in (y, y+t/2)$ that

$$
\begin{aligned}
|u(x_i, s) - &u(x_i, s+t/2) - u(x_{i+1}, s) + u(x_{i+1}, s+t/2)| \\
&\leq |u(x_i, s) - u(x_i, s+t/2) - (1-2\theta)\gamma t/2| + |u(x_{i+1}, s) - (1-2\theta)\gamma s| \\
&\quad + |u(x_{i+1}, s+t/2) - (1-2\theta)\gamma(s+t/2)| \\
&\leq 3\gamma\theta t.
\end{aligned}
\tag{17}
$$

On the other hand, it holds by our assumption that

$$
\begin{aligned}
\|u(x_i, s) - &u(x_i, s+t/2) - u(x_{i+1}, s) + u(x_{i+1}, s+t/2)\|_{L^1(y, y+t/2)} \\
&\geq \|u(x_i, s) - u(x_i, s+t/2) - (1-2\theta)\gamma t/2\|_{L^1(y, y+t/2)} \\
&\quad - \|u(x_{i+1}, s) - (1-2\theta)\gamma s\|_{L^1(y, y+t/2)} - \|u(x_{i+1}, s+t/2) - (1-2\theta)\gamma\|_{L^1(y, y+t/2)} \\
&\geq \frac{1}{4}\gamma\theta t^2 - \frac{1}{8}\gamma\theta t^2 \\
&= \frac{1}{8}\gamma\theta t^2.
\end{aligned}
\tag{18}
$$

Now, consider the set

$$
\mathcal{S} := \left\{ s \in (y, y+t/2) : \frac{1}{8}\gamma\theta t \leq v(s) \leq 3\gamma\theta t \right\},
$$

where $v(s) = |u(x_i, s) - u(x_i, s+t/2) - u(x_{i+1}, s) + u(x_{i+1}, s+t/2)|$. We denote by $\mathcal{L}^1(\mathcal{S})$ its $1-$ dimensional Lebesgue measure, and find with (17) and (18)

$$
\frac{1}{8}\gamma\theta t^2 \leq \mathcal{L}^1(\mathcal{S}) \cdot 3\gamma\theta t + \mathcal{L}^1((y, y+t/2) \setminus \mathcal{S}) \cdot \frac{1}{8}\gamma\theta t \leq \mathcal{L}^1(\mathcal{S}) \cdot 3\gamma\theta t + \frac{t}{2} \cdot \frac{1}{8}\gamma\theta t,
$$

which implies

$$
\mathcal{L}^1(\mathcal{S}) \geq \frac{t}{48}.
$$

Using that $\frac{1}{4}\gamma\theta(\gamma^2\theta)^i \leq x_i - x_{i+1} \leq \frac{3}{2}\gamma\theta(\gamma^2\theta)^i$ this means that for a subset $(y, y+t/2)$ of size at least $\frac{t}{48}$ it holds

$$
10\,\gamma^2\theta \leq \left| \frac{u(x_i, s) - u(x_{i+1}, s)}{x_i - x_{i+1}} - \frac{u(x_i, s+t/2) - u(x_{i+1}, s+t/2)}{x_i - x_{i+1}} \right| \leq 12 \cdot 120\,\gamma^2\theta.
$$

Therefore there exists a universal constant $c > 0$ such that it holds for all s from a subset of $(y, y+t)$ whose measure is at least $\frac{t}{48}$ that

$$
\left| \left| \frac{u(x_i, s) - u(x_{i+1}, s)}{x_i - x_{i+1}} \right| - 1 \right| \geq c\gamma^2\theta.
$$

Now one can argue as in Step 4 of the proof of [27, Lemma 6] to conclude. We recall the argument in our setting for the convenience of the reader. We assume that for a point s as above there holds $|\partial_{11}u(\cdot, s)|((x_{i+1}, x_i)) < \frac{1}{2}$. Without loss of generality, this implies for almost all $t \in (x_{i+1}, x_i)$,

$$
|\partial_1 u(t, s) - 1| \leq 3 \min \{|\partial_1 u(t, s) - 1|, |\partial_1 u(t, s) + 1|\}.
$$

Then

$$\int\limits_{x_{i+1}}^{x_i} \min\{|\partial_1 u(t,s) - 1|, |\partial_1 u(t,s) + 1|\}^2 \, dt$$

$$\geq \frac{1}{9} \int\limits_{x_{i+1}}^{x_i} |\partial_1 u(t,s) - 1|^2 \, dt$$

$$\geq \frac{1}{9(x_i - x_{i+1})} \left(\int\limits_{x_{i+1}}^{x_i} (\partial_1 u(t,s) - 1) dt \right)^2$$

$$= \frac{1}{9}(x_i - x_{i+1}) \left(\frac{u(x_i, s) - u(x_{i+1}, s)}{x_i - x_{i+1}} - 1 \right)^2$$

$$\geq \frac{1}{9} \cdot \frac{1}{4} \gamma\theta(\gamma^2\theta)^i \cdot c^2\gamma^4\theta^2 \geq \frac{c^2}{36}(\gamma^2\theta)^{i+3} \geq \frac{c^2}{36}\sigma$$

since $i + 3 < k$ and $\sigma < \gamma^4\theta^2(\gamma^2\theta)^k$.

Hence, it follows $E_{\sigma,\gamma,\theta}(u; (x_{i+1}, x_i) \times (y, y+t)) \geq \frac{t}{48} \min\{\frac{c^2}{36}, \frac{1}{2}\}\sigma$, which implies the claim. This concludes the proof of Lemma 2. □

Finally, we consider the parameter regime, in which branching is expected but $|\log(\gamma^2\theta)|$ is of order 1 and does therefore not appear in the energy scaling, c.f. [27, Lemma 4].

Lemma 3 Let $\alpha_0 \in (0, 1)$. Then there exists $c > 0$ such that for all $\gamma^2\theta \geq \alpha_0$ and $\sigma \leq (\gamma^2\theta)^{14}$ it holds

$$E_{\sigma,\gamma,\theta}(u) \geq c\sigma (|\log\sigma| + 1).$$

Proof Assume that $E_{\sigma,\gamma,\theta}(u) \leq \frac{1}{2}\sigma (|\log\sigma| + 1)$ (otherwise there is nothing to show). Let $x \in (0, \gamma\theta/4)$ and $t := \frac{4x}{\gamma\theta} \leq 1$. Let $I_t \subseteq (0, 1)$ be an interval of length t such that $E_{\sigma,\gamma,\theta}(u; \{x\} \times I_t) \leq Ct E_{\sigma,\gamma,\theta}(u; \{x\} \times (0, 1))$ and $E_{\sigma,\gamma,\theta}(u; (0, 1) \times I_t) \leq Ct E_{\sigma,\gamma,\theta}(u)$. Then one of the following statements is true on I_t:

(a) $|\partial_2 \nabla u(x, \cdot)|(I_t) \geq \gamma/2$,
(b) $\min\{|\partial_2 u(x, y) - \gamma|^2, |\partial_2 u(x, y) + \gamma|^2\} \geq \gamma^2/4$ for almost every $y \in I_t$,
(c) $|\partial_2 u(x, y) - \gamma| \leq |\partial_2 u(x, y) + \gamma|$ for almost every $y \in I_t$, or
(d) $|\partial_2 u(x, y) + \gamma| \leq |\partial_2 u(x, y) - \gamma|$ for almost every $y \in I_t$.

If (a) or (b) is true then $E_{\sigma,\gamma,\theta}(u; \{x\} \times (0, 1)) \geq c \min\{\sigma\gamma/t, \gamma^2\theta^2\}$. Assume now that (c) is true. Then it follows from the triangle inequality that

$$\frac{1}{2}\gamma\theta t^2 \leq \min_{a\in\mathbb{R}} \|u(x, y) - \gamma y - a\|_{L^1(I_t)} + \|u(x, y) - (1 - 2\theta)\gamma y\|_{L^1(I_t)}$$

$$\leq t\|\partial_2 u(x, \cdot) - \gamma\|_{L^1(I_t)} + \|u(x, \cdot) - u(0, \cdot)\|_{L^1(I_t)}$$

$$\leq t^{3/2} E_{\sigma,\gamma,\theta}(u; \{x\} \times I_t)^{1/2} + tx + \|\min\{|\partial_1 u - 1|, |\partial_1 u + 1|\}\|_{L^1((0,x)\times I_t)}$$

$$\leq Ct^2 E_{\sigma,\gamma,\theta}(u; \{x\} \times (0, 1))^{1/2} + tx + tx^{1/2} E_{\sigma,\gamma,\theta}(u)^{1/2}.$$

Hence,

$$\frac{1}{4}\gamma\theta t^2 = \frac{1}{2}\gamma\theta t^2 - tx \leq Ct^2 E_{\sigma,\gamma,\theta}(u; \{x\} \times (0, 1))^{1/2} + tx^{1/2} E_{\sigma,\gamma,\theta}(u)^{1/2}.$$

If $\frac{1}{4}\gamma\theta t^2 \leq 2Ct^2 E_{\sigma,\gamma,\theta}(u; \{x\} \times (0,1))^{1/2}$ then $E_{\sigma,\gamma,\theta}(u; \{x\} \times (0,1)) \geq c\gamma^2\theta^2$. On the other hand, if $\frac{1}{4}\gamma\theta t^2 \leq 2tx^{1/2}E_{\sigma,\gamma,\theta}(u)^{1/2}$ then $x \leq 4E_{\sigma,\gamma,\theta}(u) \leq 2\sigma(|\log\sigma|+1)$.

Hence, we have for all $x \in (2\sigma(|\log\sigma|+1), \frac{\gamma\theta}{4})$ that $E_{\sigma,\gamma,\theta}(u; \{x\} \times (0,1)) \geq c\min\{\sigma\gamma^2\theta/(4x), \gamma^2\theta^2\}$. Next, we note that $\sigma\gamma^2\theta/(4x) \leq \gamma^2\theta^2$ if and only if $x \geq \sigma/(4\theta)$. Moreover, observe that $2\sigma(|\log\sigma|+1) \leq \frac{\sigma}{\theta}(|\log\sigma|+1)$. Thus,

$$E_{\sigma,\gamma,\theta}(u) \geq c \int_{\frac{\sigma}{\theta}(|\log\sigma|+1)}^{\gamma\theta/4} \frac{\sigma\gamma^2\theta}{4x}\, dx$$

$$\geq c \int_{\frac{\sigma}{\theta}(|\log\sigma|+1)}^{\gamma\theta/4} \frac{\sigma}{x}\, dx$$

$$\geq c\sigma\left(-\log\left(\frac{4\sigma}{\gamma\theta^2}(|\log\sigma|+1)\right)\right).$$

Note that by assumption we have $\theta \leq 1/2$ and $\sigma \leq \sigma^{1/2}(\gamma^2\theta)^7 \leq \sigma^{1/2}(\gamma\theta)^2\frac{1}{32}$. Thus,

$$\frac{4\sigma}{\gamma\theta^2}(|\log\sigma|+1) \leq \frac{\sigma^{1/2}}{8}(4|\log\sigma^{1/4}|+1) \leq \frac{\sigma^{1/2}}{8}(4\sigma^{-1/4}+1) \leq \sigma^{1/4}.$$

Consequently,

$$E_{\sigma,\gamma,\theta}(u) \geq \frac{c}{4}\sigma|\log\sigma| \geq \frac{c}{8}\sigma(|\log\sigma|+1),$$

since $|\log\sigma| \geq 14 \cdot |\log\theta| \geq 14 \cdot \log 2 \geq 1$. This concludes the proof of Lemma 3. $\qquad\square$

3 Proof of Theorem 2

We now turn to the case of small x-derivatives and prove the scaling laws in Theorem 2.

3.1 Upper bound

To prove the upper bound, we first present all constructions used in the proof and show afterwards in Remark 1 how this result implies the upper bound stated in Theorem 2. Some test functions show similarities in structure with those used in [7, 16, 27, 34].

Before we present the precise statement, let us briefly discuss the heuristics for the upper bound constructions. As in the heuristics in the setting of a small y-derivative, the main ways to meet the boundary conditions are y-derivative $(1-2\theta)$ or quick oscillations of y-derivative $+1$ and -1 with volume fractions $1-\theta$ and θ, respectively, close to $x = 0$. Again, the first option is penalized by the first term of the energy $F_{\sigma,\gamma,\theta}$, whereas the second term penalizes oscillations of the y-derivative. Hence, again for $\sigma > 0$ relatively large uniform structures such as $u(x,y) = \gamma x + (1-2\theta)y$ are energetically favorable, see construction (a) below. Moreover, if θ is much smaller than γ the gradient $(\gamma, 1-2\theta)$ is rank-1 connected to the gradient $(-\gamma, 1) \in \mathcal{M}_\gamma$ over an almost vertical interface, see Fig. 9 and construction (d) below, giving rise to another competitor with y-derivative $1-2\theta$ close to $x = 0$ which turns out to be energetically favorable for moderate values of $\sigma > 0$. The remaining constructions will exploit oscillations close to $x = 0$ and yield low energies for small values of $\sigma > 0$.

Formally, as γ approaches 0, the set $\mathcal{M}_\gamma$ of four preferred gradients collapses to a set with only two preferred gradients. Consequently, it is to be expected that a version of the optimal (in the sense of scaling) constructions for two preferred gradients from [7, 34] play a role in a regime where $\gamma > 0$ is small, see Fig. 6. This construction uses anisotropic rescalings of the building block sketched in Fig. 7 to increase the number of oscillations of the y-derivative towards $x = 0$. In this construction it does not hold $\nabla u \in \mathcal{M}_\gamma$, essentially balancing the two terms in $F_{\sigma,\gamma,\theta}$. On the other hand, isotropic rescalings of the building block lead to $\nabla u \in \mathcal{M}_\gamma$ and hence lower the energy contribution from the first term in $F_{\sigma,\gamma,\theta}$ while increasing the contribution from the second term of $F_{\sigma,\gamma,\theta}$ per refinement step. The construction (c) below exploits that in the way that it starts with refinements through isotropic rescaling of the building block and switches to the anisotropic rescaling of the building block when this is energetically preferable, cf. the energy estimates (22) and (23) and the comment below. A sketch of this construction can be found in Fig. 8. In particular, depending on the parameters this construction transitions either into the Kohn-Müller like construction in (b) or into a construction which mainly uses $\nabla u \in \mathcal{M}_\gamma$, which is closer to the construction from [27]. In parameter regimes where $\gamma > 0$ is larger than $\theta > 0$ it turns out that the anisotropic rescaling of the building block will not play a role leading to branching constructions which essentially satisfy $\nabla u \in \mathcal{M}_\gamma$. Similarly to the heuristics for a small y-derivative it follows that the number of jumps of the y-derivative should be of order $\frac{\theta}{\gamma x}$. This will be satisfied at the end of the construction steps in constructions (e) and (f). The main difference between the remaining constructions (e) and (f) is the following: Glueing different construction steps leads to a contribution from the term $|\partial_1 \partial_1 u|$ of order γ, cf. Fig. 10 (left). On the other hand, the energy contribution of $|\partial_2 \partial_2 u|$ is of order $\frac{\theta}{\gamma}$, cf. Fig. 7. Hence, as long as $\theta \geq \gamma^2$ the energy contribution from $|\partial_2 \partial_2 u|$ necessarily dominates. This leads to construction (e). If $\theta \leq \gamma^2$, we modify the building block in order to balance the energy contributions from $|\partial_1 \partial_1 u|$ and $|\partial_2 \partial_2 u|$ to obtain construction (f), see Fig. 10 (right). Again, the proof of the lower bound suggests that this is necessary.

The following proposition collects the test functions used to prove the upper bound. We label the test functions according to the scaling regimes in Fig. 2. Even if they have the same names, the functions are different from the ones in Proposition 5.

Proposition 5 *Let $c_0 > 0$. Then there is a constant $C > 0$ with the following property: For all $\theta \in (0, 1/2]$, $\gamma \in (0, 1]$, and $\sigma > 0$ the following holds:*

(a) *There is a function $u_{UC} \in \mathcal{B}_\theta$ such that $F_{\sigma,\gamma,\theta}(u_{UC}) \leq C\theta^2$.*

(b) *If $\sigma \leq \theta^2$ and $\gamma \leq c_0(\sigma\theta)^{1/3}$ then there is a function $u_{TG} \in \mathcal{B}_\theta$ such that $F_{\sigma,\gamma,\theta}(u_{TG}) \leq C\sigma^{2/3}\theta^{2/3}$.*

(c) *If $\sigma\theta \leq \gamma^3 \leq \theta^3$ then there is a function $u_{IB} \in \mathcal{B}_\theta$ such that $F_{\sigma,\gamma,\theta}(u_{IB}) \leq C\frac{\sigma\theta}{\gamma}\left(\log\frac{\gamma^3}{\sigma\theta} + 1\right)$.*

(d) *If $\theta \leq \gamma$ then there is a function $u_{RI} \in \mathcal{B}_\theta$ such that $F_{\sigma,\gamma,\theta}(u_{RI}) \leq C\left(\sigma\gamma + \frac{\theta^3}{\gamma}\right)$.*

(e) *If $\sigma \leq \theta^2$ and $\gamma \leq \frac{\theta}{\gamma}$ then there is a function $u_{FG1} \in \mathcal{B}_\theta$ such that $F_{\sigma,\gamma,\theta}(u_{FG1}) \leq C\frac{\sigma\theta}{\gamma}\left(\log\frac{\theta^2}{\sigma} + 1\right)$.*

(f) *If $\theta \leq \gamma^2/2$ and $\sigma \leq 2\theta^3/\gamma^2$ then there is a function $u_{FG2} \in \mathcal{B}_\theta$ such that $F_{\sigma,\gamma,\theta}(u_{FG2}) \leq C\sigma\gamma\left(\frac{|\log(\sigma\gamma^2/\theta^3)|}{|\log(\gamma^2/\theta)|} + 1\right)$.*
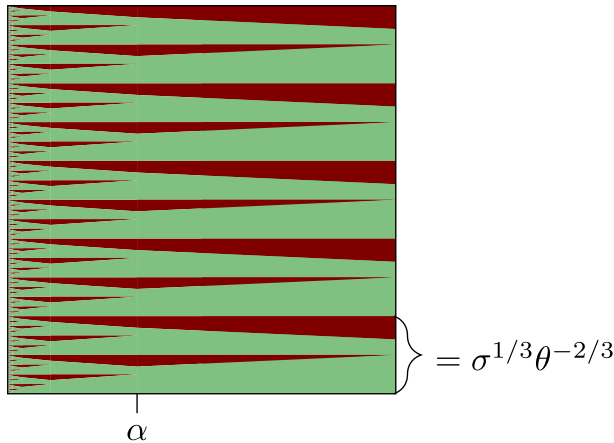
$$= \sigma^{1/3}\theta^{-2/3}$$

$$\alpha$$

Fig. 6 Sketch of the Kohn–Müller like branching construction used in (b)

Proof *(a) Uniform configuration (UC).* Define $u_{\mathrm{UC}} : (0, 1)^2 \to \mathbb{R}$ as $u_{\mathrm{UC}}(x, y) = (1 - 2\theta)y + \gamma x$. Then $u_{\mathrm{UC}} \in \mathcal{B}_\theta$ and

$$F_{\sigma,\gamma,\theta}(u_{\mathrm{UC}}) \le 4\theta^2.$$

(b) Kohn-Müller-type two-gradient branching (TG).

We assume $\sigma \le \theta^2$. Let $\delta = \sigma^{1/3}\theta^{-2/3}$, $\alpha = 2^{-3/2}$ and $N = \left\lceil \frac{\log(\theta^{1/3}\sigma^{-2/3})}{\log 2} \right\rceil \ge 1$. Note that $2\theta^{1/3}\sigma^{-2/3} \ge 2^N \ge \theta^{1/3}\sigma^{-2/3}$ since $\theta^{1/3}\sigma^{-2/3} \ge \theta^{-1} \ge 2$. Consider the function $W := (W_1, W_2)^T : (\alpha, 1] \times \mathbb{R} \to \mathbb{R}^2$ defined as

$$W(x, y) = \begin{cases} \begin{pmatrix} -\frac{\delta\theta}{2(1-\alpha)} \\ 1 \end{pmatrix} & \text{if } 0 \le y \le \frac{\delta}{2} + (x - 1)\frac{\delta\theta}{2(1-\alpha)} \\[2mm] \begin{pmatrix} \frac{\delta\theta}{2(1-\alpha)} \\ -1 \end{pmatrix} & \text{if } \frac{\delta}{2} + (x - 1)\frac{\delta\theta}{2(1-\alpha)} \le y \le \frac{\delta}{2} \\[2mm] \begin{pmatrix} \frac{\delta\theta}{2(1-\alpha)} \\ 1 \end{pmatrix} & \text{if } \frac{\delta}{2} \le y \le \delta(1 - \theta) - (x - 1)\frac{\delta\theta}{2(1-\alpha)} \\[2mm] \begin{pmatrix} -\frac{\delta\theta}{2(1-\alpha)} \\ -1 \end{pmatrix} & \text{if } \delta(1 - \theta) - (x - 1)\frac{\delta\theta}{2(1-\alpha)} \le y \le \delta, \end{cases}$$

and extended periodically to $\mathbb{R}$ in the y-component. Then $W_2(\alpha, y) = W_2(1, 2y)$ for all $y \in \mathbb{R}$. We define $U : (\alpha^N, 1) \times (0, 1) \to \mathbb{R}^2$ as

$$U(x, y) = \begin{pmatrix} 2^{-k}\alpha^{-k} W_1(\alpha^{-k}x, 2^k y) \\ W_2(\alpha^{-k}x, 2^k y) \end{pmatrix} \qquad \text{if } x \in (\alpha^{k+1}, \alpha^k].$$

Then U is a gradient field, and we and denote by $\tilde{u} : (\alpha^N, 1) \times (0, 1) \to \mathbb{R}$ the corresponding primitive with $\tilde{u}(\alpha^N, 0) = 0$. Eventually define $u_{\mathrm{TG}} : (0, 1)^2 \to \mathbb{R}$ as

$$u_{\mathrm{TG}}(x, y) = \begin{cases} \tilde{u}(x, y) & \text{if } x \ge \alpha^N, \\ (x - \alpha^N)\alpha^{-N}(1 - 2\theta)y - x\alpha^{-N}\tilde{u}(\alpha^N, y) & \text{else.} \end{cases}$$

We remark that it holds $|\tilde{u}(\alpha^N, y) - (1 - 2\theta)y| \leq \theta\delta 2^{-N}$. Consequently, we find for $x \leq \alpha^N$

$$|\partial_1 u_{\text{TG}}(x, y)| \leq \frac{\theta\delta}{2^N \alpha^N} \leq 2^{N/2}\theta\delta \leq 2\theta^{1/6}\sigma^{-1/3}\theta\delta = 2\theta^{1/2}. \tag{19}$$

Additionally, we estimate for $1 \leq k \leq N$, using $(2\alpha)^k = 2^{-k/2}$ and $\gamma \leq c_0(\sigma\theta)^{1/3}$

$$\int_{\alpha^k}^{\alpha^{k-1}} \int_0^1 \text{dist}^2(\nabla u_{\text{TG}}, \mathcal{M}_\gamma)\, dx\, dy + \sigma|D^2 u_{\text{TG}}|([\alpha^k, \alpha^{k-1}] \times (0, 1))$$

$$\leq \int_{\alpha^k}^{\alpha^{k-1}} \int_0^1 \text{dist}^2(\nabla u_{\text{TG}}, \mathcal{M}_\gamma)\, dx\, dy$$

$$+ C\sigma\left(|\partial_{12} u_{\text{TG}}| + |\partial_{11} u_{\text{TG}}| + |\partial_{22} u_{\text{TG}}|\right)([\alpha^k, \alpha^{k-1}] \times (0, 1))$$

$$\leq \int_{\alpha^k}^{\alpha^{k-1}} \int_0^1 |U_1(x, y) \pm \gamma|^2\, dx\, dy + C\sigma\left(\theta + 2^{k/2}\delta\theta + (1 - \alpha)\alpha^{k-1}2^k\delta^{-1}\right)$$

$$\leq (1 - \alpha)\alpha^{k-1}\gamma^2\left(2^{k/2}\frac{\delta\theta}{2\gamma}(1 - \alpha) - 1\right)^2 + C\sigma\left(\theta + 2^{k/2}\sigma^{1/3}\theta^{1/3} + 2^{-k/2}\sigma^{-1/3}\theta^{2/3}\right)$$

$$\leq C\left(2^{-k/2}\sigma^{2/3}\theta^{2/3} + 2^{k/2}\sigma^{4/3}\theta^{1/3} + \sigma\theta\right)$$

$$\leq C 2^{-k/2}\sigma^{2/3}\theta^{2/3},$$

where we used in the last step that by the definition of N, we have $2^{k/2} \leq 2^{-k/2}2^N \leq 2 \cdot 2^{-k/2}\theta^{1/3}\sigma^{-2/3}$, and $2^{-k/2}\sigma^{2/3}\theta^{2/3} \geq 2^{-N/2}\sigma^{2/3}\theta^{2/3} \geq \sigma\theta^{1/2}/2 \geq \sigma\theta/2$. Hence, we obtain, using the definitions of N, α and δ, as well as the estimates $\gamma \leq c_0(\sigma\theta)^{1/3} \leq c_0\theta \leq c_0\theta^{1/2}$ and $\sigma \leq \theta^2$, and (19)

$$E_{\sigma,\gamma,\theta}(u_{\text{TG}})$$

$$\leq C\sum_{k=1}^N 2^{-k/2}\sigma^{2/3}\theta^{2/3} + \int_0^1 \int_0^{\alpha^N} \text{dist}(\nabla u_{\text{TG}}, \mathcal{M}_\gamma)^2\, dx\, dy$$

$$+ \sigma|D^2 u_{\text{TG}}|((0, \alpha^N] \times (0, 1))$$

$$\leq C\sum_{k=1}^N 2^{-k/2}\sigma^{2/3}\theta^{2/3} + 2\int_0^1 \int_0^{\alpha^N} |\partial_1 u_{\text{TG}}|^2 + \gamma^2 + \min\left\{|\partial_2 u_{\text{TG}} \pm 1|^2\right\}\, dx\, dy$$

$$+ \sigma|D^2 u_{\text{TG}}|((0, \alpha^N] \times (0, 1))$$

$$\leq C\sigma^{2/3}\theta^{2/3} + C\alpha^N\theta + C\alpha^N\gamma^2 + C\alpha^N\theta^2 + C\alpha^N\theta$$

$$+ C\sigma\left(1 + 2^{N/2}\delta\theta + \alpha^N 2^N + C\alpha^N\theta^{1/2}\right)$$

$$\leq C\sigma^{2/3}\theta^{2/3} + C\theta(2^N)^{-3/2} + C\sigma$$

$$\leq C\sigma^{2/3}\theta^{2/3}.$$

(c) Intermediate Branching (IB).
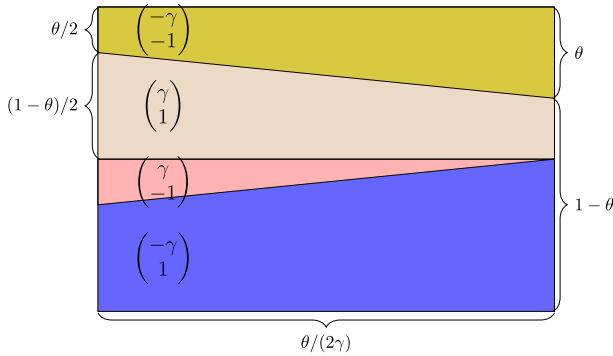We assume now that $\sigma\theta \leq \gamma^3 \leq \theta^3$, and define a branching construction as follows.

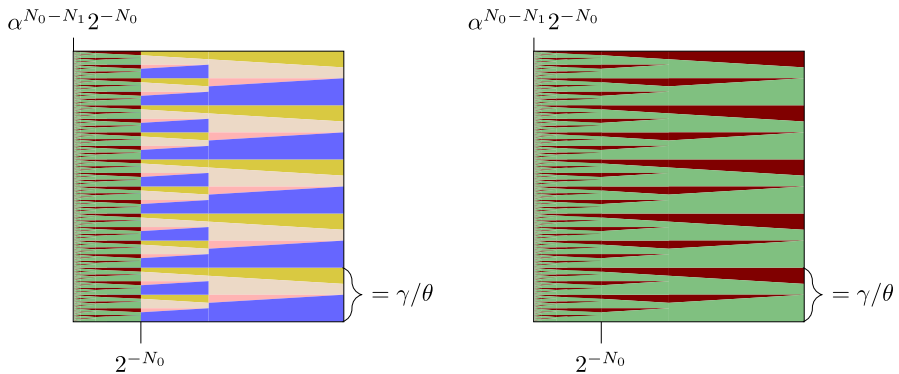**Fig. 7** Sketch of the building block in construction (c) an (e)

Fig. 8 Left: Sketch of the branching construction in the intermediate regime: The regions of the four preferred gradients in the isotropically rescaled building blocks are colored in blue, pink, yellow and beige. In the anisotropically rescaled building blocks, $\partial_1 u \neq \pm \gamma$ but $\partial_2 u = 1$ (dark red) or $\partial_2 u = -1$ (light green). Right: Same construction: $\partial_2 u = 1$ (dark red) and $\partial_2 u = -1$ (light green)

Step 1: Definition of the building block

  Consider the function $V : [\frac{\theta}{2\gamma}, \frac{\theta}{\gamma}] \times \mathbb{R} \to \mathbb{R}^2$ defined for $(x, y) \in [\frac{\theta}{2\gamma}, \frac{\theta}{\gamma}] \times (0, 1)$ as (see Fig. 7)

$$V(x, y) = \begin{cases} \begin{pmatrix} -\gamma \\ -1 \end{pmatrix} & \text{if } y \geq 1 - \theta/2 - \gamma(x - \frac{\theta}{2\gamma}), \\[2mm] \begin{pmatrix} \gamma \\ 1 \end{pmatrix} & \text{if } 1 - \theta/2 - \gamma(x - \frac{\theta}{2\gamma}) \geq y \geq 1/2, \\[2mm] \begin{pmatrix} \gamma \\ -1 \end{pmatrix} & \text{if } 1/2 \geq y \geq 1/2 - \theta/2 + \gamma(x - \frac{\theta}{2\gamma}), \\[2mm] \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} & \text{if } y \leq 1/2 - \theta/2 + \gamma(x - \frac{\theta}{2\gamma}). \end{cases} \tag{20}$$

and extended periodically in the y-component. Then V is a gradient field and it holds for the second component V_2 that $V_2(\theta/(2\gamma), y) = V_2(\theta/\gamma, 2y)$ for all $y \in \mathbb{R}$.

Step 2: Definition branching gradient Let $N_0 = \left\lceil \dfrac{\log\left(\frac{\gamma^3}{\sigma\theta}\right)}{\log 2} \right\rceil \geq 1$ and $N_1 = \left\lceil \dfrac{\left(\log\left(\frac{\theta}{\gamma^2}\right)\right)}{\log 2} \right\rceil +$

$N_0 \geq N_0 + 1$. Note that $\frac{\gamma^3}{\sigma\theta} \leq 2^{N_0} \leq 2\frac{\gamma^3}{\sigma\theta}$ and $\frac{\theta}{\gamma^2} \leq 2^{N_1-N_0} \leq 2\frac{\theta}{\gamma^2}$. In addition, set $\alpha = 2^{-3/2}$ as in (b). Then we define the function $U : \left(\alpha^{N_1-N_0}2^{-N_0}, 1\right) \times \mathbb{R} \to \mathbb{R}^2$ as follows, see Fig. 8. If $x \in (2^{-N_0}, 1)$ we set

$$U(x, y) = V(2^N \theta\gamma^{-1}x, 2^N\theta\gamma^{-1}y) \text{ if } x \in (2^{-N-1}, 2^{-N}].$$

Moreover, for $N_1 > N \geq N_0$ we define for $x \in \left(2^{-N_0}\alpha^{N-N_0+1}, 2^{-N_0}\alpha^{N-N_0}\right)$

$$U(x, y) = \begin{pmatrix} \alpha^{-N+N_0}2^{-N+N_0}V_1\left(\alpha^{-N+N_0}2^{N_0}\frac{\theta}{\gamma}x, 2^{N-N_0}\frac{\theta}{\gamma}y\right) \\ V_2\left(\alpha^{N-N_0}2^{N_0}\frac{\theta}{\gamma}x, 2^{N-N_0}\frac{\theta}{\gamma}y\right) \end{pmatrix}.$$

Note that for $2^{-N_0}\alpha^{N-N_0+1} \leq x \leq 2^{-N_0}\alpha^{N-N_0}$ we have

$$U_1(x, y) = \pm 2^{\frac{N-N_0}{2}}\gamma. \tag{21}$$

In addition we always have that $U_2(x, y) \in \{\pm 1\}$. Moreover, note that U is a gradient.

Step 3: Definition of the branching:

Let $\tilde{u} : \left(2^{-N_0}\alpha^{N_1-N_0}, 1\right) \times (0, 1) \to \mathbb{R}$ be a corresponding primitive such that $\tilde{u}(2^{-N_0}\alpha^{N_1-N_0}, 0) = 0$. Note that $2^{-N_0}\alpha^{N_1-N_0} \gtrsim \sigma/(2^{5/2}\theta^{1/2}) \geq \sigma/8$. Eventually, we define $u_{\text{IB}} : (0, 1)^2 \to \mathbb{R}$ as

$$u_{\text{IB}}(x, y) = \begin{cases} \tilde{u}(x, y) & \text{if } x \geq 2^{-N_0}\alpha^{N_1-N_0}, \\ (x - \sigma)\frac{1}{\sigma}(1 - 2\theta)y - \frac{x}{\sigma}\tilde{u}\left(2^{-N_0}\alpha^{N_1-N_0}, y\right) & \text{if } x \leq 2^{-N_0}\alpha^{N_1-N_0}. \end{cases}$$

Step 4: Energy estimates:

For $N_0 \geq N$, we note that $\nabla u_{\text{IB}} \in \mathcal{K}_\gamma$ a.e. in $[2^{-N}, 2^{-N+1}] \times (0, 1)$, and thus

$$\int_{2^{-N}}^{2^{-N+1}} \int_0^1 \text{dist}(\nabla u_{\text{IB}}, \mathcal{M}_\gamma)^2 \, dy \, dx + \sigma|D^2 u_{\text{IB}}|([2^{-N}, 2^{-N+1}] \times (0, 1))$$

$$\leq C \left(|\partial_{11}u_{\text{IB}}| + |\partial_{12}u_{\text{IB}}| + |\partial_{22}u_{\text{IB}}|\right)([2^{-N}, 2^{-N+1}] \times (0, 1))$$

$$\leq C\sigma\left(\gamma + \theta + \frac{\theta}{\gamma}\right)$$

$$\leq C\frac{\sigma\theta}{\gamma}. \tag{22}$$

For $N_0 < N \leq N_1$ we compute using (21) and $\gamma \leq \theta$

$$\int_{2^{-N_0}\alpha^{N-N_0}}^{2^{-N_0}\alpha^{N-N_0-1}} \int_0^1 \text{dist}(\nabla u_{\text{IB}}, \mathcal{M}_\gamma)^2 \, dy \, dx$$

$$+ \sigma|D^2 u_{\text{IB}}|\left(\left[2^{-N_0}\alpha^{N-N_0}, 2^{-N_0}\alpha^{N-N_0-1}\right] \times (0, 1)\right)$$

$$\leq 2^{-N_0}\alpha^{N-N_0}(1-\alpha)2^{N-N_0}\gamma^2 + C\sigma\left((2\alpha)^{-N+N_0}\gamma + \theta + 2^{-N_0}\alpha^{N-N_0}(1-\alpha)2^N\frac{\theta}{\gamma}\right)$$

$$\leq C(2\alpha)^{N-N_0}\frac{\sigma\theta}{\gamma} + C\sigma\theta, \tag{23}$$

where we used that $(2\alpha)^{-N+N_0}\gamma \leq (2\alpha)^{-N_1+N_0}\gamma \leq C\frac{\gamma^2}{\theta^{1/2}} \leq \theta$. Comparing (22) and (23) we note that for $N > N_0$ the anisotropic rescaling of the building block yields a smaller energy per refinement step than the isotropic rescaling.

Next, note that $|u_{\text{IB}}(2^{-N_0}\alpha^{N_1-N_0}, y) - (1-2\theta)y| \leq 2 \cdot 2^{-N_1}\frac{\gamma}{\theta}\theta$. Consequently, we have for $x \in (0, 2^{-N_0}\alpha^{N_1-N_0})$ the bounds

$$|\partial_1 u_{\text{IB}}(x, y)| \leq \frac{2 \cdot 2^{-N_1}\gamma}{2^{-N_0}\alpha^{N_1-N_0}} = 2 \cdot (2\alpha)^{N_0-N_1}\gamma.$$

Hence, since $2 \cdot (2\alpha)^{N_0-N_1}\gamma \geq \gamma$,

$$\int_0^{2^{-N_0}\alpha^{N_1-N_0}}\int_0^1 \text{dist}(\nabla u_{\text{IB}}, \mathcal{M}_\gamma)^2\, dy\, dx + \sigma|D^2 u_{\text{IB}}|((0, 2^{-N_0}\alpha^{N_1-N_0}))$$

$$\leq \left((2 \cdot (2\alpha)^{N_0-N_1}\gamma)^2 + 4\theta^2(1-\theta) + 2\theta\right)(2^{-N_0}\alpha^{N_1-N_0})$$

$$+ C\sigma\left(\theta + 2^{-N_0}\alpha^{N_1-N_0}2^{N_1}\frac{\theta}{\gamma}\right)$$

$$\leq C\sigma\theta^{1/2} \leq C\frac{\sigma\theta}{\gamma}.$$

Combining the various estimates we obtain

$$F_{\sigma,\gamma,\theta}(u_{\text{IB}}) \leq C\left(N_0\frac{\sigma\theta}{\gamma} + \frac{\sigma\theta}{\gamma}\sum_{N=N_0+1}^{N_1}2^{(-N+N_0)/2} + (N_1 - N_0)\sigma\theta\right)$$

$$\leq C\frac{\sigma\theta}{\gamma}\left(\log\frac{\gamma^3}{\sigma\theta} + 1\right),$$

where we used that $N_1 - N_0 \leq C2^{(N_1-N_0)/2} \leq C\frac{\theta^{1/2}}{\gamma} \leq C/\gamma$.

(d): Rotated interface (RI). We assume $\theta \leq \gamma$, and use the construction sketched in Fig. 9. Precisely, we set

$$u_{\text{RI}}(x, y) = \begin{cases} \gamma x + (1-2\theta)y & \text{if } \theta y \geq \gamma x, \\ -\gamma x + y & \text{if } \theta y \leq \gamma x. \end{cases}$$

Then

$$F_{\sigma,\gamma,\theta}(u_{\text{RI}}) \leq 2\frac{\theta^3}{\gamma} + C\sigma\left(\gamma + \theta + \frac{\theta^2}{\gamma}\right) \leq C\left(\frac{\theta^3}{\gamma} + \sigma\gamma\right),$$

where we used that $\theta \leq \gamma$.

(e): Four-gradient branching without linear interpolation (FG1). We assume that $\sigma \leq \theta^2$. We use a branching construction and a variant of the construction in (d) instead of interpolation, see Fig. 10. Precisely, we consider the functions $V : (\frac{\theta}{2\gamma}, \frac{\theta}{\gamma}) \times \mathbb{R} \to \mathbb{R}^2$ as

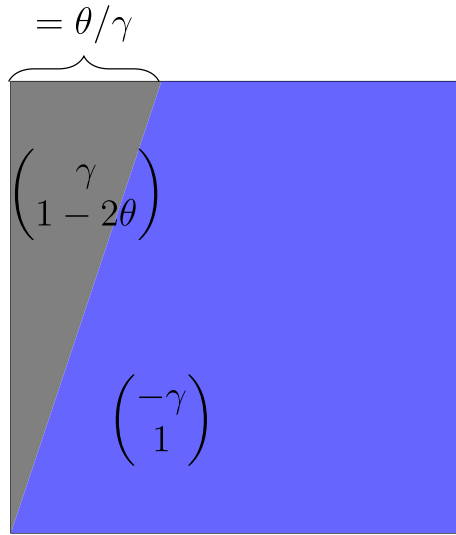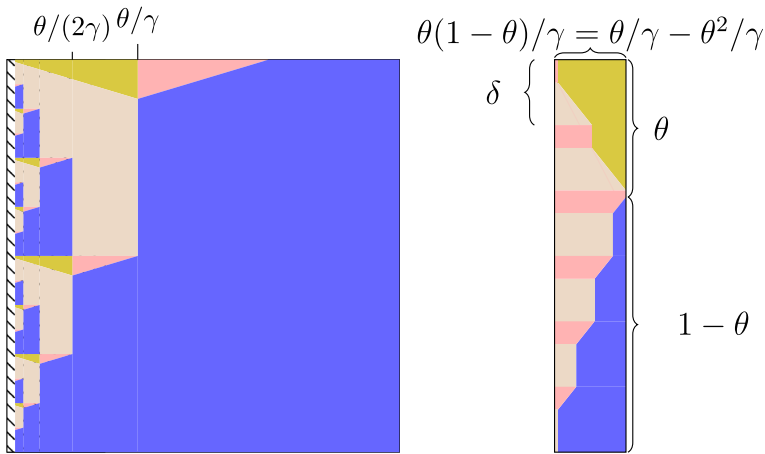**Fig. 9** Sketch of the construction with a rotated interface used in (d)

Fig. 10 Sketch of the branching construction as described in (e) and sketch of the building block as described in (f)

defined in (20). Additionally, consider $W : (0, \frac{\theta}{\gamma}) \times \mathbb{R} \to \mathbb{R}^2$ defined as

$$
W(x, y) = \begin{cases} \begin{pmatrix} -\gamma \\ -1 \end{pmatrix} & \text{if } y \geq 1 - \gamma x, \\ \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} & \text{if } y \leq (1 - \theta)\frac{\gamma x}{\theta}, \\ \begin{pmatrix} (1 - 2\theta)\gamma \\ 1 - 2\theta \end{pmatrix} & \text{else,} \end{cases} \tag{24}
$$

and extend W periodically to $\mathbb{R}$ in the y-variable. Note that W is a gradient field.

For $N \in \mathbb{N}$ define $U_N : (0,1)^2 \to \mathbb{R}^2$ as

$$U_N(x,y) = \begin{cases} W(2^N x, 2^N y) & \text{if } x \le 2^{-N}\frac{\theta}{\gamma} \\ V(2^k x, 2^k y) & \text{if } x \in \left(2^{-k-1}\frac{\theta}{\gamma}, 2^{-k}\frac{\theta}{\gamma}\right), 0 \le k \le N-1 \\ \begin{pmatrix} \gamma \\ -1 \end{pmatrix} & \text{if } x \ge \frac{\theta}{\gamma}, y \ge 1-\theta + \gamma\left(x - \frac{\theta}{\gamma}\right), \\ \begin{pmatrix} -\gamma \\ 1 \end{pmatrix} & \text{if } x \ge \frac{\theta}{\gamma}, y \le 1-\theta + \gamma\left(x - \frac{\theta}{\gamma}\right). \end{cases}$$

Note that also U_N is a gradient field. Let $u_{FG1} : (0,1)^2 \to \mathbb{R}$ be a corresponding primitive such that $u_{FG1}(0,0) = 0$. Then $u_{FG1} \in \mathcal{B}_\theta$ and, using $\gamma \le \theta$,

$$F_{\sigma,\gamma,\theta}(u_{FG1}) \le C 2^{-N}\left(\frac{\theta^3}{\gamma} + \frac{\gamma^2\theta^3}{\gamma}\right) + C\sigma N\left(\gamma + \theta + \frac{\theta}{\gamma}\right) \le C\left(2^{-N}\frac{\theta^3}{\gamma} + \frac{\sigma\theta}{\gamma}N\right).$$

Choosing $N = \left\lceil \frac{\log\frac{\theta^2}{\sigma}}{\log 2} \right\rceil$ leads to $F_{\sigma,\gamma,\theta}(u_{FG1}) \le C\frac{\sigma\theta}{\gamma}\left(\log\frac{\theta^2}{\sigma} + 1\right)$.

(f) Variant of four-gradient branching (FG2). We assume that $\theta \le \gamma^2/2$ and $\sigma \le \theta^3/(2\gamma^2)$. Let $\delta = (\lceil\frac{\gamma^2}{\theta}\rceil)^{-1}$. Similarly to the branching construction in Proposition 3, one can construct a function $V : (\theta^2/\gamma, \theta/\gamma) \times \mathbb{R} \to \mathbb{R}^2$ such that (see Fig. 10)

1. V is 1-periodic in the second variable,
2. V is a gradient field,
3. $V(x,y) \in \mathcal{K}_\gamma$ for a.e. $(x,y) \in (\theta^2/\gamma, \theta/\gamma) \times \mathbb{R}$,
4. $|D^2 V|((\theta^2/\gamma, \theta/\gamma) \times (0,1)) \le C\gamma$,
5. $V(\frac{\theta}{\gamma}, y) = \chi_{\{0 \le y \le 1-\theta\}}(y) - \chi_{\{1-\theta \le y \le 1\}}(y)$ for $y \in (0,1)$, and
1. $V_2(\theta^2/\gamma, y) = V_2(\theta/\gamma, \delta^{-1}y)$ for all $y \in \mathbb{R}$.

Then we define for $N \in \mathbb{N}$ the function $V_N : (0,1)^2 \to \mathbb{R}^2$ as

$$V_N(x,y) = \begin{cases} \begin{pmatrix} (1-2\theta)\gamma \\ 1-2\theta \end{pmatrix} & \text{if } x \in (0, \delta^N\frac{\theta}{\gamma}\frac{\delta-\theta}{1-\delta}), \\ W(\delta^{-N}x - \frac{\theta}{\gamma}\frac{\delta-\theta}{1-\delta}, \delta^{-N}y) & \text{if } x \in \left(\delta^N\frac{\theta}{\gamma}\frac{\delta-\theta}{1-\delta}, \delta^N\frac{\theta(1-\theta)}{\gamma(1-\delta)}\right), \\ V(\delta^{-k+1}x - \frac{\theta}{\gamma}\frac{\delta-\theta}{1-\delta}, \delta^{-k+1}y) & \text{if } x \in \left(\delta^k\frac{\theta(1-\theta)}{\gamma(1-\delta)}, \delta^{k-1}\frac{\theta(1-\theta)}{\gamma(1-\delta)}\right) \\ & \text{for } 1 \le k \le N, \\ \begin{pmatrix} \gamma \\ -1 \end{pmatrix} & \text{if } x \ge \frac{\theta(1-\theta)}{\gamma(1-\delta)} \text{ and } y \ge 1-\theta, \\ \begin{pmatrix} \gamma \\ 1 \end{pmatrix} & \text{if } x \ge \frac{\theta(1-\theta)}{\gamma(1-\delta)} \text{ and } y \le 1-\theta, \end{cases}$$

where the function W is defined in (24). As before note that V_N is a gradient field. Then let $u_{FG2} : (0,1)^2 \to \mathbb{R}$ be a corresponding primitive with $u_{FG2}(0,0) = 0$. Note that $u_N \in \mathcal{B}_\theta$. Moreover, we estimate the corresponding energy

$$F_{\sigma,\gamma,\theta}(u_{FG2}) \le C\theta^2\delta^N\frac{\theta(1-\theta)}{\gamma(1-\delta)} + C\sigma\left(\gamma N + \frac{\theta}{\gamma} + \theta\right) \le C\left(\frac{\theta^3}{\gamma}\delta^N + \gamma\sigma N\right).$$

Choosing $N = \lceil \frac{|\log \sigma \gamma^2/\theta^3|}{|\log \gamma^2/\theta|} \rceil$ yields the estimate

$$F_{\sigma,\gamma,\theta}(u_{\text{FG2}}) \leq C\gamma\sigma \left(\frac{|\log \sigma \gamma^2/\theta^3|}{|\log \gamma^2/\theta|} + 1 \right).$$

□

We are now in the position to prove the upper bound in Theorem 2.

Corollary 1 *There is a constant $C > 0$ such that the following assertions hold:*

1. *If $\gamma \leq \theta/8$ then*

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \leq C \min \left\{ \theta^2, \sigma^{2/3}\theta^{2/3}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma\theta}{\gamma^3} \right| + 1 \right) \right\}.$$

2. *If $0 < \gamma^2/2 \leq \theta/8 \leq \gamma$ then*

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \leq C \min \left\{ \theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma}{\theta^2} \right| + 1 \right) \right\}.$$

3. *If $0 < \theta/8 \leq \gamma^2/2$ then*

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \leq C \min \left\{ \theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \sigma\gamma \left(\frac{|\log \sigma \gamma^2/\theta^3|}{|\log \gamma^2/\theta|} + 1 \right) \right\}.$$

Proof

1. Consider $\gamma \leq \theta/8$.

   - If $\min \left\{ \theta^2, \sigma^{2/3}\theta^{2/3}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma\theta}{\gamma^3} \right| + 1 \right) \right\} = \theta^2$, then the assertion follows from Proposition 5(a).
   - If $\min \left\{ \theta^2, \sigma^{2/3}\theta^{2/3}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma\theta}{\gamma^3} \right| + 1 \right) \right\} = \sigma^{2/3}\theta^{2/3}$ then $\sigma \leq \theta^2$ (since $\sigma^{2/3}\theta^{2/3} \leq \theta^2$) and $\gamma^3 \leq \sigma\theta$ (since $\sigma^{2/3}\theta^{2/3} \leq \frac{\sigma\theta}{\gamma} \left(|\log \frac{\sigma\theta}{\gamma^3}| + 1 \right)$ implies that $\frac{\sigma\theta}{\gamma^3} \left(|\log \frac{\sigma\theta}{\gamma^3}| + 1 \right)^3 \geq 1$). Hence, the assertion follows from Proposition 5(b).
   - If $\min \left\{ \theta^2, \sigma^{2/3}\theta^{2/3}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma\theta}{\gamma^3} \right| + 1 \right) \right\} = \frac{\sigma\theta}{\gamma} \left(|\log \frac{\sigma\theta}{\gamma^3}| + 1 \right)$ then $\sigma\theta \leq \gamma^3$ (since $\frac{\sigma\theta}{\gamma} \left(|\log \frac{\sigma\theta}{\gamma^3}| + 1 \right) \leq \sigma^{2/3}\theta^{2/3}$) and $\gamma \leq \theta/8 \leq \theta$, and hence the assertion follows from Proposition 5(c).

2. Consider $\gamma^2/2 < \theta/8 \leq \gamma$.

   - If $\min \left\{ \theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma}{\theta^2} \right| + 1 \right) \right\} = \theta^2$, the assertion follows from Proposition 5(a).
   - If $\min \left\{ \theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma}{\theta^2} \right| + 1 \right) \right\} = \sigma\gamma + \theta^3/\gamma$ then $\theta \leq \gamma$ (since $\theta^3/\gamma \leq \theta^2$), and the assertion follows from Proposition 5(d).
   - If $\min \left\{ \theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma}{\theta^2} \right| + 1 \right) \right\} = \frac{\sigma\theta}{\gamma} \left(|\log \frac{\sigma}{\theta^2}| + 1 \right)$ then $\sigma \leq 2\theta^2$ (since $\frac{\sigma\theta}{\gamma} \left(|\log \frac{\sigma}{\theta^2}| + 1 \right) \leq \sigma\gamma + \theta^3/\gamma$ implies that $\frac{1}{2}\frac{\sigma\theta}{\gamma} \leq \theta^3/\gamma$). If $\sigma \leq \theta^2$ then the assertion follows from Proposition 5(e). If $\theta^2 \leq \sigma \leq 2\theta^2$ and $\theta \leq \gamma$ then the assertion follows from Proposition 5(d) since $\sigma\gamma + \frac{\theta^3}{\gamma} \leq 2\frac{\sigma\theta}{\gamma}$. Eventually, if $\theta^2 \leq \sigma \leq 2\theta^2$ and $\gamma \leq \theta$ then the assertion follows from 5(a) since $\theta^2 \leq \frac{\sigma\theta}{\gamma}$.

3. Consider $\theta/8 < \gamma^2/2$.

   - If $\min\left\{\theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \sigma\gamma\left(\frac{|\log\sigma\gamma^2/\theta^3|}{|\log\gamma^2/\theta|} + 1\right)\right\} = \theta^2$ then the assertion follows from Proposition 5(a).
   - If $\min\left\{\theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \sigma\gamma\left(\frac{|\log\sigma\gamma^2/\theta^3|}{|\log\gamma^2/\theta|} + 1\right)\right\} = \sigma\gamma + \frac{\theta^3}{\gamma}$ then $\theta \leq \gamma$ (since $\theta^3/\gamma \leq \theta^2$) and the assertion follows from Proposition 5(d).
   - If $\min\left\{\theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \sigma\gamma\left(\frac{|\log\sigma\gamma^2/\theta^3|}{|\log\gamma^2/\theta|} + 1\right)\right\} = \sigma\gamma\left(\frac{|\log\sigma\gamma^2/\theta^3|}{|\log\gamma^2/\theta|} + 1\right)$ then $\sigma \leq \theta^3/\gamma^2$ (since $\sigma\gamma \leq \theta^3/\gamma$). If $\sigma \leq \theta^3/(2\gamma^2)$ and $\theta \leq \gamma^2/2$ then the assertion follows from Proposition 5(f). Note that the assumption $\theta \leq 4\gamma^2$ always implies that $\theta \leq \gamma$ (if $4\gamma^2 \geq \gamma$ then $\gamma \geq 1/2 \geq \theta$). Therefore, if $\theta^3/(2\gamma^2) \leq \sigma \leq \theta^3/\gamma^2$ then the assertion follows from Proposition 5(d).

$\square$

3.2 Lower bound

The proof of the lower bound is again split in several steps. In the following proposition, we outline how they imply the assertion in all parameter regimes.

Proposition 6 *There is a constant $c > 0$ such that for all $\sigma \in (0, \infty)$, all $\gamma \in (0, 1)$, and $\theta \in (0, 1/2]$, the following statements hold:*

1. *If $\gamma \leq \theta/8$, then*

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \geq c \min\left\{\theta^2, \sigma^{2/3}\theta^{2/3}, \frac{\sigma\theta}{\gamma}\left(\left|\log\frac{\sigma\theta}{\gamma^3}\right| + 1\right)\right\}.$$

2. *If $\gamma^2/2 \leq \theta/8 < \gamma$, then*

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \geq c \min\left\{\theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \frac{\sigma\theta}{\gamma}\left(\left|\log\frac{\sigma}{\theta^2}\right| + 1\right)\right\}.$$

3. *If $\theta/8 < \gamma^2/2$, then*

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \geq c \min\left\{\theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \sigma\gamma\left(\frac{|\log(\sigma\gamma^2/\theta^3)|}{|\log(\gamma^2/\theta)|} + 1\right)\right\}.$$

Proof

1. The first statement is proven in Lemma 5(1).
2. For the second statement, we consider the cases $\sigma \leq \theta^2/\gamma$ and $\sigma > \theta^2/\gamma$ separately. If $\sigma \leq \theta^2/\gamma$ then $\sigma\gamma + \theta^3/\gamma \leq 9\theta^2$, and the assertion follows from the estimate proven in Lemma 5(2), namely

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \geq c \min\left\{\sigma\gamma + \frac{\theta^3}{\gamma}, \frac{\sigma\theta}{\gamma}\left(\left|\log\frac{\sigma}{\theta^2}\right| + 1\right)\right\}.$$

If $\sigma \geq \theta^2/\gamma$, then $\theta^2 \leq \sigma\gamma \leq \sigma\gamma + \theta^3/\gamma$, and $\sigma\theta/\gamma \geq \frac{\theta^2}{\gamma}\frac{\theta}{\gamma} \geq (\gamma\theta)\frac{\theta}{\gamma} = \theta^2$. Hence, the assertion follows from the lower bound in Lemma 4, namely

$$\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \geq c \min\left\{\theta^2, \sigma\gamma\right\} = c\theta^2.$$

3. For the third statement, we consider three cases separately, depending on the size of σ.

- If $\sigma < (\theta^3/\gamma^2)(\theta/\gamma^2)^{32}$ then there exists $k \in \mathbb{N}$, $k \geq k_0 = 32$ such that

$$
\sigma \in \left[\frac{\theta^3}{\gamma^2} \left(\frac{\theta}{\gamma^2} \right)^{k+1}, \frac{\theta^3}{\gamma^2} \left(\frac{\theta}{\gamma^2} \right)^k \right).
$$

By Lemma 6, we obtain the lower bound $\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \geq ck\sigma\gamma$, which yields the claimed lower bound, observing that $2k > k + 2 \geq \log(\sigma\gamma^2/\theta^3)/\log(\theta/\gamma^2) + 1$.

- If $(\theta^3/\gamma^2)(\theta/\gamma^2)^{32} \leq \sigma < \theta^3/\gamma^2$ then we have by Lemma 4 the lower bound $\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \geq c \min\{\theta^2, \sigma\gamma\} = c\sigma\gamma$. This concludes the proof in this case since

$$
\min \left\{ \theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \sigma\gamma \left(\frac{|\log(\sigma\gamma^2/\theta^3)|}{|\log(\gamma^2/\theta)|} + 1 \right) \right\} \leq \sigma\gamma \left(\frac{|\log(\sigma\gamma^2/\theta^3)|}{|\log(\gamma^2/\theta)|} + 1 \right)
$$

$$
\leq \sigma\gamma \left(\frac{32|\log(\gamma^2/\theta)|)}{|\log(\gamma^2/\theta)|} + 1 \right) \leq 33\sigma\gamma.
$$

- Consider finally the case $\theta^3/\gamma^2 \leq \sigma$. If $\sigma\gamma \geq \theta^2$ then the assertion follows from 4, using that $\min_{\mathcal{B}_\theta} \geq c \min\{\theta^2, \sigma\gamma\} = c\theta^2$. On the other hand, if $\sigma\gamma < \theta^2$ then we obtain by Lemma 4 that $\min_{\mathcal{B}_\theta} F_{\sigma,\gamma,\theta} \geq c \min\{\sigma\gamma, \theta^2\} = c\sigma\gamma$ which concludes the proof since

$$
\min \left\{ \theta^2, \sigma\gamma + \frac{\theta^3}{\gamma}, \sigma\gamma \left(\frac{|\log(\sigma\gamma^2/\theta^3)|}{|\log(\gamma^2/\theta)|} + 1 \right) \right\} \leq \sigma\gamma + \theta^3/\gamma \leq 2\sigma\gamma.
$$

$\square$

Similarly to Sect. 2, we start with a rough lower bound without the logarithmic terms. The following can be seen as an analogue to Lemma 1.

Lemma 4 *There exists a constant $c > 0$ such that for all $\gamma \geq \theta/8$ it holds*

$$
\min F_{\sigma,\gamma,\theta}(u) \geq c \min\{\theta^2, \sigma\gamma\}.
$$

Proof Let $u \in \mathcal{B}_\theta$ and assume that $F_{\sigma,\gamma,\theta}(u) \leq \frac{1}{256} \min\{\theta^2, \sigma\gamma\}$. Then there exist $y_1, y_2 \in (0, 1)$ such that $y_2 - y_1 \geq \frac{1}{2}$ and $F_{\sigma,\gamma,\theta}(u; (0, 1) \times \{y_i\}) \leq 4F_{\sigma,\gamma,\theta}(u)$. Additionally, there exists $\bar{x} \in (0, 1)$ such that $F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0, 1)) \leq F_{\sigma,\gamma,\theta}(u)$. Then there exists $\bar{y} \in (0, 1)$ such that $\operatorname{dist}^2(\nabla u(\bar{x}, \bar{y}), \mathcal{M}_\gamma) \leq \frac{1}{256}\theta^2$. In particular, there exists $M \in \mathcal{M}_\gamma$ with $|\nabla u(\bar{x}, \bar{y}) - M|^2 \leq \frac{1}{256}\theta^2$. Then we obtain for almost every $y \in (0, 1)$

$$
|\nabla u(\bar{x}, y) - M| \leq |\nabla u(\bar{x}, \bar{y}) - M| + |\nabla u(\bar{x}, \bar{y}) - \nabla u(\bar{x}, y)|
$$

$$
\leq \frac{1}{16}\theta + |\partial_2 \nabla u(\bar{x}, \cdot)|((0, 1))
$$

$$
\leq \frac{1}{16}\theta + \frac{1}{256}\gamma \leq \frac{\gamma}{2} + \frac{1}{64}\gamma
$$

for almost all $y \in (0, 1)$, and hence $|\nabla u(x, y_i) - M| \leq \frac{1}{2}\gamma + \frac{1}{256}\gamma + \frac{4}{256}\gamma \leq \gamma$ for almost all $x \in (0, 1)$ and $i = 1, 2$. Since the points in $\mathcal{M}_\gamma$ have a distance of at least 2γ, we obtain

$$
\begin{aligned}
|u(\bar{x}, y_2) - u(\bar{x}, y_1) - M_2(y_2 - y_1)|^2 &\leq \int_{y_1}^{y_2} |\partial_2 u(\bar{x}, y) - M_2|^2 \, dy \\
&\leq \int_{y_1}^{y_2} \operatorname{dist}(\nabla u(\bar{x}, y), \mathcal{M}_\gamma)^2 \, dy \\
&\leq F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0, 1)) \leq F_{\sigma,\gamma,\theta}(u).
\end{aligned}
$$

On the other hand, we estimate

$$
\begin{aligned}
|u(\bar{x}, y_2) &- u(\bar{x}, y_1) - M_2(y_2 - y_1)| \\
&\geq |(1 - 2\theta - M_2)(y_2 - y_1)| - |u(\bar{x}, y_2) - u(\bar{x}, y_1) - (1 - 2\theta)(y_2 - y_1)| \\
&\geq \theta - \sum_{i=1}^{2} \int_0^{\bar{x}} |\partial_1 u(x, y_i) - M_1| \, dx \\
&= \theta - \sum_{i=1}^{2} \int_0^{\bar{x}} \operatorname{dist}(\nabla u(x, y_i), \mathcal{M}_\gamma) \, dx \\
&\geq \theta - \sum_{i=1}^{2} \left(\int_0^{\bar{x}} \operatorname{dist}(\nabla u(x, y_i), \mathcal{M}_\gamma)^2 \, dx \right)^{1/2} \\
&\geq \theta - 4 F_{\sigma,\gamma,\theta}(u)^{1/2} \geq \frac{1}{2}\theta.
\end{aligned}
$$

Hence, combining the two estimates, we obtain

$$
F_{\sigma,\gamma,\theta}(u) \geq \frac{1}{4}\theta^2.
$$

$\square$

We now turn to the treatment of the remaining logarithmic terms.

Lemma 5 *There exists $c > 0$ such that the following lower bounds hold:*

1. *If $\gamma \leq \theta/8$ then*

$$
\min F_{\sigma,\gamma,\theta} \geq c \min \left\{ \theta^2, \sigma^{2/3}\theta^{2/3}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma\theta}{\gamma^3} \right| + 1 \right) \right\}.
$$

2. *If $\gamma \geq \theta/8$ and $\theta^2/\gamma \geq \sigma$ then*

$$
\min F_{\sigma,\gamma,\theta} \geq c \min \left\{ \sigma\gamma + \frac{\theta^3}{\gamma}, \frac{\sigma\theta}{\gamma} \left(\left| \log \frac{\sigma}{\theta^2} \right| + 1 \right) \right\}.
$$

Proof We first introduce a slicing argument that is close to the argument in the proof of Lemma 3 which is needed for both statements.

Step 1. Preparations.

Let $\bar{x} \in (0, 1)$. Let $t \in (0, 1)$ and consider the intervals $I_l = (lt, (l+1)t)$ for $l = 0, \ldots, \lfloor 1/t \rfloor$.

Choose an interval I_l such that

$$F_{\sigma,\gamma,\theta}(u; (0,1) \times I_l) \le 4|I_l|F_{\sigma,\gamma,\theta}(u) \text{ and } F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times I_l) \le 4|I_l|F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0,1)).$$

Then one of the following statements is true on I_l:

(a) $|\partial_2 \partial_2 u(\bar{x}, \cdot)|(I_l) \ge \frac{1}{2}$,
(b) $\min\{|\partial_2 u(\bar{x}, y) + 1|, |\partial_2 u(\bar{x}, y) - 1|\}^2 \ge \frac{1}{4}$ for almost all $y \in I_l$,
(c) $|\partial_2 u(\bar{x}, y) - 1| \le |\partial_2 u(\bar{x}, y) + 1|$ for almost all $y \in I_l$,
(d) $|\partial_2 u(\bar{x}, y) + 1| \le |\partial_2 u(\bar{x}, y) - 1|$ for almost all $y \in I_l$.

We consider the cases separately.
If (a) is true then $F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0,1)) \ge \frac{1}{8t}\sigma$.
If (b) is true then $F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0,1)) \ge \frac{1}{16}$.
If (c) is true then by the triangle inequality

$$\begin{aligned}
\frac{1}{2}\theta t^2 &\le \min_{a\in\mathbb{R}} \|u(\bar{x},y) - y - a\|_{L^1(I_l)} + \|u(\bar{x},y) - (1-2\theta)y\|_{L^1(I_l)} \\
&\le t\|\partial_2 u(\bar{x},\cdot) - 1\|_{L^1(I_l)} + \|u(\bar{x},\cdot) - u(0,\cdot)\|_{L^1(I_l)} \\
&\le t^{3/2} F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times I_l) + \gamma\bar{x}t + t^{1/2}\bar{x}^{1/2}F_{\sigma,\gamma,\theta}(u; (0,1) \times I_l)^{1/2} \\
&\le 2t^2 F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0,1))^{1/2} + \gamma\bar{x}t + 2t\bar{x}^{\frac{1}{2}}F_{\sigma,\gamma,\theta}(u)^{1/2}
\end{aligned}$$

Hence, it follows $64F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0,1)) \ge \theta^2$ or $\frac{1}{4}t\theta - \gamma\bar{x} \le 2\bar{x}^{1/2}F_{\sigma,\gamma,\theta}(u)^{1/2}$.
If (d) is true the same conclusion follows from the stronger estimate

$$\frac{1}{2}t^2 \le \min_{a\in\mathbb{R}} \|u(\bar{x},y) + y - a\|_{L^1(I_l)} + \|u(\bar{x},y) - (1-2\theta)y\|_{L^1(I_l)}.$$

Consequently, we obtain from (a) - (d) that

$$F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0,1)) \ge c\min\{\sigma/t, \theta^2\} \qquad \text{or} \qquad \frac{1}{4}t\theta - \gamma\bar{x} \le 2\bar{x}^{1/2}F_{\sigma,\gamma,\theta}(u)^{1/2}. \quad (25)$$

Step 2. Proof of (1): The regime: $\gamma \le \frac{1}{8}\theta$.
Kohn-Müller regime: Let us first assume that $\gamma \le \frac{\theta}{8}$ and $\gamma \le \frac{1}{8}\sigma^{1/3}\theta^{1/3}$. Then $\sigma\theta/\gamma \ge 8\sigma^{2/3}\theta^{2/3}$. In this case we choose $t := \min\{1, \sigma^{-1/3}\theta^{2/3}\}$. If $t = 1$ then $\frac{1}{4}t\theta - \gamma \ge \frac{1}{8}t\theta$. If $t = \theta^{2/3}\sigma^{-1/3}$ then $\sigma \le \theta^2$. It follows that

$$\frac{1}{4}t\theta - \gamma\bar{x} \ge \frac{1}{4}\sigma^{-1/3}\theta^{5/3} - \frac{1}{8}\sigma^{1/3}\theta^{1/3} \ge \frac{1}{8}\sigma^{-1/3}\theta^{5/3} = \frac{1}{8}t\theta.$$

Hence, we conclude from (25) that

$$F_{\sigma,\gamma,\theta}(u) \ge c\int_{1/2}^{1} \min\{\sigma/t, \theta^2\}\,dx = \frac{c}{2}\min\{\sigma/t, \theta^2\} \qquad \text{or} \qquad \frac{t^2}{64}\theta^2 \le 4F_{\sigma,\gamma,\theta}(u).$$

In the first case, we obtain $F_{\sigma,\gamma,\theta}(u) \ge \frac{c}{2}\min\{\sigma^{2/3}\theta^{2/3}, \theta^2\}$. In the latter case, there are two possibilities: If $t = 1$ then $F_{\sigma,\gamma,\theta}(u) \ge c\theta^2$, and if $t = \sigma^{-1/3}\theta^{2/3}$ (i.e., if $\sigma \le \theta^2$) then $F_{\sigma,\gamma,\theta}(u) \ge c\sigma^{-2/3}\theta^{10/3} \ge c\sigma^{2/3}\theta^{2/3}$. Putting things together, we obtain

$$\min F_{\sigma,\gamma,\theta} \ge c\min\{\theta^2, \sigma^{2/3}\theta^{2/3}\},$$

which concludes the proof in this case.

Intermediate regime : Let us now assume that $\gamma \leq \frac{\theta}{8}$ and $\gamma \geq \frac{1}{8}\sigma^{1/3}\theta^{1/3}$. In addition, we may assume that $F_{\sigma,\gamma,\theta}(u) \leq c_1\frac{\sigma\theta}{\gamma}\left(|\log\frac{\sigma\theta}{\gamma^3}| + 1\right)$ for $c_1 > 0$ to be chosen later. Choose $\bar{x} \in (0, 1/16)$ and $t := \frac{8\gamma\bar{x}}{\theta}$. Then

$$\frac{1}{4}t\theta - \gamma\bar{x} = 2\gamma\bar{x} - \gamma\bar{x} = \gamma\bar{x}.$$

Hence, we obtain from (25) that

$$F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0,1)) \geq c\min\left\{\frac{\sigma\theta}{\gamma\bar{x}}, \theta^2\right\} \text{ or } \bar{x} \leq \frac{4}{\gamma^2}F_{\sigma,\gamma,\theta}(u) \leq 4c_1\frac{\sigma\theta}{\gamma^3}\left(\left|\log\frac{\sigma\theta}{\gamma^3}\right| + 1\right).$$

In particular, we have

$$F_{\sigma,\gamma,\theta}(u) \geq c \int_{c_1\frac{\sigma\theta}{\gamma^3}\left(|\log\frac{\sigma\theta}{\gamma^3}|+1\right)}^{1/16} \min\left\{\frac{\sigma\theta}{\gamma x}, \theta^2\right\}dx.$$

Note that $c_1\frac{\sigma\theta}{\gamma x} \leq \theta^2$ if $x \geq c_1\frac{\sigma}{\theta\gamma}$. Since $\gamma \leq \theta$ it holds $c_1\frac{\sigma}{\theta\gamma} \leq c_1\frac{\sigma\theta}{\gamma^3} \leq c_1\frac{\sigma\theta}{\gamma^3}\left(\left|\log\frac{\sigma\theta}{\gamma^3}\right| + 1\right)$. Hence,

$$F_{\sigma,\gamma,\theta}(u) \geq cc_1 \int_{c_1\frac{\sigma\theta}{\gamma^3}\left(|\log\frac{\sigma\theta}{\gamma^3}|+1\right)}^{1/16} \frac{\sigma\theta}{\gamma x}dx$$

$$= cc_1\frac{\sigma\theta}{\gamma}\left(\log 1/16 - \log\frac{\sigma\theta}{\gamma^3} - \log\left(\left|\log\frac{\sigma\theta}{\gamma^3}\right| + 1\right) - \log c_1\right)$$

$$\geq cc_1\frac{\sigma\theta}{\gamma}\left(\left|\log\frac{\sigma\theta}{\gamma^3}\right| + 1\right)$$

if the universal constant $c_1 > 0$ is chosen small enough. This concludes the proof of the lower bound in this case, and hence the proof of (1).

Step 3. Proof of (2): The regime : $\gamma > \frac{1}{8}\theta$ and $\sigma \leq \theta^2/\gamma$. Let $\bar{x} \in (0, \frac{\theta}{16\gamma})$ and $t := 4\frac{\gamma\bar{x}}{\theta}$. Then

$$\frac{1}{2}t\theta - \gamma\bar{x} = \gamma\bar{x} = \frac{1}{4}t\theta.$$

Hence we obtain again from (25) that

$$F_{\sigma,\gamma,\theta}(u; \{\bar{x}\} \times (0,1)) \geq c\min\left\{\frac{\sigma\theta}{\gamma\bar{x}}, \theta^2\right\} \quad \text{or} \quad \bar{x} \leq \frac{4}{\gamma^2}F_{\sigma,\gamma,\theta}(u).$$

In particular,

$$F_{\sigma,\gamma,\theta}(u) \geq c \int_{\frac{4}{\gamma^2}F_{\sigma,\gamma,\theta}(u)}^{\theta/(16\gamma)} \min\left\{\frac{\sigma\theta}{\gamma x}, \theta^2\right\}dx.$$

We consider the two possibilities $\sigma \leq \theta^2$ and $\sigma > \theta^2$ separately.

- Consider first the case $\sigma \leq \theta^2$. We assume that $F_{\sigma,\gamma,\theta}(u) \leq c_2\frac{\sigma\theta}{\gamma}\left(|\log\frac{\sigma}{\theta^2}| + 1\right)$ for some $c_2 > 0$ fixed below (otherwise we are done.). We observe that $c_2\frac{\sigma}{\theta\gamma}\left(|\log\sigma/\theta^2| + 1\right) \geq$

$c_2 \frac{\sigma\theta}{64\gamma^3} \left(|\log \sigma/\theta^2| + 1 \right)$. Additionally, we note that $c_2 \frac{\sigma\theta}{\gamma \bar{x}} \leq \theta^2$ if and only if $\bar{x} \geq c_2 \frac{\sigma}{\theta\gamma}$ and $c_2 \frac{\sigma}{\theta\gamma} \left(|\log \sigma/\theta^2| + 1 \right) \leq c_2 \frac{\theta}{\gamma} \leq \theta/(16\gamma)$ if $c_2 \leq 1/16$. Consequently,

$$
\begin{aligned}
F_{\sigma,\gamma,\theta}(u) &\geq cc_2 \int_{c_2 \frac{\sigma}{\theta\gamma} (|\log \sigma/\theta^2|+1)}^{\theta/(16\gamma)} \frac{\sigma\theta}{\gamma x} \, dx \\
&= cc_2 \frac{\sigma\theta}{\gamma} \left(\log \theta/\gamma - \log 16 - \log \sigma/(\gamma\theta) - \log c_2 - \log \left(|\log \sigma/\theta^2| + 1 \right) \right) \\
&= cc_2 \frac{\sigma\theta}{\gamma} \left(\log \theta^2/\sigma - \log 16c_2 - \log \left(|\log \sigma/\theta^2| + 1 \right) \right) \\
&\geq cc_2 \frac{\sigma\theta}{\gamma} \left(|\log \sigma/\theta^2| + 1 \right)
\end{aligned}
$$

for $c_2 > 0$ small enough. This shows the lower bound in this case.

- Consider now the case $\sigma > \theta^2$. We assume that $F_{\sigma,\gamma,\theta}(u) \leq c_3 \left(\frac{\theta^3}{\gamma} + \gamma\sigma \right)$ for $c_3 = 1/256$ (otherwise we are done.). Note that by Lemma 4 we already know that $F_{\sigma,\gamma,\theta}(u) \geq c \min\{\theta^2, \sigma\gamma\} = c\sigma\gamma$. In particular, if $\sigma \leq \theta^3/\gamma^2$ then $F_{\sigma,\gamma,\theta}(u) \geq \frac{c}{2}(\sigma\gamma + \frac{\theta^3}{\gamma})$, and we are done. Hence, we may assume $\sigma \leq \frac{\theta^3}{\gamma^2}$. Since we consider the regime $\sigma < \theta^2$, we have $\frac{\sigma}{\theta\gamma} \geq \frac{\theta}{\gamma}$, which implies that $\min\left\{ \frac{\sigma\theta}{\gamma x}, \theta^2 \right\} = \theta^2$ for all $x \leq \frac{\theta}{16\gamma}$. Consequently, we obtain

$$
\begin{aligned}
F_{\sigma,\gamma,\theta}(u) &\geq c \int_{c_3(\gamma\sigma+\theta^3/\gamma)}^{\theta/(16\gamma)} \theta^2 \, dx \\
&= c\theta^3/(16\gamma) - c_3 c(\theta^5/\gamma^3 + \sigma\theta^2/\gamma) \geq \frac{c}{16} \left(\frac{\theta^3}{\gamma} - \frac{\theta^3}{4\gamma} - \frac{1}{4}\sigma\gamma \right) \\
&\geq \frac{c\theta^3}{32\gamma} \geq \frac{c}{64} \left(\frac{\theta^3}{\gamma} + \sigma\gamma \right).
\end{aligned}
$$

This concludes the proof in the regime $\gamma \geq \theta/8$ and $\sigma \leq \theta^2/\gamma$. $\qquad\square$

We finally turn to the parameter regime in which the logarithmic terms in the third regime occur. We proceed similarly to Lemma 2.

Lemma 6 *There exist $\alpha_0 > 0$ and $c > 0$ such that for all $k \geq k_0 = 32$, all $\gamma \in (0, 1)$, all $\theta \in (0, \alpha_0\gamma^2]$ and all*

$$
\sigma \in \left[\frac{\theta^3}{\gamma^2} \left(\frac{\theta}{\gamma^2} \right)^{k+1}, \frac{\theta^3}{\gamma^2} \left(\frac{\theta}{\gamma^2} \right)^{k} \right)
$$

there holds

$$
F_{\sigma,\gamma,\theta}(u) \geq ck\sigma\gamma.
$$

Proof Similarly to the proof of Lemma 2 we set $k_0 := 32$, $0 < \alpha_0 < 1/(63)^2$ such that $2 \cdot 64 \cdot 21^2 k\alpha_0^{k/4} \leq 1$ for all $k \geq k_0$. We assume $F_{\sigma,\gamma,\theta}(u) \leq k\sigma$ and that $k \geq k_0$.

Let $\sigma \in \left(\frac{\theta^3}{\gamma^2} \left(\frac{\theta}{\gamma^2} \right)^{k+1}, \frac{\theta^3}{\gamma^2} \left(\frac{\theta}{\gamma^2} \right)^{k} \right)$. Then find $x_i \in \left(\frac{1}{2} \frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^{i}, \frac{3}{2} \frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^{i} \right)$ such that

$$F_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0,1)) \leq \frac{\gamma}{\theta} \left(\frac{\gamma^2}{\theta} \right)^{i} F_{\sigma,\gamma,\theta} \left(u; \left(\frac{1}{2} \frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^{i}, \frac{3}{2} \frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^{i} \right) \times (0,1) \right).$$

for $i = 1, \ldots, k$.

Claim: There exists a constant $c > 0$ such that for all $i = 1, \ldots, \lfloor k/2 \rfloor$ it holds

$$\frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^{i} F_{\sigma,\gamma,\theta}(u, \{x_i\} \times (0,1)) + F_{\sigma,\gamma,\theta}(u; (x_{i+1}, x_i) \times (0,1)) \geq c\sigma\gamma.$$

We first show how to derive the lower bound from the claim. We have

$$2 F_{\sigma,\gamma,\theta}(u)$$

$$\geq \sum_{i=1}^{\lfloor k/2 \rfloor} F_{\sigma,\gamma,\theta} \left(u; \left(\frac{1}{2} \frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^{i}, \frac{3}{2} \frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^{i} \right) \times (0,1) \right) + F_{\sigma,\gamma,\theta}(u; (x_{i+1}, x_i) \times (0,1))$$

$$\geq \sum_{i=1}^{\lfloor k/2 \rfloor} \frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^{i} F_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0,1)) + F_{\sigma,\gamma,\theta}(u; (x_{i+1}, x_i) \times (0,1))$$

$$\geq c \frac{k}{4} \gamma \sigma.$$

Proof of claim: The claim can be obtained following the arguments in the proof of Proposition 2. We sketch it here for the sake of completeness.

First, define $N_i := \{s \in (0,1) : |\partial_2 u(x_i, s) + 1| \leq 3|\partial_2 u(x_i, s) - 1|\}$ and assume for a contradiction that $\mathcal{L}^1(N_i) > 2/3$. Then one can show with the analogous definitions of $y_1, y_2 \in (0,1)$ along the lines of the proof of Lemma 2 that

$$\int_{(y_1,y_2) \cap \{\partial_2 u \geq 1/2\}} \partial_2 u(x_i, s) \, ds \leq \frac{1}{3} + F_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0,1))^{1/2} < 1/2$$

and

$$\int_{(y_1,y_2) \cap \{\partial_2 u \geq 1/2\}} \partial_2 u(x_i, s) \, ds$$

$$\geq (1 - 2\theta)(y_2 - y_1) - 4\gamma x_i - 3 F_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0,1))^{1/2} + \frac{2}{3} - \frac{1}{12} \geq \frac{1}{2}.$$

This shows that $\mathcal{L}^1(N_i) \leq 2/3$.

Next, let $t = 120 \left(\frac{\theta}{\gamma^2} \right)^{i+1}$. Again, we find $(y, y+t) \subseteq (0,1)$ such that $(y, y+t) \cap N_i^c \neq \emptyset$ and

$$F_{\sigma,\gamma,\theta}(u; (0,1) \times (y, y+t)) \leq 48t \, F_{\sigma,\gamma,\theta}(u),$$
$$F_{\sigma,\gamma,\theta}(u; (x_{i+1}, x_i) \times (y, y+t)) \leq 48t \, F_{\sigma,\gamma,\theta}(u; (x_{i+1}, x_i) \times (0,1)),$$
$$F_{\sigma,\gamma,\theta}(u; \{x_i\} \times (y, y+t)) \leq 48t \, F_{\sigma,\gamma,\theta}(u; \{x_i\} \times (0,1)),$$
$$F_{\sigma,\gamma,\theta}(u; \{x_{i+1}\} \times (y, y+t)) \leq 48t \, F_{\sigma,\gamma,\theta}(u; \{x_{i+1}\} \times (0,1)).$$

Moreover, we observe that on $(y, y+t)$ one of the following three assertions has to hold

1. $|\partial_2 \partial_2 u(x_i, \cdot)|(y, y+t) \geq 1/2$,

2. $|\partial_2 u(x_i, s) - 1| \le |\partial_2 u(x_i, s) + 1|$ for almost all $s \in (y, y + t)$,
3. $|\partial_2 u(x_i, s) + 1| \le 3|\partial_2 u(x_i, s) - 1|$ for almost all $s \in (y, y + t)$.

If (1) is true then the estimate follows immediately. Moreover, (3) cannot be true by our choice of $(y, y + t)$. Hence, from now on, we assume that (2) is true. By the triangle inequality, it holds

$$\frac{1}{2}\theta t^2 \le \|u(x_i, \cdot + t/2) - u(x_i, \cdot) - t/2\|_{L^1(y, y+t/2)}$$
$$+ \|u(x_i, \cdot + t/2) - u(x_i, \cdot) - (1 - 2\theta)t/2\|_{L^1(y, y+t/2)}.$$

First we assume that $\frac{1}{4}\theta t^2 \le \|u(x_i, \cdot + t/2) - u(x_i, \cdot) - t/2\|_{L^1(y, y+t/2)}$ and define $a = \int_y^{y+t/2} (u(x_i, s + t/s) - u(x_i, s) - t/2)\, ds$. Then one shows as in the proof of Lemma 2 that

$$|a| \le 21t^2 F_{\sigma, \gamma, \theta}(u; \{x_i\} \times (0, 1))^{1/2} \le \frac{1}{8}t^2\theta,$$

where we used that $\frac{\theta}{\gamma}\left(\frac{\gamma^2}{\theta}\right)^i k\sigma \le \theta^2 k\left(\frac{\theta}{\gamma^2}\right)^{k-i} \le \frac{1}{21^2 \cdot 64}\theta^2$. Then it follows similarly to (15)

$$21t^4 F_{\sigma, \gamma, \theta}(u; \{x_i\} \times (0, 1)) \ge \left(\frac{1}{4}\theta t^2 - \frac{t}{2}|a|\right)^2 \ge \frac{t^4}{64}\theta^2 \ge \frac{t^4}{64}\gamma\frac{\theta^3}{\gamma^2}\left(\frac{\theta}{\gamma^2}\right)^k\frac{\gamma}{\theta}\left(\frac{\gamma^2}{\theta}\right)^i$$
$$\ge \frac{t^4}{64}\sigma\gamma\frac{\gamma}{\theta}\left(\frac{\gamma^2}{\theta}\right)^i,$$

which yields the claim.

Next, assume that $\frac{1}{4}\theta t^2 \le \|u(x_i, \cdot + t/2) - u(x_i, \cdot) - (1 - 2\theta)t/2\|_{L^1(y, y+t)}$. Along the lines of the proof of Lemma 2 one shows for

$$S := \left\{s \in (y, y + t/2) : \frac{1}{8}\theta t \le w(s) \le 3\theta t\right\},$$

where $w(s) = |u(x_i, s) - u(x_i, s + t/2) - u(x_{i+1}, s) + u(x_{i+1}, s + t/2)|$, that $\mathcal{L}^1(S) \ge \frac{t}{48}$. Since $\frac{1}{4}\frac{\theta}{\gamma}\left(\frac{\theta}{\gamma^2}\right)^i \le x_i - x_{i+1} \le \frac{3}{2}\frac{\theta}{\gamma}\left(\frac{\theta}{\gamma^2}\right)^i$, we find for $s \in S$

$$10\frac{\theta}{\gamma} \le \left|\frac{u(x_i, s) - u(x_i, s + t/2) - u(x_{i+1}, s) + u(x_{i+1}, s + t/2)}{x_i - x_i + 1}\right| \le 12 \cdot 120\frac{\theta}{\gamma},$$

which implies since $\frac{\theta}{\gamma} \le \alpha_0\gamma \le (12 \cdot 120)^{-1}\gamma$ that there exists a subset of $(y, y + t)$ whose measure is at least $\frac{t}{48}$ such that

$$\left|\left|\frac{u(x_i, s) - u(x_{i+1}, s)}{x_i - x_{i+1}}\right| - \gamma\right| \ge c\frac{\theta}{\gamma}.$$

Fix $s \in (y, y + t)$ with the above property and assume that $|\partial_1 \partial_1 u(\cdot, s)|((x_{i+1}, x_i)) < \gamma/2$. Moreover, we may assume without loss of generality that $|\partial_1 u(t, s) - \gamma| \le 3|\partial_1 u(t, s) + \gamma|$

for almost all $t \in (x_{i+1}, x_i)$. Then it follows similarly to the proof of Lemma 2 that

$$
\int_{x_{i+1}}^{x_i} \min \{|\partial_1 u(t, s) + \gamma|, |\partial_1 u(t, s) - \gamma|\}^2 \, dt
$$

$$
\geq \frac{1}{9}(x_i - x_{i+1}) \left(\frac{u(x_i, s) - u(x_{i+1}, s)}{x_i - x_{i+1}} - \gamma \right)^2
$$

$$
\geq \frac{1}{36} \frac{\theta}{\gamma} \left(\frac{\theta}{\gamma^2} \right)^i c^2 \frac{\theta^2}{\gamma^2}
$$

$$
\geq \frac{c^2}{36} \gamma \frac{\theta^3}{\gamma^2} \left(\frac{\theta}{\gamma^2} \right)^{i+1} \geq \frac{c^2}{36} \gamma \sigma.
$$

Consequently, we find

$$
48t \, F_{\sigma, \gamma, \theta}(u; (x_{i+1}, x_i) \times (0, 1)) \geq F_{\sigma, \gamma, \theta}(u; (x_{i+1}, x_i) \times (y, y + t)) \geq \min \left\{ \frac{1}{2}, \frac{c^2}{36} \right\} \frac{t}{48} \gamma \sigma.
$$

Dividing by t on both sides of the inequality, this concludes the proof. $\square$

Data availability All data generated or analysed during this study are included in the main article file.

References

1. Ball, J.M., James, R.D.: Fine phase mixtures as minimizers of energy. Arch. Rational Mech. Anal. **100**, 13–52 (1987)
2. Conti, S., Zwicknagl, B.: Low volume-fraction microstructures in martensites and crystal plasticity. Math. Models Methods App. Sci. **26**, 1319–1355 (2016)
3. Giuliani, A., Müller, S.: Striped periodic minimizers of a two-dimensional model for martensitic phase transitions. Commun. Math. Phys. **309**, 313–339 (2012)
4. Conti, S., Diermeier, J., Koser, M., Zwicknagl, B.: Asymptotic self-similarity of minimizers and local bounds in a model of shape-memory alloys. J. Elast. **147**, 149–200 (2021)
5. Conti, S.: Branched microstructures: scaling and asymptotic self-similarity. Commun. Pure Appl. Math. **53**, 1448–1474 (2000)
6. Kohn, R.V., Müller, S.: Branching of twins near an austenite-twinned martensite interface. Phil. Mag. A **66**, 697–715 (1992)
7. Kohn, R.V., Müller, S.: Surface energy and microstructure in coherent phase transitions. Commun. Pure Appl. Math. **47**(4), 405–435 (1994). https://doi.org/10.1002/cpa.3160470402
8. Capella, A., Otto, F.: A rigidity result for a perturbation of the geometrically linear three-well problem. Commun. Pure Appl. Math. **62**(12), 1632–1669 (2009). https://doi.org/10.1002/cpa.20297

9. Capella, A., Otto, F.: A quantitative rigidity result for the cubic-to-tetragonal phase transition in the geometrically linear theory with interfacial energy. Proc. R. Soc. Edinburgh Sect. A **142**(2), 273–327 (2012). https://doi.org/10.1017/S0308210510000478

10. Knüpfer, H., Kohn, R.V., Otto, F.: Nucleation barriers for the cubic-to-tetragonal phase transformation. Commun. Pure Appl. Math. **66**(6), 867–904 (2013). https://doi.org/10.1002/cpa.21448

11. Chan, A., Conti, S.: Energy scaling and branched microstructures in a model for shape-memory alloys with SO(2) invariance. Math. Models Methods App. Sci. **25**, 1091–1124 (2015). https://doi.org/10.1142/S0218202515500281

12. Bella, P., Goldman, M.: Nucleation barriers at corners for cubic-to-tetragonal phase transformation. Proc. R. Soc. Edinburgh A **145**, 715–724 (2015)

13. Simon, T.M.: Rigidity of branching microstructures in shape memory alloys. Arch. Ration. Mech. Anal. **241**, 1707–1783 (2021)

14. Simon, T.M.: Quantitative aspects of the rigidity of branching microstructures in shape memory alloys via H-measures. SIAM J. Math. Anal. **53**, 4537–4567 (2021)

15. Knüpfer, H., Otto, F.: Nucleation barriers for the cubic-to-tetragonal phase transformation in the absence of self-accommodation. ZAMM Z. Angew. Math. Mech. **99**, 201800179 (2019)

16. Conti, S., Diermeier, J., Melching, C., Zwicknagl, B.: Energy scaling laws for geometrically linear elasticity models for microstructures in shape memory alloys. ESAIM: COCV **26**, 115–164 (2020). https://doi.org/10.1051/cocv/2020020

17. Seiner, H., Plucinsky, P., Dabade, V., Benešová, B., James, R.D.: Branching of twins in shape memory alloys revisited. J. Mech. Phys. Solids **141**, 103961 (2020)

18. Rüland, A., Tribuzio, A.: On the energy scaling behaviour of a singularly perturbed tartar square. Arch. Rational Mech. Anal. **243**, 401–431 (2022). https://doi.org/10.1007/s00205-021-01729-1

19. Choksi, R., Kohn, R.V.: Bounds on the micromagnetic energy of a uniaxial ferromagnet. Commun. Pure Appl. Math. **51**(3), 259–289 (1998). https://doi.org/10.1002/(SICI)1097-0312(199803)51:3

20. Choksi, R., Kohn, R.V., Otto, F.: Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy. Commun. Math. Phys. **201**(1), 61–79 (1999)

21. DeSimone, A., Kohn, R.V., Müller, S., Otto, F.: A reduced theory for thin-film micromagnetics. Commun. Pure Appl. Math. **55**, 1408–1460 (2002)

22. Choksi, R., Kohn, R.V., Otto, F.: Energy minimization and flux domain structure in the intermediate state of a type-I superconductor. J. Nonlinear Sci. **14**, 119–171 (2004)

23. Otto, F., Viehmann, T.: Domain branching in uniaxial ferromagnets: asymptotic behavior of the energy. Calc. Var. PDE **38**(1–2), 135–181 (2010)

24. Knüpfer, H., Muratov, C.B.: Domain structure of bulk ferromagnetic crystals in applied fields near saturation. J. Nonlinear Sci. **21**, 921–962 (2011)

25. Venkatraman, R., Dabade, V., James, R.D.: Bounds on the energy of a soft cubic ferromagnet with large magnetostriction. J. Nonlinear Sci. **30**(6), 3367–3388 (2020). https://doi.org/10.1007/s00332-020-09653-6

26. Dabade, V., Venkatraman, R., James, R.D.: Micromagnetics of Galfenol. J. Nonlinear Sci. **29**, 415–460 (2019)

27. Ginster, J., Zwicknagl, B.: Energy scaling law for a singularly perturbed four-gradient problem in helimagnetism. J. Nonlinear Sci. **33**, 20 (2023)

28. Ginster, J.: On the formation of microstructure for singularly perturbed problems with 2,3 or 4 preferred gradients. cvgmt preprint (2023). http://cvgmt.sns.it/paper/6022/

29. Diep, H.T.: Frustrated Spin Systems. World Scientific, Singapore (2013)

30. Cicalese, M., Solombrino, F.: Frustrated ferromagnetic spin chains: a variational approach to chirality transitions. J. Nonlinear Sci. **25**(2), 291–313 (2015). https://doi.org/10.1007/s00332-015-9230-4

31. Cicalese, M., Forster, M., Orlando, G.: Variational analysis of a two-dimensional frustrated spin system: emergence and rigidity of chirality transitions. SIAM J. Math. Anal. **51**, 4848–4893 (2019)

32. Ginster, J., Koser, M., Zwicknagl, B.: Microstructures in a two dimensional frustrated spin system: scaling regimes and a discrete-to-continuum limit. in preparation

33. Conti, S.: A lower bound for a variational model for pattern formation in shape-memory alloys. Cont. Mech. Thermodyn. **17**(6), 469–476 (2006)

34. Zwicknagl, B.: Microstructures in low-hysteresis shape memory alloys: Scaling regimes and optimal needle shapes. Arch. Rational Mech. Anal. **213**(2), 355–421 (2014)

35. Conti, S., Ortiz, M.: Optimal scaling in solids undergoing ductile fracture by crazing. Arch. Rational Mech. Anal. **219**, 607–636 (2016)

36. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford University Press, Oxford (2000)

37. Conti, S., Diermeier, J., Zwicknagl, B.: Deformation concentration for martensitic microstructures in the limit of low volume fraction. Calc. Var. PDE **56**, 16 (2017)