Calc. Var. (2023) 62:266
https://doi.org/10.1007/s00526-023-02596-2

Calculus of Variations

Regularity theory and Green's function for elliptic equations with lower order terms in unbounded domains

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Received: 18 October 2022 / Accepted: 4 October 2023 / Published online: 2 November 2023

Abstract

We consider elliptic operators in divergence form with lower order terms of the form $Lu = -\mathrm{div}(A\cdot\nabla u + bu) - c\cdot\nabla u - du$, in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with possibly infinite Lebesgue measure. We assume that the $n \times n$ matrix A is uniformly elliptic with real, merely bounded and possibly non-symmetric coefficients, and either $b, c \in L^{n,\infty}_{\mathrm{loc}}(\Omega)$ and $d \in L^{\frac{n}{2},\infty}_{\mathrm{loc}}(\Omega)$, or $|b|^2, |c|^2, |d| \in \mathcal{K}_{\mathrm{loc}}(\Omega)$, where $\mathcal{K}_{\mathrm{loc}}(\Omega)$ stands for the local Stummel–Kato class. Let $\mathcal{K}_{\mathrm{Dini}}(\Omega)$ be a variant of $\mathcal{K}(\Omega)$ satisfying a Carleson-Dini-type condition. We develop a De Giorgi/Nash/Moser theory for solutions of $Lu = f - \mathrm{div} g$, where f and $|g|^2 \in \mathcal{K}_{\mathrm{Dini}}(\Omega)$ if, for $q \in [n, \infty)$, any of the following assumptions holds: (i) $|b|^2, |d| \in \mathcal{K}_{\mathrm{Dini}}(\Omega)$ and either $c \in L^{n,q}_{\mathrm{loc}}(\Omega)$ or $|c|^2 \in \mathcal{K}_{\mathrm{loc}}(\Omega)$; (ii) $\mathrm{div} b + d \leq 0$ and either $b + c \in L^{n,q}_{\mathrm{loc}}(\Omega)$ or $|b + c|^2 \in \mathcal{K}_{\mathrm{loc}}(\Omega)$; (iii) $-\mathrm{div} c + d \leq 0$ and $|b + c|^2 \in \mathcal{K}_{\mathrm{Dini}}(\Omega)$. We also prove a Wiener-type criterion for boundary regularity. Assuming global conditions on the coefficients, we show that the variational Dirichlet problem is well-posed and, assuming $-\mathrm{div} c + d \leq 0$, we construct the Green's function associated with L satisfying quantitative estimates. Under the additional hypothesis $|b + c|^2 \in \mathcal{K}'(\Omega)$, we show that it satisfies global pointwise bounds and also construct the Green's function associated with the formal adjoint operator of L. An important feature of our results is that all the estimates are scale invariant and independent of Ω, while we do not assume smallness of the norms of the coefficients or coercivity of the associated bilinear form.

Mathematics Subject Classification 35A08 · 35B50 · 35B51 · 35B65 · 35J08 · 35J15 · 35J20 · 35J25 · 35J67 · 35J86

Communicated by A. Mondino.

The author was supported by IKERBASQUE and partially supported by the grants MTM-2017-82160-C2-2-P and PID2020-118986GB-I00 of the Ministerio de Economía y Competitividad (Spain) and by the grant IT-1615-22 (Basque Government).

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Contents

1 Introduction

In the present paper we will deal with elliptic equations of the form

$$Lu = -\mathrm{div}(A \cdot \nabla u + bu) - c \cdot \nabla u - du = 0 \tag{1.1}$$

in an open set $\Omega \subset \mathbb{R}^n$, $n \geq 3$, where $A(x) = (a_{ij}(x))_{i,j=1}^n$ is a matrix with entries $a_{ij} : \Omega \to \mathbb{R}$, for $i, j \in \{1, 2, \dots, n\}$, $b, c : \Omega \to \mathbb{R}^n$ are vector fields, and $d : \Omega \to \mathbb{R}$ a real-valued function. Our standing assumptions are the following:

There exist $0 < \lambda < \Lambda < \infty$, so that

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle, \quad \text{for all } \xi \in \mathbb{R}^n \text{ and a.e. } x \in \Omega, \tag{1.2}$$

$$\langle A(x)\xi, \eta\rangle \leq \Lambda|\xi||\eta|, \quad \text{for all } \xi, \eta \in \mathbb{R}^n \text{ and a.e. } x \in \Omega, \tag{1.3}$$

$$|b|^2, |c|^2, |d| \in \mathcal{K}_{\mathrm{loc}}(\Omega) \quad \text{or} \quad b, c \in L^{n,\infty}_{\mathrm{loc}}(\Omega), d \in L^{\frac{n}{2},\infty}_{\mathrm{loc}}(\Omega), \tag{1.4}$$

where $\mathcal{K}_{\mathrm{loc}}(\Omega)$ and $L^{n,\infty}_{\mathrm{loc}}(\Omega)$ stand for the local Stummel-Kato class and the local weak-L^n space respectively (see Definitions 2.9 and 2.22)[1]. In several cases, we will also need to assume one of the following negativity conditions:

$$\int_\Omega (d\,\varphi - b \cdot \nabla\varphi) \leq 0, \quad \text{for all } 0 \leq \varphi \in C_0^\infty(\Omega), \tag{1.5}$$

or

$$\int_\Omega (d\,\varphi + c \cdot \nabla\varphi) \leq 0, \quad \text{for all } 0 \leq \varphi \in C_0^\infty(\Omega). \tag{1.6}$$

[1] Our original assumptions were $b, c \in L^n(\Omega)$ and $d \in L^{\frac{n}{2}}(\Omega)$. The extension to weak Lebesgue spaces is due to an observation of G. Sakellaris in [28]; a more detailed discussion can be found at the end of the introduction.

If (1.5) (resp. (1.6)) holds we will say that the bd (resp. cd) negativity condition is satisfied. If we reverse the inequality signs we will say that the bd or cd positivity condition is satisfied.

The objective of the current manuscript is to generalize the standard theory of elliptic PDE of the form $-\text{div}A\nabla u = 0$ in open sets $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with possibly infinite Lebesgue measure, to equations of the form (1.1) under the aforementioned standing assumptions. In particular, we aim to show **scale invariant** *a priori* local estimates (Caccioppoli inequality, local boundedness and weak Harnack inequality), interior and boundary regularity for solutions of (1.1), the weak maximum principle, well-posedness of the Dirichlet and obstacle problems, and finally to construct the Green's function for our operator satisfying several quantitative estimates. It is important to highlight that neither the bilinear form associated with the elliptic equation is coercive, nor the norms of the coefficients are small, which is one of the main technical difficulties.

We would like to point out that we will only state the theorems in the main body of the paper, just before their proofs. Nevertheless, the reader can find a detailed description of our results in the introduction.

Let us give a brief overview of our results. In Sect. 3.1 we prove the standard interior and boundary Caccioppoli's inequality under either negativity condition (Theorems 3.1, 3.2, and 3.3), while, in Sect. 5, having global assumptions on the coefficients, we show the well-posedness of the generalized Dirichlet problem (5.2) satisfying the estimate (5.11), as well as the validity of the weak maximum principle (Theorem 5.1). This maximum principle allows us to solve the obstacle problem in bounded domains (Theorem 5.6). Then we assume that one of the following conditions hold:

(1) $|b|^2, |d| \in \mathcal{K}_{\text{Dini}}(\Omega)$ and either $|c|^2 \in \mathcal{K}_{\text{loc}}(\Omega)$ or $c \in L^{n,q}_{\text{loc}}(\Omega)$, for $q \in [n, \infty)$;
(2) $\text{div}b + d \leq 0$ and either $|b+c|^2 \in \mathcal{K}_{\text{loc}}(\Omega)$ or $b + c \in L^{n,q}_{\text{loc}}(\Omega)$, for $q \in [n, \infty)$;
(3) $-\text{div}c + d \leq 0$ and $|b+c|^2 \in \mathcal{K}_{\text{Dini}}(\Omega)$ (see Definition 2.11).

In Sect. 3.2, we demonstrate that the refined Caccioppoli inequality holds in the interior and the boundary (Theorems 3.5 and 3.8), which leads to the local boundedness of subsolutions (Theorem 4.4) and the weak Harnack inequality for non-negative supersolutions (Theorem 4.5) both in the interior and at the boundary. In Sect. 4.2 we prove interior and boundary regularity for solutions and finally, assuming the cd-negativity condition and either $b + c \in L^{n,q}(\Omega)$, for $q \in [n, \infty)$, or $|b+c|^2 \in \mathcal{K}'(\Omega)$, we use the aforementioned results to construct the Green's function associated with the operator L satisfying several quantitative estimates. Under the additional hypothesis $|b + c|^2 \in \mathcal{K}'(\Omega)$, we show global pointwise bounds and construct the Green's function associated with the formal adjoint operator of L. All our estimates are scale invariant and independent of the Lebesgue measure of the domain.

We now briefly review the history of work in this area for linear elliptic equations in divergence form with merely bounded leading coefficients and singular lower order terms. The generalized Dirichlet problem in the Sobolev space $W^{1,2}$ is well-posed if there exists a unique $u \in W^{1,2}(\Omega)$ such that $Lu = f + \text{div}g$ and $u - \phi \in W^{1,2}_0(\Omega)$ for fixed $\phi \in W^{1,2}(\Omega)$ and $f, g_i \in L^2(\Omega)$. Moreover, there exists a constant $C_{\phi,f,g}$ so that the global estimate $\|u\|_{W^{1,2}(\Omega)} \lesssim C_{\phi,f,g}$ holds. For operators without lower order terms this problem has a long history and we refer to [9, p.214] and the references therein for details. In *bounded* domains, in the presence of lower order terms, Ladyzhenskaya and Ural'tseva [19] and Stampacchia [34] proved well-posedness of the generalized Dirichlet problem assuming conditions related to the coercivity of the operator or smallness of the norms of the lower order coefficients. This was quite restrictive as, for example, the "bad" terms coming from the lower order coefficients can be absorbed in view of smallness. Gilbarg and Trudinger [9] gave an extension of the previous results replacing the smallness conditions by the assumptions $b, c, d \in L^\infty(\Omega)$

assuming either (1.5) or (1.6). In fact, they only need $b, c \in L^s(\Omega)$ and $d \in L^{s/2}(\Omega)$, for some $s > n$. Recently, Kim and Sakellaris [16], generalized it to operators whose coefficients are in the critical Lebesgue space. Unfortunately, in all those results, the implicit constant in the global estimate depends on the Lebesgue measure of Ω and thus, they cannot be extended to unbounded domains by approximation. On the other hand, in unbounded domains with possibly infinite Lebesgue measure, already in 1976, Bottaro and Marina [2] proved that, if $b, c \in L^n(\Omega)$, $d \in L^{n/2}(\Omega) + L^\infty(\Omega)$, and $\operatorname{div} b + d \leq \mu < 0$, then the generalized Dirichlet problem is well-posed. To our knowledge, this was the first paper establishing well-posedness in such generality. Using the same method, Vitanza and Zamboni [36], showed well-posedness of the same problem when $|b|^2, |c|^2, |d| \in \mathcal{K}'(\Omega)$.

The local pointwise estimates find their roots in De Giorgi's celebrated paper [7] on the Hölder continuity of solutions of elliptic equations of the form $-\operatorname{div} A\nabla u = 0$, where Theorems 4.4 (i) and 4.12 were proved in this special case (see also [25]). A few years later, Moser gave a new proof of De Giorgi's theorem in [23]. The same results were extended in equations of the form (1.1) by Morrey [22] when $b, c \in L^q$ and $d \in L^{q/2}$, for $q > n$ and Stampacchia [32] (in more special cases). Moser also established the weak Harnack inequality for solutions of $-\operatorname{div} A\nabla u = 0$ in [24], while Stampacchia [34] proved all the a priori estimates for equations of the form (1.1) with $c \in L^n$ and $|b|^2, d \in L^s$, $s > n/2$, assuming that (1.5) holds and the radius of the balls are sufficiently small so that the respective norms of the lower order coefficients on those balls are small themselves. If the lower order coefficients are in the Stummel-Kato class $\mathcal{K}(\Omega)$ with sufficiently small norms, one can find such results in [3] and [18] (see the references therein as well). Under the assumptions $b, c \in L^n$, and $d \in L^{\frac{n}{2}}$, Kim and Sakellaris [16] also established local boundedness for subsolutions of the equation (1.1) satisfying either (1.5) or (1.6) and $b + c \in L^s$, $s > n$ (with implicit constants dependent on the Lebesgue measure of Ω). They also constructed a counterexample showing that if (1.6) holds, it is necessary to have an additional hypothesis on $b + c$ (see [16, Lemma 7.4]).

Proving the boundary regularity of solutions to the generalized Dirichlet problem with data $\phi \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ has been an important problem in the area and stems back to the work of Wiener for the Laplace operator [37]. Wiener characterized the points $\xi \in \partial\Omega$ that a solution converges continuously to the boundary in terms of the capacity of the complement of the domain in the balls centered at ξ. The proof was tied to the pointwise bounds of the Green's function and so were its generalizations to elliptic equations. In particular, Littman, Stampacchia and Weinberger [20] constructed the Green's function in a bounded domain for equations $-\operatorname{div} A\nabla u = 0$, where A is real and symmetric, proving such a criterion and later, Grüter and Widman [11] extended their results to operators with possibly non-symmetric A. For equations with lower order coefficients in bounded domains, Stampacchia [34] showed a Wiener-type criterion in sufficiently small balls centered at the boundary of Ω. On the other hand, Kim and Sakellaris [16] succeeded to construct the Green's function with pointwise bounds (which was their main goal) following the method of Grüter and Widman, assuming either (1.6) and $b + c \in L^n$, or (1.5) and $b + c \in L^s$, $s > n$. This is the best known result in this setting in domains with finite Lebesgue measure. In this case though, the construction of the Green's function was not used to conclude boundary regularity. For elliptic systems in unbounded domains, Hofmann and Kim [13] constructed the Green's function assuming that their solutions satisfy the interior *a priori* estimates of De Giorgi/Nash/Moser. They also showed boundary Hölder continuity of the solution of the Dirichlet problem with $C^\alpha(\overline{\Omega})$ data under the stronger assumption of Lebesgue measure density condition of the complement of Ω in the balls centered at $\partial\Omega$ (see also [15]). Recently, Davey, Hill and Mayboroda [6] extended [13] to systems with lower order terms in $b \in L^q$, $c \in L^s$ and $d \in L^{t/2}$, with

$\min\{q, s, t\} > n$, whose associated bilinear form is coercive. For lower order coefficients in the Stummel-Kato class in domains with $C^{1,1}$ boundary, the Green's function was constructed in [14], while in [39], elliptic systems were considered, assuming though smallness on the norms and coercivity.

Let us now discuss our methods. Inspired by the treatment of the Dirichlet problem in [2] and specifically the use of Lemma 2.34, we are able to extend their results to operators with either negativity assumption (as opposed to $-\text{div}b + d \leq \mu < 0$) by requiring solvability in the Sobolev space $Y^{1,2}$ instead of $W^{1,2}$ with non-divergence interior data in $L^{\frac{2n}{n+2}}$ instead of L^2. This is the "correct" Sobolev space in unbounded domains and had already appeared in [21] and in connection with the Green's function in [13]. The main difficulty lies on the fact that when we are proving the global bounds for the solution of the Dirichlet problem, we arrive to an estimate where the term

$$\|b + c\|_{L^{n,q}(\Omega)} \|\nabla u\|^2_{L^2(\Omega)}$$

should be absorbed. But unless one has smallness of $\|b + c\|_{L^{n,q}(\Omega)}$ this is impossible. To deal with this issue, we use Lemma 2.34 and split the domain in a finite number of subsets Ω_i where the norm $L^{n,q}(\Omega_i)$ norm of $b+c$ becomes small. We also write u as a finite sum of u_i so that $(\text{supp}\nabla u_i) \subset \Omega_i$ and, loosely speaking, the term above can be hidden. An iteration argument is then required, which concludes the desired result. An approximation argument on the data and the domain yields the desired well-posedness. The same considerations apply to prove the weak maximum principle for subsolutions with either negativity condition, which, in turn, allows us to solve the unilateral variational poblem and thus, the obstacle problem in bounded domains. As a corollary we obtain that the minimum of two subsolutions of the inhomogeneouus equation $Lu = f - \text{div}g$ is also a subsolution.

Moving further to the proof of Caccioppoli inequality, some serious difficulties arise. Up to now, Caccioppoli's inequality was unknown with so general conditions, since it could be solved only for balls $r \leq 1$ and then rescale. This resulted to the appearance of the Lebesgue measure of the domain in the constants and so, it could not serve our purpose for scale invariant estimates. To overcome this important obstacle, we had to make a technically challenging adaptation of the method that solves the Dirichlet problem. The idea to use this iteration method to prove standard and refined Caccoppoli inequalities is novel and turns out to be the most important ingredient that overcomes the necessity for smallness of the norms of the coefficients in order to develop a De Giorgi/Nash/Moser theory for so general operators.

To prove local boundedeness, weak Harnack inequality, interior and boundary regularity, we have to make a non-trivial adaptation of the arguments of Gilbarg and Trudinger [9, pp. 194–209]. To do so, we are required to prove a refined version of Caccioppoli inequality (Theorems 3.5–3.8), which in [9] was immediate. This turns out to be an even more demanding task than the proof of Caccioppoli inequality itself. Once we obtain them, we show Lemma 4.1, which is the building block of a Moser-type iteration argument. For this lemma, we need an embedding inequality (see Corollary 2.17) with constants independent of the domain, which we prove, since we were not able to find it in the literature (with constants independent of the domain). The use of the Stummel-Kato class $\mathcal{K}(\Omega)$ as an appropriate class of functions for the interior data and the lower order coefficients is not new and has its roots to Schrödinger operators with singular potentials (see [18] and the references therein). Although, in our case, due to the counterexample of Kim and Sakellaris [16] (see Example 4.8), $|b + c|^2$ should be in appropriate subspace of it satisfying a Carleson-Dini-type condition. In fact, a $\frac{1}{2}$-Dini condition on the Stummel-Kato modulus was imposed in [26] to prove local boundedness of subsolutions for certain quasi-linear equations, but their constants depended on Ω. Our Moser-type iteration argument in the proof of Theorem 4.4 follows their ideas, but to get scale invariant estimates, it is necessary to come up with the condition (2.4) and deal with some technical details that required attention already in the original proof. In Example 4.9, we also show that a negativity condition is necessary to obtain local boundedness.

Regarding interior and boundary regularity, as is customary, we go through an application of the weak Harnack inequality. But for this, we need the positivity condition to hold which would force us to assume $L1 = 0$, or equivalently $-\mathrm{div} b + d = 0$. But since this would lead to a significant restriction on the class of operators that our theorems would apply, we incorporate $-\mathrm{div}(bu)$ and du to the interior data $-\mathrm{div} g$ and f respectively. The "new" equation has the form

$$\widetilde{L}u = -\mathrm{div}(A\nabla u) - c\nabla u = (f + du) - \mathrm{div}(g - bu),$$

for which it is true that $\widetilde{L}1 = 0$. The price we have to pay is to impose the additional assumptions $|b|^2$ and $|d| \in \mathcal{K}_{\mathrm{Dini}}(\Omega)$ (for interior regularity and boundary regularity under the CDC condition). Of course, we require u to be locally bounded as well and thus, we need to assume one of the Assumptions (1)–(3). It is interesting to see that the proof of Theorem 4.14 (ii), where we are proving a Wiener type criterion for boundary regularity, is quite laborious as it requires a modification of the original argument in [9] (which is not obvious without the capacity density condition) and a new way of handling the second term Σ_2 in the iteration scheme. Moreover we have to assume a slightly stronger condition, i.e., that $|f|, |d| \in \mathcal{K}_{\mathrm{Dini},\delta}(\Omega)$ and $|b|^2, |g|^2 \in \mathcal{K}_{\mathrm{Dini},\delta/2}(\Omega)$ for some $\delta \in (0, 1)$. To our knowledge, this is the first Wiener-type criterion for boundary regularity of solutions for equations with lower order coefficients with so general assumptions. Moreover, the interior regularity is also new in the case that the radii of the balls we consider are not small (and thus, we do not have smallness of the norms of the coefficients). Let us comment here that one could try to prove boundary regularity following [11] or even [12], but in both cases, there would only be treated solutions of equations with no right hand-side and $b_i = d = 0$, $1 \leq i \leq n$. This is because of the need of lower pointwise bounds for the Green's function or equivalently a Harnack inequality, which, in this situation, only holds for equations of the form $Lu = -\mathrm{div} A\nabla u - c\nabla u = 0$.

Finally, having proved all the results above, we are in a position to construct the Green's function using the method of Hofmann and Kim [13] along with its variant of Kang and Kim [15], where the main ingredients are the well-posedness of the Dirichlet problem, local boundedness, Caccioppoli's inequality, and maximum principle, while, for the approximating operators, we also use the interior continuity for solutions of equations with lower order coefficients that satisfy $|b|^2, |c|^2, |d| \in \mathcal{K}_{\mathrm{Dini}}(\Omega)$. We would not need an approximation argument if it wasn't for the lack of continuity in the general case. This creates some trouble in the proof of $G(x, y) = G^t(y, x)$ (and nowhere else), where G^t stands for the Green's function associated with L^t, the formal adjoint of L. It is important to point out that the pointwise bounds for G do not hold unless local boundedness of subsolutions of $L^t u = 0$ is true; in view of Example 4.8, an additional condition on $b + c$ is necessary. In our case, this will be $|b + c|^2 \in \mathcal{K}_{\mathrm{Dini}}(\Omega)$ as before. Remark that, since Ω may have infinite Lebesgue measure, we can assume $\Omega = \mathbb{R}^n$ and construct the fundamental solution.

Related results: An interesting result, which is very related to our work, was obtained simultaneously and independently by Georgios Sakellaris. The first version of our paper and [28] were uploaded on ArXiv.org the same day (9th of April 2019). His primary goal was to construct Green's functions for elliptic operators of the form (1.1) in general domains under either negativity condition that satisfy scale invariant pointwise bounds. Then, he applies them to obtain global and local boundedness for solutions to equations with interior data in the case (1.6). To do this, it was required $b+c$ to be in a scale invariant space, which for the author was the Lorentz space $L^{n,1}(\Omega)$ (as opposed to $|b+c|^2 \in \mathcal{K}_{\text{Dini}}(\Omega)$ we identified). His method is totally different than ours and is based on delicate estimates for decreasing rearrangements. In fact, he first proves the existence of Green's functions via various approximations and then uses their properties to obtain *a priori estimates*; our method follows the exact opposite direction. Our paper and [28] are complementary since, apart from the major differences in the approach, the conditions $|b+c|^2 \in \mathcal{K}_{\text{Dini}}(\Omega)$ and $|b+c| \in L^{n,1}(\Omega)$ are not comparable. Indeed, if $g(x) = |x|^{-1}(-\log|x|)^{-3}\mathbf{1}_B(x)$, where $B := B(0, \frac{1}{e})$, then $g \in L^{n,1}(B)$ such that $g^2 \notin \mathcal{K}_{\text{Dini},\alpha}(B)$ for any $\alpha > 0$ (see [28]), while, in Example 2.23, we show that there exists a non-negative function $f \in \mathcal{K}_{\text{Dini},\alpha}(\mathbb{R}^n_+)\setminus L^{p,q}(\mathbb{R}^n_+)$ for any $\alpha > 0$, $p > 0$ and $q \in (0,\infty]$, and so $h := \sqrt{f} \notin L^{n,1}(\mathbb{R}^n_+)$ and $h^2 \in \mathcal{K}_{\text{Dini},\alpha}(\mathbb{R}^n_+)$. We would like to note here that Sakellaris observed that, due to a Lorentz-Sobolev embedding theorem and density, (1.5) or (1.6) can be applied assuming that $b, c \in L^{n,\infty}(\Omega)$, $d \in L^{n/2,\infty}(\Omega)$. Although our original assumptions were $b, c \in L^n(\Omega)$, $d \in L^{n/2}(\Omega)$, and the constants depended on $\|b+c\|_{L^n(\Omega)}$ (the same dependence as in [28]), while working the details of the case $|b+c|^2 \in \mathcal{K}(\Omega)$, we realized that our method extends almost unchanged when $b+c \in L^{n,q}(\Omega)$, for $q \in [n,\infty)$, which is a slight improvement compared to our previous results and the ones in [28]. We claim no credit though for the idea to use the Lorentz-Sobolev embedding theorem, which we learned from [28].

Around a year after the last version of the present manuscript was uploaded on ArXiv.org (26th of April 2019), Sakellaris uploaded [29] on the same preprint server (28th of May 2020), where, under the assumptions of [28], he obtains interior and boundary Harnack inequalities and, under smallness assumptions on the norms of the coefficients, he further proves interior and boundary Moser's estimates as well as interior local continuity.

2 Preliminaries

We will write $a \lesssim b$ if there is $C > 0$ so that $a \leq Cb$ and $a \lesssim_t b$ if the constant C depends on the parameter t. We write $a \approx b$ to mean $a \lesssim b \lesssim a$ and define $a \approx_t b$ similarly. If $B_r(x)$ is a ball of radius r and center $x \in \overline{\Omega}$, we will denote $\Omega_r(x) = B_r(x) \cap \Omega$.

2.1 Sobolev space

Definition 2.1 If $1 \leq p < n$ and $p^* = \frac{np}{n-p}$, we define the Sobolev spaces $Y^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ to be the space of all weakly differentiable functions $u \in L^{p^*}(\Omega)$ and $L^p(\Omega)$ respectively, whose weak derivatives are functions in $L^p(\Omega)$. We endow these spaces with the respective norms

$$\|u\|_{Y^{1,p}(\Omega)} = \|u\|_{L^{p^*}(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$$
$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

We say that $u \in Y^{1,2}_{\text{loc}}(\Omega)$ (resp. $u \in W^{1,2}_{\text{loc}}(\Omega)$) if $u \in Y^{1,2}(K)$ (resp. $u \in W^{1,2}(K)$) for any compact $K \subset \Omega$. We also define $Y^{1,p}_0(\Omega)$ and $W^{1,p}_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ in $Y^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ respectively, and denote their dual spaces by $Y^{-1,p'}(\Omega)$ and $W^{-1,p'}(\Omega)$, where p' is the Hölder conjugate of p.

By Sobolev embedding theorem, it is clear that $W_0^{1,p}(\Omega) \subset Y_0^{1,p}(\Omega)$, while if Ω has finite Lebesgue measure they are in fact equal. See, for instance, Theorem 1.56 and Corollary 1.57 in [21]. Moreover, $Y_0^{1,p}(\mathbb{R}^n) = Y^{1,p}(\mathbb{R}^n)$ (see e.g. Lemma 1.76 in [21]). We will denote by $2_* = \frac{2n}{n+2}$ the dual Sobolev exponent for $p = 2$.

For $u \in Y^{1,2}_{\text{loc}}(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$, the bilinear form which corresponds to the elliptic operator (1.1) is given by

$$\mathcal{L}(u, \varphi) = \int_\Omega (A\nabla u + du) \cdot \nabla\varphi - (c \cdot \nabla u + du)\,\varphi.$$

which, by the embedding given in (2.15) or the one in [28, p.6 and Lemma 2.2], is well-defined if (1.4) holds. For the same reasons we can use (1.5) and (1.6) with $Y_0^{1,\frac{n}{n-1}}(\Omega)$ functions.

When we write $Lu = f - \operatorname{div} g$, where $f \in L^1_{\text{loc}}(\Omega)$ and $g \in L^1_{\text{loc}}(\Omega; \mathbb{R}^n)$, we mean that it holds "in the weak sense", i.e.,

$$\mathcal{L}(u, v) = \int_\Omega f v + g \cdot \nabla v, \quad \text{for all } v \in C_c^\infty(\Omega).$$

If $f \in L^{2_*}(\Omega)$ and $g \in L^2(\Omega)$, we can extend it by density to $v \in Y_0^{1,2}(\Omega)$.

In the sequel we will require a notion of supremum and infimum of a function in $Y^{1,2}(\Omega)$ at the boundary of an open set $\Omega \subset \mathbb{R}^n$ since such a function is not necessarily continuous all the way to the boundary. Let Y denote either $Y^{1,2}(\Omega)$ or $W^{1,2}(\Omega)$ and Y_0 be either $Y_0^{1,2}(\Omega)$ or $W_0^{1,2}(\Omega)$.

Definition 2.2 Given a function $u \in Y$, we say that $u \leq 0$ on $\partial\Omega$ if $u^+ \in Y_0$. If u is continuous in a neighborhood of $\partial\Omega$ then $u \leq 0$ on $\partial\Omega$ in the Sobolev sense if $u \leq 0$ in the pointwise sense. In the same way $u \geq 0$ if $-u \leq 0$ and $u \leq w$ if $u - w \leq 0$. We define the boundary supremum and infimum of u as

$$\sup_{\partial\Omega} u = \inf\{k \in \mathbb{R} : (u - k)^+ \in Y_0\} \quad \text{and} \quad \inf_{\partial\Omega} u = -\sup_{\partial\Omega}(-u).$$

Definition 2.3 Let $E \subset \overline{\Omega}$ and $u \in Y$. We say that $u \leq 0$ on E if u^+ is the limit in Y-norm of a sequence of $C_c^\infty(\overline{\Omega} \setminus E)$. Then $u \geq 0$ and $u \leq v$ can be defined naturally. Moreover, if Ω has finite Lebesgue measure.

$$\sup_E u = \inf\{k \in \mathbb{R} : u \leq k \text{ on } E\} \quad \text{and} \quad \inf_E u = -\sup_{\partial\Omega}(-u).$$

If $E = \partial\Omega$ the two definitions above coincide.

We record some results for Sobolev functions that we will need later. Their proofs can be found in [21] and/or in [12] for functions in $W^{1,2}(\Omega)$ or $W_0^{1,2}(\Omega)$. Although, one can make the obvious modifications to prove them for $Y^{1,2}(\Omega)$ or $Y_0^{1,2}(\Omega)$.

Lemma 2.4 *If $\Omega \subset \mathbb{R}^n$ is open and connected, $u \in Y$ and $\nabla u = 0$ a.e. in Ω, then u is a constant in Ω. If we also assume $u \in Y_0$, then $u = 0$.*

Proof The fact that u is a constant can be found in [21, Corollary 1.42], while the second part can proved by a slight modification of the proof of [12, Lemma 1.17]. □

Lemma 2.5 ([21], Corollary 1.43) *If $u, v \in Y$ (resp. Y_0) then* $\max(u, v)$ *and* $\min(u, v)$ *are in Y (resp. Y_0) and*

$$\nabla \max(u, v)(x) = \begin{cases} \nabla u & , \text{if } u \geq v \\ \nabla v & , \text{if } v \geq u \end{cases},$$

$$\nabla \min(u, v)(x) = \begin{cases} \nabla v & , \text{if } u \geq v \\ \nabla u & , \text{if } v \geq u \end{cases}.$$

In particular, $\nabla u = \nabla v$ a.e. on the set $\{x \in \Omega : u(x) = v(x)\}$.

Theorem 2.6 ([21], Theorem 1.74) *Let $\Omega \subset \mathbb{R}^n$ be an open set and let f be a Lipschitz function such that $f(0) = 0$.*

(i) *If $u \in W^{1,1}_{loc}(\Omega)$ then $f \circ u \in W^{1,1}_{loc}(\Omega)$. Moreover, for a.e. $x \in \Omega$, we have that either*

$$\nabla(f \circ u)(x) = f'(u(x))\nabla u(x),$$

or

$$\nabla(f \circ u)(x) = \nabla u(x) = 0.$$

(ii) *If $u \in Y_0$, then $f \circ u \in Y_0$ and*

$$\|f \circ u\|_Y \leq \|f'\|_{L^\infty(\Omega)} \|u\|_Y.$$

Remark that it is necessary to have $f(0) = 0$ when Ω is unbounded. For example, if $f(t) = 1$, then $f \circ u \notin Y^{1,2}(\Omega)$.

Lemma 2.7 *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f : \mathbb{R} \to \mathbb{R}$ be a function in $Lip(\mathbb{R})$. If $u \in Y$, then $f \circ u \in Y_{loc}$.*

Lemma 2.8 ([12], Theorem 1.25) *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in Y$.*

(i) *If u has compact support, then $u \in Y_0$.*
(ii) *If $v \in Y_0$ and $0 \leq u \leq v$ a.e.in Ω, then $u \in Y_0$.*
(iii) *If $v \in Y_0$ and $|u| \leq |v|$ a.e. in $\Omega \backslash K$, where K is a compact subset of Ω, then $u \in Y_0$.*

2.2 Stummel–Kato class

Definition 2.9 Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and set

$$\vartheta(f, r) := \sup_{x \in \mathbb{R}^n} \left(\int_{B_r(x)} \frac{|f(y)|}{|x - y|^{n-2}} \, dy \right), \quad \text{for } r > 0,$$

We will denote by $\vartheta_\Omega(f, r) := \vartheta(f \chi_\Omega, r)$, for $r > 0$. We define the Stummel-Kato class $\mathcal{K}$ and its variant $\mathcal{K}'$ as follows:

$$\begin{aligned} \widehat{\mathcal{K}}(\Omega) &= \{f \in L^1_{\text{loc}}(\Omega) : \vartheta_\Omega(f, r) < \infty, \ \text{for each } r > 0\}, \\ \mathcal{K}(\Omega) &= \{f \in L^1_{\text{loc}}(\Omega) : \lim_{r \to 0} \vartheta_\Omega(f, r) = 0 \text{ and } \vartheta_\Omega(f, r) < \infty, \ \text{for } r > 0\}, \\ \mathcal{K}'(\Omega) &= \{f \in L^1(\Omega) : \lim_{r \to 0} \vartheta_\Omega(f, r) = 0 \text{ and } \vartheta_\Omega(f) := \sup_{r > 0} \vartheta_\Omega(f, r) < \infty\}. \end{aligned} \tag{2.1}$$

We will write that $f \in \widehat{\mathcal{K}}_{\mathrm{loc}}(\Omega)$ (resp. $\mathcal{K}_{1,\mathrm{loc}}(\Omega)$) if $f \in \widehat{\mathcal{K}}(D)$ (resp. $\mathcal{K}(D)$) for any bounded open set $D \subset \mathbb{R}^{n+1}$ so that $\overline{D} \subset \Omega$. If Ω is bounded,

$$\vartheta_\Omega(f) = \sup_{r \in (0, 2\,\mathrm{diam}(\Omega))} \vartheta_\Omega(f, r),$$

and so $\mathcal{K}(\Omega) = \mathcal{K}'(\Omega)$.

It is easy to see that, by a simple covering argument, there exists a dimensional constant $C_{\mathrm{db}} > 0$ so that

$$\vartheta_\Omega(f, r) \leq C_{\mathrm{db}} \vartheta_\Omega(f, r/2) \quad \text{for every } r > 0. \tag{2.2}$$

Therefore, since $\vartheta_\Omega(f, r)$ is non-decreasing in r, there exists $c > 0$ so that

$$c := \frac{\ln 2}{C_{\mathrm{db}}} \leq \frac{1}{\vartheta_\Omega(f, r)} \int_{r/2}^{r} \vartheta_\Omega(f, t) \frac{dt}{t} \leq \frac{1}{\vartheta_\Omega(f, r)} \int_0^r \vartheta_\Omega(f, t) \frac{dt}{t}.$$

Let us recall that that a function $f \in L^1_{\mathrm{loc}}(\Omega)$ is in the Morrey space $\mathcal{M}^\lambda(\Omega)$, if

$$\sup_{r>0} \sup_{B_r \subset \mathbb{R}^n} \frac{1}{r^\lambda} \int_{B_r \cap \Omega} |f(x)|\, dx < \infty.$$

that a function $f \in L^1_{\mathrm{loc}}(\Omega)$ is in the generalized Morrey space $\mathcal{M}^\varphi(\Omega)$ with modulus φ if

$$\sup_{r>0} \sup_{B_r \subset \mathbb{R}^n} \frac{1}{\varphi(r)} \frac{1}{r^{n-2}} \int_{B_r \cap \Omega} |f(x)|\, dx < \infty \quad \text{and} \quad \int_0^1 \varphi(t) \frac{dt}{t} < \infty.$$

By [27, Lemma 1.1], $\mathcal{M}^{n-2+\varepsilon}(\Omega) \subset \mathcal{K}(\Omega)$, for any $\varepsilon \in (0, 2)$, since for every $f \in \mathcal{M}^{n-2+\varepsilon}(\Omega)$, it holds that

$$\vartheta_\Omega(f, r) \lesssim r^{n-2+\varepsilon} \|f\|_{\mathcal{M}^{n-2+\varepsilon}(\Omega)},$$

while, if $f \in \mathcal{K}(\Omega)$ and $\int_0^1 \vartheta_\Omega(f, t) \frac{dt}{t} < \infty$, then it is straightforward to see that $f \in \mathcal{M}^{\vartheta_\Omega(f,\cdot)}(\Omega)$ since

$$\int_{B(x,r) \cap \Omega} |f(y)|\, dy \leq r^{n-2} \vartheta_\Omega(f, r).$$

For fixed $r > 0$, we define the space

$$L^1_{\mathrm{loc},r}(\Omega) = \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|_{L^1_{\mathrm{loc},r}(\Omega)} := \sup_{x \in \mathbb{R}^n} \|f\|_{L^1(B(x,r) \cap \Omega)} < \infty \right\},$$

which clearly contains $\widehat{\mathcal{K}}(\Omega)$. One case see that $\|\cdot\|_{L^1_{\mathrm{loc},r}(\Omega)}$ is a norm on $L^1_{\mathrm{loc},r}(\Omega)$ and $\vartheta_\Omega(\cdot, r)$ is a norm on $\widehat{\mathcal{K}}(\Omega)$ and $\mathcal{K}(\Omega)$. Analogously, $\vartheta_\Omega(\cdot)$ is a norm on $\mathcal{K}'(\Omega)$. In the next lemma we provide an elementary proof of the fact that those spaces are complete.

Lemma 2.10 *$L^1_{loc,r}(\Omega)$, $\widehat{\mathcal{K}}(\Omega)$, $\mathcal{K}(\Omega)$, and $\mathcal{K}'(\Omega)$ are Banach spaces.*

Proof To simplify our notation, for fixed $r > 0$, we will denote

$$X_1 = L^1_{\mathrm{loc},r}(\Omega), \quad X_2 = \widehat{\mathcal{K}}(\Omega), \quad X_3 = \mathcal{K}(\Omega), \quad \text{and} \quad X_4 = \mathcal{K}'(\Omega).$$

We first prove that X_1 is complete. Indeed, there exists $k \in \mathbb{Z}$ such that $2^k < r \leq 2^{k+1}$, and let $Q \in \mathcal{D}_k(\mathbb{R}^n)$ be the dyadic grid in $\mathbb{R}^n$ that consists of cubes of sidelength 2^k and notice that, by easy geometric considerations,

$$\|f\|_{L^1_{\mathcal{D}_k}(\Omega)} := \sup_{Q \in \mathcal{D}_k} \|f\|_{L^1(Q\cap\Omega)} \approx_n \|f\|_{X_1}.$$

In addition, $L^1_{\mathcal{D}_k}(\Omega)$ is the direct sum $\bigoplus_{Q\in\mathcal{D}_k} X_Q$ of the Banach spaces $X_Q = L^s(Q \cap \Omega)$ with norm $\sup_Q \|\cdot\|_{L^1(Q\cap\Omega)}$. In this case, the completeness is preserved and thus, $L^1_{\mathcal{D}_k}(\Omega)$ is a Banach space as well, which readily implies that X_1 is a Banach space.

Now, we will show that X_2 is a Banach space. Let

$$B_{X_2} = \{f \in X_2 : \|f\|_{X_2} \leq 1\}$$

be the closed unit ball in X_2, and let f_k be a Cauchy sequence in X_2. It is easy to see that $\|f\|_{X_1} \leq r^{\frac{n-2}{s}} \vartheta_\Omega(f, r) = r^{n-2}\|f\|_{X_2}$, and by the completeness of X_1, there exists $f \in X_1$ such that $f_k \to f$ in X_1. By Fatou's lemma,

$$\vartheta_\Omega(f, r) \leq \liminf_{k\to\infty} \vartheta_\Omega(f_k, r) \leq 1,$$

and so $f \in B_{X_2}$. Therefore, since X_1 is a Banach space and the embedding of X_2 in X_1 is continuous, by [8, Proposition 14.2.3], we deduce that X_2 is Banach as well. It is easy to see that X_3 is a closed subspace of X_2, and thus, Banach, while, if we replace X_1 by X_2 and X_2 by X_4 in the argument above, we infer that X_4 is Banach space as well. □

2.3 Carleson-Dini Stummel–Kato class

For any $\epsilon > 0$, we define

$$\vartheta_{\epsilon,\Omega}(f, r) = \vartheta_\Omega(f, r) + \epsilon\, r, \tag{2.3}$$

which is strictly increasing, continuous, and satisfies the same properties as $\vartheta_\Omega(f, r)$. Therefore, it is invertible with continuous and strictly increasing inverse $\vartheta^{-1}_{\epsilon,\Omega}(f, r)$. It is clear that $\vartheta_{\epsilon,\Omega}(f, \cdot)$ also satisfies the doubling condition (2.2) with constant $\max(C_{db}, 2)$.

Definition 2.11 If $\alpha > 0$, we say that a function $f \in \widehat{\mathcal{K}}(\Omega)$ is in the *Careslon-Dini Stummel-Kato class* $\mathcal{K}_{\mathrm{Dini},\alpha}(\Omega)$ if it satisfies

$$\int_0^r \vartheta_\Omega(f, t)^\alpha \,\frac{dt}{t} \leq C\, \vartheta_\Omega(f, r)^\alpha, \tag{2.4}$$

for every $r > 0$. and we denote

$$C_{f,\Omega,\alpha} := \sup_{r>0} \frac{1}{\vartheta_\Omega(f, r)^\alpha} \int_0^r \vartheta_\Omega(f, t)^\alpha \,\frac{dt}{t}. \tag{2.5}$$

If $\alpha = 1$ then we write that $f \in \mathcal{K}_{\mathrm{Dini}}(\Omega)$ and $C_{f,\Omega} := C_{f,\Omega,1}$.

Example 2.12 Let $e_j = (\delta_{1j}, \ldots, \delta_{nj})$, for $j \in \{1, \ldots, n\}$ be the orthonormal basis of $\mathbb{R}^n$ and, for any $k \in \{1, 2, \ldots, 2^n\}$, let us denote $\vec{\lambda}_k = (\lambda^1_k, \ldots, \lambda^n_k) \neq 0$ to be the distinct vectors such that $\lambda_k^i = 0$ or 1 for $i \in \{1, \dots, n\}$. For $k \in \{1, 2, \dots, 2^n\}$ and $j \in \mathbb{N}$, define the distinct points in $\mathbb{R}^n$ by

$$x_k := \sum_{i=1}^{n} \lambda_k^i \, e_i \quad \text{and} \quad y_k^j := 2^j x_k.$$

Define now

$$f(x) = \mathbf{1}_{B(0,\frac{1}{8})}(x) + \sum_{j=1}^{\infty} \sum_{k=1}^{2^n} \mathbf{1}_{B(y_k^j, 2^{j-3})}(x).$$

Note that the balls $B(y_k^j, 2^{j-3})$ are mutually disjoint and thus, $|f(x)| \leq 1$ for any $x \in \mathbb{R}^n$. So, for fixed $r > 0$ and every $x \in \mathbb{R}^n$,

$$\int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-2}} dy \leq \int_{B(x,r)} \frac{1}{|x-y|^{n-2}} dy = c_n r^2,$$

which implies that $\vartheta_{\mathbb{R}^n}(f, r) \lesssim r^2$. For the reverse inequality, if $r \geq 1$ remark that there exists a positive integer j_0 such that $2^{j_0 - 1} \leq r < 2^{j_0}$. Then if we set $x_1 = y_1^{j_0+3}$,

$$\vartheta_{\mathbb{R}^n}(f, r) \geq \int_{B(x_1,r)} \frac{|f(y)|}{|x_1-y|^{n-2}} dy \geq \int_{B(x_1,r)} \frac{1}{|x_1-y|^{n-2}} dy = c_n \, r^2.$$

For $r < 1$,

$$\vartheta_{\mathbb{R}^n}(f, r) \geq \int_{B(0,r/8)} \frac{|f(y)|}{|y|^{n-2}} dy = \frac{c_n}{64} r^2.$$

Therefore, $\vartheta_{\mathbb{R}^n}(f, r) \approx r^2$ for any $r > 0$, and so, for any $\alpha > 0$, it holds

$$\int_0^r \vartheta_{\mathbb{R}^n}(f, t)^{\alpha} \frac{dt}{t} \approx \int_0^r t^{2\alpha - 1} \, dt = r^{2\alpha} \approx \vartheta_{\mathbb{R}^n}(f, r)^{\alpha},$$

which implies that $f \in \mathcal{K}_{\text{Dini},\alpha}(\mathbb{R}^n)$. If $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}$, by similar arguments, we can show that $f \in \mathcal{K}_{\text{Dini},\alpha}(\mathbb{R}^n_+)$ for any $\alpha > 0$,

The next lemma is easy to prove by a simple change of variables and we leave the routine details to the interested reader.

Lemma 2.13 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in \mathcal{K}_{Dini}(\Omega)$. For $\rho > 0$, set $f_\rho(x) = \rho \, f(\rho \, x)$ for any $x \in D_\rho := \rho^{-1}\Omega$. Then the following hold:*

(i) *If $\lambda > 0$, then $\vartheta_\Omega(\lambda \, f, t) = \lambda \vartheta_\Omega(f, t)$, for any $t > 0$ and $C_{\lambda f,\Omega} = C_{f,\Omega}$.*
(ii) *$\vartheta_{D_\rho}(f_\rho, t) = \vartheta_\Omega(f, \rho \, t)$, for any $t > 0$.*
(iii) *$C_{f_\rho, D_\rho} = C_{f,\Omega}$.*

Moreover, if $g \in \mathcal{K}_{Dini}(\Omega)$, and we set $g_\rho(x) = \rho \, g(\rho \, x)$, $V = |f| + |g|$, and $V_\rho = |f_\rho| + |g_\rho|$, then $V \in \mathcal{K}_{Dini}(\Omega)$ and

$$C_{V_\rho, D_\rho} = C_{V,\Omega} \leq 2C_{f,\Omega} + 2C_{g,\Omega}.$$

2.4 Sobolev embedding and Interpolation inequalities

The following considerations can be found in [18, p.416] and are based on an inequality proved by Simon in [33, p.455]. Assume that $f \in \mathcal{K}(\Omega)$ and let

$$\psi \in C_c^\infty(\mathbb{R}^n), \quad 0 \le \psi \le 1, \quad \psi = 0 \text{ in } \mathbb{R}^n \setminus B(0,1), \quad \text{and} \quad \int \psi = 1. \tag{2.6}$$

For $\delta > 0$, set $\psi_\delta(x) = \delta^{-n}\psi(\delta^{-1}x)$ and define

$$f_\delta = f * \psi_\delta. \tag{2.7}$$

Then, if $G \subset \Omega$, $r > 0$ and $0 < \delta \le r$, we have

$$\begin{aligned}\vartheta_G((f\mathbf{1}_G)_\delta, r) &\le \vartheta((f\mathbf{1}_G)_\delta, r) \le \vartheta(f\mathbf{1}_G, r) + \vartheta(f\mathbf{1}_G, \delta)\\ &\le 2\vartheta(f\mathbf{1}_G, r) \le 2\vartheta(f, r).\end{aligned} \tag{2.8}$$

Thus, for a ball B_r so that $B_{2r} \subset \Omega$ and $0 < \delta < r$, we also obtain

$$\vartheta_{B_r}(f_\delta, r) \le \vartheta_{B_r}((f\mathbf{1}_{B_{2r}})_\delta, r) \le 2\vartheta_{B_{2r}}(f, r). \tag{2.9}$$

Moreover, if $|g|^2 \in \mathcal{K}(\Omega)$,

$$\vartheta(|g_\delta|^2, r) \le \vartheta(|g|^2, r) + \vartheta(|g|^2, \delta) \le 2\vartheta(|g|^2, r). \tag{2.10}$$

It is useful to remark that if

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > \delta\} \cap B(0, \delta^{-1}), \tag{2.11}$$

then $\vartheta((f\mathbf{1}_{\Omega_\delta})_\delta, r) = \vartheta_\Omega((f\mathbf{1}_{\Omega_\delta})_\delta, r)$.

In the next lemma we use an argument from [36].

Lemma 2.14 *If $f \in \mathcal{K}(\Omega)$ and $\rho > 0$, it holds that $\vartheta_\Omega((f\mathbf{1}_{\Omega_\delta})_\delta - f), \rho) \to 0$, as $\delta \to 0$. If $f \in \mathcal{K}'(\Omega)$, then $\vartheta_\Omega((f\mathbf{1}_{\Omega_\delta})_\delta - f) \to 0$, as $\delta \to 0$.*

Proof Fix $\rho > 0$ and note that by (2.1), for $\varepsilon > 0$, we can find $r_0 < \rho$, so that $\vartheta_\Omega(f, r_0) < \frac{\varepsilon}{6}$. Note that by (2.8), for $0 < \delta < r_0$, we have that $\vartheta_\Omega((f\mathbf{1}_{\Omega_\delta})_\delta - f, r_0) \le 3\vartheta_\Omega(f, r_0)$. Thus,

$$\begin{aligned}\vartheta_\Omega((f\mathbf{1}_{\Omega_\delta})_\delta - f, \rho) &\le \vartheta_\Omega((f\mathbf{1}_{\Omega_\delta})_\delta - f, r_0)\\ &\quad + \sup_{x\in\mathbb{R}^n} \int_{(B(x,r)\setminus B(x,r_0))\cap\Omega} \frac{|(f\mathbf{1}_{\Omega_\delta})_\delta(y) - f(y)|}{|x-y|^{n-2}}\,dy\\ &\le \varepsilon/2 + r_0^{2-n} \sup_{x\in\mathbb{R}^n} \int_{B(x,\rho)\cap\Omega} |(f\mathbf{1}_{\Omega_\delta})_\delta(y) - f(y)|\,dy\\ &=: \varepsilon/2 + r_0^{2-n} I_\rho.\end{aligned}$$

As $\vartheta_\Omega((f\mathbf{1}_{\Omega_\delta})_\delta - f, \rho) \le 3\vartheta_\Omega(f, \rho) < \infty$, for $0 < \delta < \rho$, there exists $x_0 \in \mathbb{R}^n$ such that

$$I_\rho \le 2\int_{B(x_0,\rho)\cap\Omega} |(f\mathbf{1}_{\Omega_\delta})_\delta(y) - f(y)|\,dy.$$

Now, using that $(f\mathbf{1}_{\Omega_\delta})_\delta \to f$ in $L^1_{\text{loc}}(\Omega)$, there exists $\delta > 0$ such that $\delta < \min(r_0, \rho)$ and

$$\int_{B(x_0,\rho)\cap\Omega} |(f\mathbf{1}_{\Omega_\delta})_\delta(y) - f(y)|\,dy < 4^{-1} r_0^{n-2}\varepsilon.$$

Collecting all the estimates we obtain that $\vartheta_\Omega((f\mathbf{1}_{\Omega_\delta})_\delta - f, \rho) < \varepsilon$. The proof for $f \in \mathcal{K}'(\Omega)$ is the same. □

Lemma 2.15 *If $f \in \mathcal{K}(B_r)$, there exists a constant $c_1 > 0$ depending only on n such that for any $r > 0$ and $u \in W^{1,2}(B_r)$, it holds*

$$\int_{B_r} |u|^2 f \leq c_1\, \vartheta_{B_r}(f, r) \left(\|\nabla u\|^2_{L^2(B_r)} + \frac{1}{r^2} \|u\|^2_{L^2(B_r)} \right). \tag{2.12}$$

Proof This inequality can be found in the proof of Lemma 2.1 in [18] (display (12), p. 416). It is stated with slightly different assumptions but an inspection of the proof reveals that (2.12) is also true. For a similar inequality see Lemma 7.3 in [31]. □

Note that if we set $f = f_\delta$ in (2.12) and use (2.9), we can see that for $0 < \delta < r$,

$$\int_{B_r} |u|^2 f_\delta \leq 2c_1\, \vartheta_{B_{2r}}(f, r) \left(\|\nabla u\|^2_{L^2(B_r)} + \frac{1}{r^2} \|u\|^2_{L^2(B_r)} \right), \tag{2.13}$$

where c_1 is independent of δ.

Lemma 2.16 *If $f \in \mathcal{K}(\mathbb{R}^n)$, then, there exists a constant $c_2 > 0$ depending only on n such that for any $\varepsilon > 0$ and $u \in W^{1,2}(\mathbb{R}^n)$, it holds*

$$\int_{\mathbb{R}^n} |u|^2 f \leq \varepsilon\, \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \frac{\varepsilon}{\vartheta^{-1}_{\epsilon,\mathbb{R}^n}(f, c_2^{-1}\varepsilon)^2} \|u\|^2_{L^2(\mathbb{R}^n)}. \tag{2.14}$$

Proof We cover $\mathbb{R}^n$ with balls $B(z_j, r)$, with center all the points z_j so that nz_j/r have integer coordinates. It is clear that each point $x \in \mathbb{R}^n$ is contained in at most N balls $B(z_j, 2r)$, where N is a positive constant depending only on the dimension n. Fix $\varepsilon > 0$ and choose $r > 0$ small enough so that $\vartheta_{\epsilon,\mathbb{R}^n}(f, r) = (Nc_1)^{-1}\varepsilon$, where c_1 is the constant in (2.12). Thus, using $\vartheta_{B_r}(f, r) \leq \vartheta_{\epsilon,\mathbb{R}^n}(f, r)$ and (2.12), we have that

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^2 f &\leq \sum_{j=1}^{\infty} \int_{B(z_j,r)} |u|^2 f \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{N} \left(\int_{B(z_j,r)} |\nabla u|^2 + \frac{1}{r^2} \int_{B(z_j,r)} |u|^2 \right) \\ &\leq \varepsilon \int_{\mathbb{R}^n} |\nabla u|^2 + \frac{\varepsilon}{r^2} \int_{\mathbb{R}^n} |u|^2, \end{aligned}$$

which, if we set $c_2 = Nc_1$, implies (2.14). □

An immediate corollary of the latter theorem, which will be used in Sect. 4, is the following:

Corollary 2.17 *If $f \in \mathcal{K}(\Omega)$, then, there exists a constant $c_2 > 0$ depending only on n such that for any $\varepsilon > 0$ and $u \in W_0^{1,2}(\Omega)$, it holds*

$$\int_{\Omega} |u|^2 f \leq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} + \frac{\varepsilon}{\vartheta^{-1}_{\epsilon,\Omega}(f, c_2^{-1}\varepsilon)^2} \|u\|^2_{L^2(\Omega)}. \tag{2.15}$$

Remark 2.18 In view of (2.13), it is easy to see that (2.14) and (2.15) still hold if we replace f by f_δ on the left hand-side and keep the same term on the right hand-side.

The remark above, combined with (2.10) and (the proofs of) Lemmas 2.15 and 2.16, and Corollary 2.17, leads to the following corollary which will be crucial in an approximation argument we will need later.

Corollary 2.19 *If $|g|^2 \in \mathcal{K}(\Omega)$, then there exists a constant $c_2' > 0$ depending only on n such that for any $\varepsilon > 0$ and $u \in W_0^{1,2}(\Omega)$ it holds*

$$\int_\Omega |u|^2 |(g\mathbf{1}_{\Omega_\delta})_\delta|^2 \leq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} + \frac{\varepsilon}{\vartheta^{-1}_{\epsilon,\Omega}(|g|^2, {c_2'}^{-1}\varepsilon)^2} \|u\|^2_{L^2(\Omega)}.$$

Lemma 2.20 *If f is supported in a ball B_r and $f \in \mathcal{K}(\mathbb{R}^n)$, there exists a constant $C_s' > 0$ depending only on n such that, if $u \in Y^{1,2}(\mathbb{R}^n)$, it holds*

$$\int_{\mathbb{R}^n} |u|^2 f \leq C_s' \, \vartheta_{\mathbb{R}^n}(f, r) \|\nabla u\|^2_{L^2(\mathbb{R}^n)}. \tag{2.16}$$

Proof This follows from the combination of [21, Theorem 1.79] and the proof of [38, Lemma 3]. □

Lemma 2.21 *If $f \in \mathcal{K}'(\Omega)$, there exists a constant $C_s' > 0$ depending only on n such that, if $u \in Y_0^{1,2}(\Omega)$, it holds*

$$\int_\Omega |u|^2 f \leq C_s' \, \vartheta_\Omega(f) \|\nabla u\|^2_{L^2(\Omega)}. \tag{2.17}$$

Proof Let $B_k := B(0, k)$ and $f_k = f \mathbf{1}_{B_k}$. Then, since $|f_k| \leq |f|$ and $f_k \to f$ pointwisely, by Lemma 2.20, we have that

$$\int_\Omega |u|^2 f_k \leq C_s' \, \vartheta_\Omega(f, k) \|\nabla u\|^2_{L^2(\Omega)} \leq C_s' \, \vartheta_\Omega(f) \|\nabla u\|^2_{L^2(\Omega)},$$

which, by the dominated convergence theorem, concludes the proof of (2.17). □

2.5 Lorentz spaces

Definition 2.22 If f is a measurable function we define the distribution function

$$d_{f,\Omega}(t) = |\{x \in \Omega : |f(x)| > t\}|, \quad t > 0,$$

and its decreasing rearrangement by

$$f^*(t) = \inf\{s > 0 : d_{f,\Omega}(t) \leq s\}.$$

If $p \in (0, \infty)$ and $q \in (0, \infty]$, we can define the Lorentz semi-norm

$$\|f\|_{L^{p,q}(\Omega)} = \begin{cases} p^{\frac{1}{q}} \left(\int_0^\infty \left(t \, d_{f,\Omega}(t)^{\frac{1}{p}} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & , \text{if } q < \infty \\ \sup_{t>0} t \, d_{f,\Omega}(t)^{\frac{1}{p}} & , \text{if } q = \infty. \end{cases},$$

If $\|f\|_{L^{p,q}(\Omega)} < \infty$, we will say that f is in the Lorentz space (p, q) and write $f \in L^{p,q}(\Omega)$. This is quasi-norm and $(L^{p,q}(\Omega), \|\cdot\|_{L^{p,q}(\Omega)})$ is a quasi-Banach space.

We can also define

$$\|f\|_{L^{(p,q)}(\Omega)} = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} \, f^{**}(t) \right)^q \right)^{\frac{1}{q}} \frac{dt}{t} & , \text{if } q < \infty \\ \sup_{t>0} t^{\frac{1}{p}} \, f^{**}(t) & , \text{if } q = \infty. \end{cases},$$

which, for $p \in (1, \infty)$ and $q \in [1, \infty]$, is a norm and it holds that

$$\|f\|_{L^{p,q}(\Omega)} \leq \|f\|_{L^{(p,q)}(\Omega)} \leq \frac{p}{p-1}\|f\|_{L^{p,q}(\Omega)}.$$

If we equip $L^{p,q}(\Omega)$ with this norm, it becomes a Banach space (see [1, Lemma 4.5 and Theorem 4.6]). We will write $f \in L^{p,q}_{\mathrm{loc}}(\Omega)$ if $f \in L^{p,q}(\Omega')$ for any bounded open set $\Omega' \subset \Omega$.

We record that

(1) If $0 < p, r \leq \infty$ and $0 < q \leq \infty$,

$$\||f|^r\|_{L^{p,q}(\Omega)} = \|f\|^r_{L^{pr,qr}(\Omega)};$$

(2) If $0 < p \leq \infty$ and $0 < q_2 < q_1 \leq \infty$,

$$\|f\|_{L^{p,q_1}(\Omega)} \lesssim_{p,q_1,q_2} \|f\|_{L^{p,q_2}(\Omega)}; \tag{2.18}$$

(3) If $0 < p, q, r \leq \infty$, $0 < s_1, s_2 \leq \infty$, $1/p + 1/q = 1/r$, and $1/s_1 + 1/s_2 = 1/s$,

$$\|fg\|_{L^{r,s}(\Omega)} \lesssim_{p,q,s_1,s_2} \|f\|_{L^{p,s_1}(\Omega)}\|g\|_{L^{q,s_2}(\Omega)}. \tag{2.19}$$

We refer to [1, Chapter 4] and [10, Chapter 1] for the proofs. It is worth noting that

$$L^{\frac{n}{2},1}(\Omega) \subset \mathcal{K}'(\Omega),$$

while, for $n \geq 3$, $\mathcal{K}(\Omega)$ and $L^{\frac{n}{2},q}(\Omega)$, $q \geq n$, are not comparable.

Example 2.23 If f is the function of Example 2.12, then it is easy to see that

$$d_{f,\mathbb{R}^n}(t) = \begin{cases} 0 & \text{, if } t > 1 \\ +\infty & \text{, if } t \in (0,1]. \end{cases}$$

and, by definition, for every $p > 0$ and $q \in (1, \infty)$,

$$\|f\|^q_{L^{p,q}(\mathbb{R}^n)} = p\int_0^1 d_{f,\mathbb{R}^n}(t)^{\frac{q}{p}} t^{q-1}\, dt \geq 2^{q-1} p \int_{1/2}^1 d_{f,\mathbb{R}^n}(t)^{\frac{q}{p}}\, dt = +\infty.,$$

while for every $p > 0$ and $q \in (0, 1]$,

$$\|f\|^q_{L^{p,q}(\mathbb{R}^n)} = p\int_0^1 d_{f,\mathbb{R}^n}(t)^{\frac{q}{p}} t^{q-1}\, dt \geq p \int_0^1 d_{f,\mathbb{R}^n}(t)^{\frac{q}{p}}\, dt = +\infty.$$

It is clear that $\|f\|_{L^{p,\infty}(\mathbb{R}^n)} = +\infty$. Therefore, $f \in \mathcal{K}_{\mathrm{Dini},\alpha}(\mathbb{R}^n)\backslash L^{p,q}(\mathbb{R}^n)$ for any $\alpha > 0$, $p > 0$, and $q \in (0, \infty]$. Similarly, one can show that $f \in \mathcal{K}_{\mathrm{Dini},\alpha}(\mathbb{R}^n_+)\backslash L^{p,q}(\mathbb{R}^n_+)$ for any $\alpha > 0$, $p > 0$, and $q \in (0, \infty]$.

Definition 2.24 Let $\{E_k\}_{k=1}^\infty$ be a sequence of measurable subsets of Ω. We will write $E_k \to \emptyset$ a.e. if $\mathbf{1}_{E_k} \to 0$ a.e. in Ω, which is equivalent to $|\limsup_{k\to\infty} E_k| = 0$.

We will say that a function f in a Banach function space X (see [30, Definition 6.5]) has *absolutely continuous norm* in X if $\|f\mathbf{1}_{E_k}\|_X \to 0$ for every sequence $\{E_k\}_{k\geq 1}$ such that $E_k \to \emptyset$ a.e. The set of all functions in X of absolutely continuous norm is denoted by X_a. If $X_a = X$, then the space itself is said to have *absolutely continuous norm*. In this case, simple functions supported on a set of finite Lebesgue measure are dense in X.

Record that $L^{p,q}(\Omega)$, for $p \in (1, \infty)$ and $q \in [1, \infty)$, is a Banach function space (see [1, p.219, Theorem 4.6]).

Lemma 2.25 *Let $f \in X$ where $X = \mathcal{K}'(\Omega)$ or $L^{p,q}(\Omega)$, $1 < p < \infty$ and $1 \le q < \infty$. If $\|\cdot\|_X$ stands for either $\vartheta_\Omega(\cdot)$ or $\|\cdot\|_{L^{p,q}(\Omega)}$, then X has absolutely continuous norm. In fact, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\text{if } E \subset \Omega \text{ with } |E| < \delta, \text{ then } \|f\mathbf{1}_E\|_X < \varepsilon.$$

Proof For $\mathcal{K}'(\Omega)$ this was proved in [36, Lemma 2.2], while for $L^{p,q}(\Omega)$ it follows from [1, p. 23, Corollary 4.3] and [1, p. 221, Corollary 4.8]. □

Lemma 2.26 ([4], Theorem V4) *Let $f \in L^{p,q}(\Omega)$, with $p \in (1,\infty)$ and $q \in [1,\infty)$, and for $\delta > 0$, let Ω_δ be as in (2.11). Then, it holds that*

$$\|(f\mathbf{1}_{\Omega_\delta})_\delta\|_{L^{p,q}(\Omega)} \le C_{p,q}\|f\|_{L^{p,q}(\Omega)} \quad \text{and} \quad \|(f\mathbf{1}_{\Omega_\delta})_\delta - f\|_{L^{p,q}(\Omega)} \to 0.$$

In the following definitions and lemmas we follow [28].

Definition 2.27 We define $Y_0^{1,(p,q)}(\Omega)$, for $1 < p < n$ and $1 \le q \le \infty$, to be the closure of $C_c^\infty(\Omega)$ under the semi-norm

$$\|u\|_{Y_0^{1,(p,q)}(\Omega)} = \|u\|_{L^{\frac{np}{n-p},q}(\Omega)} + \|\nabla u\|_{L^{p,q}(\Omega)}.$$

Lemma 2.28 *If $u \in Y_0^{1,(p,q)}(\Omega)$, there exists a constant $C_s > 0$ depending on n such that*

$$\|u\|_{L^{\frac{np}{n-p},q}(\Omega)} \le C_s\|\nabla u\|_{L^{p,q}(\Omega)}. \tag{2.20}$$

If $u \in Y_0^{1,2}(\Omega)$, the same is true for $p = q = 2$.

Proof The proof of the first part can be found in [5, Theorem 4.2(i)] and of the second one in [28, Lemma 2.2]. □

Lemma 2.29 *If $u, w \in Y_0^{1,2}(\Omega)$, then $uw \in Y_0^{1,(\frac{n}{n-1},1)}(\Omega)$ and, in particular, it holds that*

$$\|uw\|_{L^{\frac{n}{n-2},1}(\Omega)} \le 2C_s^2\|\nabla u\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}. \tag{2.21}$$

Proof Here we follow the scheme of the proof of [28, Lemma 2.2]. Since both u and w belong to $Y_0^{1,2}(\Omega)$, we can use (2.20) and (2.19) to deduce that

$$\|w\nabla u\|_{L^{\frac{n}{n-1},1}(\Omega)} \le \|\nabla u\|_{L^2(\Omega)}\|w\|_{L^{\frac{2n}{n-2},2}(\Omega)} \le C_s\|\nabla u\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}. \tag{2.22}$$

The analogous estimate holds if we switch the roles of w and u. Since $u, w \in Y_0^{1,2}(\Omega)$, there exist sequences $\{\phi_k\}_{k\ge1}, \{\psi_k\}_{k\ge1} \subset C_c^\infty(\Omega)$ such that $\phi_k \to u$ and $\psi_k \to w$ in $Y_0^{1,2}(\Omega)$. By Lemma 2.28, we can find a subsequence of $\phi_k\psi_k$ that is weakly-* convergent in $Y_0^{1,(\frac{n}{n-1},1)}(\Omega)$ to some $v \in Y_0^{1,(\frac{n}{n-1},1)}(\Omega)$. But since $v \in L^{\frac{n}{n-2},1}(\Omega) \subset L^{\frac{n}{n-2}}(\Omega)$, it holds that $v = uw$ in $L^{\frac{n}{n-2},1}(\Omega)$. Thus,

$$\begin{aligned}\|uw\|_{L^{\frac{n}{n-2},1}(\Omega)} &\le \liminf_{k\to\infty}\|\phi_k\psi_k\|_{L^{\frac{n}{n-2},1}(\Omega)} \\ &\le C_s \liminf_{k\to\infty}\|\nabla(\phi_k\psi_k)\|_{L^{\frac{n}{n-1},1}(\Omega)} \le 2C_s^2\|\nabla u\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)},\end{aligned}$$

where in the last step we used the same argument as in (2.22) and the strong convergence of ϕ_k and ψ_k in $Y_0^{1,2}(\Omega)$. □

Lemma 2.30 (Embedding inequality) *Let $h \in L^{n,q}(\Omega)$, for $q \in [n, \infty]$, $u \in Y^{1,2}(\Omega)$ and $w \in Y_0^{1,2}(\Omega)$. Then if $D \subset \Omega$ is a Borel set, there exists a constant $C_{s,q} > 0$ (depending only on n and q) such that*

$$\left| \int_D h \nabla u w \right| \leq C_{s,q} \|h\|_{L^{n,q}(D)} \|\nabla u\|_{L^2(D)} \|\nabla w\|_{L^2(\Omega)}. \tag{2.23}$$

Proof This follows from (2.19), (2.20), and (2.18). □

Remark 2.31 In [28, eq. (2.9)], it was observed that if $b, c \in L^{n,\infty}(\Omega)$ and $d \in L^{\frac{n}{2},\infty}(\Omega)$, (1.5) and (1.6) hold if $\varphi \in Y_0^{1,(\frac{n}{n-1},1)}(\Omega)$.

2.6 Two auxiliary lemmas

The next lemma was stated in [26]. The proof as written in [26] is not totally correct since ω is not absolutely continuous. We overcome this obstacle by an approximation argument.

Lemma 2.32 *Let $\omega : (0, \infty) \to (0, \infty)$ be a strictly increasing and continuous function such that $\lim_{r \to 0^+} \omega(r) = 0$ and $\lim_{r \to \infty} \omega(r) = +\infty$. Let $\tau \in (0.1)$, $c > 0$, and $q \geq 1$, and set*

$$b_k = c\, \tau^{kq} \quad \text{and} \quad a_k = b_k^{1/q} \log \omega^{-1}(b_k). \tag{2.24}$$

Then it holds

$$-\sum_{k=1}^{\infty} a_k \leq \frac{1}{1-\tau} \int_0^{\omega^{-1}(c)} \omega(t)^{1/q} \frac{dt}{t}. \tag{2.25}$$

Proof Note that ω is one-to-one and its inverse ω^{-1} is also strictly increasing and continuous. If we define ω_δ as in (2.7) in $\mathbb{R}$, then ω_δ is strictly increasing and smooth satisfying

$$\lim_{t \to 0} \omega_\delta(t) = \int \psi_\delta(-s)\, \omega(s)\, ds =: \alpha_\delta \in [0, \omega(\delta)].$$

Therefore, ω_δ^{-1} is also strictly increasing and smooth on $\mathrm{Ran}(\omega_\delta)$, the range of ω_δ. As $\lim_{\delta \to 0} \omega_\delta(t) = \omega(t)$ locally uniformly in $(0, \infty)$,[2] it is not hard to show that $\lim_{\delta \to 0} \omega_\delta^{-1}(r) = \omega^{-1}(r)$ for all $r \in \mathrm{Ran}(\omega) = (0, \infty)$. Indeed, let $\varepsilon > 0$ and $r > 0$. Then, by the continuity of ω in $(0, \infty)$, there exists $\delta' = \delta'(\varepsilon, r) > 0$ such that

$$|\omega^{-1}(r + \delta') - \omega^{-1}(r)| < \varepsilon \quad \text{and} \quad |\omega^{-1}(r - \delta') - \omega^{-1}(r)| < \varepsilon.$$

For any sequence $\{\delta_n\}_{n=1}^{\infty}$ such that $\delta_n \to 0$ as $n \to \infty$, it holds that $\lim_{n \to \infty} \omega_{\delta_n} = \omega$, and so there exists $n_0 > 0$ such that for every $n > n_0$,

$$|\omega_{\delta_n}(\omega^{-1}(r + \delta')) - (r + \delta')| < \delta' \quad \text{and} \quad |\omega_{\delta_n}(\omega^{-1}(r - \delta')) - (r - \delta')| < \delta'.$$

Therefore,

$$\omega_{\delta_n}(\omega^{-1}(r + \delta')) > r \quad \text{and} \quad \omega_{\delta_n}(\omega^{-1}(r - \delta')) < r,$$

which, using that $\omega_{\delta_n}^{-1}$ is strictly increasing in $(0, \infty)$, implies that

$$\omega_{\delta_n}^{-1}(r) \in [\omega^{-1}(r - \delta'), \omega^{-1}(r + \delta')]$$

[2] Just pointwise convergence is enough here.

and thus, $|\omega_{\delta_n}^{-1}(r) - \omega^{-1}(r)| < \varepsilon$. This concludes the proof of $\lim_{\delta\to 0}\omega_\delta^{-1} = \omega$ pointwisely.

For any fixed positive $N \in \mathbb{N}$, it holds that

$$\begin{aligned}
\sum_{k=0}^{N}(\tau a_k - a_{k+1}) &= \sum_{k=0}^{N} b_{k+1}^{1/q}(\log\omega^{-1}(b_k) - \log\omega^{-1}(b_{k+1})) \\
&= \lim_{\delta\to 0}\sum_{k=0}^{N} b_{k+1}^{1/q}(\log\omega_\delta^{-1}(b_k) - \log\omega_\delta^{-1}(b_{k+1})) \\
&= \lim_{\delta\to 0}\sum_{k=0}^{N} b_{k+1}^{1/q}\int_{b_{k+1}}^{b_k}\frac{1}{\omega_\delta^{-1}(t)}\frac{1}{\omega_\delta'(\omega_\delta^{-1}(t))}\,dt \\
&\le \lim_{\delta\to 0}\sum_{k=0}^{N}\int_{b_{k+1}}^{b_k}\frac{t^{1/q}}{\omega_\delta^{-1}(t)}\frac{1}{\omega_\delta'(\omega_\delta^{-1}(t))}\,dt \\
&= \lim_{\delta\to 0}\int_{b_{N+1}}^{c}\frac{t^{1/q}}{\omega_\delta^{-1}(t)}\frac{1}{\omega_\delta'(\omega_\delta^{-1}(t))}\,dt \\
&= \lim_{\delta\to 0}\int_{\omega_\delta^{-1}(b_{N+1})}^{\omega_\delta^{-1}(c)}\omega_\delta(t)^{1/q}\frac{dt}{t}.
\end{aligned}$$

Remark that $\omega^{-1}(b_{N+1}) > 0$. For $\eta > 0$, there exists $\delta_0 = \delta(\eta, c, b_{N+1}) > 0$ such that for every $\delta < \delta_0$,

$$|\omega_\delta^{-1}(c) - \omega^{-1}(c)| < \eta \quad\text{and}\quad |\omega_\delta^{-1}(b_{N+1}) - \omega^{-1}(b_{N+1})| < \eta.$$

Therefore, for $\delta < \delta_0$,

$$\int_{\omega_\delta^{-1}(b_{N+1})}^{\omega_\delta^{-1}(c)}\omega_\delta(t)^{1/q}\frac{dt}{t} \le \int_{\omega^{-1}(b_{N+1})-\eta}^{\omega^{-1}(c)+\eta}\omega_\delta(t)^{1/q}\frac{dt}{t}.$$

Now, by the local uniform convergence of ω_δ, we can find $0 < \delta_1 \le \delta_0$ such that for every $\delta < \delta_1$, it holds that $|\omega_\delta(t) - \omega(t)| < \eta$ for every $t \in [\omega^{-1}(b_{N+1}) - \eta, \omega^{-1}(c) + \eta]$. Therefore, for $\delta < \delta_1$, we infer that

$$\begin{aligned}
\sum_{k=0}^{N} b_{k+1}^{1/q}(\log\omega_\delta^{-1}(b_k) - \log\omega_\delta^{-1}(b_{k+1})) &= \int_{\omega^{-1}(b_{N+1})-\eta}^{\omega^{-1}(c)+\eta}\omega_\delta(t)^{1/q}\frac{dt}{t} \\
&\le \eta\,\log\frac{\omega^{-1}(c)+\eta}{\omega^{-1}(b_{N+1})-\eta} + \int_{\omega^{-1}(b_{N+1})-\eta}^{\omega^{-1}(c)+\eta}\omega(t)^{1/q}\frac{dt}{t},
\end{aligned}$$

which, by taking $\delta \to 0$, implies that

$$\sum_{k=0}^{N}(\tau a_k - a_{k+1}) \le \eta\,\log\frac{\omega^{-1}(c)+\eta}{\omega^{-1}(b_{N+1})-\eta} + \int_{\omega^{-1}(b_{N+1})-\eta}^{\omega^{-1}(c)+\eta}\omega(t)^{1/q}\frac{dt}{t}.$$

Since η is arbitrary, we may take $\eta \to 0$ and deduce that

$$\sum_{k=0}^{N}(\tau a_k - a_{k+1}) \le \int_{\omega^{-1}(b_{N+1})}^{\omega^{-1}(c)}\omega(t)^{1/q}\frac{dt}{t} \le \int_{0}^{\omega^{-1}(c)}\omega(t)^{1/q}\frac{dt}{t}.$$

If we take limits as $N \to \infty$, we get

$$\sum_{k=0}^{\infty}(\tau a_k - a_{k+1}) \leq \int_0^{\omega^{-1}(c)} \omega(t)^{1/q}\,\frac{dt}{t},$$

which, combined with the equality

$$\sum_{k=0}^{\infty}(\tau a_k - a_{k+1}) = (\tau - 1)\sum_{k=0}^{\infty} a_k,$$

shows (2.25). □

Lemma 2.33 *Let* $\omega : (0, \infty) \to (0, \infty)$ *be a strictly increasing and continuous function such that* $\lim_{r\to 0^+} \omega(r) = 0$. *Assume that*

$$C_\omega := \sup_{r>0} \frac{1}{\omega(r)} \int_0^r \omega(t)\,\frac{dt}{t} < \infty \quad \textit{and} \quad \omega(2r) \leq c_0\omega(r), \; \textit{for any } r > 0.$$

Then

$$\sup_{t\in(0,\infty)} \frac{\omega(t)}{\omega(2t)} < 1.$$

Proof Since ω is strictly increasing and doubling, we have that

$$c_0^{-1} \leq \frac{\omega(t)}{\omega(2t)} < 1, \quad \text{for every } t > 0.$$

This inequality and the continuity of ω in $(0, \infty)$ imply that

$$\sup_{t\in(0,\infty)} \frac{\omega(t)}{\omega(2t)} = 1 \Leftrightarrow \lim_{t\to 0} \frac{\omega(t)}{\omega(2t)} = 1.$$

Assume that $\lim_{t\to 0} \frac{\omega(t)}{\omega(2t)} = 1$. Then, by continuity, if we fix $\varepsilon < (4\, c_0\, C_\omega)^{-1}$, there exists $\rho > 0$ such that for $t < \rho$ it holds that $\omega(t) > (1-\varepsilon)\,\omega(2t)$. If we apply this for $t_m = 2^{-m}\rho$, $m = 0, 1, \ldots, N-1$, the Dini condition yields

$$\frac{1-(1-\varepsilon)^N}{\varepsilon}\omega(\rho) = \sum_{m=0}^{N-1}(1-\varepsilon)^m\omega(\rho) < \sum_{m=0}^{N-1}\omega(2^{-m}\rho) \leq 2\, c_0\, C_\omega\, \omega(\rho).$$

Letting $N \to \infty$, we get $\varepsilon^{-1} \leq 2\, c_0\, C_\omega$ which is a contradiction. □

2.7 The splitting lemmas

The following lemma will be used repeatedly in this manuscript and for the case $p = q = n$ was proved in [2]. We extend it to the case of Lorentz spaces $L^{p,q}(\Omega)$ with $1 < p \leq q < \infty$.

Lemma 2.34 *Let* $\Omega \subset \mathbb{R}^n$ *be an open set,* $u \in Y^{1,2}(\Omega)$, $h \in L^{p,q}(\Omega)$, *for* $1 < p \leq q < \infty$ *and* $a > 0$. *Then there exist mutually disjoint measurable sets* $\Omega_i \subset \Omega$ *and functions* $u_i \in Y^{1,2}(\Omega)$ *for* $1 \leq i \leq \kappa$ *with the following properties:*

(1) $\|h\|_{L^{p,q}(\Omega_i)} = a$, *for* $1 \le i \le \kappa - 1$, *and* $\|h\|_{L^{p,q}(\Omega_\kappa)} \le a$,
(2) $\{x \in \Omega : \nabla u_i \neq 0\} \subset \Omega_i$,
(3) $\nabla u = \nabla u_i$ *in* Ω_i,
(4) $|u_i| \le |u|$,
(5) $u u_i \ge 0$,
(6) $u = \sum_{i=1}^m u_i$,
(7) $u_i \nabla u = \left(\sum_{j=1}^i \nabla u_j\right) u_i$,
(8) $u \nabla u_i = \left(\sum_{j=i}^\kappa u_j\right) \nabla u_i$,

and κ *has the upper bound*

$$\kappa \le a^{-q} \|h\|_{L^{p,q}(\Omega)}^q + 1.$$

If $u \in Y_0^{1,2}(\Omega)$, *then* $u_i \in Y_0^{1,2}(\Omega)$ *for* $1 \le i \le \kappa$.

Proof If $0 \le k < t \le \infty$, we define

$$\Omega(k, t) := \{x \in \Omega : k < |u| \le t, \nabla u \neq 0\},$$

and by Chebyshev's inequality, for $k > 0$, it holds

$$|\Omega(k, t)| \le |\Omega(k, \infty)| \le k^{-2^*} \|u\|_{L^{2^*}}^{2^*} < \infty.$$

Let us define the function $f : [0, \infty]^2 \to [0, \infty)$ by

$$f(k, t) = |\{k < |u| \le t, \nabla u \neq 0\}|.$$

We will show that $f(\cdot, t)$ is continuous in $[0, \infty)$ for any fixed $t \in (0, \infty]$.

To this end, fix $t \in (0, \infty]$ and $k < t$, and let $\{k_\ell\}_{\ell \in \mathbb{N}}$ be a positive decreasing sequence so that $k_\ell \to k$. Thus,

$$f(k, t) = |\Omega(k, t)| = \Big|\bigcup_{\ell=1}^\infty \Omega(k_\ell, t)\Big| = \lim_{\ell \to \infty} f(k_\ell, t),$$

which gives right continuity. Consider now an increasing sequence of positive numbers $\{k_l\}_{l \in \mathbb{N}}$ so that $k_l \to k$. Then

$$\bigcap_{l=1}^\infty \Omega(k_l, t) = \Omega(k, t) \cup \{x \in \Omega : |u| = k, \nabla u \neq 0\}.$$

By Lemma 2.5, we get $\big|\{x \in \Omega : |u| = t, \nabla u \neq 0\}\big| = 0$, and thus, since $|\Omega(k_1, \infty)| < \infty$, we infer that

$$f(k, t) = |\Omega(k, t)| = \Big|\bigcap_{l=1}^\infty \Omega(k_l, t)\Big| = \lim_{l \to \infty} \big|\Omega(k_l, t)\big|,$$

which implies left continuity of $f(\cdot, t)$ and consequently continuity.

If we set

$$\sigma(x) = \begin{cases} 1 & \text{, if } x > 0 \\ -1 & \text{, if } x < 0 \end{cases},$$

we define

$$F_{k,t}(u)=\begin{cases}(t-k)\sigma(u), & |u|>t\\ u-k\sigma(u), & k<|u|\le t\\ 0, & |u|\le k,\end{cases} \quad \text{and} \quad F_{k,\infty}(u)=\begin{cases}u-k\sigma(u), & |u|>k\\ 0, & |u|\le k\end{cases}.$$

For fixed $k,t\in[0,\infty]$, $F_{k,t}\in \mathrm{Lip}(\mathbb{R})$ and $F_{k,t}(0)=0$, and thus, since $u\in Y^{1,2}(\Omega)$ (resp. $Y_0^{1,2}(\Omega)$), by Lemma 2.6, $F_{k,t}(u)\in Y^{1,2}(\Omega)$ (resp. $Y_0^{1,2}(\Omega)$).

Recall that the $L^{p,q}$-norm is absolutely continuous by Lemma 2.25 and thus, since, for any fixed $t\in[0,\infty]$, $\mathbf{1}_{\Omega(k,t)}\to 0$ a.e. as $k\to t$, we will have that $\|h\mathbf{1}_{\Omega(k,t)}\|_{L^{p,q}(\Omega)}\to 0$. For $1<p\le q<\infty$, let us define

$$H(k,t):=\int_0^\infty s^q\, d_{h\mathbf{1}_{\Omega(k,t)}}(s)^{\frac{q}{p}}\,\frac{ds}{s}.$$

If $H(0,\infty)\le a^q$, then we set $\Omega_1=\{x\in\Omega:\nabla u\neq 0\}$ and $u_1=u$. Suppose now that $H(0,\infty)>a^q$, and thus, by the absolute continuity of $L^{p,q}$, there exists $k_1>0$ such that

$$H(k_1,\infty)=a^q.$$

If $H(0,k_1)\le a^q$, we set $\Omega_1=\Omega(k_1,\infty)$ and $\Omega_2=\Omega(0,k_1)$, and $u_1=F_{k_1,\infty}(u)$ and $u_2=F_{0,k_1}(u)$. If, on the other hand, $H(0,k_1)\ge a^q$, there exists $k_2\ge 0$ so that

$$H(k_2,k_1)=a^q.$$

If we iterate, there exists $j_0\in\mathbb{N}$ so that $H(k_i,k_{i-1})=a^q$, if $1\le i<j_0$, and $H(0,k_{j_0})\le a^q$, where $k_0=+\infty$. Indeed, if there were infinitely many i so that $H(k_i,k_{i-1})=a^q$, then, since $\{\Omega(k_i,k_{i-1})\}_{i\ge1}$ are disjoint, we would have

$$\infty=\sum_{i=1}^\infty a^q=\sum_{i=1}^\infty H(k_i,k_{i-1})\le\int_0^\infty s^q\, d_{h\mathbf{1}_{\Omega(0,\infty)}}(s)^{\frac{q}{p}}\,\frac{ds}{s}\le\|h\|^q_{L^{p,q}(\Omega)}<\infty,$$

which is a contradiction. Here we used that $p\le q$ and that for disjoint sets A and B it holds that

$$|\{x\in A:|f|>t\}|+|\{x\in B:|f|>t\}|\le|\{x\in A\cup B:|f|>t\}|.$$

The same argument gives us $j_0\,a^q\le\|h\|_{L^{p,q}(\Omega)}$, that is, $j_0\le a^{-q}\|h\|^q_{L^{p,q}(\Omega)}$.

If we set $\kappa=j_0+1$ and $k_\kappa=0$, for $i\in\{1,\dots,\kappa\}$, we define

$$\Omega_i=\Omega(k_i,k_{i-1}) \quad \text{and} \quad u_i=F_{k_i,k_{i-1}}(u).$$

We have already shown (1), so it remains to prove that (2)–(8) hold as well.

Firstly, (2), (3), and (4) are clear by definition, while (5) follows by simple computations; indeed, note first that $uu_i=0$ whenever $|u|<k_i$. In the set where $|u|>k_{i-1}>k_i$, we have that

$$uu_i=u\,\sigma(u)\,(k_{i-1}-k_i)=|u|\,(k_{i-1}-k_i)\ge 0,$$

while, when $k_i<|u|\le k_{i-1}$,

$$uu_i=u^2-\sigma(u)\,u\,k_i=|u|(|u|-k_i)\ge 0.$$

This concludes the proof of (5).

For (6) and (7), we may rewrite $u_j = u_{k_j,\infty} - u_{k_{j-1,\infty}}$, in view of which, we have

$$\sum_{j=1}^{i} u_j = F_{k_1,\infty}(u) + \sum_{j=2}^{i}(F_{k_j,\infty}(u) - F_{k_{j-1},\infty}(u)) = F_{k_i,\infty}(u). \tag{2.26}$$

In the case $i = \kappa$, we have

$$\sum_{j=1}^{\kappa} u_j = F_{k_\kappa,\infty}(u) = u,$$

yielding (6). By definition, $\nabla u_{k_i,\infty} = \nabla u$, when $|u| > k_i$ (i.e., in the support of u_i), while $u_i = 0$, whenever $|u| \le k_i$. and so, (7) follows from (2.26). Since $\{\nabla u_i \neq 0\} \subset \Omega_i$ we can use (6) to get

$$u\nabla u_i = u_i \sum_{j=1}^{\kappa} \nabla u_j = \nabla u_i \sum_{j=i}^{\kappa} u_j.$$

This concludes the proof of the lemma. □

The direct analogue of this lemma for the space $\mathcal{K}'(\Omega)$ was proved in [36] but it is not stated as such. For the reader's convenience we will give a sketch of the proof.

Lemma 2.35 *Let $\Omega \subset \mathbb{R}^n$ be an open set, $u \in Y^{1,2}(\Omega)$ (resp. $Y_0^{1,2}(\Omega)$), $h \in \mathcal{K}'(\Omega)$ and $a > 0$. Then, there exist mutually disjoint measurable sets $\Omega_i \subset \Omega$ and functions $u_i \in Y^{1,2}(\Omega)$ (resp. $Y_0^{1,2}(\Omega)$), for $1 \le i \le \kappa$, satisfying (2)–(8), so that*

$$\vartheta_\Omega(h\mathbf{1}_{\Omega_i}) = a^2, \ \text{for } 1 \le i \le \kappa - 1, \quad \text{and} \quad \vartheta_\Omega(h\mathbf{1}_{\Omega_\kappa}) \le a^2.$$

If $\rho_0 > 0$ is such that $\vartheta_\Omega(h, \rho_0) = a^2/4$, then κ has the upper bound

$$\kappa \le 1 + 2\, a^{-2}\, \rho_0^{2-n} \|h\|_{L^1(\Omega)}.$$

If Ω is a bounded open set contained in a ball B_r, we can assume $h \in \mathcal{K}(\Omega)$ replacing $\vartheta_\Omega(\cdot)$ by $\vartheta_\Omega(\cdot, r)$.

Proof Using the same notation as before, we define

$$H(k,t) = \vartheta_\Omega(h\mathbf{1}_{\Omega(k,t)}).$$

Making the same stopping time argument with respect to the condition $h(k,t) = a^2$ and noticing that we only used the absolute continuity of the norm, we can reason as in the proof of Lemma 2.34. The only difference lies on the estimate of κ since we cannot linearize it as we did in the previous case.

Let us first show that the stopping process results to a finite number of sets. Indeed, arguing as in the proof of Lemma 2.14, we can find ρ_0 so that $\vartheta_\Omega(h, \rho_0) = a^2/4$ so that

$$a^2 = \vartheta_\Omega(h\mathbf{1}_{\Omega_i}) \le 2\vartheta_\Omega(h\mathbf{1}_{\Omega_i}, \rho_0) + \rho_0^{2-n} \int_{\Omega_i} |h|\mathbf{1}_{\Omega_i}\, dy \le \frac{a^2}{2} + \rho_0^{2-n} \int_{\Omega_i} |h|\mathbf{1}_{\Omega_i}\, dy.$$

So, if assume that there infinite many Ω_i, we can sum in i as before and get

$$\infty \le \rho_0^{2-n} \sum_i \int_{\Omega_i} |h|\mathbf{1}_{\Omega_i}\, dy \le \rho_0^{2-n} \|h\|_{L^1(\Omega)},$$

which is a contradiction. If j_0 is the number of i's for which $\vartheta_\Omega(h\mathbf{1}_{\Omega_i}) = a^2$, the same argument will give the bound

$$j_0 \leq 2a^{-2}\rho_0^{2-n}\|h\|_{L^1(\Omega)}.$$

□

Remark 2.36 It is interesting to see that the bound on κ, although at a first glace does not seem to be scale invariant, in fact it is (with the correct scaling). Indeed, let $h_r = r^2 h(rx)$ in the open set $\Omega_r = r^{-1}\Omega$. Then, by making the change of variables $y = rx$ we have that

$$\rho_0^{2-n}\|h_r\|_{L^1(\Omega_r)} = (\rho_0\, r)^{2-n}\|h\|_{L^1(\Omega)}.$$

Now, recall that ρ_0 was chosen so that $\vartheta_{\Omega_r}(h_r, \rho_0) = a^2/4$, which, by the same change of variables, implies that $\vartheta_\Omega(h, \rho_0\, r) = a^2/4$. Note that if $\vartheta_\Omega(h, \cdot)$ is invertible, we have that $\rho_0\, r = \vartheta_\Omega^{-1}(h, a^2/4)$.

2.8 Variational capacity

Definition 2.37 Let $\Omega \subset \mathbb{R}^n$ be open and $E \subset \Omega$. If we set

$$\mathbb{K}_E(\Omega) := \{w \in Y_0^{1,2}(\Omega) : E \subset \{w \geq 1\}^{\circ}\}$$

then we define the *(variational) capacity of the condenser* (E, Ω) as

$$\operatorname{Cap}(E, \Omega) = \inf_{w \in \mathbb{K}_E} \int_\Omega |\nabla w|^2.$$

The following properties of capacity verify that it is a Choquet capacity and satisfies the axioms considered by Brelot. A proof can be found for instance in Theorem 2.3 in [21].

(i) If $E \subset \Omega$ is compact,

$$\operatorname{Cap}(E, \Omega) = \inf \left\{ \int_\Omega |\nabla w|^2 : w \in C_c^\infty(\Omega),\ u \geq 1 \text{ in } E \right\}.$$

(ii) If $E \subset \Omega$ is open,

$$\operatorname{Cap}(E, \Omega) = \sup_{\text{compact } K \subset E} \operatorname{Cap}(K, \Omega).$$

(iii) If $E_1 \supset E_2 \supset \dots$ is a sequence of compact subsets of Ω,

$$\operatorname{Cap}\Big(\bigcap_{j\geq 1} E_j, \Omega\Big) = \lim_{j\to\infty} \operatorname{Cap}(E_j, \Omega).$$

(iv) If $E_1 \subset E_2 \subset \dots$ is a sequence of arbitrary subsets of Ω,

$$\operatorname{Cap}\Big(\bigcup_{j\geq 1} E_j, \Omega\Big) = \lim_{j\to\infty} \operatorname{Cap}(E_j, \Omega).$$

(v) If $E_1, E_2 \subset \dots$ are arbitrary subsets of Ω, then

$$\operatorname{Cap}\Big(\bigcup_{j\geq 1} E_j, \Omega\Big) \leq \sum_{j\geq 1} \operatorname{Cap}(E_j, \Omega).$$

3 Interior and boundary Caccioppoli inequality

In Sects. 3–5 we will be dealing with subsolutions and supersolutions of the equation

$$Lu = -\mathrm{div}(A\nabla u + bu) - c\nabla u - du = f - \mathrm{div} g, \tag{3.1}$$

where $f \in L^1_{\mathrm{loc}}(\Omega)$ and $g \in L^2_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$.

3.1 Standard Caccioppoli inequality

Theorem 3.1 (Caccioppoli inequality I) *Let $u \in Y^{1,2}_{loc}(\Omega)$ be either a solution or a non-negative subsolution of (3.1) and $f \in L^{2_*}_{loc}(\Omega)$. Assume also that (1.5) is satisfied and either (i) $b + c \in L^{n,q}_{loc}(\Omega)$, for $q \in [n,\infty)$, or (ii) $|b + c|^2 \in \mathcal{K}_{loc}(\Omega)$. For a non-negative function $\eta \in C^\infty_c(\Omega)$, we let Ω' be a bounded open set such that* $\mathrm{supp}\,\eta \subset \Omega' \Subset \Omega$. *Then it holds*

$$\|\eta\nabla u\|^2_{L^2(\Omega')} \lesssim \|u\nabla\eta\|^2_{L^2(\Omega')} + \|f\eta\|^2_{L^{2_*}(\Omega')} + \|g\eta\|^2_{L^2(\Omega')},$$

where the implicit constant depends only on λ, Λ, and also either on $C_{s,q}$ and $\|b+c\|_{L^{n,q}(\Omega')}$, for $q \geq n$ under assumption (i), or C'_s and $\vartheta_{\Omega'}(|b + c|^2, 2\,\mathrm{diam}\Omega')$ under assumption (ii)[3].

Proof We will only treat the case that u is a non-negative subsolution of (3.1) as the proof when u is a solution is almost identical and is omitted. Notice that since $K := \mathrm{supp}\,\eta$ is a compact subset of Ω, we can always find a bounded open set Ω' such that $K \subset \Omega' \Subset \Omega$, and as $u \in Y^{1,2}_{\mathrm{loc}}(\Omega)$, it holds that $u \in Y^{1,2}(\Omega')$. Working in Ω' instead of Ω, we may assume, without loss of generality, that $u \in Y^{1,2}(\Omega)$. Moreover, u is clearly a subsolution in any open subset of Ω. For simplicity, let us preserve the notation Ω instead of Ω'.

We first assume that $b + c \in L^{n,q}(\Omega')$. Apply Lemma 2.34 to the function u, for $p = n$, $q \geq n$, $h = b + c$, and $a = \frac{\lambda}{8C_{s,q}}$, where $C_{s,q}$ is the constant in (2.23), to find $\Omega_i \subset \Omega$ and $u_i \in Y^{1,2}(\Omega)$, $1 \leq i \leq \kappa$, satisfying (1)–(8). Note that (5) tells us that u_i and u have the same sign, and so, the functions $\eta^2 u_i \in Y^{1,2}_0(\Omega)$ are non-negative. Thus, using that u is a subsolution for (3.1) we have

$$\begin{aligned}\int_\Omega f(\eta^2 u_i) + \int_\Omega g\nabla(\eta^2 u_i) &\geq \int_\Omega A\nabla u\nabla(\eta^2 u_i) + bu\nabla(\eta^2 u_i) - c\nabla u(\eta^2 u_i) - du(\eta^2 u_i)\\ &= \int_\Omega A\nabla u\nabla(\eta^2 u_i) + b\nabla(\eta^2 u u_i) - (b+c)\nabla u\eta^2 u_i - d\eta^2 u u_i\\ &\geq \int_\Omega A\nabla u\nabla(\eta^2 u_i) - (b+c)\nabla u\eta^2 u_i,\end{aligned}$$

where in the last inequality we used (5), Lemma (2.29), Remark 2.31, and (1.5). In view of (3) and (6), the latter inequality can be written as

$$\begin{aligned}\int_{\Omega_i} A\nabla u_i\nabla u_i\eta^2 &\leq -2\int_\Omega A\nabla u\nabla\eta u_i\eta + \sum_{j=1}^{i}\int_{\Omega_j}(b+c)\nabla u_j\eta^2 u_i\\ &+ \int_\Omega f(\eta^2 u_i) + \int_\Omega g\nabla(\eta^2 u_i) =: \mathrm{I}_1(i) + \mathrm{I}_2(i) + \mathrm{I}_3(i) + \mathrm{I}_4(i).\end{aligned} \tag{3.2}$$

By (1.2) we get

$$\lambda\|\eta\nabla u_i\|^2_{L^2} \leq \int_{\Omega_i} A\nabla u_i\nabla u_i\eta^2, \tag{3.3}$$

[3] Recall that C'_s and $C_{s,q}$ are the constants in Lemmas 2.20 and 2.30 respectively.

while, by Hölder's inequality,

$$|\mathrm{I}_1(i)| \leq 2\Lambda \|\eta \nabla u\|_{L^2} \|u_i \nabla \eta\|_{L^2}. \tag{3.4}$$

If we apply (2.23) and Young's inequality, along with the fact that $\|b+c\|_{L^{n,q}(\Omega_j)} \leq \frac{\lambda}{8C_{s,q}}$ for any $1 \leq j \leq \kappa$, we get that

$$\begin{aligned}
\mathrm{I}_2(i) &= \int_{\Omega_i} (b+c)\nabla u_i \eta^2 u_i + \sum_{j=1}^{i-1} \int_{\Omega_j} (b+c)\nabla u_j \eta^2 u_i \\
&\leq C_{s,q} \frac{\lambda}{8C_{s,q}} \|\eta \nabla u_i\|_{L^2} \|\nabla(u_i \eta)\|_{L^2} + C_{s,q} \frac{\lambda}{8C_{s,q}} \sum_{j=1}^{i-1} \|\eta \nabla u_j\|_{L^2} \|\nabla(u_i \eta)\|_{L^2} \\
&\leq \frac{3\lambda}{16} \|\eta \nabla u_i\|_{L^2}^2 + \frac{\lambda}{16} \|u_i \nabla \eta\|_{L^2}^2 + \frac{\lambda}{16} (\|u_i \nabla \eta\|_{L^2} + \|\eta \nabla u_i\|_{L^2})^2 \\
&\quad + \frac{\lambda}{16} \left(\sum_{j=1}^{i-1} \|\eta \nabla u_j\|_{L^2} \right)^2 \\
&\leq \frac{5\lambda}{16} \|\eta \nabla u_i\|_{L^2}^2 + \frac{3\lambda}{16} \|u_i \nabla \eta\|_{L^2}^2 + \frac{\lambda}{16} \left(\sum_{j=1}^{i-1} \|\eta \nabla u_j\|_{L^2} \right)^2.
\end{aligned} \tag{3.5}$$

By Hölder's, Sobolev's and Young's inequalities we obtain

$$\mathrm{I}_3(i) + \mathrm{I}_4(i) \leq \frac{C_{s,q}^2}{4\delta} \|f\eta\|_{L^{2_*}}^2 + \frac{1}{2\delta} \|g\eta\|_{L^2}^2 + 2\delta \|u_i \nabla \eta\|_{L^2}^2 + 2\delta \|\eta \nabla u_i\|_{L^2}^2. \tag{3.6}$$

Choosing $\delta = \frac{\lambda}{32}$ in (3.6), we can combine (3.2), (3.3), (3.4), and (3.5) and infer that

$$\begin{aligned}
\frac{3\lambda}{8} \|\eta \nabla u_i\|_{L^2}^2 \leq \left(\frac{4\Lambda^2}{\lambda} + \frac{\lambda}{4} \right) \|u_i \nabla \eta\|_{L^2}^2 + \frac{\lambda}{16} \left(\sum_{j=1}^{i-1} \|\eta \nabla u_j\|_{L^2} \right)^2 + \frac{16}{\lambda} \|g\eta\|_{L^2}^2 \\
+ \frac{16\, C_{s,q}^2}{\lambda} \|f\eta\|_{L^{2_*}}^2,
\end{aligned}$$

which implies that there exist positive constants C_1, C_2 and C_3 depending on λ, Λ and $C_{s,q}$ so that

$$\begin{aligned}
\|\eta \nabla u_i\|_{L^2} \leq C_1 \|u_i \nabla \eta\|_{L^2} + C_2 \left(\|f\eta\|_{L^{2_*}} + \|g\eta\|_{L^2} \right) + \sum_{j=1}^{i-1} \|\eta \nabla u_j\|_{L^2}. \\
+ C_3 \|\eta \nabla u\|_{L^2}^{1/2} \|u_i \nabla \eta\|_{L^2}^{1/2}.
\end{aligned}$$

Note that the constant the sum is multiplied with is indeed 1, which is convenient in the iteration argument below. If we denote $C_0 := \max(C_1, C_2, C_3)$,

$$x_j := \|\eta \nabla u_j\|_{L^2}, \text{ and } y_0 := \|u \nabla \eta\|_{L^2} + \|\eta \nabla u\|_{L^2}^{1/2} \|u \nabla \eta\|_{L^2}^{1/2} + \|f\eta\|_{L^{2_*}} + \|g\eta\|_{L^2},$$

and use that (4), the latter inequality can be written as

$$
\begin{aligned}
x_1 &\le C_0\, y_0,\\
x_i &\le C_0\, y_0 + \sum_{j=1}^{i-1} x_j, \quad \text{for } i = 2, \cdots, \kappa.
\end{aligned} \tag{3.7}
$$

By induction, we get

$$x_i \le 2^{i-1}\, C_0\, y_0. \tag{3.8}$$

Indeed, for $i = 1$, it holds $x_1 \le C_0\, y_0$. Assume now that $x_j \le 2^{j-1}\, C_0\, y_0$ for all $1 \le j \le i-1$. Then, by (3.7) and the induction hypothesis,

$$x_i \le C_0\, y_0 + C_0\, y_0 \sum_{j=1}^{i-1} 2^{j-1} = 2^{i-1}\, C_0\, y_0.$$

Summing (3.8) in $i \in \{1, \dots, \kappa\}$ we obtain

$$\sum_{i=1}^{\kappa} x_i \le 2^{\kappa}\, C_0\, y_0, \tag{3.9}$$

which, in light of (6), (3.9) and Young's inequality (with a small constant), implies that

$$\|\eta \nabla u\|_{L^2} \le \sum_{i=1}^{\kappa} \|\eta \nabla u_i\|_{L^2} \le 4^{\kappa}\, C_0^2\, \left(\|u \nabla \eta\|_{L^2} + \|f \eta\|_{L^{2_*}} + \|g\eta\|_{L^2} \right).$$

This concludes our proof when $b + c \in L^{n,q}(\Omega;\, \mathbb{R}^n)$, since κ depends only λ, Λ, $C_{s,q}$, and also on $\|b + c\|_{L^{n,q}(\Omega;\, \mathbb{R}^n)}$.

Let us now prove the same result in the case $|b + c|^2 \in \mathcal{K}(\Omega')$. We apply Lemma 2.35 to the function u, for $h = b + c$, and $a = \frac{\lambda}{8C'_s}$, where C'_s is the constant in (2.16), to find $\Omega_i \subset \Omega$ and $u_i \in Y^{1,2}(\Omega)$, $1 \le i \le \kappa$, satisfying (1)–(8). The main argument will be exactly the same as in the previous case will not be repeated. Although, there is a difference coming from the embedding theorem we apply, which is Lemma 2.20 as opposed to Lemma 2.30 we used before. Taking this under consideration, it is enough to handle the term $I_2(i)$.

To this end, apply Cauchy-Scwharz's inequality, (2.17), Sobolev's and Young's inequalities, along with the fact that for any $1 \le j \le m$ it holds $\vartheta_{\Omega'}(|b + c|^2 \mathbf{1}_{\Omega_j}, 2\,\text{diam}\Omega') \le \frac{\lambda}{8C'_s}$, and get that

$$
\begin{aligned}
\mathrm{I}_2(i) &= \int_{\Omega_i} (b + c)\nabla u_i \eta^2 u_i + \sum_{j=1}^{i-1} \int_{\Omega_j} (b + c)\nabla u_j \eta^2 u_i\\
&\le C'_s \|\nabla(u_i\eta)\|_{L^2} \left(\vartheta_{\Omega'}^{1/2}(|b + c|^2 \mathbf{1}_{\Omega_i}) \|\eta \nabla u_i\|_{L^2} + \sum_{j=1}^{i-1} \vartheta_{\Omega'}^{1/2}(|b + c|^2 \mathbf{1}_{\Omega_j}) \|\eta \nabla u_j\|_{L^2} \right)\\
&\le \frac{\lambda}{8} \|\eta \nabla u_i\|_{L^2} \|\nabla(u_i\eta)\|_{L^2} + \frac{\lambda}{8} \sum_{j=1}^{i-1} \|\eta \nabla u_j\|_{L^2} \|\nabla(u_i\eta)\|_{L^2}\\
&\le \frac{\lambda}{16} \|\eta \nabla u_i\|_{L^2}^2 + \frac{\lambda}{16} \|u_i \nabla \eta\|_{L^2}^2 + \frac{\lambda}{16} (\|u_i \nabla \eta\|_{L^2} + \|\eta \nabla u_i\|_{L^2})^2
\end{aligned}
$$

$$+\frac{\lambda}{16}\left(\sum_{j=1}^{i-1}\|\eta\nabla u_j\|_{L^2}\right)^2$$

$$\leq \frac{3\lambda}{16}\|\eta\nabla u_i\|_{L^2}^2+\frac{3\lambda}{16}\|u_i\nabla\eta\|_{L^2}^2+\frac{\lambda}{16}\left(\sum_{j=1}^{i-1}\|\eta\nabla u_j\|_{L^2}\right)^2.$$

This concludes the proof the Theorem. □

Theorem 3.2 (Caccioppoli inequality II) *Let $u \in Y^{1,2}_{loc}(\Omega)$ be either a solution or a non-negative subsolution of (3.1) and $f \in L^{2_*}_{loc}(\Omega)$. Assume also that (1.6) is satisfied and either (i) $b+c \in L^{n,q}_{loc}(\Omega)$, for $q \in [n,\infty)$, or (ii) $|b+c|^2 \in \mathcal{K}_{loc}(\Omega)$. For a non-negative function $\eta \in C^\infty_c(\Omega)$, we let Ω' be a bounded open set such that* $\operatorname{supp}\eta \subset \Omega' \Subset \Omega$. *Then it holds*

$$\|\eta\nabla u\|^2_{L^2(\Omega')} \lesssim \|u\nabla\eta\|^2_{L^2(\Omega')}+\|f\eta\|^2_{L^{2_*}(\Omega')}+\|g\eta\|^2_{L^2(\Omega')},$$

where the implicit constant depends only on λ, Λ, and also either on $C_{s,q}$ and $\|b+c\|_{L^{n,q}(\Omega')}$, for $q \geq n$ under assumption (i), or C'_s and $\vartheta_{\Omega'}(|b+c|^2, 2\operatorname{diam}\Omega')$ under assumption (ii).

Proof We only deal with the case that u is a non-negative subsolution (3.1). As seen in Theorem 3.1, we may assume that $u \in Y^{1,2}(\Omega)$ and apply Lemma 2.34 to the function u, for $p=n$, $q \geq n$, $h=b+c$, and $a=\frac{\lambda}{8C_{s,q}}$, where $C_{s,q}$ is the constant in (2.23). Using that $\eta^2 u_i \in Y^{1,2}_0(\Omega)$ and non-negative, along with the fact that u is a subsolution, we have

$$\begin{aligned}\int_\Omega f(\eta^2u_i)+\int_\Omega g\nabla(\eta^2u_i) &\geq \int_\Omega A\nabla u\nabla(\eta^2u_i)+bu\nabla(\eta^2u_i)-c\nabla u(\eta^2u_i)-du(\eta^2u_i)\\ &\geq \int_\Omega A\nabla u\nabla(\eta^2u_i)-(b+c)u\nabla(\eta^2u_i),\end{aligned}$$

where in the last inequality we used (5), Lemma (2.29), Remark 2.31, and (1.6). In view of (3) and (6), the latter inequality can be written as

$$\begin{aligned}\int_\Omega A\nabla u_i\nabla u_i\eta^2 \leq -2\int_\Omega A\nabla u\nabla\eta u_i\eta+\int_{\Omega_i}(b+c)\nabla u_i\eta^2u+2\int_\Omega(b+c)\nabla\eta u_iu\eta\\ +\int_\Omega f(\eta^2u_i)+\int_\Omega g\nabla(\eta^2u_i) =: -2\mathrm{I}_1(i)+\mathrm{I}_2(i)+2\mathrm{I}_3(i)+\mathrm{I}_4(i)+\mathrm{I}_5(i).\end{aligned} \tag{3.10}$$

By Hölder's inequality,

$$\mathrm{I}_1(i) \leq \Lambda\|\eta\nabla u\|_{L^2}\|u_i\nabla\eta\|_{L^2}. \tag{3.11}$$

Using (8) and the fact that $\|b+c\|_{L^{n,q}(\Omega_j)} \leq \frac{\lambda}{8C_{s,q}}$ for all $1 \leq j \leq \kappa$, along with (2.23) and Young's inequality, we have

$$\begin{aligned}\mathrm{I}_2(i) &= \int_{\Omega_i}(b+c)\nabla u_i\eta^2u_i+\sum_{j=i+1}^{\kappa}\int_{\Omega_j}(b+c)\nabla u_j\eta^2u_i\\ &\leq C_{s,q}\frac{\lambda}{8C_{s,q}}\|\eta\nabla u_i\|_{L^2}\|\nabla(u_i\eta)\|_{L^2}+C_{s,q}\frac{\lambda}{8C_{s,q}}\sum_{j=i+1}^{\kappa}\|\eta\nabla u_j\|_{L^2}\|\nabla(u_i\eta)\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\lambda}{8}\|\eta\nabla u_i\|_{L^2}^2 + \frac{\lambda}{8}\|\eta\nabla u_i\|_{L^2}\|u_i\nabla\eta\|_{L^2} \\
&\quad + \frac{\lambda}{16}(\|u_i\nabla\eta\|_{L^2}^2 + \|\eta\nabla u_i\|_{L^2}^2) + \frac{\lambda}{16}\left(\sum_{j=i+1}^{\kappa}\|\eta\nabla u_j\|_{L^2}\right)^2 \\
&\le \frac{\lambda}{4}\|\eta\nabla u_i\|_{L^2}^2 + \frac{\lambda}{8}\|u_i\nabla\eta\|_{L^2}^2 + \frac{\lambda}{16}\left(\sum_{j=i+1}^{\kappa}\|\eta\nabla u_j\|_{L^2}\right)^2. \qquad (3.12)
\end{aligned}
$$

If $\delta > 0$ is small enough to be chosen, then by similar (but easier) considerations we get

$$
\begin{aligned}
\mathrm{I}_3(i) &\le C_{s,q}\|b+c\|_{L^{n,q}}\|u\nabla\eta\|_{L^2}\|\nabla(u_i\eta)\|_{L^2} \qquad (3.13)\\
&\le \frac{C_{s,q}^2}{4\delta}\|b+c\|_{L^{n,q}}^2\|u\nabla\eta\|_{L^2}^2 + \delta\,\|\eta\nabla u_i\|_{L^2}^2 + \delta\,\|u_i\nabla\eta\|_{L^2}^2. \qquad (3.14)
\end{aligned}
$$

If we apply Hölder's, Sobolev's and Young's inequalities we get

$$
\begin{aligned}
\mathrm{I}_4(i) + \mathrm{I}_5(i) \le \frac{C_{s,q}^2}{4\rho}\|f\eta\|_{L^{2_*}}^2 + \left(1+\frac{1}{4\rho}\right)\|g\eta\|_{L^2}^2 \\
+ (1+2\rho)\|u_i\nabla\eta\|_{L^2}^2 + 2\rho\|\eta\nabla u_i\|_{L^2}^2. \qquad (3.15)
\end{aligned}
$$

Choose now $\delta = \frac{\lambda}{16}$ and $\rho = \frac{\lambda}{8}$. Combining (3.10), (1.2), (3.11), (3.12), (3.13), and (3.15), and using (4), we can find positive constants $C_1 = C_1(\lambda, C_{s,q}, \|b+c\|_{L^{n,q}})$, $C_2 = C_2(\lambda, C_{s,q})$ and $C_3 = C_3(\lambda)$ so that

$$
\begin{aligned}
\|\eta\nabla u_i\|_{L^2}^2 \le 2\Lambda\|\eta\nabla u\|_{L^2}\|u\nabla\eta\|_{L^2} + C_1\|u\nabla\eta\|_{L^2}^2 + C_2\|f\eta\|_{L^{2_*}}^2 \\
+ C_3\|g\eta\|_{L^2}^2 + \frac{\lambda}{16}\left(\sum_{j=i+1}^{\kappa}\|\eta\nabla u_j\|_{L^2}\right)^2.
\end{aligned}
$$

For $j \in \{1, \dots, \kappa\}$, we set

$$
x_j := \|\eta\nabla u_j\|_{L^2}
$$

and

$$
y_0 := \sqrt{2\Lambda}\|\eta\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u\nabla\eta\|_{L^2(\Omega)}^{\frac{1}{2}} + \sqrt{C_1}\|u\nabla\eta\|_{L^2} + \sqrt{C_2}\|f\eta\|_{L^{2_*}} + \sqrt{C_3}\|g\eta\|_{L^2},
$$

and so, the latter inequality can be written as

$$
x_\kappa \le y_0 \text{ and } x_i \le y_0 + \sum_{j=i+1}^{\kappa} x_j, \text{ for } i = 1, 2, \cdots, \kappa-1. \qquad (3.16)
$$

By induction, (3.16) yields $x_i \le 2^{\kappa-i}y_0$ for $i = 1, 2, \cdots, \kappa-1$, and thus, summing over all such i, we infer

$$
\begin{aligned}
\|\eta\nabla u\|_{L^2} \le \sum_{i=1}^{\kappa}\|\eta\nabla u_i\|_{L^2} \le 2^\kappa\sqrt{\Lambda}\|\eta\nabla u\|_{L^2}^{\frac{1}{2}}\|u\nabla\eta\|_{L^2}^{\frac{1}{2}} \\
+ 2^\kappa\left(\sqrt{C_1}\|u\nabla\eta\|_{L^2} + \sqrt{C_2}\|f\eta\|_{L^{2_*}} + \sqrt{C_3}\|g\eta\|_{L^2}\right),
\end{aligned}
$$

where in the first inequality we used (6). The theorem readily follows from another application of Young's inequality. This finishes the proof in the case $b + c \in L^{n,q}(\Omega')$, while the modifications to obtain the result the case $|b+c|^2 \in \mathcal{K}(\Omega')$ are identical to the ones presented in the proof of Theorem 3.1 and are omitted. □

The proofs of Theorems 3.1 and 3.2 can easily be adapted to prove the following Caccioppoli inequality at the boundary.

Theorem 3.3 (Caccioppoli inequality at the boundary) *If B_r is a ball such that $B_r \cap \Omega \neq \emptyset$, set $\Omega_r = B_r \cap \Omega$ and assume that $u \in Y^{1,2}(\Omega_r)$ vanishing on $\partial\Omega \cap B_r$ in the sense of definition 2.3. Assume that $f \in L^{2_*}(\Omega_r)$, $g \in L^2(\Omega_r)$ and either (1.5) or (1.6) holds. If either $b + c \in L^{n,q}(\Omega_r)$, $q \in [n, \infty)$, or $|b + c|^2 \in \mathcal{K}(\Omega_r)$, and u is either a solution or a non-negative subsolution of (3.1) in Ω_r, then for any non-negative function $\eta \in C_c^\infty(B_r)$ it holds*

$$\|\eta \nabla u\|^2_{L^2(\Omega_r)} \lesssim \|u \nabla \eta\|^2_{L^2(\Omega_r)} + \|f\eta\|^2_{L^{2_*}(\Omega_r)} + \|g\eta\|^2_{L^2(\Omega_r)}, \tag{3.17}$$

where the implicit constant depends only on λ, Λ, and also either on $C_{s,q}$ and $\|b+c\|_{L^{n,q}(\Omega_r)}$, for $q \geq n$, or C'_s and $\vartheta_{\Omega_r}(|b + c|^2, r)$.

Proof We follow the same strategy as before and apply either Lemma 2.34 to the function u in $\Omega_r(x)$, for $p = n$, $q \geq n$, $h = b + c$, and $a = \frac{\lambda}{8C_{s,q}}$, where $C_{s,q}$ is the constant in (2.23), or apply Lemma 2.35 to the function u in $\Omega_r(x)$, for $h = b + c$, and $a = \frac{\lambda}{8C'_s}$, where C'_s is the constant in (2.16). Thus, we find $\Omega_i \subset \Omega_r(x)$ and $u_i \in Y^{1,2}(\Omega_r)$ that vanishes on $B_r \cap \partial\Omega$, for $1 \leq i \leq \kappa$, satisfying (1)–(8). Using that the non-negative function $\eta^2 u_i$ is in $Y_0^{1,2}(\Omega_r(x))$, along with the fact that u is either a solution or a non-negative subsolution of (3.1) in Ω_r, we may proceed as in the proofs of Theorems 3.1 and 3.2 to obtain (3.17). We skip the details. □

Remark 3.4 We would like to note that if $b + c \in \mathcal{K}'(\Omega)$, we can dominate $\vartheta_{\Omega_r}(|b + c|^2, 2r)$ by $\vartheta_\Omega(|b + c|^2)$.

3.2 Refined Caccioppoli inequality

Let $m = \inf_{\partial\Omega \cap B_r} u$ and $M = \sup_{\partial\Omega \cap B_r} u$ in the sense of Definition 2.2. Define

$$u_m^-(x) := \begin{cases} \inf(u(x), m) & , x \in \Omega \\ m & , x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

and

$$u_M^+(x) := \begin{cases} \sup(u(x), M) & , x \in \Omega \\ M & , x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

Theorem 3.5 *Let B_r be a ball such that $\Omega_r = B_r \cap \Omega \neq \emptyset$ and assume that either $b + c \in L^{n,q}(\Omega_r)$, $q \in [n, \infty)$, or $|b+c|^2 \in \mathcal{K}(\Omega_r)$. We also assume that one of the following holds:*

(1) $\operatorname{div} b + d \geq 0$, *$\beta \in (-\infty, 0)$ and $u \in Y^{1,2}(\Omega_r)$ is a non-negative L-supersolution of (3.1) in Ω_r;*

(2) $\operatorname{div} b + d \leq 0$, $\beta \in (0, \infty)$ *and* $u \in Y^{1,2}(\Omega_r)$ *is a non-negative L-subsolution of (3.1) in* Ω_r.

If we set

$$\widehat{\Omega}_r = \begin{cases} \Omega_r^m := \{x \in \Omega_r : u < m\} & , \textit{in Case } (1), \\ \Omega_r^M := \{x \in \Omega_r : u > M\} & , \textit{in Case } (2), \end{cases}$$

and for $k > 0$ *we define*

$$\bar{u} = \begin{cases} u_m^- + k & , \textit{in Case } (1), \\ u_M^+ + k & , \textit{in Case } (2), \end{cases} \quad \textit{and} \quad \widetilde{\Omega}_r = \begin{cases} \{x \in \Omega_r : \nabla u_m^-(x) \neq 0\} & , \textit{in Case } (1), \\ \{x \in \Omega_r : \nabla u_M^+(x) \neq 0\} & , \textit{in Case } (2), \end{cases}$$

then, there exist constants C_0, C_1, C_2 *depending on* β, *such that for any non-negative function* $\eta \in C_c^\infty(B_r)$ *we have*

$$\|\eta\, \bar{u}^{\frac{\beta-1}{2}} \nabla u\|^2_{L^2(\widetilde{\Omega}_r)} \lesssim C_0 \|\bar{u}^{\frac{\beta+1}{2}} \nabla \eta\|^2_{L^2(\widehat{\Omega}_r)} + \int_{\widehat{\Omega}_r} \left(C_1 |\bar{f}| + C_2 |\bar{g}|^2\right) \bar{u}^{\beta+1} \eta^2, \quad (3.18)$$

where $\bar{f} = |f|/\bar{u}$, $\bar{g} = |g|/\bar{u}$, *and the implicit constant depends on* λ, Λ, *and also either on* $C_{s,q}$ *and* $\|b+c\|_{L^{n,q}(\Omega_r)}$, *for* $q \geq n$, *or* C_s' *and* $\vartheta_{\Omega_r}(|b+c|^2, r)$. *When* $|\beta| > 1$, $C_0 = |\beta+1|^{-2}$, $C_1 = |\beta + 1|^{-1}$, *and* $C_2 = 1 + |\beta + 1|^{-2}$, *while when* $|\beta| < 1$, $C_0 = 4^\kappa |\beta|^{-2}$ *and* $C_1 = C_2 = 2^\kappa |\beta|^{-1}$, *where either* $\kappa \leq 1 + \frac{1}{C|\beta|^n} \|b+c\|^n_{L^{n,q}(\Omega_r)}$ *or* $\kappa \leq 1 + 2\, a^{-2}\, \rho_0^{2-n} \|h\|_{L^1(\Omega_r)}$. *In the case* $\beta = -1$, $C_0 = C_1 = C_2 = 1$.

Proof We first assume that u is a non-negative supersolution of (3.1) and $\beta < -1$.

For $k > 0$ we define the auxiliary function

$$w = \bar{u}^{\frac{\beta+1}{2}} - (m+k)^{\frac{\beta+1}{2}}.$$

It is clear that $w \in Y^{1,2}(\Omega_r)$ vanishing on $\partial\Omega \cap B_r$ and so, we can apply Lemma 2.34 to w and Ω_r with $p = n, q \geq n, h = b+c$, and $a = \frac{\lambda}{8C_{s,q}}$, where $C_{s,q}$ is the constant in Sobolev's inequality, to find $w_i \in Y^{1,2}(\Omega_r)$ that vanishes on $\partial\Omega \cap B_r$ and $\Omega_i \subset \widetilde{\Omega}_r$, $1 \leq i \leq \kappa$, so that (1)–(8) hold.

Since w_i vanishes on $\partial\Omega \cap B_r$. there is a sequence $\phi_k \in C_c^\infty(\bar{\Omega} \setminus (\partial\Omega \cap B_r))$ such that $\phi_k \to w_i$ in $Y^{1,2}(\Omega)$. Thus, the sequence $\eta^2 \phi_k \in C_c^\infty(\Omega_r)$ converges to $\eta^2 w_i$ in $Y^{1,2}(\Omega_r)$, which implies that $\eta^2 w_i \in Y_0^{1,2}(\Omega_r)$. Note also that, by (5), $\eta^2 w_i$ is non-negative. Thus, for $i = 1, 2, \ldots \kappa$,

$$\begin{aligned} \lambda \int_{\Omega_i} |\nabla w|^2\, \eta^2 &= \lambda \int_{\Omega_r} |\nabla w_i|^2\, \eta^2 \\ &\leq \int_{\Omega_r} A \nabla w_i \nabla w_i\, \eta^2 = \frac{\beta+1}{2} \int_{\Omega_r} A \nabla u \nabla w_i\, \bar{u}^{\frac{\beta-1}{2}}\, \eta^2 \\ &= \frac{\beta+1}{2} \left(\int_{\Omega_r} A \nabla u \nabla (w_i\, \bar{u}^{\frac{\beta-1}{2}} \eta^2) - 2 \int_{\Omega_r} A \nabla u \nabla \eta\, \eta w_i \bar{u}^{\frac{\beta-1}{2}} \right) \\ &\quad - \frac{\beta+1}{2} \left(\int_{\Omega_r} A \nabla u \nabla \bar{u}^{\frac{\beta-1}{2}} w_i\, \eta^2 \right) =: \frac{\beta+1}{2} \left(J_1 - J_2 - J_3\right). \end{aligned} \quad (3.19)$$

Let us point out that

$$0 \leq w_i \leq w \leq \bar{u}^{\frac{\beta+1}{2}} \quad (3.20)$$

and

$$\nabla \bar{u}\, \mathbf{1}_{\Omega_r} = \nabla u\, \mathbf{1}_{\Omega_r^m} \quad \text{and} \quad \{x \in \Omega_r : w_i \neq 0\} \subset \{x \in \Omega_r : w \neq 0\} = \Omega_r^m. \tag{3.21}$$

Recalling that $\beta < -1$ and using (3.21), (1.2), and that $\bar{u} > 0$, we get that

$$J_3 = \frac{\beta - 1}{2} \int_{\Omega_r^m} A \nabla u \nabla u\, \bar{u}^{\frac{\beta-3}{2}} \eta^2 \leq \lambda \frac{\beta - 1}{2} \int_{\Omega_r^m} |\nabla u|^2\, \bar{u}^{\frac{\beta-3}{2}} \eta^2 \leq 0, \tag{3.22}$$

and thus, $-\frac{\beta+1}{2} J_3 \leq 0$. Moreover, by (1.3), Hölder's inequality, (3.20), and (3.21),

$$\begin{aligned} |J_2| &\leq 2\Lambda \|\eta \nabla \bar{u}^{\frac{\beta+1}{2}}\|_{L^2(\Omega_r^m)} \|w_i \nabla \eta\|_{L^2(\Omega_r^m)} \\ &\leq 2\Lambda \|\eta \nabla \bar{u}^{\frac{\beta+1}{2}}\|_{L^2(\Omega_r^m)} \|\bar{u}^{\frac{\beta+1}{2}} \nabla \eta\|_{L^2(\Omega_r^m)}. \end{aligned} \tag{3.23}$$

Since u is a supersolution of (3.1), $\beta + 1 < 0$, and $\operatorname{div} b - d \geq 0$, we obtain

$$\begin{aligned} J_1 &\geq \int_{\Omega_r} (b + c)\, \nabla u\, w_i\, \bar{u}^{\frac{\beta-1}{2}} \eta^2 + \int_{\Omega_r} f w_i\, \bar{u}^{\frac{\beta-1}{2}} \eta^2 + \int_{\Omega_r} g \nabla \left(w_i\, \bar{u}^{\frac{\beta-1}{2}} \eta^2 \right) \\ &=: I_1 + I_2 + I_3, \end{aligned} \tag{3.24}$$

and so $\frac{\beta+1}{2} J_1 \leq \frac{\beta+1}{2}(I_1 + I_2 + I_3)$. As

$$\nabla \left(w_i\, \bar{u}^{\frac{\beta-1}{2}}\, \eta^2 \right) = \nabla w_i\, \bar{u}^{\frac{\beta-1}{2}}\, \eta^2 + 2 \nabla \eta\, w_i\, \eta\, \bar{u}^{\frac{\beta-1}{2}} + \nabla \bar{u}^{\frac{\beta-1}{2}}\, w_i \eta^2,$$

we may write I_3 as the sum of three integrals I_{31}, I_{32}, I_{33} that correspond to the terms on the right hand-side of the latter equality. So, by Young's inequality (for ε small enough to be chosen) along with (3.20) and (3.21), we get

$$\frac{|\beta + 1|}{2} |I_{31}| \leq \varepsilon \|\nabla w_i\, \eta\|^2_{L^2(\Omega_r)} + \frac{|\beta + 1|^2}{16 \varepsilon} \int_{\Omega_r^m} |g|^2 \bar{u}^{\beta-1} \eta^2, \tag{3.25}$$

$$\frac{|\beta + 1|}{2} |I_{32}| \leq \|\bar{u}^{\frac{\beta+1}{2}} \nabla \eta\|^2_{L^2(\Omega_r^m)} + \frac{|\beta + 1|^2}{4} \int_{\Omega_r^m} |g|^2 \bar{u}^{\beta-1} \eta^2. \tag{3.26}$$

$$\frac{|\beta + 1|}{2} |I_{33}| \leq \varepsilon \frac{|\beta + 1|^2}{4} \|\bar{u}^{\frac{\beta-1}{2}}\, \nabla \bar{u}\, \eta\|^2_{L^2(\Omega_r^m)} + \frac{|\beta - 1|^2}{16\, \varepsilon} \int_{\Omega_r^m} |g|^2 \bar{u}^{\beta-1} \eta^2 \tag{3.27}$$

$$|I_2| \leq \int_{\Omega_r^m} |f| \bar{u}^{\beta} \eta^2. \tag{3.28}$$

Moreover, by (7),

$$\begin{aligned} \frac{\beta + 1}{2} I_1 &= \int_{\Omega_r} (b + c) \nabla w\, w_i\, \eta^2 \\ &= \int_{\Omega_i} (b + c) \nabla w_i\, w_i\, \eta^2 + \sum_{j=1}^{i-1} \int_{\Omega_j} (b + c) \nabla w_j\, w_i\, \eta^2 =: I_1^i + \sum_{j=1}^{i-1} I_1^j. \end{aligned} \tag{3.29}$$

If we apply (2.23) and Young's inequality,

$$\begin{aligned} |I_1^i| &\leq C_{s,q} \|b + c\|_{L^{n,q}(\Omega_i)} \|\eta \nabla w_i\|_{L^2(\Omega_r)} \|\nabla(\eta w_i)\|_{L^2(\Omega_r)} \\ &\leq \frac{3 a C_{s,q}}{2} \|\eta \nabla w_i\|^2_{L^2(\Omega_r)} + \frac{a C_{s,q}}{2} \|w_i \nabla \eta\|^2_{L^2(\Omega_r^m)}. \end{aligned} \tag{3.30}$$

Similarly,

$$\sum_{j=1}^{i-1} |I_1^j| \le C_{s,q} \|b+c\|_{L^{n,q}(\Omega_i)} \|\nabla(\eta w_i)\|_{L^2(\Omega_r^m)} \sum_{j=1}^{i-1} \|\eta \nabla w_j\|_{L^2(\Omega_r)}$$
$$\le a C_{s,q} \|\eta \nabla w_i\|^2_{L^2(\Omega_r)} + a C_{s,q} \|w_i \nabla \eta\|^2_{L^2(\Omega_r^m)} + \frac{a C_{s,q}}{2} \left(\sum_{j=1}^{i-1} \|\eta \nabla w_j\|_{L^2(\Omega_r)} \right)^2 . \tag{3.31}$$

Let us set

$$x_0 = \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla \bar{u}\|_{L^2(\Omega_r^m)}, \;\; x_j = \|\eta \nabla w_j\|_{L^2(\Omega_r)}, \;\; y_0 = \|\bar{u}^{\frac{\beta+1}{2}} \nabla \eta\|_{L^2(\Omega_r^m)},$$

and also, if $\gamma_0 := |\beta+1|/2$, set

$$z_0 = \||f|^{\frac{1}{2}} \bar{u}^{\frac{\beta}{2}} \eta\|_{L^2(\Omega_r^m)}, \qquad z_1 = \||g| \bar{u}^{\frac{\beta-1}{2}} \eta\|_{L^2(\Omega_r^m)}, \text{ and}$$
$$C(\varepsilon, \gamma_0) := \left[\left((4\varepsilon)^{-1} + 1 \right) \gamma_0^2 + (4\varepsilon)^{-1} (1+\gamma_0)^2 \right]^{\frac{1}{2}} .$$

Then, using this notation, $|\beta-1|/2 \le 1+\gamma_0$, and choosing α small enough, depending on λ, Λ, $\|b+c\|_{L^n(\Omega)}$, and $C_{s,q}$, we can collect the inequalities (3.19)–(3.31) and find a constant C_0 (depending on λ, Λ and $C_{s,q}$) so that

$$x_i \le C_0(\sqrt{\gamma_0} z_0 + C(\varepsilon, \gamma_0) z_1 + \sqrt{\varepsilon}\, \gamma_0\, x_0 + y_0) + \sum_{j=1}^{i-1} x_j .$$

By the induction argument that appeared in the proof of Theorem 3.1 and (3.21), we can show that

$$\gamma_0\, x_0 = \|\eta \, \nabla w\|^2_{L^2(\Omega_r)} \le C_1(\sqrt{\gamma_0} z_0 + C(\varepsilon, \gamma_0) z_1 + \sqrt{\varepsilon}\, \gamma_0 \;\; x_0 + y_0),$$

where C_1 depends on λ, Λ, $\|b+c\|_{L^{n,q}(\Omega)}$ and $C_{s,q}$. We may choose ε small enough compared to C_1^{-2} and use Young's inequality with ε to deduce

$$\gamma_0\, x_0 \le C_2\, (y_0 + \sqrt{\gamma_0} z_0 + (1+\gamma_0^2)^{1/2} z_1)$$

in order to show (3.18). The details are omitted.

We turn our attention to the case that u is a non-negative supersolution of (3.1) and $\beta \in [-1, 0)$. For $k > 0$ we define the auxiliary function

$$w = \bar{u}^{\beta} - (m+k)^{\beta}.$$

Since $w \in Y^{1,2}(\Omega)$ and vanishes on $\partial\Omega \cap B_r$, we apply Lemma 2.34 as in the previous case to w and Ω_r, for $p = n$, $h = b + c$, and a small enough depending on $\lambda, \beta, C_{s,q}$ (to be picked later), to find $w_i \in Y^{1,2}(\Omega)$ that also vanishes on $\partial\Omega \cap B_r$ and $\Omega_i \subset \widetilde{\Omega}_r$, $1 \le i \le m$, satisfying (1)–(8). By (5) we see that $\eta^2 w_i \in Y_0^{1,2}(\Omega)$ is non-negative and we may use it as a test function. Therefore,

$$\begin{aligned}\int_{\Omega_r} f(\eta^2 w_i) + \int_{\Omega_r} g\nabla(\eta^2 w_i) \\ &\leq \int_{\Omega_r} A\nabla u \nabla(\eta^2 w_i) + bu\nabla(\eta^2 w_i) - c\nabla u(\eta^2 w_i) - du(\eta^2 w_i) \\ &= \int_{\Omega_r} A\nabla u \nabla(\eta^2 w_i) + b\nabla(\eta^2 u w_i) - (b+c)\nabla u \eta^2 w_i - d\eta^2 u w_i \\ &\leq \int_{\Omega_r} A\nabla u \nabla(\eta^2 w_i) - (b+c)\nabla u \eta^2 w_i, \end{aligned} \tag{3.32}$$

where in the last inequality we used (1.5).

At this point let us recall (3.21) and also record that

$$0 \leq w_i \leq w \leq \bar{u}^\beta \tag{3.33}$$

and

$$\nabla w_i = \beta \bar{u}^{\beta-1} \nabla u \, \mathbf{1}_{\Omega_i}. \tag{3.34}$$

Therefore, by (3.34) and $\beta < 0$, (3.32) can be written

$$\begin{aligned}\lambda|\beta| \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla u\|^2_{L^2(\Omega_i)} &\leq |\beta| \int_{\Omega_i} A\nabla u \cdot \nabla u \eta^2 \bar{u}^{\beta-1} \leq 2\int_{\Omega_r} A\nabla u \cdot \nabla \eta \, w_i \eta \\ &- \int_{\Omega_r} (b+c)\nabla u \eta^2 w_i - \int_{\Omega_r} f(\eta^2 w_i) - \int_{\Omega_r} g\nabla(\eta^2 w_i) = \sum_{i=1}^4 I_i. \end{aligned} \tag{3.35}$$

We apply Hölder's inequality along with (3.21) and (3.33) to get

$$|I_1| \leq 2\Lambda \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla u\|_{L^2(\Omega_r^m)} \|\bar{u}^{\frac{\beta+1}{2}} \nabla \eta\|_{L^2(\Omega_r^m)}. \tag{3.36}$$

By Young's inequality, (3.33), and (3.34), it is easy to see that

$$\begin{aligned}|I_3| + |I_4| &\leq \int_{\Omega_r^m} |f| \bar{u}^\beta \eta^2 + \left(1 + \frac{|\beta|}{4\varepsilon}\right) \int_{\Omega_r^m} |g|^2 \bar{u}^{\beta-1} \eta^2 \\ &+ |\beta| \, \varepsilon \int_{\Omega_i} \bar{u}^{\beta-1} |\nabla u|^2 \eta^2 + \int_{\Omega_r^m} \bar{u}^{\beta+1} |\nabla \eta|^2. \end{aligned} \tag{3.37}$$

It only remains to handle I_2. At this point we cannot use (6) or (7) as in previous arguments. The reason why is that we do not have u and u_i but rather two different functions u and w_i. Although, we can recall that $\{x \in \Omega_r : w_i \neq 0\} = \cup_{j=1}^i \Omega_j$ and thus, using (2.23), (3.21), (3.33), $\|b+c\|_{L^{n,q}(\Omega_j)} \leq a$ for any $j \in \{1, 2, \cdots m\}$, and $w_i \bar{u}^{\frac{1-\beta}{2}} \eta \in Y_0^{1,2}(\Omega_r)$, we get

$$\begin{aligned}|I_2| &\leq C_{s,q} \|b+c\|_{L^{n,q}(\Omega_i)} \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla u\|_{L^2(\cup_{j=1}^i \Omega_j)} \|\nabla(w_i \bar{u}^{\frac{1-\beta}{2}} \eta)\|_{L^2(\Omega_r)} \\ &\leq a C_{s,q} \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla u\|_{L^2(\cup_{j=1}^i \Omega_j)} \|\nabla(w_i \bar{u}^{\frac{1-\beta}{2}} \eta)\|_{L^2(\Omega_r)}. \end{aligned} \tag{3.38}$$

Note that

$$\nabla(w_i \bar{u}^{\frac{1-\beta}{2}} \eta) \, \mathbf{1}_{\Omega_r} = \beta \eta \bar{u}^{\frac{\beta-1}{2}} \nabla u \, \mathbf{1}_{\Omega_i} + w_i \bar{u}^{\frac{1-\beta}{2}} \nabla \eta + \frac{1-\beta}{2} w_i \bar{u}^{-\frac{\beta+1}{2}} \eta \nabla u.$$

Also, for $\beta \in [-1, 0)$, it holds $\frac{\beta-1}{2\beta} > 0$ and $\frac{\beta+1}{2} > 0$. Thus, by (3.33),

$$w_i \bar{u}^{-\frac{\beta+1}{2}} \le w_i^{\frac{\beta-1}{2\beta}} \le \bar{u}^{\frac{\beta-1}{2}} \mathbf{1}_{\cup_{j=1}^i \Omega_j} \quad \text{and} \quad w_i \bar{u}^{\frac{1-\beta}{2}} \le \bar{u}^{\beta} \bar{u}^{\frac{1-\beta}{2}} \mathbf{1}_{\cup_{j=1}^i \Omega_j} \le \bar{u}^{\frac{\beta+1}{2}} \mathbf{1}_{\Omega_r^m},$$

which, in turn, implies that

$$\begin{aligned} \|\nabla(w_i \bar{u}^{\frac{1-\beta}{2}} \eta)\|_{L^2(\Omega_r)} \le |\beta| \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla \bar{u}\|_{L^2(\Omega_i)} + \|\bar{u}^{\frac{\beta+1}{2}} \nabla \eta\|_{L^2(\Omega_r^m)} \\ + \frac{1-\beta}{2} \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla \bar{u}\|_{L^2(\cup_{j=1}^i \Omega_j)}. \end{aligned} \tag{3.39}$$

Set now

$$x_0 = \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla \bar{u}\|_{L^2(\Omega_r^m)}, \ \ x_j = \|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla \bar{u}\|_{L^2(\Omega_j)}, \ \ y_0 = \|\bar{u}^{\frac{\beta+1}{2}} \nabla \eta\|_{L^2(\Omega_r^m)},$$
$$z_0 = \||f|^{1/2} \eta \bar{u}^{\frac{\beta}{2}}\|_{L^2(\Omega_r^m)} \quad \text{and} \quad z_1 = \||g| \eta \bar{u}^{\frac{\beta-1}{2}}\|_{L^2(\Omega_r^m)}.$$

With this notation, we can write

$$\|\eta \bar{u}^{\frac{\beta-1}{2}} \nabla \bar{u}\|^2_{L^2(\cup_{j=1}^i \Omega_j)} = x_i^2 + \sum_{j=1}^{i-1} x_j^2$$

$$\|\nabla(w_i \bar{u}^{\frac{1-\beta}{2}} \eta)\|_{L^2(\Omega_r)} \le |\beta| x_i + y_0 + \frac{1+|\beta|}{2} \left(x_i^2 + \sum_{j=1}^{i-1} x_j^2 \right)^{1/2},$$

which, in combination with inequalities (1.2) and (3.35)–(3.39), and $|\beta| \le 1$, implies

$$\begin{aligned} |\beta| \lambda x_i^2 \le 2\Lambda x_0 y_0 + a C_{s,q} \Big(x_i^2 + \sum_{j=1}^{i-1} x_j^2 \Big)^{1/2} \Big(|\beta| x_i + y_0 + \Big(x_i^2 + \sum_{j=1}^{i-1} x_j^2 \Big)^{1/2} \Big) \\ + \Big(|\beta| \varepsilon\, x_0^2 + y_0^2 + z_0^2 + \Big(1 + \frac{|\beta|}{4\varepsilon} \Big) z_1^2 \Big). \end{aligned}$$

Therefore, if we choose α small enough (depending linearly on $|\beta|$), by Young's inequality, we can find a positive constant C_0 depending only on λ, Λ, and $C_{s,q}$ so that

$$x_i \le \frac{C_0}{\sqrt{|\beta|}} \Big((x_0 y_0)^{1/2} + \sqrt{|\beta| \varepsilon}\, x_0 + (1 + \sqrt{|\beta|}) y_0 + z_0 + (1 + \sqrt{|\beta|}) z_1 \Big) + \sum_{j=1}^{i-1} x_j,$$

The proof of (3.18) is concluded by the same iteration argument as in the proof of Theorem 3.1 along with the facts that $\cup_{i=1}^{\kappa} \Omega_i = \widetilde{\Omega}_r$ and $|\beta| < 1$ obtaining

$$x_0 \le \frac{C_0 2^{\kappa}}{\sqrt{|\beta|}} \Big((x_0 y_0)^{1/2} + \sqrt{|\beta| \varepsilon}\, x_0 + y_0 + z_0 + 2 z_1 \Big),$$

where $\kappa \le 1 + \frac{1}{C|\beta|^n} \|b+c\|^n_{L^{n,q}(\Omega_r)}$. By Young's inequality and if we choose ε small enough (depending on λ, Λ, C_0, and κ), we obtain (3.18). The case $\beta > 0$ and u positive subsolution of (3.1) is almost identical and we will not repeat it.

The same reasoning shows (3.18) when $|b+c|^2 \in \mathcal{K}(\Omega_r)$ if we use Lemma 2.35. The only difference lies on the manipulation of the terms that include $b + c$ and a similar argument can be found at the end of the proof of Theorem 3.1. The details are omitted. □

In fact, if we incorporate $-\text{div}(bu)$ and du into the interior data, the same proof gives the following theorem:

Theorem 3.6 *If we use the same notation as in Theorem 3.5 and either $c \in L^{n,q}(\Omega_r)$, for $q \in [n, \infty)$ or $|c|^2 \in \mathcal{K}(\Omega_r)$, then for any non-negative function $\eta \in C_c^\infty(B_r)$, we have*

$$\|\eta\, \bar{u}^{\frac{\beta-1}{2}}\, \nabla u\|^2_{L^2(\widetilde{\Omega}_r)} \lesssim C_0 \|\bar{u}^{\frac{\beta+1}{2}}\nabla\eta\|^2_{L^2(\widehat{\Omega}_r)} + \int_{\widehat{\Omega}_r} (C_1 \bar{f} + C_1|d| + C_2\bar{g}^2 + C_2|b|^2)\bar{u}^{\beta+1}\eta^2, \tag{3.40}$$

where $\bar{f} = |f|/\bar{u}$, $\bar{g} = |g|/\bar{u}$, and C_0, C_1, and C_2 are the constants given in Theorem 3.5. The implicit constant depends on λ, Λ, and either on $C_{s,q}$ and $\|c\|_{L^{n,q}(\Omega_r)}$, or C_s' and $\vartheta_{\Omega_r}(|c|^2, r)$.

The analogue of Theorem 3.5 for the case $-\mathrm{div} c + d \geq 0$ (or $-\mathrm{div} c + d \leq 0$) will be a lot easier to prove, as one does not need to handle either the $L^{n,q}$-norm of $b + c$ or the $\mathcal{K}$-norm of $|b + c|^2$ in a delicate way as before. Instead, we will incorporate $|b + c|^2$ into the interior data side (as in Theorem 3.6). It may look surprising bearing in mind the special case $\beta = 1$ we proved in Theorem 3.2, but (3.18) cannot hold in this case. The reason is that it is the main ingredient of the proof of local boundedness and weak Harnack inequality and, by Example (4.8), we know that if $b + c$ does not have any additional hypothesis, solutions may not be locally bounded.

Theorem 3.7 *If we replace $\mathrm{div} b + d \geq 0$ (or $\mathrm{div} b + d \leq 0$) with $-\mathrm{div} c + d \geq 0$ (or $-\mathrm{div} c + d \leq 0$) in the assumptions of Theorem 3.5 and use the same notation, we can find constants C_0, C_1, C_2 depending on β, such that for any non-negative function $\eta \in C_c^\infty(B_r)$ we have*

$$\|\eta\, \bar{u}^{\frac{\beta-1}{2}}\, \nabla u\|^2_{L^2(\widetilde{\Omega}_r)} \lesssim C_0 \|\bar{u}^{\frac{\beta+1}{2}}\nabla\eta\|^2_{L^2(\widehat{\Omega}_r)} + \int_{\widehat{\Omega}_r} (C_1 \bar{f} + C_2\bar{g}^2 + C_2|b+c|^2)\bar{u}^{\beta+1}\eta^2, \tag{3.41}$$

where $\bar{f} = |f|/\bar{u}$, $\bar{g} = |g|/\bar{u}$, and the implicit constant depends λ and Λ. When $|\beta| > 1$, $C_0 = |\beta+1|^{-2}$, $C_1 = |\beta+1|^{-1}$, and $C_2 = 1 + |\beta+1|^{-2}$, while when $|\beta| < 1$, $C_0 = |\beta|^{-2}$ and $C_1 = C_2 = |\beta|^{-1}$. When $\beta = -1$, $C_0 = C_1 = C_2 = 1$.

Proof We will only give a sketch of the proof. Let us assume that $\beta \in [-1, 0)$. For $k > 0$ we define the auxiliary function

$$w = \bar{u}^\beta - (m + k)^\beta.$$

Since $\eta^2 w \in Y_0^{1,2}(\Omega_r)$, arguing as in Case $\beta > -1$ in the proof of the previous theorem and using $-\mathrm{div} c + d \geq 0$, we get

$$\int_{\Omega_r} f(\eta^2 w) + \int_{\Omega_r} g\nabla(\eta^2 w) \leq \int_{\Omega_r} A\nabla u \nabla(\eta^2 w) - (b + c)u\nabla(\eta^2 w).$$

Because $\beta < 0$ and $\{x \in \Omega_r : w \neq 0\} = \Omega_r^m$, the latter inequality can be written as

$$|\beta| \int_{\Omega_r} A\nabla u \cdot \nabla u \eta^2 \bar{u}^{\beta-1} \leq 2\int_{\Omega_r} A\nabla u \cdot \nabla\eta\, w\eta - \int_{\Omega_r} (b + c)u\nabla(\eta^2 w)$$
$$- \int_{\Omega_r} f(\eta^2 w) - \int_{\Omega_r} g\nabla(\eta^2 w) = \sum_{i=1}^4 I_i.$$

Note that if we use $0 \le u \le \bar{u}$, then I_1, I_3 and I_4 can be bounded as in (3.36) and (3.37). So, it only remains to handle I_2. But as we do not need to use Lemma 2.34 it will be fairly easy to do so. Indeed,

$$I_2 = -2 \int_{\Omega_r^m} (b+c)\nabla\eta w u \eta + |\beta| \int_{\Omega_r^m} (b+c)\nabla u \eta^2 \bar{u}^{\beta-1} u,$$

which, in light of Young's inequality with ε small (to be picked), $w \le \bar{u}^\beta \mathbf{1}_{\Omega_r^m}$ and $\beta \in [-1, 0)$, implies

$$|I_2| \le (1 + |\beta|(4\varepsilon)^{-1}) \int_{\Omega_r^m} |b+c|^2 \bar{u}^{\beta+1}\eta^2 + \int_{\Omega_r^m} |\nabla\eta|^2 \bar{u}^{\beta+1} + \varepsilon|\beta| \int_{\Omega_r^m} |\nabla u|^2 \bar{u}^{\beta-1}\eta^2.$$

If we choose ε small enough we conclude our result. We may handle the case $\beta < -1$ and $\beta \ge 0$ for subsolutions in a similar fashion adapting the argument in the proof of Theorem 3.5. We omit the routine details. □

Moreover, the proofs of Theorems 3.5, 3.6, and 3.7 can be easily adapted to get a refined version of Theorems 3.1 and 3.2. We only state the first one.

Theorem 3.8 *Let B_r be a ball of radius $r > 0$ so that $\overline{B}_r \subset \Omega$ and assume that either $b+c \in L^{n,q}(B_r)$, $q \in [n, \infty)$, or $|b+c|^2 \in \mathcal{K}(B_r)$. If $u \in Y^{1,2}(B_r)$ and one of the following holds:*

(1) $\operatorname{div} b + d \le 0$ *and u is L-subsolution in B_r and $\beta \in (0, +\infty)$;*
(2) $\operatorname{div} b + d \le 0$ *and u is L-supersolution in B_r and $\beta \in (0, +\infty)$;*
(3) $\operatorname{div} b + d \ge 0$ *and u is a non-negative L-supersolution in B_r and $\beta \in (-\infty, 0)$.*

For $k > 0$, we set

$$\bar{u} = \begin{cases} u^+ + k & , \text{in Case } (1), \\ u^- + k & , \text{in Case } (2), \\ u + k & , \text{in Case } (3). \end{cases}$$

Then, there exist constants C_0, C_1, C_2 depending on β, such that for any non-negative function $\eta \in C_c^\infty(B_r)$ we have

$$\|\eta\, \bar{u}^{\frac{\beta-1}{2}} \nabla u\|^2_{L^2(B_r)} \lesssim C_0 \|\bar{u}^{\frac{\beta+1}{2}} \nabla\eta\|^2_{L^2(B_r)} + \int_{B_r} \left(C_1|\bar{f}| + C_2|\bar{g}|^2\right) \bar{u}^{\beta+1}\eta^2, \quad (3.42)$$

where $\bar{f} = |f|/\bar{u}$, $\bar{g} = |g|/\bar{u}$, and the implicit constant depends on λ, Λ, and also either on $C_{s,q}$ and $\|b+c\|_{L^{n,q}(B_r)}$, or C_s' and $\vartheta_{B_r}(|b+c|^2, r)$. When $|\beta| > 1$, $C_0 = |\beta+1|^{-2}$, $C_1 = |\beta+1|^{-1}$, and $C_2 = 1 + |\beta+1|^{-2}$, while when $|\beta| < 1$, $C_0 = 4^\kappa|\beta|^{-2}$ and $C_1 = C_2 = 2^\kappa|\beta|^{-1}$, where either $\kappa \le 1 + \frac{1}{C|\beta|^n}\|b+c\|^n_{L^{n,q}(B_r)}$ or $\kappa \le 1 + 2\,a^{-2}\,\rho_0^{2-n}\|h\|_{L^1(B_r)}$. In the case $\beta = -1$, $C_0 = C_1 = C_2 = 1$.

4 Local estimates and regularity of solutions up to the boundary

In this part we will present the iterating method of Moser to obtain the following results:

- Local boundedness for subsolutions;
- Weak Harnack inequality for supersolutions;
- Hölder continuity in the interior for solutions;
- A Wiener criterion for continuity of solutions at the boundary.

4.1 Local boundedness and weak Harnack inequality

Denote $\Omega_{r_0} = B_{r_0} \cap \Omega \neq \emptyset$, where $r_0 \in (0, \infty]$, and let $f \in \mathcal{K}(\Omega_{r_0})$ and $|g|^2 \in \mathcal{K}(\Omega_{r_0})$. Set

$$\gamma := \beta + 1$$

and

$$k(r) := \vartheta_{\Omega_{r_0}}(|f|, r) + \vartheta_{\Omega_{r_0}}(|g|^2, r)^{1/2}, \quad \text{for any } r \in (0, r_0].$$

Define

$$w = \begin{cases} \bar{u}^{\frac{\beta+1}{2}}, & \text{if } \beta \neq -1 \\ \log \bar{u}, & \text{if } \beta = -1, \end{cases} \tag{4.1}$$

where $\bar{u}$ is either the one given in Theorem 3.5 or in Theorem 3.8, with

$$k = k(r).$$

Here B_r is a ball of radius $r \in (0, r_0]$ which is either centered at the boundary (as in Theorem 3.5) or such that $B_r \subset \Omega$ (as in Theorem 3.8). We will handle both cases simultaneously and it should be understood from the context what kind of balls we are referring to. Set

$$\tilde{f} = \frac{|f|}{k(r)}, \quad \tilde{g} = \frac{|g|}{k(r)}, \quad \text{and} \quad V = \tilde{f} + \tilde{g}^2.$$

Notice that for $k = k(r)$, we have that $|\bar{f}| \leq |\tilde{f}|$ and $|\bar{g}| \leq |\tilde{g}|$ and so (3.18), (3.40), (3.41), and (3.42) hold for $\tilde{f}$ and $\tilde{g}$ instead of $\bar{f}$ and $\bar{g}$. Moreover,

$$\vartheta_{\Omega_{r_0}}(V, r) = \frac{1}{k(r)} \sup_{x \in \mathbb{R}^n} \int_{B(x,r) \cap \Omega_{r_0}} \frac{|f(y)|}{|x-y|^{n-2}}\, dy + \frac{1}{k(r)^2} \sup_{x \in \mathbb{R}^n} \int_{B(x,r) \cap \Omega_{r_0}} \frac{|g(y)|^2}{|x-y|^{n-2}}\, dy \leq 2. \tag{4.2}$$

Lemma 4.1 *Assume that B_r be a ball such that $\Omega_r = B_r \cap \Omega \neq \emptyset$, $r \leq r_0$, and that either $b + c \in L^{n,q}(\Omega_{r_0})$, $q \in [n, \infty)$, or $|b + c|^2 \in \mathcal{K}(\Omega_{r_0})$. If w is defined in (4.1), and $\eta \in C_c^\infty(B_r)$ is non-negative, then the following hold: If $|\beta| > 1$, there exist constants $c_3' > 1$ and $c_4' \in (0, 1)$ so that for any $0 < \epsilon \leq 1$,*

$$\|\eta w\|_{L^{2^*}(B_r)} \leq \frac{c_3'(1 + |\gamma|^{-2})}{\vartheta^{-1}_{\epsilon, \Omega_{r_0}}(V, \epsilon\, c_4'\,(1 + |\gamma|^{-2})^{-1})} \|(\eta + |\nabla \eta|) w\|_{L^2(B_r)}. \tag{4.3}$$

and if, in addition, $|\gamma| > \frac{1}{2}$, there exist $c_3 > 1$ and $c_4 \in (0, 1)$ such that

$$\|\eta w\|_{L^{2^*}(B_r)} \leq \frac{c_3}{\vartheta^{-1}_{\epsilon, \Omega_{r_0}}(V, \epsilon\, c_4\, |\gamma|^{-1})} \|(\eta + |\nabla \eta|) w\|_{L^2(B_r)}. \tag{4.4}$$

If there exists $\beta_0 \in (0, 1)$ such that $\beta_0 \leq |\beta| < 1$, then there exist constants $c_5 > 1$ and $c_6 = c_6(\beta_0) \in (0, 1)$ so that

$$\|\eta w\|_{L^{2^*}(B_r)} \leq \frac{c_5}{\vartheta^{-1}_{\epsilon, \Omega_{r_0}}(V, \epsilon\, c_6\, |\gamma|)} \|(\eta + |\nabla \eta|) w\|_{L^2(B_r)}. \tag{4.5}$$

The implicit constants are independent of ϵ and gamma.

Proof If $|\beta| > 1$, for ε to be chosen, by (2.15) we have that

$$\int_{\Omega_r} (|\tilde{f}| + |\tilde{g}|^2) w^2 \eta^2 \leq c_1 \varepsilon \left(\int_{\Omega_r} |\nabla (w\eta)|^2 + \frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_2^{-1}\varepsilon)^2} \int_{\Omega_r} |w\eta|^2 \right). \tag{4.6}$$

By (4.6), we may rewrite (3.18) or (3.42),

$$\int_{\Omega_r} |\eta \nabla w|^2 \leq C\,|\gamma|^{-2} \int_{\Omega_r} |\nabla \eta|^2 w^2 + 2\,\varepsilon\, C\, c_1\, (1 + |\gamma|^{-2}) \int_{\Omega_r} |\nabla (w\eta)|^2$$
$$+\, \varepsilon\, C\, c_1\, (1 + |\gamma|^{-2}) \frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_2^{-1}\varepsilon)^2} \int_{\Omega_r} |w\eta|^2.$$

Therefore, if we choose $\varepsilon = \frac{\epsilon}{10 C c_1 (1+|\gamma|^{-2})} < 0.1$, we deduce

$$\int_{\Omega_r} |\eta \nabla w|^2 \leq C\,|\gamma|^{-2} \int_{\Omega_r} |\nabla \eta|^2 w^2 + \frac{1}{5} \int_{\Omega_r} |\nabla (w\eta)|^2 + \frac{1}{10\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_2^{-1}\varepsilon)^2} \int_{\Omega_r} |w\eta|^2,$$

which, in turn, since $C > 1$, implies

$$\int_{B_r} |\nabla (w\eta)|^2 \leq \frac{10C(1 + |\gamma|^{-2})}{3} \int_{\Omega_r} |\nabla \eta|^2 w^2 + \frac{1}{3\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_2^{-1}\varepsilon)^2} \int_{\Omega_r} |w\eta|^2. \tag{4.7}$$

Notice that $\epsilon < \vartheta_{\epsilon,\Omega_{r_0}}(V, 1)$ and so $\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon) \leq 1$. Thus

$$\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_2^{-1}\varepsilon) = \vartheta^{-1}_{\epsilon,\Omega_{r_0}}\left(V, \epsilon \left(10 C c_1 c_2 (1 + |\gamma|^{-2})\right)^{-1}\right) \leq \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon) \leq 1,$$

which, if we set $c_4' := (10 C c_1 c_2)^{-1} < \frac{1}{10}$, in light of (4.7), gives

$$\|\nabla (w\eta)\|_{L^2(\Omega_r)} \leq \frac{(11C/3)(1 + |\gamma|^{-2})}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}\left(V, \epsilon\, c_4' \left(1 + |\gamma|^{-2}\right)^{-1}\right)} \|(\eta + |\nabla \eta|) w\|_{L^2(\Omega_r)}.$$

Moreover, if $|\gamma| > \frac{1}{2}$, it holds that $\frac{|\gamma|^2}{1+|\gamma|^2} \geq \frac{1}{10|\gamma|}$, and, if we set $c_4 := \frac{c_4'}{10}$, we can deduce that

$$\|\nabla (w\eta)\|_{L^2(\Omega_r)} \leq 20C (\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon\, c_4 |\gamma|^{-1}))^{-1} \|(\eta + |\nabla \eta|) w\|_{L^2(\Omega_r)}.$$

Since $\eta w \in Y_0^{1,2}(B_r)$, (4.3) and (4.4) follow by Sobolev's inequality.

In a similar fashion, for $0 < |\beta| < 1$, if we choose $\varepsilon = \frac{\epsilon |\beta|^2}{10 C c_1} < \frac{1}{10}$, since $4^{\kappa} \geq 1$, we obtain

$$\int_{\Omega_r} |\eta \nabla w|^2 \leq \frac{C}{|\beta|^2} \int_{\Omega_r} |\nabla \eta|^2 w^2 + \frac{1}{5} \int_{\Omega_r} |\nabla (w\eta)|^2 + \frac{1}{10\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_2^{-1}\varepsilon)^2} \int_{\Omega_r} |w\eta|^2.$$

which entails

$$\int_{B_r} |\nabla (w\eta)|^2 \leq \frac{10C}{3} \left(1 + \frac{1}{|\beta|^2}\right) \int_{\Omega_r} |\nabla \eta|^2 w^2 + \frac{1}{3\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon\, c_4' |\beta|^2)^2} \int_{\Omega_r} |w\eta|^2.$$

Thus, as $0 < \beta_0 \le |\beta| < 1$, we have that $c_2^{-1}\,\varepsilon \ge \epsilon\,\beta_0^2 c_4' \ge \epsilon\,|\gamma|\beta_0^2 c_4'/2$ and so, if we set $c_6 := \beta_0^2 c_4'/2$, since $\epsilon\, c_6 < \vartheta_{\epsilon,\Omega_{r_0}}(V, c_6)$ and so $\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon\, c_6) < c_6$, there exists $c_5 > 1$ (independent of β_0) such that

$$\|\nabla(\eta w)\|_{L^2(\Omega_r)} \le \frac{c_5}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon\, c_6\,|\gamma|)} \|(\eta + |\nabla \eta|) w\|_{L^2(B_r)}.$$

We conclude the proof of (4.5) by Sobolev's inequality. □

Remark 4.2 Lemma 4.1 can be proved in the cases

(1) $-\mathrm{div} c + d \le 0$ (or ≥ 0) and $|b+c|^2 \in \mathcal{K}(\Omega_{r_0})$,
(2) $|b|^2 \in \mathcal{K}(\Omega_{r_0})$ and $d \in \mathcal{K}(\Omega_{r_0})$, and either $c \in L^{n,q}(\Omega_{r_0})$, $q \in [n, \infty)$, or $|c|^2 \in \mathcal{K}(\Omega_{r_0})$.

We set

$$k(r) = \begin{cases} \vartheta_{\Omega_{r_0}}(|f|, r) + \vartheta_{\Omega_{r_0}}(|g|^2, r)^{1/2} + \vartheta_{\Omega_{r_0}}(|b+c|^2, r)^{1/2} & \text{, in Case (1),} \\ \vartheta_{\Omega_{r_0}}(|f|, r) + \vartheta_{\Omega_{r_0}}(|g|^2, r)^{1/2} + \vartheta_{\Omega_{r_0}}(|b|^2, r)^{1/2} + \vartheta_{\Omega_{r_0}}(|d|, r) & \text{, in Case (2),} \end{cases} \tag{4.8}$$

For k as in (4.8), we use Theorem 3.7 and Theorem 3.6 respectively, and set

$$V = \begin{cases} |\tilde{f}| + |\tilde{g}|^2 + |b+c|^2 & \text{, in Case (1),} \\ |\tilde{f}| + |\tilde{g}|^2 + |b|^2 + |d| & \text{, in Case (2),} \end{cases}$$

in order to obtain the same results as in Lemma 4.1.

We are now ready to prove the local boundedness of subsolutions.

Definition 4.3 We will say that *the condition* $(\mathrm{N})_{r_0}$ is satisfied if one the following conditions hold:

(1) $\mathrm{div} b + d \le 0$ and $b + c \in L^{n,q}(\Omega_{r_0})$, $q \in [n, \infty)$ or $|b+c|^2 \in \mathcal{K}(\Omega_{r_0})$;
(2) $-\mathrm{div} c + d \le 0$ and $|b+c|^2 \in \mathcal{K}_{\mathrm{Dini}}(\Omega_{r_0})$.

Analogously, we will say that *the condition* $(\mathrm{P})_{r_0}$ is satisfied if we reverse the inequalities in condition (N). Here, (N) and (P) stand for the negativity and positivity condition respectively. We will also say that *the condition* $(\mathrm{D})_r$ is satisfied if $|b|^2 \in \mathcal{K}_{\mathrm{Dini}}(\Omega_r)$, $d \in \mathcal{K}_{\mathrm{Dini}}(\Omega_r)$, and either $c \in L^{n,q}(\Omega_{r_0})$, $q \in [n, \infty)$, or $|c|^2 \in \mathcal{K}(\Omega_{r_0})$. If the above conditions hold globally, i.e., for $r_0 = \infty$ and Ω instead of Ω_{r_0}, we will drop the subscript r_0 and simply write (N), (P), and (D).

In the next theorem we borrow ideas from [26], although, some details are different in our case. For example, we had to introduce the auxiliary modulus ϑ'_{Ω_r} to be able to use Lemma 2.32 and define the appropriate Dini condition that gives constants independent of Ω.

Theorem 4.4 (Local boundedness) *Let B_r be a ball such that $B_r \cap \Omega \ne \emptyset$, for $r \le r_0$, and assume that $f, |g|^2 \in \mathcal{K}_{Dini}(\Omega_{r_0})$. If $\sigma \in (0,1)$, then the following hold:*
(1) If u is a subsolution of (3.1) in $B_r \cap \Omega$ and the condition $(\mathrm{N})_{r_0}$ or $(\mathrm{D})_{r_0}$ holds, then

(i) if $B_r \subset \Omega$

$$\sup_{B_{\sigma r}} u^+ \lesssim (1-\sigma)^{-n/p} \left(r^{-n/p}\, \|u^+\|_{L^p(B_r)} + k(r) \right); \tag{4.9}$$

(ii) if B_r is centered at a point $\xi \in \partial\Omega$,

$$\sup_{B_{\sigma r}} u_M^+ \lesssim (1-\sigma)^{-n/p}\left(r^{-n/p}\,\|u_M^+\|_{L^p(B_r)} + k(r)\right). \tag{4.10}$$

(2) If u is a supersolution of (3.1) in $B_r \subset \Omega$ and the condition $(\mathrm{P})_{r_0}$ or $(\mathrm{D})_{r_0}$ holds, then

(i) if $B_r \subset \Omega$

$$\sup_{B_{\sigma r}} u^- \lesssim (1-\sigma)^{-n/p}\left(r^{-n/p}\,\|u^-\|_{L^p(B_r)} + k(r)\right). \tag{4.11}$$

(ii) if B_r is centered at a point $\xi \in \partial\Omega$,

$$\sup_{B_{\sigma r}}(-u_m^-) \lesssim (1-\sigma)^{-n/p}\left(r^{-n/p}\,\|u_m^-\|_{L^p(B_r)} + k(r)\right). \tag{4.12}$$

The implicit constants depend only on $p, \sigma, n, \lambda, \Lambda, C_{|f|,\Omega_{r_0}}, C_{|g|^2,\Omega_{r_0}}$ and according to our assumptions, on the following: a) $C_{s,q}$ and $\|b+c\|_{L^{n,q}(\Omega_{r_0})}$, or C_s' and $\vartheta_{\Omega_{r_0}}(|b+c|^2, r)$, b) $C_{|b+c|^2,\Omega_{r_0}}$, and c) $C_{|b|^2,\Omega_{r_0}}, C_{|d|,\Omega_{r_0}}$, and either $C_{s,q}$ and $\|c\|_{L^{n,q}(\Omega_{r_0})}$, or C_s' and $\vartheta_{\Omega_{r_0}}(|c|^2, r)$.

Proof Let us now pick η so that, for $0 \le \sigma_1 < \sigma_2 \le \frac{1}{2}$,

$$0 \le \eta \le 1, \quad \eta = 1 \text{ in } B_{\sigma_1 r}, \quad \eta = 0 \text{ in } B_{\sigma_2 r}, \quad \|\nabla\eta\|_\infty \le 2/(\sigma_2-\sigma_1)r.$$

If we set $\chi = \frac{n}{n-2}$ and $k = k(r)$, then (4.4) for $r \le 1$ can be written as

$$\|w\|_{L^{2\chi}(B_{\sigma_1 r})} \le \frac{2c_3}{(\sigma_2-\sigma_1)r}\,\frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon\, c_4\,|\gamma|^{-1})}\,\|w\|_{L^2(B_{\sigma_2 r})},$$

which, in turn, implies that

$$\|\bar u\|_{L^{\gamma\chi}(B_{\sigma_1 r})} \le \left(\frac{2c_3}{(\sigma_2-\sigma_1)r}\right)^{2/\gamma}\frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon\, c_4\,|\gamma|^{-1})^{2/\gamma}}\,\|\bar u\|_{L^\gamma(B_{\sigma_2 r})}, \tag{4.13}$$

if $|\gamma| > \frac{1}{2}$ and u is a subsolution.

For $p > 1$ and any non-negative integer i, we set

$$\gamma_i := \chi^i\, p = (1+\tfrac{2}{n-2})^i\, p \ge p > 1 \quad \text{and} \quad \sigma_i := \frac{1}{2} + \frac{1}{2^{i+1}},$$

and apply (4.13) with $\gamma = \gamma_i$, $\sigma_1 = \sigma_{i+1}$ and $\sigma_2 = \sigma_i$ to obtain

$$\begin{aligned}\|\bar u\|_{L^{\gamma_{i+1}}(B_{\sigma_{i+1} r})} &\le (2c_3 2^{i+2}/r)^{2/\gamma_i}\,\frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, \epsilon\, c_4\gamma_i^{-1})^{2/\gamma_i}}\,\|\bar u\|_{L^{\gamma_i}(B_{\sigma_i r})}\\ &=: (K_1/r^{2/p})^{1/\chi^i}\, K_2^{i/\chi^i}\,\frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_7\,\chi^{-i})^{2/p\chi^i}}\,\|\bar u\|_{L^{\gamma_i}(B_{\sigma_i r})},\end{aligned}$$

where $K_1 = (8\,c_3)^{2/p}$ and $K_2 = 2^{2/p}$ and $c_7 := \epsilon\, c_4\, p < 1$ (we can choose c_4 so that $p\,c_4 < 1$). Iteration of this inequality leads to

$$\sup_{B_{r/2}} \bar u \le (K_1 r)^{\sum_i \frac{1}{\chi^i}}\, K_2^{\sum_i \frac{i}{\chi^i}} \prod_{i=0}^{\infty} \frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_7\,\chi^{-i})^{2/p\chi^i}}\,\|\bar u\|_{L^p(B_r)}. \tag{4.14}$$

Thus, since

$$\log \prod_{i=0}^{\infty} \frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_7\, \chi^{-i})^{2/p\chi^i}} = -\frac{2}{\epsilon\, c_4} \sum_{i=0}^{\infty} \frac{c_7}{\chi^i} \log \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_7\, \chi^{-i}),$$

we may apply Lemma 2.32 for $\tau = \chi^{-1}$ and $c = c_7$, and by Lemma 2.13, we obtain

$$\begin{aligned}
-\frac{2}{\epsilon\, c_4} \sum_{i=0}^{\infty} \frac{c_7}{\chi^i} \log \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_7\, \chi^{-i}) &\leq \frac{2\chi}{(\chi-1)\epsilon\, c_4} \int_0^{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7)} \vartheta_{\epsilon,\Omega_{r_0}}(V,t) \frac{dt}{t} \\
&= \frac{2\chi}{(\chi-1)\epsilon\, c_4} \int_0^{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7)} \vartheta_{\Omega_{r_0}}(V,t) \frac{dt}{t} + \frac{2\chi\, \epsilon}{(\chi-1)\epsilon\, c_4} \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7) \\
&\leq \frac{2\chi}{(\chi-1)\epsilon\, c_4} \Big(C_{|f|,\Omega_{r_0}} \vartheta_{\Omega_{r_0}}(|\tilde{f}|, \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7)) + C_{|g|^2,\Omega_{r_0}} \vartheta_{\Omega_{r_0}}(|\tilde{g}|^2, \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7)) \Big) \\
&\quad + \frac{2\chi\, \epsilon}{(\chi-1)\epsilon\, c_4} \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7) \\
&\leq \frac{2\chi}{(\chi-1)\epsilon\, c_4} \Big((C_{|f|,\Omega_{r_0}} + C_{|g|^2,\Omega_{r_0}}) \vartheta_{\Omega_{r_0}}(V, \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7)) + \epsilon \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7) \Big) \\
&\leq \frac{2\chi}{(\chi-1)\epsilon\, c_4} \max\Big((C_{|f|,\Omega_{r_0}} + C_{|g|^2,\Omega_{r_0}}), 1 \Big) \vartheta_{\epsilon,\Omega_{r_0}}(V, \vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V,c_7)) \\
&\leq \frac{2\chi\, c_7}{(\chi-1)\epsilon\, c_4} \max\Big(C_{|f|,\Omega_{r_0}} + C_{|g|^2,\Omega_{r_0}}, 1 \Big) = \frac{2\chi\, p}{(\chi-1)} \max\Big(C_{|f|,\Omega_{r_0}} + C_{|g|^2,\Omega_{r_0}}, 1 \Big),
\end{aligned}$$

where $C_{|f|,\Omega_{r_0}}$ and $C_{|g|^2,\Omega_{r_0}}$ stand for the Carleson-Dini constants (2.5).

By the definition of $\bar{u}$, we get

$$\sup_{B_{r/2}} u_M^+ \leq \sup_{B_{r/2}} u_M^+ + k(r) \lesssim r^{-n/p}\, \|\bar{u}\|_{L^p(B_r)} \lesssim r^{-n/p} \|u_M^+\|_{L^p(B_r)} + k(r),$$

from which, (4.10) for $r \leq 1$ follows. Replacing u_M^+ by u^+, the same argument shows (4.9) for $r \leq 1$.

To obtain the desired estimates in any ball of arbitrary radius $r > 1$ we use a rescaling argument. Note that $u_r = u(rx)$ is a subsolution (resp. supersolution) of the equation

$$-\mathrm{div}(A_r \nabla w + b_r w) - c_r \nabla w - d_r w = f_r - \mathrm{div} g_r,$$

where

$$\begin{aligned}
&A_r(x) = A(rx), \quad b_r(x) = rb(rx), \quad c_r(x) = rc(rx), \quad d_r(x) = r^2 d(rx), \\
&f_r(x) = r^2 f(rx), \quad g_r(x) = rg(rx).
\end{aligned}$$

If we set $D_r = \frac{1}{r}\Omega_{r_0}$, by Lemma 2.13, we get that

$$\begin{aligned}
\|b_r + c_r\|_{L^{n,q}(D_r)} &= \|b + c\|_{L^{n,q}(\Omega_{r_0})}, \\
\vartheta_{D_r}(f_r, 1) = \vartheta_{\Omega_{r_0}}(f, r) \quad \vartheta_{D_r}(|g_r|^2, 1) &= \vartheta_{\Omega_{r_0}}(|g|^2, r),
\end{aligned}$$

and since the Dini condition is scale invariant, we have

$$C_{f_r,D_r} = C_{f,\Omega_{r_0}} \quad C_{|g_r|^2,D_r} = C_{|g|^2,\Omega_{r_0}}.$$

If we apply (4.14) to u_r in the domain D_r, by the change of variables $y = rx$, we obtain the following estimate:

$$\sup_{B_{r/2}} \bar{u} = \sup_{B_{1/2}} \bar{u}_r \lesssim \|\bar{u}_r\|_{L^p(B_1)} \approx r^{-n/p} \|\bar{u}\|_{L^p(B_r)}.$$

Remark that the implicit constants do not depend on r.

Moreover, if $0 < \sigma < 1/2$,

$$\sup_{B_{\sigma r}} \bar{u} \le \sup_{B_{r/2}} \bar{u} \lesssim r^{-n/p} \|\bar{u}\|_{L^p(B_r)} \lesssim (1-\sigma)^{-n/p} r^{-n/p} \|\bar{u}\|_{L^p(B_r)}.$$

and if $1/2 < \sigma < 1$, then for any ball $B(z, (1-\sigma)r) \subset B_{\sigma r}$, we get

$$\sup_{B(z,(1-\sigma)r)} \bar{u} \lesssim (1-\sigma)^{-n/p} r^{-n/p} \|\bar{u}\|_{L^p(B(z,2(1-\sigma)r)} \le (1-\sigma)^{-n/p} r^{-n/p} \|\bar{u}\|_{L^p(B_r)}.$$

Thus, for any $\sigma \in (0, 1)$, we have shown that

$$\sup_{B_{\sigma r}} \bar{u} \lesssim (1-\sigma)^{-n/p} r^{-n/p} \|\bar{u}\|_{L^p(B_r)},$$

which trivially implies (4.9) and (4.10). To show (4.11) and (4.12) it suffices to notice that $w = -u$ is a subsolution of $Lw = -f + \text{div} g$ and use (4.9) and (4.10) as $\text{div} b + d \le 0$ still holds.

Using Remark 4.2 we can prove the same result under either condition (D) or $-\text{div} c + d \le 0$ and $|b+c|^2 \in \mathcal{K}_{\text{Dini}}(\Omega_r)$. We omit the details. □

We turn our attention to the weak Harnack inequality.

Theorem 4.5 (Weak Harnack inequality) *Let B_r be a ball such that $B_r \cap \Omega \neq \emptyset$, for $r \le r_0$, and assume that $f, |g|^2 \in \mathcal{K}_{Dini}(\Omega_{r_0})$. If u is a supersolution of (3.1) in $B_r \cap \Omega$ and the condition (P)$_r$ or (D)$_r$ is satisfied, then for $0 < s < p < \chi = n/n-2$, the following hold:*

(i) if $B_r \subset \Omega$

$$r^{-n/p} \|u\|_{L^p(B_{r/2})} \lesssim r^{-n/q} \|u\|_{L^s(B_r)} + k(r), \tag{4.15}$$

$$r^{-n/p} \|u\|_{L^p(B_r)} \lesssim \inf_{B_{r/2}} u + k(r/2). \tag{4.16}$$

(ii) if B_r is centered at a point $\xi \in \partial\Omega$,

$$r^{-n/p} \|u_m^-\|_{L^p(B_{r/2})} \lesssim r^{-n/s} \|u_m^-\|_{L^s(B_r)} + k(r), \tag{4.17}$$

$$r^{-n/p} \|u_m^-\|_{L^p(B_r)} \lesssim \inf_{B_{r/2}} u_m^- + k(r/2), \tag{4.18}$$

The implicit constants depend only on p, s, σ, n, λ, Λ, $C_{|f|,\Omega_{r_0}}$, $C_{|g|^2,\Omega_{r_0}}$ and according to our assumptions, on the following: a) $C_{s,q}$ and $\|b+c\|_{L^{n,q}(\Omega_{r_0})}$, or C_s' and $\vartheta_{\Omega_{r_0}}(|b+c|^2, r)$, b) $C_{|b+c|^2,\Omega_{r_0}}$, and c) $C_{|b|^2,\Omega_{r_0}}$, $C_{|d|,\Omega_{r_0}}$, and either $C_{s,q}$ and $\|c\|_{L^{n,q}(\Omega_{r_0})}$, or C_s' and $\vartheta_{\Omega_{r_0}}(|c|^2, r)$.

Proof We shall first prove the reverse Hölder inequality for $\bar{u}$. Recall first that $\gamma = \beta + 1$. If $p < \chi$, there exists $\delta \in (0, 1)$ such that $p = \delta\chi$. For any non-negative integer i, we let

$$\gamma_i = \chi^{-i} p \quad \text{and} \quad \sigma_i = 1 - \frac{1}{2^{i+1}},$$

and apply (4.13) (which is still true as $\beta < 0$ when $0 < \gamma = \beta + 1 < 1$) with $\gamma = \gamma_i$, $\sigma_1 = \sigma_i$ and $\sigma_2 = \sigma_{i+1}$. If we argue as in the proof of the previous theorem we obtain

$$\|\bar{u}\|_{L^{\gamma_i}(B_{\sigma_i})} \leq K_1^{1/\chi^i} \, K_2^{i/\chi^i} \, \frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_6\chi^{-i})^{2/p\chi^i}} \, \|\bar{u}\|_{L^{\gamma_{i+1}}(B_{\sigma_{i+1}})},$$

where $K_1 = (4\,c_5)^{2/p}$ and $K_2 = 2^{2/p}$ and $c_6 < 1$. As $q < p$, there exists $i_0 \in \mathbb{N}$ such that $\gamma_{i_0-1} \leq q < \gamma_{i_0-2}$. Thus, if we iterate the latter inequality i_0 times we get

$$\|\bar{u}\|_{L^p(B_{1/2})} \lesssim \|\bar{u}\|_{L^q(B_1)}. \tag{4.19}$$

If u is a supersolution, then (4.5) for $r = 1$ implies

$$\|\bar{u}\|_{L^{-q}(B_{\sigma_2})} \leq \|\bar{u}\|_{L^{-\gamma_{i_0-1}}(B_{\sigma_2})} \leq K_1^{1/\chi^{i_0}} \, K_2^{i/\chi^{i_0}} \, \frac{1}{\vartheta^{-1}_{\epsilon,\Omega_{r_0}}(V, c_6\chi^{-i_0})^{2/p\chi^{i_0}}} \, \|\bar{u}\|_{L^{\gamma_{i_0}}(B_{\sigma_1})}.$$

By a similar iteration argument as above we can show that for any $q \in (0, \chi)$,

$$\|\bar{u}\|_{L^{-q}(B_1)} \lesssim \inf_{B_{1/4}} \bar{u}. \tag{4.20}$$

Set now $w = \log \bar{u}$ and let $B_r(x)$ a ball centered at x of radius $r \leq 1/2$ so that $B_{2r}(x) \subset B_1$. Let also $\eta \in C_c^\infty(B_{2r}(x))$ so that $\eta = 1$ in $B_r(x)$, $\eta = 0$ outside $B_{2r}(x)$ and $\|\nabla\eta\|_\infty \lesssim 1/r$. Then, by Poincaré and Hölder inequalities, along with (3.18) or (3.42) for $\beta = -1$ and the fact that $|\bar{f} \leq |\tilde{f}|$, $|\bar{g}| \leq |\tilde{g}|$, and $k = k(1)$, we get

$$\begin{aligned}
\int_{B_r(x)} \Big| w - \fint_{B_r} w \Big| &\lesssim r \int_{B_r(x)} |\nabla w| \lesssim r r^{n/2} \left(\int_{B_r(x)} |\nabla w|^2 \right)^{1/2} \\
&\leq r r^{n/2} \left(\int_{B_{2r}(x)} |\eta \nabla w|^2 \right)^{1/2} \lesssim r r^{n/2} \left[\int_{B_{2r}(x)} |\nabla \eta|^2 + \int_{B_{2r}(x)} (|\tilde{f}| + |\tilde{g}|^2) \right]^{1/2} \\
&\lesssim r r^{n/2} \left[\int_{B_{2r}(x)} |\nabla \eta|^2 + r^{n-2} \int_{B_{2r}(x)} \frac{|\tilde{f}(y)| + |\tilde{g}(y)|^2}{|x-y|^{n-2}} \, dy \right]^{1/2} \\
&\lesssim r^n \left[1 + \vartheta_{\Omega_{r_0}}(|\tilde{f}|, r) + \vartheta_{\Omega_{r_0}}(|\tilde{g}|^2, r) \right]^{1/2} \\
&= r^n \left[1 + \frac{\vartheta_{\Omega_{r_0}}(|f|, r)}{k(1)} + \frac{\vartheta_{\Omega_{r_0}}(|g|^2, r)}{k(1)} \right]^{1/2} \leq 2r^n.
\end{aligned}$$

This shows that, $w \in \mathrm{BMO}(B_1)$ and thus, there exists $s \in (0, 1)$ such that e^{sw} is in the class of A_2 Muckenhoupt weights in B_1. That is,

$$\left(\fint_{B_1} \bar{u}^s \right)^{1/s} \lesssim \left(\fint_{B_1} \bar{u}^{-s} \right)^{-1/s}.$$

This, combined with (4.19) and (4.20), implies that, for any $0 < p < \chi$,

$$\|\bar{u}\|_{L^p(B_{1/2})} \lesssim \inf_{B_{1/4}} \bar{u}$$

and so (4.15)-(4.18) hold for $r = 1$. The general case follows by rescaling. □

Remark 4.6 If we impose global assumptions (e.g. $|c|^2 \in \mathcal{K}'(\Omega)$ and $|b|^2, |d| \in \mathcal{K}_{\text{Dini}}(\Omega)$) on the coefficients and the interior data, then we may take $r_0 = \infty$ and all of the constants in Theorems 4.4 and 4.5 are independent of r. In particular, the implicit constants depend on $p, \sigma, n, \lambda, \Lambda, C_{s,q}, C_{|f|,\Omega}, C_{|g|^2,\Omega}$ and according to our assumptions, on the following: a) $C_{s,q}$ and $\|b+c\|_{L^{n,q}(\Omega)}$, for $q \in [n,\infty)$, or C_s' and $\vartheta_\Omega(|b+c|^2)$, b) $C_{|b+c|^2,\Omega}$, and c) $C_{s,q}$ and $\|c\|_{L^{n,q}(\Omega)}$, for $q \in [n,\infty)$, or C_s' and $\vartheta_\Omega(|b+c|^2)$, $C_{|b|^2,\Omega}$, and $C_{|d|,\Omega}$.

Remark 4.7 Let $\delta > 0$, ψ_δ be as in (2.6), and $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\} \cap B(0, \delta^{-1})$. Define $b_\delta = (b\mathbf{1}_{\Omega_\delta}) * \psi_\delta$, $c_\delta = (c\mathbf{1}_{\Omega_\delta}) * \psi_\delta$, and $d_\delta = (d\mathbf{1}_{\Omega_\delta}) * \psi_{\Omega_\delta}$. Let us also define $L_\delta u = -\operatorname{div} A\nabla u - \operatorname{div}(b_\delta u) - c_\delta \nabla u - d_\delta u$. If (1.5) (resp. (1.6)) holds for b, c and d in Ω, then (1.5) (resp. (1.6)) holds in Ω_δ. For a proof see Lemma 6.9 in [16]. Moreover, $\|b_\delta + c_\delta\|_{L^{n,q}(\Omega)}$ is dominated by $2\|b+c\|_{L^{n,q}((\Omega)}$ and so, all the constants in the theorems of Sect. 3 are independent of δ. In the case that (1.5) holds, everything works exactly as before. On the other hand, if (1.6) is satisfied and $|b+c|^2 \in \mathcal{K}_{\text{Dini}}(\Omega)$, we should use Corollary 2.17 in the proof of Lemma 4.1 to obtain bounds which are independent of δ. Theorems (4.4) and (4.5) for subsolutions and supersolutions of L_δ in Ω_δ will then follow from the same proofs with estimates uniform in δ.

Example 4.8 Let us now refer to the counterexample constructed in [16, Lemma 7.4]. In particular, the authors defined the operator

$$-\Delta u - \operatorname{div}(\delta b u) = 0 \text{ in } B(0, e^{-1}),$$

where $b(x) = -\frac{x}{|x|^2 |\ln|x||}$ and $\delta > 0$, and showed that the solution $u = |\ln|x||^\delta \in Y^{1,2}(B(0,e^{-1}))$ does not satisfy (4.9) around 0. They proved that $b \in L^q(B(0,e^{-1}))$ for any $q > n$ but not in $L^n(B(0,e^{-1}))$. It is not hard to see that $|b|^2 \in \mathcal{K}(B(0,e^{-1}))$ but not in $\mathcal{K}_{\text{Dini}}(B(0,e^{-1}))$, and thus, assuming $|b+c|^2 \in \mathcal{K}(\Omega)$ does not suffice to establish local boundedness. A modification of this example shows that (4.16) does not hold when $|b+c|^2 \in \mathcal{K}(\Omega)$. It is important to note that, since δ can be taken as small as we want, this example shows that assuming the norms to be small is not enough either.

Example 4.9 Adjusting the previous example we can find an operator which does not satisfy neither (1.5) nor (1.6), for which there exists a non-bounded solution in the ball $B(0,e^{-1})$. Indeed, let

$$-\Delta u - du = 0 \text{ in } B(0,e^{-1}), \quad \text{where } d(x) = \frac{n-2}{|x|^2 |\ln|x||}. \tag{4.21}$$

It is not hard to see that $d \geq 0$ is in the Lorentz space $L^{n/2,q}(B(0,e^{-1}))$, for any $q > 1$. But notice that $u = |\ln|x||$ is a solution of (4.21) and $u \in Y^{1,2}(B(0,e^{-1}))$. Since u fails to be bounded around 0, the necessity of either (1.5) or (1.6) to prove local boundedness is established. It is interesting to see that d is not in $\mathcal{K}(B(0,e^{-1}))$ (and thus, it is not in $L^{n/2,1}(B(0,e^{-1}))$ either).

4.2 Interior and boundary regularity

Theorem 4.10 *Let u be a supersolution of (3.1) in Ω with $\sup_\Omega u < \infty$ and assume that the condition (P) or (D) holds. Then u has a lower semi-continuous representative satisfying*

$$u(x) = ess \liminf_{y \to x} u(y) = \lim_{r \to 0} \fint_{B(x,r)} u(y)\, dy, \quad \textit{for all } x \in \Omega. \tag{4.22}$$

Proof We follow the proof of [12, Theorem 3.66]. Fix a ball B_r centered at $x \in \Omega$ so that $B_{2r} \subset \Omega$ and apply Theorem 4.5 (i) to $u - m_r$, where $m_r = \operatorname{ess\,inf}_{B_r} u$. Then, we have

$$0 \leq \fint_{B_r} (u - m_r) \leq C((m_{r/2} - m_r) + k(r)).$$

Since C is either a constant independent of r and $(m_{r/2} - m_r) + k(r) \to 0$ as $r \to 0$, by taking limits in the inequality above as $r \to 0$, we obtain

$$\lim_{r \to 0} \fint_{B_r} (u - m_r) = \operatorname{ess} \liminf_{y \to x} (u - m_r) = 0,$$

which implies (4.22). □

Let us now introduce some notation that we will use in the rest of this section. For $r \leq r_0/2$ and $r_0 \in (0, \infty]$, set

$$k_{\epsilon,1}(r) := \vartheta_{\Omega_{r_0}}(|f|, r) + \big(\sup_{\Omega_r} |u|\big)\, \vartheta_{\Omega_{r_0}}(|d|, r) + \epsilon r, \tag{4.23}$$

$$\lim_{\epsilon \to 0} k_{\epsilon,1}(r) = k_1(r) := \vartheta_{\Omega_{r_0}}(|f|, r) + \big(\sup_{\Omega_r} |u|\big)\, \vartheta_{\Omega_{r_0}}(|d|, r), \tag{4.24}$$

$$k_{\epsilon,2}(r) := \vartheta_{\Omega_{r_0}}(|g|^2, r)^{1/2} + \big(\sup_{\Omega_r} |u|\big)\, \vartheta_{\Omega_{r_0}}(|b|^2, r)^{1/2} + \epsilon r, \tag{4.25}$$

$$\lim_{\epsilon \to 0} k_{\epsilon,2}(r) = k_2(r) := \vartheta_{\Omega_{r_0}}(|g|^2, r)^{1/2} + \big(\sup_{\Omega_r} |u|\big)\, \vartheta_{\Omega_{r_0}}(|b|^2, r)^{1/2}, \tag{4.26}$$

$$k_{\epsilon,3}(r) := \vartheta_{\Omega_{r_0}}(|b|^2, r)^{1/2} + \vartheta_{\Omega_{r_0}}(|d|, r) + \epsilon r, \tag{4.27}$$

$$\lim_{\epsilon \to 0} k_{\epsilon,3}(r) = k_3(r) := \vartheta_{\Omega_{r_0}}(|b|^2, r)^{1/2} + \vartheta_{\Omega_{r_0}}(|d|, r), \tag{4.28}$$

$$k_{\epsilon,4}(r) := \vartheta_{\Omega_{r_0}}(|f|, r) + \vartheta_{\Omega_{r_0}}(|g|^2, r)^{1/2} + \epsilon r, \tag{4.29}$$

$$\lim_{\epsilon \to 0} k_{\epsilon,4}(r) = k_4(r) := \vartheta_{\Omega_{r_0}}(|g|^2, r)^{1/2} + \vartheta_{\Omega_{r_0}}(|f|, r), \tag{4.30}$$

$$\widetilde{k}_\epsilon(r) := k_{\epsilon,1}(r) + k_{\epsilon,2}(r), \tag{4.31}$$

$$\widetilde{k}(r) := k_1(r) + k_2(r). \tag{4.32}$$

If k is defined as in Case (2) of (4.8), then $k = k_3 + k_4$. All the functions above with subscript ϵ are strictly increasing and from their very definitions we have the following:

Lemma 4.11 *If u satisfies*

$$\sup_{\Omega_r} |u| \lesssim \left(\fint_{\Omega_{2r}} |u|^2 \right)^{1/2} + k(2r), \quad \textit{for any } r \leq r_0/2. \tag{4.33}$$

then, if $0 < r_1 \leq r_0$,

$$\widetilde{k}(r) \lesssim k_3(r) \left[\left(\fint_{\Omega_{r_1}} |u|^2 \right)^{1/2} + k(r_1) \right] + k_4(r), \quad \textit{for any } r \leq r_1/2. \tag{4.34}$$

Theorem 4.12 (Modulus of continuity in the interior) *Let* $0 < r \le r_0/2$ *and* B_r *be a ball such that* $\overline{B}_r \subset \Omega$. *Assume that* $|f|$, $|d|$, $|b|^2$, *and* $|g|^2 \in \mathcal{K}_{Dini}(B_{r_0})$, *and either* $c \in L^{n,q}(B_{r_0})$, $q \in [n, \infty)$, *or* $|c|^2 \in \mathcal{K}(B_{r_0})$. *If* u *is a solution of* (3.1) *in* B_r, *then for every* $\mu \in (0, 1)$, *there exists* $\alpha \in (0, 1)$ *so that*

$$|u(x) - u(y)| \lesssim \left[\left(\frac{|x-y|}{r}\right)^{\alpha} + k_3(|x-y|^{\mu} r^{1-\mu})\right]\left[\left(\frac{1}{r^n}\int_{B_r}|u|^2\right)^{1/2} + k(r)\right] + k_4(|x-y|^{\mu} r^{1-\mu}),$$

for all $x, y \in B_{r/2}$, *where* $k_3(r)$ *and* $k_4(r)$ *are given by* (4.28). *and* (4.30). *Note that* α *and the implicit constants depend only on* λ, Λ, $C_{|f|,\Omega_{r_0}}$, $C_{|g|^2,\Omega_{r_0}}$ *and either on* $C_{s,q}$ *and* $\|c\|_{L^{n,q}(\Omega_r)}$, *or* C_s' *and* $\vartheta_{\Omega_{r_0}}(|b+c|^2, r)$.

Proof Fix $r_1 \in (0, r_0/2)$ such that $B_{r_1} \subset \Omega$ and assume that u is a weak solution of the equation $Lu = f - \operatorname{div} g$ in B_{r_1}. It is easy to see that u is also a weak solution of the equation

$$\widetilde{L}u = -\operatorname{div} A\nabla u - c\nabla u = (f + du) - \operatorname{div}(g - bu). \tag{4.35}$$

in B_{r_1}. Note that $\widetilde{L}1 = 0$ and since $\widetilde{d} = \widetilde{b}_i = 0$, $i = 1, \dots, n$, we can use Theorems 4.4 and 4.5 with $\widetilde{k}$ as in (4.32) to get

$$\sup_{B_r}(u + \widetilde{k}(r)) \lesssim \fint_{B_{2r}}(u + \widetilde{k}(r)) \lesssim \inf_{B_r}(u + \widetilde{k}(r)), \quad \text{for any } r \le r_0/2. \tag{4.36}$$

Now, let

$$M_0 = \sup_{B_{r_1}}|u|, \quad M_r = \sup_{B_r} u \quad \text{and} \quad m_r = \inf_{B_r} u,$$

and since $M_r - u$ and $u - m_r$ are non-negative solutions of (4.35) in B_{r_0}, by (4.36) for $r \le r_0/2$, we obtain

$$\fint_{B_r}(M_r - u) \le C\left(M_r - M_{r/2} + \widetilde{k}(r/2)\right),$$

$$\fint_{B_r}(u - m_r) \le C\left(m_{r/2} - m_r + \widetilde{k}(r/2)\right).$$

Summing those two inequalities we get

$$(M_r - m_r) \le C\left[(M_r - m_r) - (M_{r/2} - m_{r/2}) + 2\widetilde{k}(r/2)\right],$$

which further implies

$$(M_{r/2} - m_{r/2}) \le \frac{C-1}{C}(M_r - m_r) + 2\widetilde{k}(r/2).$$

If we set $\omega(r) = \operatorname{osc}_{B_r} u = M_r - m_r$ and $\gamma = 1 - C^{-1} \in (0, 1)$, the latter inequality can be written

$$\omega(r/2) \le \gamma\omega(r) + 2\widetilde{k}(r/2),$$

which implies that for any $\mu \in (0, 1)$ and for $\alpha = -(1-\mu)\log\gamma/\log 2 \in (0, 1)$, there exists a constant $C' > 0$ depending only on γ such that

$$\omega(r) \lesssim \left(\frac{r}{r_1}\right)^{\alpha}\omega(r_1) + \widetilde{k}(r^{\mu} r_1^{1-\mu}),$$

which, by (4.34), concludes the proof. □

The last goal of this section is to develop of a Wiener-type criterion for boundary regularity of solutions. We will follow the proof of Theorem 8.30 in [9]. Several modifications are required in our case and in particular, we would like to draw the reader's attention to the iteration argument at the end of the proof. In [9] it is claimed that the inequality (8.81) on p. 208 can be iterated to produce the desired oscillation bound at the boundary. Unless the CDC is satisfied, it is not clear to us that the second term on the right hand-side of that inequality will converge after infinitely many iterations. In fact, the exact term one picks up after m iterations is

$$\left(\chi(r/2^m) + \sum_{k=0}^{m-1} \chi(r/2^k) \prod_{j=k+1}^{m} (1-\chi(r/2^j))\right) \operatorname*{osc}_{\partial\Omega\cap B_r} u =: S_m \operatorname*{osc}_{\partial\Omega\cap B_r} u.$$

It seems that if we do not have additional information about the behavior of the sequence $a_k = \chi(r/2^k)$, we could choose different sequences a_k so that S_m either converges or diverges or even have multiple limit points. We resolve this issue by incorporating this term into the main oscillation term.

Let us first introduce some definitions.

Definition 4.13 We say that a set E is *thick* at $\xi \in E$ if

$$\int_0^1 \frac{\operatorname{Cap}(E \cap \overline{B}_r(\xi), B_{2r}(\xi))}{r^{n-2}} \frac{dr}{r} = +\infty. \tag{4.37}$$

If $\Omega \subset \mathbb{R}^n$ is an open set and for $\xi \in \partial\Omega$ it holds that

$$\operatorname{Cap}(\overline{B}_r(\xi) \setminus \Omega, B_{2r}(\xi)) \geq c_0\, r^{n-2}, \quad \text{for all } r \in (0, \operatorname{diam}\partial\Omega),$$

for some $c \in (0, 1)$ independent of r, we say that Ω satisfies the *capacity density condition* (CDC) *at* ξ. If this holds for every $\xi \in \partial\Omega$ and a uniform constant c, we say that $\partial\Omega$ has the *capacity density condition*.

Theorem 4.14 (Boundary oscillation) *Let $r \leq r_0/2$ and B_r be a ball centered at $\xi \in \partial\Omega$. Assume also that u is a solution of (3.1) and $\varphi \in Y^{1,2}(\Omega) \cap C(\overline{\Omega})$ so that $u - \varphi$ vanishes on $\partial\Omega \cap B_r$ in the Sobolev sense. Then, the following hold:*

(i) *Let $|f|$, $|d|$, $|b|^2$, and $|g|^2 \in \mathcal{K}_{Dini}(\Omega_{r_0})$, and either $c \in L^{n,q}(\Omega_{r_0})$, $q \in [n, \infty)$, or $|c|^2 \in \mathcal{K}(\Omega_{r_0})$. If Ω satisfies the (CDC) at ξ, then*

$$|u(x) - u(y)| \lesssim \left[\left(\frac{|x-y|}{r}\right)^{\alpha} + k_3(|x-y|^{\mu} r^{1-\mu})\right]\left[\left(\frac{1}{r^n}\int_{\Omega_r} |u|^2\right)^{1/2} + k(r)\right]$$
$$+ k_4(|x-y|^{\mu} r^{1-\mu}) + |\varphi(x) - \varphi(y)|, \tag{4.38}$$

for all $x, y \in B_{r/2}$ and $0 < r \leq r_0/2$. Here k_3 and $k_4(r)$ are given by (4.28). and (4.30), and the implicit constants depend on the CDC constant c_0, $C_{|f|,\Omega_{r_0}}$, $C_{|g|^2,\Omega_{r_0}}$, $C_{|b|^2,\Omega_{r_0}}$, $C_{|d|,\Omega_{r_0}}$, λ, Λ, and either C_s and $\|c\|_{L^{n,q}(\Omega_r)}$ or C_s' and $\vartheta_{\Omega_{r_0}}(|c|^2, r)$.

(ii) *Let $|f|, |d| \in \mathcal{K}_{Dini,\delta}(\Omega_{r_0})$, $|b|^2, |g|^2 \in \mathcal{K}_{Dini,\delta/2}(\Omega_{r_0})$ for some $\delta \in (0, 1)$, and either $c \in L^{n,q}(\Omega_{r_0})$, $q \in [n, \infty)$, or $|c|^2 \in \mathcal{K}(\Omega_{r_0})$. For any $0 \leq \rho \leq r/2$, it holds*

$$\operatorname*{osc}_{B_\rho(\xi)\cap\Omega} u \leq \operatorname*{osc}_{\partial\Omega\cap B_\rho(\xi)} \varphi$$
$$+ \exp\left(-\frac{1}{C}\int_{2\rho}^{r} \frac{\operatorname{Cap}(\overline{B}_s(\xi)\setminus\Omega)}{s^{n-2}} \frac{ds}{s}\right)\left(\operatorname*{osc}_{B_r(\xi)\cap\Omega} u + \left(\widetilde{k}(r) + \frac{\widetilde{k}(r_0/2)}{r_0/2} r\right)\right),$$

where $C > 0$ depends on λ, Λ, k_0 as defined in (4.43), $C_{|f|,\Omega_{r_0},\delta}$, $C_{|g|^2,\Omega_{r_0},\delta/2}$, $C_{|b|^2,\Omega_{r_0},\delta/2}$, $C_{|d|,\Omega_{r_0},\delta}$, and either C_s and $\|c\|_{L^{n,q}(\Omega_r)}$ or C_s' and $\vartheta_{\Omega_{r_0}}(|c|^2, r)$.

Proof If we set $B_r = B_r(\xi)$ we record that u is a solution of $Lu = f - \operatorname{div} g$ in $B_r \cap \Omega$ and thus, a solution of (4.35). Using the same notation as above, one can prove that for $\eta \in C_c^\infty(B_r)$,

$$\|\eta \nabla u_m^-\|_{L^2(B_r)} \lesssim \|(\eta + |\nabla \eta|)(u_m^- + \widetilde{k}_\epsilon)\|_{L^2(B_r)}. \tag{4.39}$$

This follows easily by inspection of the proofs of Theorem 3.5 and Lemma 4.1.

We fix η so that $\eta = 1$ in $B_{1/2}$, $0 \le \eta \le 1$ and $|\nabla \eta| \le 2$. If we set $w = \eta(u_m^- + \widetilde{k}_\epsilon(1))$, by (4.39) and (the proof of) (4.18), we deduce that

$$\begin{aligned}\|\nabla w\|^2_{L^2(B_1)} &\lesssim \|(\eta + |\nabla \eta|)(u_m^- + \widetilde{k}_\epsilon(1))\|^2_{L^2(B_1)}\\ &\lesssim (m + \widetilde{k}_\epsilon(1)) \int_{B_1} (u_m^- + \widetilde{k}_\epsilon(1)) \lesssim (m + \widetilde{k}_\epsilon(1))(\inf_{B_{1/2}} u_m^- + \widetilde{k}_\epsilon(1/2)).\end{aligned}$$

If we rescale, the latter inequality is written as

$$r^{2-n} \|\nabla w\|^2_{L^2(B_r)} \lesssim (m + \widetilde{k}_\epsilon(r))(\inf_{B_{r/2}} u_m^- + \widetilde{k}_\epsilon(r/2)).$$

It is easy to see that $\frac{w}{m+\widetilde{k}_\epsilon(r)}$ is a function in the convex set $\mathbb{K}_{\overline{B}_{r/2}\setminus\Omega}$ in the definition of capacity. This observation along with the latter inequality implies that

$$(m + \widetilde{k}_\epsilon(r))^2 \operatorname{Cap}(\overline{B}_{r/2} \setminus \Omega) \lesssim r^{n-2}(m + \widetilde{k}_\epsilon(r))(\inf_{B_{r/2}} u_m^- + \widetilde{k}_\epsilon(r/2)).$$

Therefore, since $\widetilde{k}_\epsilon(r) \ge 0$,

$$m \frac{\operatorname{Cap}(\overline{B}_{r/2} \setminus \Omega)}{(r/2)^{n-2}} \le C(\inf_{B_{r/2}} u_m^- + \widetilde{k}_\epsilon(r/2)). \tag{4.40}$$

If we set

$$\gamma(r/2) = \frac{\operatorname{Cap}(\overline{B}_{r/2} \setminus \Omega)}{C\,(r/2)^{n-2}}, \quad M = \sup_{B_r \cap \partial\Omega} u, \quad \text{and} \quad m = \inf_{B_r \cap \partial\Omega} u,$$

we can apply (4.40) to the functions $M_r - u$ and $u - m_r$ to obtain

$$\begin{aligned}(M_r - M)\gamma(r/2) &\le M_r - M_{r/2} + \widetilde{k}_\epsilon(r/2) = (M_r - M) - (M_{r/2} - M) + \widetilde{k}_\epsilon(r/2),\\ (m - m_r)\gamma(r/2) &\le m_{r/2} - m_r + \widetilde{k}_\epsilon(r/2) = (m - m_r) - (m - m_{r/2}) + \widetilde{k}_\epsilon(r/2).\end{aligned}$$

Set

$$\omega(r) = \operatorname*{osc}_{\Omega \cap B_r} u - \operatorname*{osc}_{\partial\Omega \cap B_r} u,$$

and sum the above inequalities to get

$$\omega(r/2) \le (1 - \gamma(r/2))\omega(r) + 2\widetilde{k}_\epsilon(r/2). \tag{4.41}$$

If $\gamma(r) > c$, for every $r > 0$, we can write (4.41) as $\omega(r/2) \le (1 - c)\omega(r) + 2\widetilde{k}_\epsilon(r/2)$ and take limits as $\epsilon \to 0$. Then, we can repeat the iteration argument in the proof of Theorem 4.12 to show (4.38).

If $\gamma(r)$ is not uniformly bounded from below, then for $m \in \mathbb{N}$, (4.41) can be iterated to obtain

$$\omega(2^{-m}r) \le \prod_{j=1}^{m}(1-\gamma(2^{-j}r))\,\omega(r) + 2\sum_{j=1}^{m}\widetilde{k}_\epsilon(2^{-j}r)\prod_{\ell=j+1}^{m}(1-\gamma(2^{-\ell}r))$$
$$=: \Sigma_1 + \Sigma_2. \tag{4.42}$$

To handle Σ_2 we adjust the argument in [21, pp. 202-203]. Let us define

$$k_0^{\frac{1}{1-\delta}} := \sup_{t\in(0,r_0)} \frac{\widetilde{k}_\epsilon(t)}{\widetilde{k}_\epsilon(2t)} < 1, \tag{4.43}$$

for some $\delta \in (0,1)$, where we used Lemma 2.33 to deduce that $k_0 < 1$. Define also

$$b(r) = \frac{\gamma(r)}{1+\gamma_1}, \quad \text{where} \quad \gamma_1 = (1-k_0)^{-1}\sup_{r\in(0,r_0)}\gamma(r).$$

Since $b(r) \le 1-k_0$ for all $r \in (0,r_0)$, $1-t \le e^{-t}$ and $b(r) \le \gamma(r)$, we have

$$\begin{aligned}
\Sigma_2 &\le 2\prod_{k=1}^{m} e^{-b(2^{-k}r)}\sum_{j=1}^{m}\widetilde{k}_\epsilon(2^{-j}r)\prod_{\ell=1}^{j}(1-b(2^{-\ell}r))^{-1}\\
&= 2\prod_{k=1}^{m} e^{-b(2^{-k}r)}\sum_{j=1}^{m}\widetilde{k}_\epsilon(2^{-j}r)k_0^{-j}\\
&\le \exp\Big(-\sum_{k=1}^{m}b(2^{-k}r)\Big)\sum_{j=1}^{m}\widetilde{k}_\epsilon(2^{-j}r)\prod_{\ell=1}^{j}\left(\frac{\widetilde{k}_\epsilon(2^{-\ell+1}r)}{\widetilde{k}_\epsilon(2^{-\ell}r)}\right)^{1-\delta}\\
&= \widetilde{k}_\epsilon(r)^{1-\delta}\exp\Big(-\sum_{k=1}^{m}b(2^{-k}r)\Big)\sum_{j=1}^{m}\widetilde{k}_\epsilon(2^{-j}r)^{\delta}\\
&\lesssim \widetilde{k}_\epsilon(r)^{1-\delta}\exp\Big(-\sum_{k=1}^{m}b(2^{-k}r)\Big)\widetilde{k}_\epsilon(r/2)^{\delta},
\end{aligned} \tag{4.44}$$

where in the last inequality we used the fact that $|f|, |d| \in \mathcal{K}_{\mathrm{Dini},\delta}(\Omega_{r_0})$ and $|b|^2, |g|^2 \in \mathcal{K}_{\mathrm{Dini},\delta/2}(\Omega_{r_0})$ and the implicit constants depend on the constants of the relevant Carleson-Dini conditions. If we choose $\epsilon = \min(2\widetilde{k}(r_0/2)/r_0, 1)$, the latter quantity is dominated by

$$\left(\widetilde{k}(r) + \frac{2\widetilde{k}(r_0/2)}{r_0}r\right)\exp\Big(-\sum_{k=1}^{m}b(2^{-k}r)\Big). \tag{4.45}$$

Arguing similarly, we get

$$\Sigma_1 \le \exp\Big(-\sum_{k=1}^{m}b(2^{-k}r)\Big)\omega(r). \tag{4.46}$$

Therefore, combining (4.42), (4.44), (4.45) and (4.46), we infer that

$$\omega(2^{-m}r) \le \left(\omega(r) + C\Big(\widetilde{k}(r) + \frac{2\widetilde{k}(r_0/2)}{r_0}r\Big)\right)\exp\Big(-\sum_{k=1}^{m}b(2^{-k}r)\Big). \tag{4.47}$$

It is easy to see that

$$\int_{2^{-m}r}^{r} b(s)\,\frac{ds}{s} \le 2^{n-2}\sum_{j=0}^{m-1} b(2^{-j}r),$$

which can be used in (4.47) along with $K_0\,\gamma(s) := \frac{1-k_0}{1-k_0+c_n}\gamma(s) \le b(s)$ (using the fact that $\mathrm{Cap}_2(\overline{B(\xi,s)}, B(\xi,2s)) = c_n s^{n-2}$ for any $s>0$) to obtain

$$\omega(2^{-m}r) \le \left(\omega(r) + 2\left(\widetilde{k}(r) + \frac{2\widetilde{k}(r_0/2)}{r_0}r\right)\right)\exp\left(-\frac{K_0}{2^{n-2}}\int_{2^{-m}r}^{r}\frac{\mathrm{Cap}(\overline{B}_s\setminus\Omega)}{s^{n-2}}\,\frac{ds}{s}\right). \tag{4.48}$$

For any $\rho \le r \le r_0/2$, there exists $m_0 \in \mathbb{N}$ such that $2^{-m_0-1}r \le \rho < 2^{-m_0}r$. Thus, by (4.48) we deduce that

$$\operatorname*{osc}_{B_\rho\cap\Omega} u \le \operatorname*{osc}_{\partial\Omega\cap B_\rho} u$$
$$+\exp\left(-\frac{K_0}{2^{n-2}}\int_{2\rho}^{r}\frac{\mathrm{Cap}(\overline{B}_s\setminus\Omega)}{s^{n-2}}\,\frac{ds}{s}\right)\left(\operatorname*{osc}_{B_r\cap\Omega} u - \operatorname*{osc}_{\partial\Omega\cap B_r} u + 2\left(\widetilde{k}(r) + \frac{2\widetilde{k}(r_0/2)}{r_0}r\right)\right),$$

which, by (4.34), concludes the proof of Theorem 4.14, since $\mathrm{osc}_{\partial\Omega\cap B_r}\, u \ge 0$ and $u = \varphi$ on $\partial\Omega \cap B_r$ in the Sobolev sense. □

As a corollary of the previous theorem we obtain the following Wiener-type criterion for continuity of solutions up to the boundary as well as a modulus of continuity under the CDC.

Theorem 4.15 (Boundary continuity) *Under the assumptions of Theorem 4.14, if u is the unique solution of 5.2 the following hold:*

(i) *If $\xi \in \partial\Omega$ and $\mathbb{R}^n\setminus\Omega$ is thick at ξ, then $\lim_{\Omega\ni x\to\xi} u(x) = \varphi(\xi)$ continuously.*
(ii) *If φ is continuous with a modulus of continuity and $\partial\Omega$ has the CDC, then u is continuous in $\overline{\Omega}$ with a modulus of continuity depending on the one of φ as well as the Stummel-Kato modulus of continuity of the data and the coefficients in the definition of $\widetilde{k}$.*

5 Dirichlet and obstacle problems in Sobolev space

In this section we will need to assume the following standing (global) assumptions:

$$|b|^2, |c|^2, |d| \in \mathcal{K}'(\Omega) \quad \text{or} \quad b, c \in L^{n,\infty}(\Omega), d \in L^{\frac{n}{2},\infty}(\Omega).$$

5.1 Weak maximum principle

Theorem 5.1 *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and assume that either $b+c \in L^{n,q}(\Omega)$, for $q \in [n,\infty)$, or $b+c \in \mathcal{K}'(\Omega)$. If $u \in Y^{1,2}(\Omega)$ is a subsolution of $Lu = 0$, then the following hold:*

(i) *If (1.5) holds then*

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+.$$

(ii) *If (1.6) holds and* $u^+ \in Y_0^{1,2}(\Omega)$, *then*

$$\sup_{\Omega} u \leq 0. \tag{5.1}$$

Proof Set $\ell = \sup_{\partial\Omega} u^+$ and define $w = (u-\ell)^+ \in Y_0^{1,2}(\Omega)$. We apply Lemma 2.34 to w, for $p = n$, $q \in [n.\infty)$, $h = b + c$, and $a = \lambda/2C_{s,q}$, to find $w_i \in Y_0^{1,2}(\Omega)$ and $\Omega_i \subset \Omega$, $1 \leq i \leq m$, satisfying (1)–(8). In light of (5), as $w \geq 0$, we have that $w_i \in Y_0^{1,2}(\Omega)$ is also non-negative. Recall also that $\nabla w_i = \nabla u$ in Ω_i. We will now proceed as usual. Indeed, using that u is a subsolution along with (1.2), (1.5), (8), and (2.23), we infer

$$\begin{aligned}
\lambda \|\nabla w_i\|^2_{L^2(\Omega)} &\leq \int_\Omega A\nabla w_i \nabla w_i = \int_\Omega A\nabla u \nabla w_i \leq \int_\Omega (b+c)\nabla u w_i \\
&= \sum_{j=1}^{i} \int_\Omega (b+c)\nabla w_j w_i \\
&\leq aC_{s,q}\|\nabla w_i\|^2_{L^2(\Omega)} + aC_{s,q}\|\nabla w_i\|_{L^2(\Omega)} \sum_{j=1}^{i-1} \|\nabla w_j\|_{L^2(\Omega)},
\end{aligned}$$

which implies

$$\|\nabla w_i\|_{L^2(\Omega)} \leq \sum_{j=1}^{i-1} \|\nabla w_j\|_{L^2(\Omega)}.$$

By the induction argument in the proof of Theorem 3.1, we get that for any $i = 1, 2, \dots, \kappa$, $\|\nabla w_i\|_{L^2(\Omega)} = 0$, which we may sum in i and use the condition (6) to obtain $\|\nabla w\|_{L^2(\Omega)} = 0$. Since $w \in Y_0^{1,2}(\Omega)$, by Lemma 2.4, $w = 0$. Therefore, $u \leq \ell$, which concludes the proof of (i).

To prove of (ii), we argue as above for $w = u^+ \in Y_0^{1,2}(\Omega)$ (i.e., $\ell = 0$) and use (1.6) instead of (1.5), to get

$$\begin{aligned}
\lambda \|\nabla w_i\|^2_{L^2(\Omega)} &\leq \int_\Omega (b+c)u\nabla w_i = \sum_{j=i}^{\kappa} \int_\Omega (b+c) w_j \nabla w_i \\
&\leq aC_{s,q}\|\nabla w_i\|^2_{L^2(\Omega)} + aC_{s,q}\|\nabla w_i\|_{L^2(\Omega)} \sum_{j=i+1}^{\kappa} \|\nabla w_j\|_{L^2(\Omega)}.
\end{aligned}$$

Thus,

$$\|\nabla w_i\|_{L^2(\Omega)} \leq \sum_{j=i+1}^{\kappa} \|\nabla w_j\|_{L^2(\Omega)},$$

which, by the induction argument in Theorem 3.2, implies $\|\nabla w\|_{L^2(\Omega)} = 0$, and so, (5.1) readily follows.

The proof when $b + c \in \mathcal{K}'(\Omega)$ is analogous and the required adjustments are the same as in the proof of Theorem 3.1. Details are omitted. □

A direct consequence of the weak maximum principles proved above is the following comparison principle:

Corollary 5.2 *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and assume that either (1.5) or (1.6) holds. Assume also either $b + c \in L^{n,q}(\Omega)$, for $q \in [n, \infty)$, or $b + c \in \mathcal{K}'(\Omega)$. If $u \in Y^{1,2}(\Omega)$ is a supersolution of (3.1) and $v \in Y^{1,2}(\Omega)$ is a subsolution of (3.1) such that $(v-u)^+ \in Y_0^{1,2}(\Omega)$, then we have that*

$$v \leq u \text{ in } \Omega.$$

Proof Since $L(v-u) \leq 0$ and $(v-u)^+ \in Y_0^{1,2}(\Omega)$, we apply Theorem 5.1 (either (i) or (ii)) and obtain

$$\sup_{\Omega}(v-u) \leq 0,$$

which concludes our proof. □

5.2 Dirichlet problem

Let $f : \Omega \to \mathbb{R}$, $g : \Omega \to \mathbb{R}^n$ and $\varphi : \Omega \to \mathbb{R}$, such that $f \in L^{2_*}(\Omega)$, $g \in L^2(\Omega)$, and $\varphi \in Y^{1,2}(\Omega)$. In this section we deal with the *Dirichlet problem*

$$\begin{cases} Lu = f - \operatorname{div} g \\ u - \varphi \in Y_0^{1,2}(\Omega). \end{cases} \tag{5.2}$$

In particular, we show that it is well-posed assuming either (1.5) or (1.6). In fact, if we set $w = u - \varphi$, then, $w \in Y_0^{1,2}(\Omega)$, and (in the weak sense) it holds

$$\begin{aligned} Lw &= Lu - L\varphi \\ &= (f - c\nabla\varphi - d\varphi) - \operatorname{div}(g + A\nabla\varphi + b\varphi) \\ &=: \hat{f} - \operatorname{div}\hat{g}. \end{aligned}$$

Thus, (5.2) is readily reduced to the following inhomogeneous Dirichlet problem with zero boundary data:

$$\begin{cases} Lu = f - \operatorname{div} g \\ u \in Y_0^{1,2}(\Omega). \end{cases} \tag{5.3}$$

Well-posedness of the Dirichlet problem (5.3) with solutions $u \in W_0^{1,2}(\Omega)$ instead of $u \in Y_0^{1,2}(\Omega)$ in unbounded domains was shown in [2, Theorem 1.4] for data $f, g \in L^2(\Omega)$, but with a stronger negativity assumption than $\operatorname{div} b + d \leq 0$. Namely, it was assumed that there exists $\mu < 0$ such that $\operatorname{div} b + d \leq \mu$. This was necessary exactly because they required the solutions to be in $W_0^{1,2}(\Omega)$ as opposed to $Y_0^{1,2}(\Omega)$. It is worth mentioning that (1.6) was not treated at all.

In the following theorem we follow the proof of [2, Theorem 1.4] adjusting the arguments to the weaker negativity assumption $\operatorname{div} b + d \leq 0$ and the Sobolev space $Y_0^{1,2}(\Omega)$. Moreover, our argument works for Lorentz spaces as well as the Stummel-Kato class.

Theorem 5.3 *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and assume that either $b + c \in L^{n,q}(\Omega)$, for $q \in [n, \infty)$, or $|b+c|^2 \in \mathcal{K}'(\Omega)$. If $g_i \in L^2(\Omega)$ for $1 \leq i \leq n$, $f \in L^{2_*}(\Omega)$, and* *either (1.5) or (1.6) holds, then the Dirichlet problem (5.3) has a unique solution* $u \in Y_0^{1,2}(\Omega)$ *satisfying*

$$\|u\|_{Y^{1,2}(\Omega)} \lesssim \|f\|_{L^{2_*}(\Omega)} + \|g\|_{L^2(\Omega)}, \tag{5.4}$$

where the implicit constant depends only on λ, Λ, *and either* $C_{s,q}$ *and* $\|b+c\|_{L^{n,q}(\Omega)}$ *or* C_s' *and* $\vartheta_\Omega(|b+c|^2)$.

Proof To demonstrate that (5.4) holds assuming that such a solution exists, it is enough to repeat the argument in the proof of Theorem 5.1 applying Lemma 2.34 to $u \in Y_0^{1,2}(\Omega)$. The difference is that we should use that u is a solution of (3.1) instead of a subsolution of $Lu = 0$ and thus, we pick up two terms related to the interior data exactly as in the proofs of Theorems 3.1 and 3.2. Similar (but easier) manipulations along with the same induction argument conclude (5.4). We omit the details.

To show that (5.3) has a unique solution it is enough to apply the comparison principle given in Corollary 5.2.

Existence of solutions of (5.3) is also based on (5.4). We first assume that Ω is a bounded domain and solve the variational problem (5.3) in $W_0^{1,2}(\Omega)$ with interior data $f \in L^2(\Omega) \cap L^{2_*}(\Omega)$ and $g \in L^2(\Omega)$.

Let $u \in W_0^{1,2}(\Omega)$ and note that by (1.2) and $\operatorname{div} b + d \leq 0$ we have

$$\mathcal{L}(u,u) = \int_\Omega A\nabla u \nabla u + (b-c)u\nabla u - du^2 \geq \lambda \|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega (b+c)\cdot \nabla u\, u. \tag{5.5}$$

If $(b+c) \in L^{n,q}(\Omega)$, for $\delta > 0$ sufficiently small to be chosen, we can find $\zeta \in L^\infty(\Omega)$ which support has finite Lebesgue measure, such that $\|(b+c)^2 - \zeta\|_{L^{n,q}(\Omega)} < \delta$. Thus, by (2.23),

$$\begin{aligned}\int_\Omega (b+c)\cdot\nabla u\, u &\leq C_{s,q}\|b+c-\zeta\|_{L^{n,q}(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^{2^*}(\Omega)} + \int_\Omega \zeta\cdot\nabla u\, u \\ &\leq \delta C_{s,q}\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega \zeta\cdot\nabla u\, u.\end{aligned} \tag{5.6}$$

If $\varepsilon > 0$ small enough to be chosen, then by (5.5), (5.6), and Young inequality, we infer

$$\mathcal{L}(u,u) \geq (\lambda - \delta C_{s,q} - \frac{\varepsilon}{2})\|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{2\varepsilon}\int_\Omega |\zeta|^2 u^2.$$

We now choose $\varepsilon = \frac{\lambda}{4}$ and $\delta = \frac{\lambda}{4C_{s,q}}$, and obtain

$$\mathcal{L}(u,u) \geq \frac{\lambda}{2}\|\nabla u\|^2_{L^2(\Omega)} - \frac{2\|\zeta\|^2_{L^\infty(\Omega)}}{\lambda}\|u\|^2_{L^2(\Omega)} =: \frac{\lambda}{2}\|\nabla u\|^2_{L^2(\Omega)} - \sigma\|u\|^2_{L^2(\Omega)}. \tag{5.7}$$

If $|b+c|^2 \in \mathcal{K}(\Omega)$, then we apply Cauchy-Schwarz and (2.15),

$$\begin{aligned}\int_\Omega (b+c)\nabla u\, u &\leq \left(\int_\Omega |b+c|^2|u|^2\right)^{1/2}\|\nabla u\|_{L^2(\Omega)} \\ &\leq \varepsilon\|\nabla u\|^2_{L^2(\Omega)} + C_\varepsilon\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\leq 2\varepsilon\|\nabla u\|^2_{L^2(\Omega)} + C_\varepsilon'\|u\|^2_{L^2(\Omega)}.\end{aligned}$$

If we choose $\varepsilon = \frac{\lambda}{4}$, we get

$$\mathcal{L}(u,u) \geq \frac{\lambda}{2}\|\nabla u\|^2_{L^2(\Omega)} - C_\varepsilon'\,\|u\|^2_{L^2(\Omega)} =: \frac{\lambda}{2}\|\nabla u\|^2_{L^2(\Omega)} - \sigma\|u\|^2_{L^2(\Omega)}.$$

Let us denote $H = L^2(\Omega)$, $V = W_0^{1,2}(\Omega)$ and its dual $V^* = W^{-1,2}(\Omega)$ and define

$$L_\sigma w := Lw + \sigma w.$$

By (5.7), its associated bilinear form is clearly coercive and bounded in V. As $f \in H$ and $g \in H$, by Lax-Milgram theorem, there exists a unique solution to the problem

$$\begin{cases} L_\sigma u = f - \mathrm{div} g \\ u \in V. \end{cases} \tag{5.8}$$

and so, L_σ has a bounded inverse $L_\sigma^{-1} : V^* \to V$.

If $J : V \to V^*$ is an embedding given by

$$Jv = \int_\Omega uv, \quad v \in V, \tag{5.9}$$

$I_2 : V \to H$ is the natural embedding and $I_1 : H \to V^*$ is an embedding given also by (5.9), we can write $J = I_1 \circ I_2$. It is clear that J is compact as I_2 is compact and I_1 is continuous.

The interior data naturally induces a linear functional on V by

$$F(v) = \int_\Omega fv + g \cdot \nabla v, \quad \text{for } v \in V,$$

so we wish to solve the equation $Lu = F$. This is is equivalent to $L_\sigma u - \sigma Ju = F$, which in turn, can be written as

$$u - \sigma L_\sigma^{-1} Ju = L_\sigma^{-1} F. \tag{5.10}$$

But $L_\sigma^{-1} J$ is compact as J is compact and L_σ^{-1} is continuous. Thus, by the Fredholm alternative, (5.10) has a unique solution if and only if $w = 0$ is the unique function in V satisfying $w - \sigma L_\sigma^{-1} Jw = 0$ (or else $Lw = 0$). But this readily follows from the weak maximum principle in Theorem 5.1 and thus, a solution of (5.3) exists in bounded domains.

If Ω be an unbounded domain, we can find a sequence of function $f_k \in C_c^\infty(\Omega)$ such that $f_k \to f$ in $L^{2_*}(\Omega)$, and then for $j \in \mathbb{N}$ define

$$\Omega_j := \{x \in \Omega \cap B(0, j) : \mathrm{dist}(x, \partial\Omega) > j^{-1}\}.$$

Since $f_k \in L^2(\Omega) \cap L^{2_*}(\Omega)$ and Ω_j is a bounded open set, by (5.8), there exists $u_{k,j} \in W_0^{1,2}(\Omega_j) = Y_0^{1,2}(\Omega_j)$ such that $Lu_{k,j} = f_k - \mathrm{div} g$ in Ω_j. If we extend $u_{k,j}$ by zero outside Ω_j, by (5.4), we will have

$$\|u_{k,j}\|_{Y^{1,2}(\Omega)} \lesssim \|f_k\|_{L^{2_*}(\Omega)} + \|g\|_{L^2(\Omega)},$$

that is, $u_{k,j}$ is a uniformly bounded sequence in $Y_0^{1,2}(\Omega)$ with bounds independent of j and k. Thus, since $Y_0^{1,2}(\Omega)$ is weakly compact, there exists a subsequence $\{u_{k,j_m}\}_{m \geq 1}$ converging weakly to a function $u_k \in Y_0^{1,2}(\Omega)$. Notice also that if $\varphi \in C_c^\infty(\Omega)$, then for j large enough, it also holds $\varphi \in C_c^\infty(\Omega_j)$. Therefore, since $Lu_{k,j} = f_k - \mathrm{div} g$ in Ω_j for any $j \geq 0$, and $u_{k,j_m} \to u_k$ weakly in $Y_0^{1,2}(\Omega)$ as $m \to \infty$, we obtain

$$\langle f_k, \varphi \rangle + \langle g, \nabla \varphi \rangle = \mathcal{L}(u_{k,j_m}, \varphi) \xrightarrow{m \to \infty} \mathcal{L}(u_k, \varphi), \quad \text{for all } \varphi \in C_c^\infty(\Omega),$$

i.e., $Lu_k = f_k - \mathrm{div} g$ in Ω. In addition, since u_k is the weak limit of u_{k,j_m}, for k large enough, it satisfies

$$\|u_k\|_{Y^{1,2}(\Omega)} \lesssim \|f_k\|_{L^{2_*}(\Omega)} + \|g\|_{L^2(\Omega)} \lesssim \|f\|_{L^{2_*}(\Omega)} + \|g\|_{L^2(\Omega)},$$

with implicit contants independent of k. Once again by the weak compactness of $Y_0^{1,2}(\Omega)$, we can find a subsequence $\{u_{k_m}\}_{m\geq 1}$ converging weakly to a function $u \in Y_0^{1,2}(\Omega)$. Thus, since $Lu_k = f_k - \operatorname{div} g$ in Ω, $u_{k_m} \to u$ weakly in $Y_0^{1,2}(\Omega)$ and $f_{k_m} \to f$ in $L^{2_*}(\Omega)$-norm, we obtain

$$\mathcal{L}(u, \varphi) = \langle f, \varphi \rangle + \langle \nabla g, \varphi \rangle, \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

The proof is now concluded. □

An immediate corollary of the last theorem in light of the considerations at the beginning of this section is the following:

Theorem 5.4 *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and assume that either $b + c \in L^{n,q}(\Omega)$, for $q \in [n, \infty)$, or $|b + c|^2 \in \mathcal{K}'(\Omega)$. If $\varphi \in Y^{1,2}(\Omega)$, $g_i \in L^2(\Omega)$ for $1 \leq i \leq n$, $f \in L^{2_*}(\Omega)$, and either (1.5) or (1.6) holds, then the Dirichlet problem (5.2) has a unique solution $u \in Y^{1,2}(\Omega)$ satisfying*

$$\|u\|_{Y^{1,2}(\Omega)} \lesssim \|\varphi\|_{Y^{1,2}(\Omega)} + \|f\|_{L^{2_*}(\Omega)} + \|g\|_{L^2(\Omega)}, \tag{5.11}$$

with the implicit constant depending only on λ, Λ, and either $C_{s,q}$ and $\|b+c\|_{L^{n,q}(\Omega)}$ or C_s' and $\vartheta_\Omega(|b+c|^2)$.

5.3 Obstacle problem

In this subsection, we let Ω be a bounded and open set, and assume that either (1.5) or (1.6) is satisfied, and also that either $b + c \in L^{n,q}(\Omega)$, for $q \in [n, \infty)$, or $|b + c|^2 \in \mathcal{K}'(\Omega)$ holds.

Definition 5.5 Let $\psi, \phi \in W^{1,2}(\Omega)$ such that $\phi \geq \psi$ on $\partial\Omega$ in the $W^{1,2}$ sense. Let us also define the convex set

$$\mathbb{K} := \{v \in W^{1,2}(\Omega) : v \geq \psi \text{ on } \Omega \text{ in the } W^{1,2} \text{ sense and } v - \phi \in W_0^{1,2}(\Omega)\}.$$

We say that u is a *solution to the obstacle problem* in Ω with obstacle ψ and boundary values ϕ and we write $u \in \mathcal{K}_{\psi,\phi}(\Omega)$, if $u \in \mathbb{K}$ and

$$\mathcal{L}(u, v - u) \geq 0, \text{ for all } v \in \mathbb{K}.$$

This problem can be reduced to the one with zero boundary data as follows: Let us define the convex set

$$\mathbb{K}_0 := \{w \in W_0^{1,2}(\Omega) : w \geq \psi - \phi \text{ on } \Omega \text{ in the } W^{1,2} \text{ sense}\}.$$

Suppose that $u \in \mathcal{K}_{\psi,\phi}(\Omega)$ and write

$$\begin{aligned} u &= u_0 + \phi, \quad \text{for } v_0 \in \mathbb{K}_0 \\ v &= v_0 + \phi, \quad \text{for } v_0 \in \mathbb{K}_0. \end{aligned}$$

Thus,

$$\mathcal{L}(u_0, v_0 - u_0) \geq \langle f, v_0 - u_0 \rangle - \mathcal{L}(\phi, v_0 - u_0),$$

and since $\langle F, \eta \rangle := \langle f, \eta \rangle - \mathcal{L}(\phi, \eta)$, $\eta \in W_0^{1,2}(\Omega)$, defines an element $F \in W^{-1,2}(\Omega)$, it is enough to prove the following theorem:

Theorem 5.6 *Let ψ be measurable such that $\psi \leq 0$ on $\partial\Omega$ in the $W^{1,2}$ sense. Define*

$$\mathbb{K}_\psi := \{w \in W_0^{1,2}(\Omega) : w \geq \psi \text{ in } \Omega \text{ in the } W^{1,2} \text{ sense}\}.$$

Given $F \in W^{-1,2}(\Omega)$, there exists a unique $u \in \mathbb{K}_\psi$ such that

$$\mathcal{L}(u, v - u) \geq \langle F, v - u \rangle, \quad \text{for all } v \in \mathbb{K}_\psi.$$

Moreover, u is the minimal among all $w \in W^{1,2}(\Omega)$ that are supersolutions of $Lw = F$ and satisfy $w \geq \psi$ in Ω and $w \geq 0$ on $\partial\Omega$ in the $W^{1,2}$ sense.

Proof By the weak maximum principle proved in Theorem 5.1, our theorem follows from Theorem 4.27 in [35] and the Corollary right after it. □

An important consequence of this theorem is the following:

Corollary 5.7 *Let $\Omega \subset \mathbb{R}^n$ be an open set (not necessarily bounded). If u and v are supersolutions of $Lw = F$ in Ω, then $\min(u, v)$ is a supersolution of the same equation.*

Proof If Ω is bounded, the proof is a consequence of Theorem 5.6 and can be found in [17, Chapter II, Theorem 6.6]. Let Ω be an unbounded open set and assume that u and v are supersolutions of $Lw = F$ in Ω. Since they are supersolutions of the same equation in any bounded open set $D \subset \Omega$, $\min(u, v)$ is a supersolution in any such D as well. Using a partition of unity, this yields that $\min(u, v)$ is a supersolution in Ω. □

The proof of the following theorem can be found for instance in [17, Chapter II, Theorem 6.9].

Theorem 5.8 *Let u be the unique solution obtained in Theorem 5.6 for $\psi \in W^{1,2}(\Omega)$. Then there exists a non-negative Radon measure so that*

$$Lu = f + \mu, \quad \text{in } \Omega,$$

with

$$\text{supp}(\mu) \subset I := \Omega \setminus \{x \in \Omega : u(x) > \psi(x)\}.$$

In particular,

$$Lu = f \quad \text{in } \{x \in \Omega : u(x) > \psi(x)\}.$$

6 Green's functions in unbounded domains

Here we construct the Green's function associated with an elliptic operator given by (1.1) satisfying either negativity assumption following the approach of Hofmann and Kim [13] along with its variation due to Kang and Kim [15].

6.1 Construction of Green's functions

Before we start, we should mention that the equation formal adjoint operator of L is given by

$$L^t u = -\text{div}(A \cdot \nabla u - cu) + b \cdot \nabla u - du = 0,$$

with corresponding bilinear form

$$\mathcal{L}^t(u,\varphi) = \int_\Omega (A^t\nabla u - cu)\nabla\varphi - (du - b\nabla u)\varphi.$$

Moreover, if $\mathcal{L}$ satisfies (1.5), then its adjoint satisfies (1.6) and vice versa.

In the current section, we will require the following conditions to hold:

$$|b|^2, |c|^2, |d| \in \mathcal{K}'(\Omega) \quad \text{or} \quad b, c \in L^{n,\infty}(\Omega), d \in L^{\frac{n}{2},\infty}(\Omega).$$

Theorem 6.1 *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and L be an operator given by (1.1) so that (1.6) holds. For a fixed $y \in \Omega$, there exists the Green's function $G(x, y) \geq 0$ for a.e. $x \in \Omega \setminus \{y\}$ with the following properties:*

(1) $G(\cdot, y) \in Y^{1,2}(\Omega \setminus B_r(y))$ for all $r > 0$ and vanishes on $\partial\Omega$.

(2) If $f \in L^{\frac{n}{2},1}(\Omega)$ and $g \in L^{n,1}(\Omega)$, we have that

$$u(x) = \int_\Omega G(y,x)\, f(y)\, dy + \int_\Omega \nabla_y G(y,x)\, g(y)\, dy, \tag{6.1}$$

is a solution of $L^t u = f - \operatorname{div} g$ in Ω and $u \in Y_0^{1,2}(\Omega)$ satisfying $\|u\|_{L^\infty(\Omega)} \lesssim \|f\|_{L^{\frac{n}{2},1}(\Omega)} + \|g\|_{L^{n,1}(\Omega)}$.

(3) For any other Green's function $\widehat{G}(x, y)$ satisfying (3), it holds $G(x, y) = \widehat{G}(x, y)$ for a.e. $x \in \Omega\setminus\{y\}$.

(4) $G(\cdot, y) \in W^{1,1}_{loc}(\Omega)$ and for any $\eta_y \in C_c^\infty(B_r(y))$ such that $\eta_y = 1$ in $B_{r/2}(y)$, for $r > 0$, it holds that

$$\mathcal{L}(G(\cdot,y), (1-\eta_y)\varphi) = 0, \quad \text{for any } \varphi \in C_c^\infty(\Omega). \tag{6.2}$$

If we set $d_y = \operatorname{dist}(y, \partial\Omega)$ ($d_y = \infty$ if $\Omega = \mathbb{R}^n$), the following bounds are satisfied:

$$\|G(\cdot,y)\|_{Y^{1,2}(\Omega\setminus B_r(y))} \lesssim r^{1-\frac{n}{2}}, \quad \text{for any } r > 0, \tag{6.3}$$

$$\|G(\cdot,y)\|_{L^p(B_r(y))} \lesssim_p r^{2-n+\frac{n}{p}}, \quad \text{for all } r < d_y \text{ and } p \in \Big[1, \frac{n}{n-2}\Big), \tag{6.4}$$

$$\|\nabla G(\cdot,y)\|_{L^p(B_r(y))} \lesssim_p r^{1-n+\frac{n}{p}}, \quad \text{for all } r < d_y, \text{ and } p \in \Big[1, \frac{n}{n-1}\Big), \tag{6.5}$$

$$|\{x \in \Omega : G(x,y) > t\}| \lesssim t^{-\frac{n}{n-2}}, \quad \text{for all } t > 0, \tag{6.6}$$

$$|\{x \in \Omega : \nabla_x G(x,y) > t\}| \lesssim t^{-\frac{n}{n-1}}, \quad \text{for all } t > 0, \tag{6.7}$$

The implicit constants depend only on λ, Λ, and either $C_{s,q}$ and $\|b + c\|_{L^{n,q}(\Omega)}$, or C'_s and $\vartheta_\Omega(|b + c|^2)$. If we also assume $|b + c|^2 \in \mathcal{K}_{Dini}(\Omega)$, then

$$G(x,y) \lesssim \frac{1}{|x-y|^{n-2}}, \quad \text{for all } x \in \Omega \setminus \{y\}. \tag{6.8}$$

where the implicit constant depends also on $C_{|b+c|^2,\Omega}$.

If $|b+c|^2 \in \mathcal{K}_{Dini}(\Omega)$, we can construct the Green's function $G^t(x, y)$ associated with the operator L^t which is non-negative for a.e. $x \in \Omega \setminus \{y\}$ and satisfies the analogous properties (1)–(4) and the bounds (6.3)–(6.8). The implicit constants depend on λ, Λ, C'_s and $C_{|b+c|^2,\Omega}$, and, in the pointwise bounds, on $\|b + c\|_{L^{n,q}(\Omega)}$, or C'_s and $\vartheta_\Omega(|b + c|^2)$ as well. Moreover, if $b, c \in L^{n,q}(\Omega)$, $d \in L^{\frac{n}{2},q}(\Omega)$, for $q \in [n, \infty)$, or $|b|^2, |c|^2, |d| \in \mathcal{K}'(\Omega)$, it holds that

$$G^t(x,y) = G(y,x), \quad \text{for a.e. } (x,y) \in \Omega^2 \setminus \{x \neq y\}, \tag{6.9}$$

and

$$u(x) = \int_\Omega G^t(x, y)\, f(y)\, dy + \int_\Omega \nabla_y G^t(x, y)\, g(y)\, dy, \ \textit{for all}\, x \in \Omega. \tag{6.10}$$

Proof Given a point $y \in \Omega$, if $\Omega_\rho(y) = \Omega \cap B_\rho(y)$, we define

$$f_\rho(x, y) = |B_\rho(y)|^{-1} \mathbf{1}_{\Omega_\rho(y)}(x), \quad x \in \Omega.$$

Since L satisfies (1.6) and $f_\rho(\cdot, y) \in L^\infty(\Omega)$ with bounded support, we may apply Theorem 5.3 (ii) to find a function $G_\rho(\cdot, y) \in Y_0^{1,2}(\Omega)$ so that

$$\mathcal{L}(G_\rho(\cdot, y), \varphi) = \int f_\rho(\cdot, y)\, \varphi, \tag{6.11}$$

for any $\varphi \in C_c^\infty(\Omega)$, with global bounds

$$\|G_\rho(\cdot, y)\|_{Y^{1,2}(\Omega)} \lesssim |B_\rho(y)|^{\frac{2-n}{2n}}. \tag{6.12}$$

Note that $G_\rho(\cdot, y) \in Y_0^{1,2}(\Omega)$ and is an L-supersolution. If we apply the maximum principle given in Theorem 5.1 (ii),we get that $G_\rho(\cdot, y) \geq 0$ in Ω.

Let now $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Omega)$ so that $|\,\mathrm{supp}(f)| + |\,\mathrm{supp}(g)| < \infty$. Then, by Theorem 5.3, there exists $u \in Y_0^{1,2}(\Omega)$ such that

$$\mathcal{L}^t(u, \psi) = \int f\psi + \int g\nabla\psi \quad \text{for all } \psi \in C_c^\infty(\Omega), \tag{6.13}$$

satisfying

$$\begin{aligned}\|u\|_{Y^{1,2}(\Omega)} &\lesssim \|f\|_{L^{2_*}(\Omega)} + \|g\|_{L^2(\Omega)} \\ &\leq |\,\mathrm{supp}(f)|^{\frac{n+2}{2n}} \|f\|_{L^\infty(\Omega)} + |\,\mathrm{supp}(g)|^{\frac{1}{2}} \|g\|_{L^\infty(\Omega)}.\end{aligned} \tag{6.14}$$

Remark here that, by the density of $C_c^\infty(\Omega)$ in $Y_0^{1,2}(\Omega)$, both (6.11) and (6.13) can be extended to test functions $\varphi \in Y_0^{1,2}(\Omega)$. So, if we set $\varphi = u$ in (6.11) and $\psi = G_\rho(\cdot, y)$ in (6.13), we obtain that

$$\int G_\rho(x, y) f(x)\, dx + \int \nabla_x G_\rho(x, y) g(x)\, dx = \fint_{\Omega_\rho(y)} u(x)\, dx. \tag{6.15}$$

For $r > 0$ fixed, assume that $\mathrm{supp}(f) \subset \Omega_r(y)$, $g = 0$, and let $\rho < r/2$. Since u_f is in $Y^{1,2}(\Omega_r(y))$, vanishes on $B_r(y) \cap \partial\Omega$, and satisfies $L^t u_f = f$ in $\Omega_r(y)$, by Theorem 4.4 (1) with $M = 0$, we obtain

$$\|u_f\|_{L^\infty(\Omega_{\frac{r}{2}}(y))} \lesssim r^{-\frac{n}{2}} \|u_f\|_{L^2(\Omega_r(y))} + r^2 \|f\|_{L^\infty(\Omega_r(y))} \lesssim r^2 \|f\|_{L^\infty(\Omega_r(y))},$$

where in the penultimate inequality we used Hölder inequality and (6.14). Similarly, if $f = 0$, $\mathrm{supp}(g) \subset \Omega_r(y)$, and $\rho < r/2$, since $u_g \in Y^{1,2}(\Omega_r(y))$ that vanishes on $B_r(y) \cap \partial\Omega$ and $L^t u_g = -\mathrm{div} g$ in $\Omega_r(y)$,

$$\|u_g\|_{L^\infty(\Omega_{\frac{r}{2}}(y))} \lesssim r^{-\frac{n}{2}} \|u_g\|_{L^2(\Omega_r(y))} + r \|g\|_{L^\infty(\Omega_r(y))} \lesssim r \|g\|_{L^\infty(\Omega_r(y))}.$$

By (6.15), duality considerations, and the latter two estimates, we have that for all $r > 0$ and $\rho < r/2$,

$$\begin{aligned}\|G_\rho(\cdot, y)\|_{L^1(\Omega_r(y))} &\lesssim r^2, \\ \|\nabla G_\rho(\cdot, y)\|_{L^1(\Omega_r(y))} &\lesssim r.\end{aligned} \tag{6.16}$$

In fact, arguing similarly, we can prove that for all $r > 0$, $\rho < r/2$, and $q \in [1, \frac{n}{n-2})$,

$$\|G_\rho(\cdot, y)\|_{L^q(\Omega_r(y))} \lesssim r^{2-n+\frac{n}{q}},$$
$$\|\nabla G_\rho(\cdot, y)\|_{L^q(\Omega_r(y))} \lesssim r^{1-n+\frac{n}{q}}.$$

To avoid an early use of the pointwise bounds and thus, of the assumption $|b + c|^2 \in \mathcal{K}_{\text{Dini}}(\Omega)$, we will need the following auxiliary lemma.

Lemma 6.2 *Let $\Omega \subset \mathbb{R}^n$ be an open set and L be the operator given by (1.1) that satisfies either (1.5) or (1.6). Let $B_s = B(x, s)$ be a ball of radius s centered at $x \in \Omega$ such that $3B_s \subset \Omega$ and $u \in Y^{1,2}(\Omega\backslash B_s)$ be a solution of $Lu = 0$ in $\Omega\backslash B_s$ that vanishes on $\partial\Omega$. Then for any $r \geq 4\,s$ we have*

$$\int_{\Omega\cap(B_{2r}\backslash B_{r/3})} |u|^2 \lesssim \frac{1}{r^n}\left(\int_{\Omega\cap(B_{3r}\backslash B_{r/4})} |u|\right)^2, \tag{6.17}$$

where the implicit constants depend only on λ, Λ, $\|b + c\|_{L^n(\Omega;\, \mathbb{R}^n)}$, and $C_{s,q}$.

Proof The proof can be found in [16, Lemma 3.19] with the difference that we use Theorems 3.3 instead of [16, Lemma 3.18] that only holds for $r \leq 1$. □

For fixed $r > 0$ and $\rho \in (0, r/6)$ we let $\eta \in C^\infty(\mathbb{R}^n)$ so that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } \mathbb{R}^n \setminus B_r(y), \quad \eta \equiv 0 \text{ on } B_{r/2}(y), \quad \text{and} \quad |\nabla\eta| \leq \frac{4}{r}.$$

Thus, by Theorem 3.3, since $LG_\rho(\cdot, y) = 0$, in $\Omega\backslash B_{r/2}(y)$,

$$\begin{aligned}\|\nabla G_\rho(\cdot, y)\|^2_{L^2(\Omega\backslash B_r(y))} &\leq \int_\Omega |\eta\nabla G_\rho(\cdot, y)|^2 \overset{(3.17)}{\lesssim} \int_\Omega |G_\rho(\cdot, y)\nabla\eta|^2 \\ &\lesssim \frac{1}{r^2}\int_{\Omega\cap(B_r(y)\backslash B_{r/2}(y))} G_\rho(\cdot, y)^2 \\ &\overset{(6.17)}{\lesssim} \frac{1}{r^{n+2}}\left(\int_{\Omega\cap(B_{2r}(y)\backslash B_{r/4}(y))} G_\rho(\cdot, y)\right)^2 \overset{(6.16)}{\lesssim} r^{2-n},\end{aligned} \tag{6.18}$$

which, in turn, by Sobolev embedding theorem, implies that for $0 < \rho < r/6$,

$$\|G_\rho(\cdot, y)\|_{L^{2^*}(\Omega\backslash B_r(y))} \leq \|G_\rho(\cdot, y)\eta\|_{L^{2^*}(\Omega)} \lesssim \|\nabla(G_\rho(\cdot, y)\eta)\|_{L^{2^*}(\Omega)} \lesssim r^{1-\frac{n}{2}}. \tag{6.19}$$

On the other hand, for $\rho \geq r/6$, by (6.12), we have that

$$\|G_\rho(\cdot, y)\|_{Y^{1,2}(\Omega\backslash B_r(y))} \leq \|G_\rho(\cdot, y)\|_{Y^{1,2}(\Omega)} \lesssim |B_{\rho/6}(y)|^{\frac{2-n}{n}} \lesssim r^{2-n}. \tag{6.20}$$

Therefore, if we apply (6.18), (6.19), and (6.20), we obtain that for any $r > 0$, there exists a constant $C(r)$ depending on r so that

$$\|G_\rho(\cdot, y)\|_{Y^{1,2}(\Omega\backslash B_r(y))} \leq C(r),$$

uniformly in $\rho > 0$. So, by a diagonalization argument and weak compactness of $Y_0^{1,2}$, there exists a sequence $\{\rho_m\}_{m=1}^\infty$ that converges to zero as $m \to \infty$ such that for all $r > 0$,

$$G_{\rho_m}(\cdot, y) \rightharpoonup G(\cdot, y) \text{ in } Y_0^{1,2}(\Omega \setminus B_r(y)), \quad \text{as } m \to \infty, \tag{6.21}$$

where $G(\cdot, y) \in Y_0^{1,2}(\Omega \setminus B_r(y))$. Moreover, by (6.20),

$$\|G(\cdot, y)\|_{Y^{1,2}(\Omega\setminus B_r(y))} \lesssim r^{2-n}, \quad \text{for all } r > 0.$$

If we follow the proof of inequalities (3.21) and (3.23) in [13] using the the same considerations that lead to the proof of the estimates for $G_\rho(\cdot, y)$ away from the pole, we can show that

$$|\{x \in \Omega : G_\rho(x, y) > s\}| \lesssim s^{-\frac{n}{n-2}}, \quad \text{for all } s > 0, \tag{6.22}$$

$$|\{x \in \Omega : \nabla_x G_\rho(x, y) > s\}| \lesssim s^{-\frac{n}{n-1}}, \quad \text{for all } s > 0, \tag{6.23}$$

uniformly in $\rho > 0$. This yields that $G_\rho(\cdot, y) \in L^{\frac{n}{n-2},\infty}(\Omega)$ and $\nabla G_\rho(\cdot, y) \in L^{\frac{n}{n-1},\infty}(\Omega)$ with bounds independent of ρ.

Moreover, in light of (6.22) and (6.23), we can mimic the proof of inequalities (3.24) and (3.26) in [13] and infer that for any $\rho > 0$ and $r < d_y$,

$$\|G_\rho(\cdot, y)\|_{L^p(B_r(y))} \lesssim r^{2-n+\frac{n}{p}}, \quad p \in (0, \tfrac{n}{n-2}),$$

$$\|\nabla G_\rho(\cdot, y)\|_{L^p(B_r(y))} \lesssim r^{1-n+\frac{n}{p}}, \quad p \in (0, \tfrac{n}{n-1}).$$

In particular,

$$\|G_\rho(\cdot, y)\|_{W^{1,p}(B_r(y))} \le C(r, p), \quad r < d_y, \;\; p \in [1, \tfrac{n}{n-1}),$$

uniformly in $\rho > 0$. Thus, fixing $p \in (1, \frac{n}{n-1})$, by a diagonalization argument, we can find a subsequence of ρ_m in (6.21) (which we still denote by ρ_m for simplicity) so that

$$G_{\rho_m}(\cdot, y) \rightharpoonup \tilde{G}(\cdot, y) \text{ in } W^{1,p}(B_r(y)) \quad \text{as } m \to \infty, \tag{6.24}$$

for all $r < d_y$. We also have that $\tilde{G}(\cdot, y)$ satisfies (6.4) and (6.5) for this particular p. Since $G(\cdot, y) = \tilde{G}(\cdot, y)$ in $B(y, d_y)\setminus B(y, d_y/2)$, we can extend $\tilde{G}(\cdot, y)$ by $G(\cdot, y)$ to the entire Ω by setting $G(\cdot, y) = \tilde{G}(\cdot, y)$.

Let $\Omega_t = \{x \in \Omega : G(x, y) > t\}$, $p = \frac{n}{n-2}$, $\varepsilon \in (0, p-1)$. If we apply Chebyshev inequality, and then use that the L^p-norms are weakly lower semicontinuous and $|\Omega_t| < \infty$, by (6.3) and (6.4), we have

$$t^{p-\varepsilon}|\Omega_t| \lesssim \|G(\cdot, y)\|^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega_t)} \le \liminf_{m\to\infty} \|G_{\rho_m}(\cdot, y)\|^{p-\varepsilon}_{L^{p-\varepsilon}(\Omega_t)}$$

$$\le \liminf_{m\to\infty} \frac{p}{\varepsilon}|\Omega_t|^{\frac{\varepsilon}{p}} \|G_{\rho_m}(\cdot, y)\|^{p-\varepsilon}_{L^{p,\infty}(\Omega)} \overset{(6.22)}{\le} \frac{p}{\varepsilon}|\Omega_t|^{\frac{\varepsilon}{p}} C^{p-\varepsilon}.$$

Letting $\varepsilon \to p-1$, we get $|\Omega_t|^{\frac{1}{p}} \lesssim 1$ which proves (6.6). A similar reasoning proves (6.7). Moreover,

$$G_{\rho_m}(\cdot, y) \overset{*}{\rightharpoonup} G(\cdot, y) \text{ in } L^{\frac{n}{n-2},\infty}(\Omega) \quad \text{as } m \to \infty, \tag{6.25}$$

$$\nabla G_{\rho_m}(\cdot, y) \overset{*}{\rightharpoonup} \nabla G(\cdot, y) \text{ in } L^{\frac{n}{n-1},\infty}(\Omega) \quad \text{as } m \to \infty. \tag{6.26}$$

Therefore, by (6.11) and (6.15), in view of (6.25), (6.26), and (6.21), we can prove (6.2) and also, (6.1) for $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Omega)$ so that $|\operatorname{supp}(f)| + |\operatorname{supp}(g)| < \infty$ (a detailed but more involved argument can be found after equation (6.33)). To show that (6.1) holds in general, it is enough to use that simple functions are dense in $L^{p,q}(\Omega)$ if $q \neq \infty$ along with (6.6) and (6.7). Details are left to the reader.

The proof of inequalities (3.30) and (3.31) in [13] gives us (6.4) and (6.5) for any p (in the stated range).

We will now demonstrate that for a fixed $y \in \Omega$, $G(\cdot, y) \geq 0$ a.e. in $\Omega \backslash \{y\}$. Assume that σ_n is the sequence converging to zero for which $G_{\sigma_n}(\cdot, y)$ converge to $G(\cdot, y)$ in the sense of (6.21) and (6.24). If necessary, we can pass to a subsequence so that $\sigma_n < \min(|x-y|, d_y)/10$. Fix $x \in \Omega$ so that $x \neq y$ and let ρ_m be a sequence converging to zero so that $\rho_m \leq \min(|x-y|, d_x)/10$. Therefore, since $G_{\sigma_n}(\cdot, y) \geq 0$ in Ω, we have that

$$0 \leq \fint_{B_{\rho_m}(x)} G_{\sigma_n}(\cdot, y) \longrightarrow \fint_{B_{\rho_m}(x)} G(\cdot, y), \text{ as } n \to \infty,$$

where we used (6.21) in the case $B_{\rho_m}(y) \subset \Omega \backslash B_r(x)$ for some $r > 0$ and (6.24) in the case $B_{\rho_m}(x) \cap B_{\sigma_n}(y) \neq \emptyset$. By Lebesgue differentiation theorem, if we let $m \to \infty$, we infer that $G(x, y) \geq 0$ for a.e. $x \in \Omega \setminus \{y\}$.

To prove uniqueness of the Green's function, we assume that $\widehat{G}(\cdot, y)$ is another Green's function for the same operator. Then for $f \in C_c^\infty(\Omega)$ and $g = 0$, we have that for fixed $y \in \Omega$,

$$\int_\Omega \widehat{G}(\cdot, y)\, f = \widehat{u}(y) \in Y_0^{1,2}(\Omega) \quad \text{and} \quad L^t \widehat{u} = f.$$

By the comparison principle Corollary 5.2, $u = \widehat{u}$ in Ω and so,

$$\int_\Omega G(\cdot, y)\, f = \int_\Omega \widehat{G}(\cdot, y)\, f.$$

Since $f \in C_c^\infty(\Omega)$ is arbitrary, this readily implies that $G(x, y) = \widehat{G}(x, y)$ for a.e. $x \in \Omega \backslash \{y\}$.

So far, we have not used the local boundedness of solutions of $L^t u = 0$ and thus, the assumption $|b+c|^2 \in \mathcal{K}_{\text{Dini}}(\Omega)$. It is only for the pointwise bounds we will need it. Indeed, let $x, y \in \Omega$, $x \neq y$ and set $r = |x-y|/4$. Then, (6.2) yields that $LG(\cdot, y) = 0$ away from y. So, by Theorem 4.4 and (6.3) for $p = 2$, we obtain

$$\begin{aligned} |G(x, y)| \leq \sup_{\Omega_r(x)} |G(\cdot, y)| &\lesssim r^{-n/2} \|G(\cdot, y)\|_{L^2(\Omega_r(x))} \\ &\lesssim r^{-n/2} r^{2-n/2} \approx |x-y|^{2-n}. \end{aligned} \tag{6.27}$$

Notice that, under the additional assumption $|b+c|^2 \in \mathcal{K}_{\text{Dini}}(\Omega)$, we can apply the previous considerations to construct the Green's function $G^t(\cdot, y)$ associated with the operator L^t with all the properties above. The only thing that remains to be shown is that $G^t(x, y) = G(y, x)$ for a.e. $(x, y) \in \Omega^2 \backslash \{x = y\}$. We will first prove it in the case that solutions of $Lu = 0$ and $L^t u = 0$ are locally Hölder continuous in $\Omega \backslash \{x\}$ and $\Omega \backslash \{y\}$ respectively. In this case, all the properties that hold a.e. in $\Omega \backslash \{\text{pole}\}$, because of the continuity therein, will actually hold everywhere in $\Omega \setminus \{\text{pole}\}$.

To this end, let σ_n and ρ_m be the sequences converging to zero for which $G_{\sigma_n}(\cdot, x)$ and $G^t_{\rho_m}(\cdot, y)$ converge to $G(\cdot, x)$ and $G^t(\cdot, y)$ in the sense of (6.21), (6.24), and (6.25). If necessary, we may further pass to subsequences so that

$$\sigma_n < \min(|x-y|, d_x)/10 \quad \text{and} \quad \rho_m \leq \min(|x-y|, d_y)/10.$$

Because $G_{\sigma_n}(\cdot, x)$ and $G^t_{\rho_m}(\cdot, y)$ are locally Hölder continuous in $\Omega \backslash \{x\}$ and $\Omega \backslash \{y\}$ respectively, with constants uniform in σ_n and ρ_m and, by Theorem 4.4, they are uniformly bounded on compact subsets of the respective domains, we may pass to subsequences so that

$$G_{\sigma_n}(\cdot, x) \to G(\cdot, x) \text{ unifomly on compact subsets of } \Omega \setminus \{x\}, \quad (6.28)$$
$$G^t_{\rho_m}(\cdot, y) \to G^t(\cdot, y) \text{ unifomly on compact subsets of } \Omega \setminus \{y\}.$$

We now use $G^t_{\rho_m}(\cdot, y)$ and $G_{\sigma_n}(\cdot, x)$ as test functions in their very definitions to obtain

$$\fint_{B_{\sigma_n}(x)} G^t_{\rho_m}(\cdot, y) = \mathcal{L}(G_{\sigma_n}(\cdot, x), G^t_{\rho_m}(\cdot, y))$$
$$= \mathcal{L}^t(G^t_{\rho_m}(\cdot, y), G_{\sigma_n}(\cdot, x)) = \fint_{B_{\rho_m}(y)} G_{\sigma_n}(\cdot, x).$$

By Lebesgue's differentiation theorem and continuity of $G_{\sigma_n}(\cdot, x)$ in $\Omega\backslash\{x\}$,

$$\lim_{m\to\infty} \fint_{B_{\rho_m}(y)} G_{\sigma_n}(\cdot, x) = G_{\sigma_n}(y, x),$$

which, in view of (6.28), yields that

$$\lim_{n\to\infty} \lim_{m\to\infty} \fint_{B_{\rho_m}(y)} G_{\sigma_n}(\cdot, x) = G(y, x) \quad \text{for all } y \in \Omega \setminus \{x\}.$$

On the other hand, the weak convergence of $G^t_{\rho_m}(\cdot, y)$ in $Y^{1,2}(\Omega\backslash B_r(y))$ for any $r > 0$ implies

$$\lim_{m\to\infty} \fint_{B_{\sigma_n}(x)} G^t_{\rho_m}(\cdot, y) = \fint_{B_{\sigma_n}(x)} G^t(\cdot, y),$$

from which, by Lebesgue differentiation theorem and the continuity of $G^t(\cdot, y)$ in $\Omega\backslash\{y\}$, we deduce that

$$\lim_{n\to\infty} \lim_{m\to\infty} \fint_{B_{\sigma_n}(x)} G^t_{\rho_m}(\cdot, y) = G^t(x, y) \quad \text{for all } x \in \Omega \setminus \{y\}.$$

Therefore, $G(x, y) = G^t(y, x)$ for all $(x, y) \in \Omega^2\backslash\{x = y\}$, which, combined with (6.1), implies (6.10).

We are now ready to remove the Hölder continuity assumption. Set

$$\Omega_k = \{x \in \Omega : d(x, \partial\Omega) > k^{-1}\} \cap B(0, k),$$

which are open sets such that $\cup_{k\geq 1}\Omega_k = \Omega$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ so that

$$0 \leq \psi \leq 1, \;\; \psi = 0 \text{ in } \mathbb{R}^n \setminus B(0, 1) \text{ and } \int \psi = 1.$$

For $k \in \mathbb{N}$, set $\psi_k(x) = k^n \psi(kx)$ and define $b_k = (b\,\mathbf{1}_{\Omega_k}) * \psi_k$, $c_k = (c\,\mathbf{1}_{\Omega_k}) * \psi_k$ and $d_k = (d\,\mathbf{1}_{\Omega_k}) * \psi_k$.

Define

$$L_k u = -\text{div} A\nabla u - \text{div}(b_k u) - c_k \nabla u - d_k u.$$

If we fix $x \neq y \in \Omega$, there exists k_0 large enough such that $x, y \in \Omega_k$ for every $k \geq k_0$ and in particular, x and y are in the same connected component of Ω_k. Therefore, Remark 4.7 applies, and since, for such k, Theorem 4.4 holds for L_k in Ω_k with bounds independent of k, we can construct the Green's functions $G_k(\cdot, y)$ and $G^t_k(\cdot, x)$ associated with L_k and L^t_k in Ω_k as above, with the additional property that $G_k(\cdot, x)$ and $G^t_k(\cdot, y)$ are locally Hölder continuous away from x and y respectively. In the last part we used Theorem 4.12, which applies in this situation, since $b_k, c_k, d_k \in L^\infty$ with compact support and thus, $|b_k|^2, |c_k|^2, |d_k| \in \mathcal{K}_{\text{Dini}}(\Omega_k)$ (with implicit constants depending in the domain). Extend both $G_k(\cdot, x)$ and $G^t_k(\cdot, y)$ by zero outside Ω_k and note that (6.3)-(6.7) hold in Ω with constants independent of k (see Remark 4.7). Therefore, repeating essentially the arguments concerning the convergence of G_ρ and the inheritance of the bounds from G_ρ, we can find $G(\cdot, y)$ which is non-negative a.e. in $\Omega \setminus \{y\}$ and vanishes on $\partial\Omega$. Additionally, it satisfies (6.3)-(6.7), and, after passing to a subsequence,

$$
\begin{aligned}
G_k(\cdot, y) &\rightharpoonup G(\cdot, y) \text{ in } Y^{1,2}(\Omega \setminus B_r(y)) \text{ for all } r > 0,\\
G_k(\cdot, y) &\rightharpoonup G(\cdot, y) \text{ in } W^{1,p}(B_r(y)), \text{ for all } r < d_y,\\
G_k(\cdot, y) &\overset{*}{\rightharpoonup} G(\cdot, y) \text{ in } L^{\frac{n}{n-2},\infty}(\Omega), \qquad (6.29)\\
\nabla G_k(\cdot, y) &\overset{*}{\rightharpoonup} \nabla G(\cdot, y) \text{ in } L^{\frac{n}{n-1},\infty}(\Omega), \qquad (6.30)\\
G_k(\cdot, y) &\to G(\cdot, y) \text{ a.e. in } \Omega. \qquad (6.31)
\end{aligned}
$$

The considerations above apply to G^t_k as well.

Let $f \in L^\infty(\Omega)$ and $g \in L^\infty_\Omega)$ which supports have finite Lebesgue measure. Thus, by virtue of (6.1), we have that

$$
u_k(y) = \int_\Omega G_k(\cdot, y)\, f + \int_\Omega \nabla G_k(\cdot, y)\, g. \tag{6.32}
$$

Since $u_k \in Y^{1,2}_0(\Omega_k)$, we can extend it by 0 outside Ω_k. Recall that u_k satisfies $L^t_k u_k = f - \operatorname{div} g$ in Ω_k and also

$$
\|u_k\|_{Y^{1,2}(\Omega)} = \|u_k\|_{Y^{1,2}(\Omega_k)} \lesssim \|f\|_{L^{2_*}(\Omega_k)} + \|g\|_{L^2(\Omega_k)} \leq \|f\|_{L^{2_*}(\Omega)} + \|g\|_{L^2(\Omega)},
$$

where the implicit constant is independent of k. If we take limits in (6.32) as $k \to \infty$ and use (6.29) and (6.30) for $G^t_k(\cdot, y)$, we can show that for all $y \in \Omega$,

$$
\begin{aligned}
\lim_{k\to\infty} u_k(y) &= \lim_{k\to\infty} \int_\Omega G_k(x, y)\, f(x)\, dx + \lim_{k\to\infty} \int_\Omega \nabla G_k(x, y)\, g(x)\, dx\\
&= \int_\Omega G(x, y)\, f(x)\, dx + \int_\Omega \nabla G(x, y)\, g(x)\, dx =: u(y).
\end{aligned}
$$

Therefore, since $u_k \to u$ pointwisely in Ω and u_k is a uniformly bounded sequence in $Y^{1,2}_0(\Omega)$, it holds that $u_k \rightharpoonup u$ in $Y^{1,2}(\Omega)$ and $u \in Y^{1,2}_0(\Omega)$. For a proof see for instance [12, Theorem 1.32]. We will show that u is the unique solution of the Dirichlet problem $L^t u = f$ and $u \in Y^{1,2}_0(\Omega)$. If $\varphi \in C^\infty_c(\Omega)$, there exists $k_1 \geq k_0$ such that $\varphi \in C^\infty_c(\Omega_k)$ for every $k \geq k_1$. Thus,

$$
\mathcal{L}^t_{k,\Omega}(u_k, \varphi) = \mathcal{L}^t_{k,\Omega_k}(u_k, \varphi) = \int_{\Omega_k} f\varphi + \int_{\Omega_k} g\nabla\varphi = \int_\Omega f\varphi + \int_\Omega g\nabla\varphi.
$$

To pass to the limit, we need to treat each of the terms of the bilinear form separately. We first write

$$
\int_\Omega b_k \nabla u_k \phi = \int_\Omega (b_k - b)\nabla u_k \phi + \int_\Omega b \nabla u_k \phi = I^k_{b,1} + I^k_{b,2}.
$$

If $b \in L^{n,q}(\Omega)$, by Lemma 2.26 we have that $b_k \to b$ in $L^{n,q}(\Omega)$, which, combined with (2.23) and the uniform $Y^{1,2}$-bound of u_k, yields that $\lim_{k\to\infty} I^k_{b,1} = 0$. To prove that

$$\lim_{k\to\infty} I^k_{b,2} = \int_\Omega b\nabla u\phi, \tag{6.33}$$

it is enough to notice that, by Hölder inequality in Lorentz spaces and Lemma 2.30, $b\phi \in L^2(\Omega)$, and then use that $\nabla u_k \rightharpoonup \nabla u$ in $L^2(\Omega)$. If $|b|^2 \in \mathcal{K}'(\Omega)$, we combine Cauchy-Schwarz inequality, Lemma 2.21, the uniform $Y^{1,2}$-bound of u_k, and Lemma 2.14, to show $I^k_{b,1} \to 0$. By (2.17), we have that $b\phi \in L^2(\Omega)$, and thus, (6.33) follows from the weak-L^2 convergence of ∇u_k to ∇u. Let us now prove the limit for the one involving d_k. To this end, write

$$\int_\Omega d_k u_k \phi = \int_\Omega (d_k - d) u_k \phi + \int_\Omega d u_k \phi = I^k_{d,1} + I^k_{d,2}.$$

If $d \in L^{\frac{n}{2},q}(\Omega)$, $d_k \to d$ in $L^{\frac{n}{2},q}(\Omega)$, which, by Hölder inequality for Lorentz spaces, (2.18), (2.21), and the uniform $Y^{1,2}$-bound of u_k, yields that $\lim_{k\to\infty} I^k_{d,1} = 0$. Moreover, as $u_k \to u$ pointwisely, we can apply the dominated convergence theorem to obtain

$$\lim_{k\to\infty} I^k_{d,2} = \int_\Omega d u\phi. \tag{6.34}$$

If $|d| \in \mathcal{K}'(\Omega)$, we first apply Cauchy-Schwarz inequality, and then use Lemma 2.21 and the uniform $Y^{1,2}$-bound of u_k. Finally, in view of Lemma 2.14, we can take limits as $k \to \infty$ to conclude that $\lim_{k\to\infty} I^k_{d,1}$. The proof of (6.34) follows by dominated convergence. The integral involving c_k can be treated very similarly and the details are left to the reader. We have thus proved that

$$\mathcal{L}^t_\Omega(u,\varphi) = \lim_{k\to\infty} \mathcal{L}^t_{k,\Omega}(u_k,\varphi) = \int_\Omega f\varphi + \int_\Omega g\nabla\varphi,$$

which, in turn, yields that u is the unique solution of the Dirichlet problem $L^t u = f - \mathrm{div} g$ and $u \in Y^{1,2}_0(\Omega)$.

Let us now recall thatfrom the first part of the proof (before the approximation) we can construct a Green's function $\widehat{G}(\cdot, y)$ associated with L so that the function

$$\widehat{u}(y) = \int_\Omega \widehat{G}(x,y)\, f(x)\, dx + \int_\Omega \nabla_x \widehat{G}(x,y)\, g(x)\, dx,$$

is also a solution of the Dirichlet problem $L^t\widehat{u} = f - \mathrm{div} g$ and $\widehat{u} \in Y^{1,2}_0(\Omega)$. But since there is only one such solution we must have $u = \widehat{u}$, which, as we showed before, implies that $G(x,y) = \widehat{G}(x,y)$, for a.e. $x \in \Omega\setminus\{y\}$. As we have shown that (6.3) holds for $\widehat{G}(x,y)$, it also holds for $G(x,y)$.

The same arguments are valid if we replace G by G^t and L by L^t (and vice versa), implying that

$$\begin{aligned}\lim_{k\to\infty} u^t_k(x) &= \lim_{k\to\infty} \int_\Omega {G^t}_k(y,x)\, f(y)\, dy + \lim_{k\to\infty} \int_\Omega \nabla_y {G^t}_k(y,x)\, f(y)\, dy \\ &= \int_\Omega G^t(y,x)\, f(y)\, dy + \int_\Omega \nabla_y G^t(y,x)\, f(y)\, dy =: u^t(x),\end{aligned}$$

and after passing to a subsequence, $u^t_k \rightharpoonup u^t$ in $Y^{1,2}(\Omega)$, $u^t \in Y^{1,2}_0(\Omega)$, and $Lu^t = f$ in Ω.

For $f, g \in C_c^\infty(\Omega)$ we set

$$u_{f,k}(y) = \int G_k(x, y)\, f(x)\, dx \quad \text{and} \quad u^t_{g,k}(x) = \int G^t_k(y, x)\, g(y)\, dy;$$
$$u_f(y) = \int G(x, y)\, f(x)\, dx \quad \text{and} \quad u^t_g(x) = \int G^t(y, x)\, g(y)\, dy.$$

Recall that

$$u_{f,k} \rightharpoonup u_f \text{ in } Y^{1,2}(\Omega) \quad \text{and} \quad u_f \in Y^{1,2}_0(\Omega),$$

and

$$u^t_{g,k} \rightharpoonup u^t_g \text{ in } Y^{1,2}(\Omega) \quad \text{and} \quad u^t_g \in Y^{1,2}_0(\Omega).$$

By Fubini theorem and $G^t_k(x, y) = G_k(y, x)$ for all $(x, y) \in \Omega^2 \setminus \{x = y\}$, we have that

$$\int u_{f,k}(y)\, g(y)\, dy = \int g(y) \int G_k(x, y)\, f(x)\, dx\, dy$$
$$= \int f(x) \int G^t_k(y, x)\, g(y)\, dy\, dx = \int u^t_{g,k}(x)\, f(x)\, dx. \tag{6.35}$$

If we take limits as $k \to \infty$ in (6.35),

$$\int u_f(y)\, g(y)\, dy = \int u^t_g(x)\, f(x)\, dx,$$

which implies

$$\int\int G(x, y)\, f(x)\, g(y)\, dx\, dy = \int\int G^t_(y, x)\, g(y)\, f(x)\, dy\, dx.$$

Since $f, g \in C_c^\infty(\Omega)$ are arbitrary, we conclude that $G^t(x, y) = G(y, x)$ for a.e. $(x, y) \in \Omega^2 \setminus \{x = y\}$.

Once we have that (6.4) holds, the proof of (6.8) is the same as in (6.27), while (6.1) follows by density. □

Remark 6.3 If $\varphi \in C_c^\infty(\Omega)$ and it holds that $b\nabla\varphi \in L^{\frac{n}{2},1}(\Omega)$, $c\varphi \in L^{n,1}(\Omega)$, and $d\varphi \in L^{\frac{n}{2},1}(\Omega)$, then we can show that

$$\mathcal{L}(G(\cdot, y), \varphi) = \varphi(y).$$

This is straightforward if we use (6.6) and (6.7).

Finally, we can prove that, under certain restrictions, the Green's function has pointwise lower bounds as well.

Lemma 6.4 *Let $\Omega \subset \mathbb{R}^n$ be an open and connected set and $Lu = -\mathrm{div}(A\nabla u + bu)$ be an elliptic operator so that $b \in \mathcal{K}_{Dini}(\Omega)$. Let $x, y \in \Omega$, $x \neq y$, such that $2|x - y| <$ $\mathrm{dist}(\{x, y\}, \partial\Omega)$. If we set $r = |x - y|/4$, then the Green's functions G constructed in Theorem 6.1 satisfy the following lower bound:*

$$G(x, y) \gtrsim \frac{1}{|x - y|^{n-2}},$$
$$G^t(x, y) \gtrsim \frac{1}{|x - y|^{n-2}}. \tag{6.36}$$

Proof Let us fix $x, y \in \Omega$ with $x \neq y$. If we set $r = \frac{|x-y|}{4}$ and let $\eta \in C_0^\infty(B_r(y))$ be a bump function so that

$$0 \leq \eta \leq 1, \quad \eta = 1 \text{ in } B_{\frac{r}{2}}(y), \text{ and } |\nabla \eta| \lesssim \frac{1}{r}.$$

Then using it as a test function we have that

$$\begin{aligned} 1 = \eta(y) = \mathcal{L}(G(\cdot, y), \eta) &= \int_\Omega A\nabla G(\cdot, y)\nabla\eta + \int_\Omega bG(\cdot, y)\nabla\eta \\ &\lesssim \frac{1}{r}\|\nabla G(\cdot, y)\|_{L^1(B_r(y)\setminus B_{\frac{r}{2}}(y))} + \frac{1}{r}\|b\|_{L^n(\Omega)}\|G(\cdot, y)\|_{L^{\frac{n}{n-1}}(B_r(y)\setminus B_{\frac{r}{2}}(y))} \\ &\lesssim \frac{1}{r^2}\|G(\cdot, y)\|_{L^1(B_{2r}(y)\setminus B_{\frac{r}{8}}(y))}, \end{aligned}$$

where we used Hölder, Sobolev and Caccioppoli inequality, along with Lemma 6.2. Thus, from (4.16), we have that $G(x, y) \gtrsim \frac{1}{|x-y|^{n-2}}$.

Let $v \in Y^{1,2}(\Omega)$ be a nonnegative function such that $Lv = 0$ and $v(y) > 0$, and let η be the bump function defined above. Then, if we assume $\rho \leq \min\left(\frac{|x-y|}{10}, \frac{d_y}{10}, \frac{d_x}{10}\right)$,

$$\begin{aligned} \fint_{B_\rho(y)} \eta\, v &= \mathcal{L}^t(G_\rho^t(\cdot, y), \eta\, v) \\ &= \int_\Omega A^t\nabla G_\rho^t(\cdot, y)\,\nabla\eta\, v - A^t\nabla\eta\nabla v\, G_\rho^t(\cdot, y) + A\nabla v\nabla(G_\rho^t(\cdot, y)\eta) \\ &\quad + \int_\Omega b\,\nabla v G_\rho^t(\cdot, y)\eta - \int_\Omega b\,\nabla\eta\, G_\rho^t(\cdot, y)\, v \\ &= \int_\Omega A^t\nabla G_\rho^t(\cdot, y)\nabla\eta\, v - A^t\nabla\eta\nabla v\, G_\rho^t(\cdot, y) - b\,\nabla\eta\, G_\rho^t(\cdot, y)\, v \\ &=: I_1 - I_2 - I_3, \end{aligned}$$

where we used that $G_\rho^t(\cdot, y)\eta$ is a test function and $Lv = 0$. We will only estimate I_3 since I_1 and I_2 can be handled similarly.

$$\begin{aligned} |I_3| &\lesssim \frac{1}{r}\|b + c\|_{L^n(B_r(y)\setminus B_{\frac{r}{2}}(y))}\,\|G_\rho^t(\cdot, y)\|_{L^2(B_r(y)\setminus B_{\frac{r}{2}}(y))}\,\|v\|_{L^{2^*}(B_r(y)\setminus B_{\frac{r}{2}}(y))} \\ &\lesssim \frac{1}{r^2}\,\|G_\rho^t(\cdot, y)\|_{L^2(B_r(y)\setminus B_{\frac{r}{2}}(y))}\,\|v\|_{L^2(B_{\frac{3r}{2}}(y)\setminus B_{\frac{3r}{8}}(y))}, \end{aligned}$$

where in the first inequality we used Hölder inequality and in the second one the local bonudedness of v. If ρ_m is the sequence obtained in (6.21), then by Rellich-Kondrachov theorem and a diagonalization argument, we may pass to a subsequence so that

$$G_{\rho_m}^t(\cdot, y) \to G^t(\cdot, y), \quad \text{strongly in } L^2(B_r(y) \setminus B_{\frac{r}{2}}(y)).$$

Thus, if we take $m \to \infty$, by Lemma 6.2, for a.e. $y \in \Omega$,

$$
\begin{aligned}
v(y) = \eta(y)\, v(y) &= \lim_{m\to\infty} \fint_{B_{\rho m}(y)} \eta\, v \\
&\lesssim \lim_{m\to\infty} \frac{1}{r^2} \|G^t_{\rho_m}(\cdot, y)\|_{L^2(B_r(y)\setminus B_{\frac{r}{2}}(y))} \|v\|_{L^2(B_{\frac{3r}{2}}(y)\setminus B_{\frac{3r}{8}}(y))} \\
&= \frac{1}{r^2} \|G^t(\cdot, y)\|_{L^2(B_r(y)\setminus B_{\frac{r}{2}}(y))} \|v\|_{L^2(B_{\frac{3r}{2}}(y)\setminus B_{\frac{3r}{8}}(y))} \\
&\lesssim \frac{1}{r^{n+2}} \|G^t(\cdot, y)\|_{L^1(B_{2r}(y)\setminus B_{\frac{r}{4}}(y))} \|v\|_{L^1(B_{2r}(y)\setminus B_{\frac{r}{4}}(y))}.
\end{aligned}
$$

So, by (4.16) and Remark (4.2), we get

$$v(y) \lesssim |x-y|^{n-2}\, G^t(x, y)\, v(y),$$

which implies (6.36). □

Acknowledgements We would like to thank Georgios Sakellaris for making his paper available to us and for helpful discussions. We are also grateful to him for spotting a gap in our previous proof of (6.9). We would also like to thank the anonymous referee for their careful reading and their suggestions, which have contributed to improve the readability of the paper.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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