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Calculus of Variations

Higher regularity for weak solutions to degenerate parabolic problems

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Abstract

In this paper, we study two related features of the regularity of the weak solutions to the following strongly degenerate parabolic equation

$$u_t - \operatorname{div}\left((|Du| - 1)_+^{p-1} \frac{Du}{|Du|} \right) = f \qquad \text{in } \Omega_T = \Omega \times (0, T),$$

where Ω is a bounded domain in $\mathbb{R}^n$ for $n \geq 2$, $p \geq$ and $T > 0$. We prove the higher differentiability of a nonlinear function of the spatial gradient of the weak solutions, assuming only that $f \in L^2_{\text{loc}}(\Omega_T)$. This allows us to establish the higher integrability of the spatial gradient under the same minimal requirement on the datum f.

Mathematics Subject Classification 35B45 · 35B65 · 35D30 · 35K10 · 35K65

1 Introduction

In this paper, we study the regularity properties of weak solutions $u : \Omega_T \to \mathbb{R}$ to the following parabolic equation

$$u_t - \operatorname{div}\left((|Du| - 1)_+^{p-1} \frac{Du}{|Du|} \right) = f \qquad \text{in } \Omega_T = \Omega \times (0, T), \tag{1.1}$$

which appears in gas filtration problems taking into account the initial pressure gradient. For a precise description of this motivation we refer to [1] and [3, Sect. 1.1].

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The main feature of this equation is that it possesses a wide degeneracy, coming from the fact that its modulus of ellipticity vanishes at all points where $|Du| \leq 1$ and hence its principal part behaves like a p-Laplacian operator only for large values of $|Du|$.
In this paper we address two interrelated aspects of the regularity theory for solutions to parabolic problems, namely the higher differentiability and the higher integrability of the gradient of the weak solutions to (1.1), with the main aim of weakening the assumption on the datum f with respect to the available literature.
These questions have been exploited in case of non-degenerate parabolic problems with quadratic growth by Campanato in [12], by Duzaar et al. in [19] in case of superquadratic growth, while Scheven in [24] faced the subquadratic growth case. In the above mentioned papers, the problem has been faced or in case of homogeneous equations or considering sufficiently regular datum. It is worth mentioning that the higher integrability of the gradient of the solution is achieved through an interpolation argument, once its higher differentiability is established.

This strategy has revealed to be successful also for widely degenerate equations as in (1.1). Indeed, the higher integrability of the spatial gradient of weak solutions to equation (1.1) has been proven in [3], under suitable assumptions on the datum f in the scale of Sobolev spaces.

We'd like to recall that a common feature for nonlinear problems with growth rate $p > 2$ is that the higher differentiability is proven for a nonlinear expression of the gradient which takes into account the growth of the principal part of the equation. Indeed, already for the non-degenerate p-Laplace equation, the higher differentiability refers to the function $V_{\mu,p}(Du) = \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du$. In case of widely degenerate problems, this phenomenon persists, and higher differentiability results, both for the elliptic and the parabolic problems, hold true for the function $H_{\frac{p}{2}}(Du) = (|Du| - 1)_+^{\frac{p}{2}} \dfrac{Du}{|Du|}$.

It is worth noticing that, as it can be expected, this function of the gradient doesn't give information on the second order regularity of the solutions in the set where the equation degenerates. Actually, since every 1-Lipschitz continuous function is a solution to the elliptic equation

$$\operatorname{div}\left(H_{p-1}(Du)\right) = 0,$$

where $H_{p-1}(Du) = (|Du| - 1)_+^{p-1} \dfrac{Du}{|Du|}$, no more than Lipschitz regularity can be expected.
Moreover, it is well known that (already for the degenerate p-Laplace equation, with $p > 2$) a Sobolev regularity is required for the datum f in order to get the higher differentiability of the solutions, when such higher differentiability comes from the existence of the weak derivatives of the function $V_{0,p}(Du) \equiv V_p(Du) = |Du|^{\frac{p-2}{2}} Du$ (see, for example, [11] for elliptic and [3] for parabolic equations). Actually, the sharp assumption for the datum in the elliptic setting has been determined in [11] as a fractional Sobolev regularity suitably related to the growth exponent p and the dimension n.

On the other hand, in [13, 14], the authors proved that the assumption $f \in L^2$ is sufficient both in the elliptic and in parabolic setting to prove the $W^{1,2}$-regularity of $|Du|^{p-2} Du$, which is a different function of the gradient with respect to $V_p(Du)$. Note, however, that $\left|D\left(V_p(Du)\right)\right| \approx |Du|^{\frac{p-2}{2}} \left|D^2 u\right|$ while $\left|D\left(|Du|^{p-2} Du\right)\right| \approx |Du|^{p-2} \left|D^2 u\right|$ and therefore, in case $p \geq 2$, if $|Du|$ approaches zero, i.e. when we are closed to the degeneracy set, the $W^{1,2}$-regularity of $V_p(Du)$ implies that of $|Du|^{p-2} Du$.

The main aim of this paper is to show that, without assuming any kind of Sobolev regularity for the datum, but assuming only $f \in L^2$, we are still able to obtain the differentiability of a function of the weak solutions that behaves as $V_p(Du)$ outside a set larger than the degeneracy set of the problem. We would like to emphasize that, while for the p-Laplace equation the degeneracy appears for $p > 2$, here, even in case $p = 2$, under a L^2 integrability assumption on the datum f, the local $W^{2,2}$ regularity of the solutions cannot be obtained. Actually, our main result is the following.

Theorem 1.1 *Let $n \geq 2$, $p \geq 2$ and $f \in L^2_{\mathrm{loc}}(\Omega_T)$. Moreover, let us assume that*

$$u \in C^0\left(0, T; L^2(\Omega)\right) \cap L^p_{\mathrm{loc}}\left(0, T; W^{1,p}_{\mathrm{loc}}(\Omega)\right)$$

is a weak solution to (1.1)*. Then, for any $\delta \in (0,1)$, we have*

$$\mathcal{G}_\delta\left((|Du| - 1 - \delta)_+\right) \in L^2_{\mathrm{loc}}\left(0, T; W^{1,2}_{\mathrm{loc}}(\Omega)\right),$$

where

$$\mathcal{G}_\delta(t) := \int_0^t \frac{s(s+\delta)^{\frac{p-2}{2}}}{\sqrt{1+\delta+s^2}}\, ds, \qquad \text{for every } t \geq 0.$$

Moreover the following estimate

$$\int_{Q_{\frac{R}{8}}} \left|D\left[\mathcal{G}_\delta\left((|Du| - \delta - 1)_+\right)\right]\right|^2 dz \leq \frac{c(n,p)}{R^2\delta^2}\left[\int_{Q_R}\left(|Du|^p + 1\right) dz + \frac{1}{\delta^p}\int_{Q_R}|f|^2\, dz\right], \tag{1.2}$$

holds for any $R > 0$ such that $Q_R = Q_R(z_0) \Subset \Omega_T$.

As already mentioned, the weak solutions of (1.1) are not twice differentiable, and hence it is not possible in general to differentiate the equation to estimate the second derivative of the solutions. We overcome this difficulty by introducing a suitable family of approximating problems whose solutions are regular enough by the standard theory [17]. The major effort in the proof of Theorem 1.1 is to establish suitable estimates for the solutions of the regularized problems that are uniform with respect to the approximation's parameter. Next, we take advantage from these uniform estimates in the use of a comparison argument aimed to bound the difference quotient of $\mathcal{G}_\delta\left((|Du| - \delta - 1)_+\right)$.
The main difference with respect to the arguments used in [3] is in the derivation of the a priori estimates for the terms involving the datum f. Indeed, if f has a Sobolev regularity, the terms of the estimate coming from the inhomogeneity of the equation can be controlled using the informations on the integrability of the gradient of the solution. Here, the terms coming from the inhomogeneity behave as the second derivatives of the solution and their L^2 integrability is known to fail already for solutions to less degenerate equations. Hence here, differently from [3], we need to construct a suitable function of the gradient with weak derivatives in L^2 that, on the one hand, vanishes on a ball larger than B_1 and on the other is comparable with the ellipticity bounds of our equation.

Hence, roughly speaking, due to the weakness of our assumption on the datum, we only get the higher differentiability of a nonlinear function of the gradient of the solutions that vanishes in a set which is larger with respect to that of the degeneracy of the problem. This is quite predictable, since the same kind of phenomenon occurs in the setting of widely degenerate elliptic problems (see, for example [15]).

Anyway, as a consequence of the higher differentiability result in Theorem 1.1, since the gradient stays bounded in the degeneracy set, we establish a higher integrability result for the spatial gradient of the solution to Eq. (1.1), which is the following.

Theorem 1.2 *Under the assumptions of Theorem 1.1, we have*

$$Du \in L^{p+\frac{4}{n}}_{\mathrm{loc}}\left(\Omega_T\right)$$

with the following estimate

$$\int_{Q_{\frac{\rho}{2}}} |Du|^{p+\frac{4}{n}}\, dz \leq \frac{c\,(n,p)}{\rho^{\frac{2(n+2)}{n}}} \left[\int_{Q_{2\rho}} \left(1+|Du|^p+|f|^2\right) dz\right]^{\frac{2}{n}+1}, \tag{1.3}$$

for every parabolic cylinder $Q_{2\rho}\left(z_0\right) \Subset \Omega_T$, with a constant $c = c(n,p)$.

The proof of previous Theorem consists in using an interpolation argument with the aim of establishing an estimate for the $L^{p+\frac{4}{n}}$ norm of the gradient of the solutions to the approximating problems that is preserved in the passage to the limit.

We would like to mention that the elliptic version of our equation naturally arises in optimal transport problems with congestion effects, and the regularity properties of its weak solutions have been widely investigated. More precisely, Lipschitz regularity results can be found in [4, 5, 9, 16, 20], while we refer to [2, 11] for higher differentiability and to [6] for a continuity result for the gradient of weak solutions.

We conclude stressing that, for sake of clarity, we confine ourselves to Eq. (1.1), but we believe that our techniques apply as well to a general class of equations with an analogous widely degenerate structure and also to solutions of systems of widely degenerate parabolic equations.

2 Notations and preliminaries

In this paper we shall denote by C or c a general positive constant that may vary on different occasions. Relevant dependencies on parameters will be properly stressed using parentheses or subscripts. The norm we use on $\mathbb{R}^n$ will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\xi, \eta \in \mathbb{R}^n$, we write $\langle \xi, \eta\rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi\rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

For points in space-time, we will use abbreviations like $z = (x,t)$ or $z_0 = (x_0, t_0)$, for spatial variables x, $x_0 \in \mathbb{R}^n$ and times t, $t_0 \in \mathbb{R}$. We also denote by $B\left(x_0, \rho\right) = B_\rho\left(x_0\right) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$ the open ball with radius $\rho > 0$ and center $x_0 \in \mathbb{R}^n$; when not important, or clear from the context, we shall omit to indicate the center, as follows: $B_\rho \equiv B\left(x_0, \rho\right)$. Unless otherwise stated, different balls in the same context will have the same center. Moreover, we use the notation

$$Q_\rho\left(z_0\right) := B_\rho\left(x_0\right) \times \left(t_0 - \rho^2, t_0\right), \qquad z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}, \quad \rho > 0,$$

for the backward parabolic cylinder with vertex (x_0, t_0) and width ρ. We shall sometimes omit the dependence on the vertex when the cylinders occurring share the same vertex. Finally, for a cylinder $Q = A \times (t_1, t_2)$, where $A \subset \mathbb{R}^n$ and $t_1 < t_2$, we denote by

$$\partial_{\mathrm{par}} Q := (A \times \{t_1\}) \cup (\partial A \times [t_1, t_2))$$

the usual parabolic boundary of Q, which is nothing but its standard topological boundary without the upper cap $A \times \{t_2\}$.
We now recall some tools that will be useful to prove our results. For the auxiliary function $H_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$H_\lambda(\xi) := \begin{cases} (|\xi| - 1)_+^\lambda \dfrac{\xi}{|\xi|} & \text{if} \quad \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if} \quad \xi = 0, \end{cases} \tag{2.1}$$

where $\lambda > 0$ is a parameter, we record the following estimates (see [10, Lemma 4.1]).

Lemma 2.1 *If $2 \le p < \infty$, then for every $\xi, \eta \in \mathbb{R}^n$ it holds*

$$\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \ge \frac{4}{p^2} \left| H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta) \right|^2,$$

$$\left| H_{p-1}(\xi) - H_{p-1}(\eta) \right| \le (p-1) \left(\left| H_{\frac{p}{2}}(\xi) \right|^{\frac{p-2}{p}} + \left| H_{\frac{p}{2}}(\eta) \right|^{\frac{p-2}{p}} \right) \left| H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta) \right|.$$

We also recall the following

Lemma 2.2 *Let $\xi, \eta \in \mathbb{R}^k$ with $|\xi| > 1$. Then, we have*

$$\left| H_{p-1}(\xi) - H_{p-1}(\eta) \right| \le c(p) \frac{\left[(|\xi| - 1) + (|\eta| - 1)_+ \right]^{p-1}}{|\xi| - 1} |\xi - \eta|$$

and

$$\langle H_{p-1}(\eta) - H_{p-1}(\xi), \eta - \xi \rangle \ge \frac{\min\{1, p-1\}}{2^{p+1}} \frac{(|\xi| - 1)^p}{|\xi| \left(|\xi| + |\eta| \right)} |\eta - \xi|^2.$$

For the proof see [6, Lemma 2.8].
Previous Lemma implies that the operator $D_\xi H_{p-1}(\xi)$ is controlled with two different functions of $(|\xi| - 1)_+$, and so our problem can be also viewed as a very particular case of parabolic problems with non-standard growth, whose study started with the papers [7, 8].

Definition 2.3 With the use of (2.1), a function $u \in C^0\left(0, T; L^2(\Omega)\right) \cap L^p\left(0, T; W^{1,p}(\Omega)\right)$ is a weak solution of Eq. (1.1) if

$$\int_{\Omega_T} \left(u \cdot \partial_t \varphi - \langle H_{p-1}(Du), D\varphi \rangle \right) dz = - \int_{\Omega_T} f \varphi \, dz \tag{2.2}$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

We shall also use the well known auxiliary function $V_{1,p} : \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$V_{1,p}(\xi) := \left(1 + |\xi|^2\right)^{\frac{p-2}{4}} \xi,$$

where $p \ge 2$, for which we have the following

Lemma 2.4 *For every $\xi, \eta \in \mathbb{R}^n$ there hold*

$$\begin{aligned} \frac{1}{c_1(p)} \left| V_{1,p}(\xi) - V_{1,p}(\eta) \right|^2 &\le \left(1 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} |\xi - \eta|^2 \\ &\le c_1(p) \left\langle \left(1 + |\xi|^2\right)^{\frac{p-2}{2}} \xi - \left(1 + |\eta|^2\right)^{\frac{p-2}{2}} \eta, \xi - \eta \right\rangle, \end{aligned}$$

We refer to [23, Chapter 12] or to [22, Lemma 9.2] for a proof of these fundamental inequalities.
For further needs, we also record the following interpolation inequality whose proof can be found in [18, Proposition 3.1].

Lemma 2.5 *Assume that the function* $v : Q_r(z_0) \cup \partial_{\text{par}} Q_r(z_0) \to \mathbb{R}$ *satisfies*

$$v \in L^\infty \left(t_0 - r^2, t_0; L^q\left(B_r(x_0)\right)\right) \cap L^p\left(t_0 - r^2, t_0; W_0^{1,p}\left(B_r(x_0)\right)\right)$$

for some exponents $1 \le p, q < \infty$. *Then the following estimate*

$$\int_{Q_r(z_0)} |v|^{p+\frac{pq}{n}}\, dz \le c \left(\sup_{s\in(t_0-r^2,t_0)} \int_{B_r(x_0)} |v(x,s)|^q\, dx \right)^{\frac{p}{n}} \int_{Q_r(z_0)} |Dv|^p\, dz$$

holds true for a positive constant c depending at most on n, p *and* q.

2.1 Difference quotients

We recall here the definition and some well known properties of the difference quotients (see, for example, [22, Chapter 8]).

Definition 2.6 For every function $F : \mathbb{R}^n \to \mathbb{R}^N$ the finite difference operator in the direction x_s is defined by

$$\tau_{s,h} F(x) = F(x + he_s) - F(x),$$

where $h \in \mathbb{R}$, e_s is the unit vector in the direction x_s and $s \in \{1, \dots, n\}$.

The difference quotient of F with respect to x_s is defined for $h \in \mathbb{R} \setminus \{0\}$ as

$$\Delta_{s,h} F(x) = \frac{\tau_{s,h} F(x)}{h}.$$

We shall omit the index s when it is not necessary, and simply write $\tau_h F(x) = F(x+h) - F(x)$ and $|\Delta_h F(x)| = \dfrac{|\tau_h F(x)|}{|h|}$ for $h \in \mathbb{R}^n$.

Proposition 2.7 *Let* $F \in W^{1,p}(\Omega)$, *with* $p \ge 1$, *and let us set*

$$\Omega_{|h|} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > |h|\}.$$

Then:
(*a*) $\Delta_h F \in W^{1,p}\left(\Omega_{|h|}\right)$ *and*

$$D_i(\Delta_h F) = \Delta_h(D_i F), \qquad \textit{for every } i \in \{1, \dots, n\}.$$

(*b*) *If at least one of the functions* F *or* G *has support contained in* $\Omega_{|h|}$, *then*

$$\int_\Omega F\, \Delta_h G\, dx = -\int_\Omega G\, \Delta_{-h} F\, dx.$$

(*c*) *We have*

$$\Delta_h(FG)(x) = F(x + he_s)\, \Delta_h G(x) + G(x) \Delta_h F(x).$$

The next result about the finite difference operator is a kind of integral version of Lagrange Theorem (see [22, Lemma 8.1]).

Lemma 2.8 *If* $0 < \rho < R$, $|h| < \dfrac{R-\rho}{2}$, $1 < p < +\infty$, *and* $F \in W^{1,p}\left(B_R, \mathbb{R}^N\right)$, *then*

$$\int_{B_\rho} |\tau_h F(x)|^p \, dx \leq c^p(n) \, |h|^p \int_{B_R} |DF(x)|^p \, dx.$$

Moreover

$$\int_{B_\rho} |F(x+he_s)|^p \, dx \leq \int_{B_R} |F(x)|^p \, dx.$$

We conclude this section with the following fundamental result, whose proof can be found in [22, Lemma 8.2]:

Lemma 2.9 *Let* $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p\left(B_R, \mathbb{R}^N\right)$ *with* $1 < p < +\infty$. *Suppose that there exist* $\rho \in (0, R)$ *and a constant* $M > 0$ *such that*

$$\sum_{s=1}^{n} \int_{B_\rho} \left|\tau_{s,h} F(x)\right|^p \, dx \leq M^p \, |h|^p$$

for every h, *with* $|h| < \dfrac{R-\rho}{2}$. *Then* $F \in W^{1,p}\left(B_\rho, \mathbb{R}^N\right)$ *and*

$$\|DF\|_{L^p(B_\rho)} \leq M.$$

Moreover

$$\Delta_{s,h} F \to D_s F \qquad \text{strongly in } L^p_{\mathrm{loc}}\left(B_R\right), \text{ as } h \to 0,$$

for each $s \in \{1, \ldots, n\}$.

2.2 Some auxiliary functions and related algebraic inequalities

In this section we introduce some auxiliary functions and we list some of their properties, that will be used in what follows. For any $k > 1$ and for $s \in [0, +\infty)$, let us consider the function

$$g_k(s) = \frac{s^2}{k+s^2}, \tag{2.3}$$

for which we record the following

Lemma 2.10 *Let* $k > 1$, *and let* g_k *be the function defined by* (2.3). *Then for every* $A, B \geq 0$ *the following Young's type inequality*

$$A \cdot B \left[s \cdot g_k'\left((s-k)_+\right)\right] \leq 2\sqrt{2}k \left[\alpha A^2 g\left((s-k)_+\right) + c_\alpha B^2\right], \tag{2.4}$$

holds for every parameter $\alpha \geq 0$ *with a constant* $c_\alpha > 0$. *Moreover, there exists a constant* $c_k > 0$, *depending on* k, *such that*

$$s g_k'\left(\left(s^2-k\right)_+\right) \leq c_k, \qquad \forall s \geq 0. \tag{2.5}$$

Proof Since

$$g_k'(s) = \frac{2ks}{\left(k+s^2\right)^2}, \tag{2.6}$$

both the conclusions trivially hold for $s \leq \sqrt{k}$. Now assume that $s > \sqrt{k}$ and note that Young's inequality implies

$$\begin{aligned} A \cdot B \left[s \cdot g_k' \left((s-k)_+\right)\right] &= A \cdot B \cdot s \cdot \frac{2k\,(s-k)_+}{\left[k+(s-k)_+^2\right]^2} \\ &\leq \alpha A^2 s \cdot \frac{2k\,(s-k)_+^2}{\left[k+(s-k)_+^2\right]^2} + c_\alpha B^2 \cdot \frac{2ks}{\left[k+(s-k)_+^2\right]^2} \\ &= \alpha A^2 g\left((s-k)_+\right) \cdot \frac{2ks}{k+(s-k)_+^2} + c_\alpha B^2 \cdot \frac{2ks}{\left[k+(s-k)_+^2\right]^2}, \end{aligned} \tag{2.7}$$

where we used the explicit expression of $g_k'(s)$ at (2.6) and the Definition (2.3).
Setting $h(s) = \dfrac{s}{k+(s-k)_+^2}$, we can easily check that

$$h(k) = 1, \quad \lim_{s\to+\infty} h(s) = 0, \quad \max_{s\in[k,+\infty)} h(s) = h\left(\sqrt{k^2+k}\right) = \frac{1}{2}\left(1+\sqrt{1+\frac{1}{k}}\right) < \sqrt{2}$$

and so

$$\frac{2ks}{k+(s-k)_+^2} \leq 2\sqrt{2}k \qquad \forall s > k.$$

Inserting this in (2.7), we get

$$\begin{aligned} &A \cdot B\left[s \cdot g_k'\left((s-k)_+\right)\right] \\ &\leq 2\sqrt{2}k\left[\alpha A^2 g\left((s-k)_+\right) + c_\alpha B^2 \cdot \frac{1}{k+(s-k)_+^2}\right], \end{aligned}$$

which, since $\dfrac{1}{k+(s-k)_+^2} < 1$ for $k > 1$, implies (2.4). In order to prove (2.5), let us notice that, recalling (2.6), we have

$$sg'\left(\left(s^2-k\right)_+\right) = \frac{2ks\left(s^2-k\right)_+}{\left(k+\left(s^2-k\right)_+^2\right)^2}.$$

So, since the function $sg'\left(\left(s^2-k\right)_+\right)$ is continuous in the interval $\left\{s \geq 0 | s^2 > k\right\} = \left(\sqrt{k}, +\infty\right)$ and

$$\lim_{s\to+\infty} \frac{2ks\left(s^2-k\right)_+}{\left(k+\left(s^2-k\right)_+^2\right)^2} = 0,$$

then there exists a constant $c_k > 0$ such that

$$sg'\left(\left(s^2-k\right)_+\right) \leq c_k \qquad \text{for every } s \geq 0,$$

which is the conclusion. □

For any $\delta > 0$, let us define

$$\mathcal{G}_\delta(t) := \int_0^t \frac{s(s+\delta)^{\frac{p-2}{2}}}{\sqrt{1+\delta+s^2}}\, ds, \qquad \text{for } t \geq 0, \tag{2.8}$$

and observe that

$$\mathcal{G}_\delta'(t) = \frac{t(t+\delta)^{\frac{p-2}{2}}}{\sqrt{1+\delta+t^2}}. \tag{2.9}$$

Next lemma relates the function $\mathcal{G}_\delta(|\xi|)$ with $H_{\frac{p}{2}}(\xi)$.

Lemma 2.11 *Let $\mathcal{G}_\delta$ be the function defined by (2.8) and $H_{\frac{p}{2}}$ be the one defined in (2.1) with $\lambda = \frac{p}{2}$. Then we have*

$$\left|\mathcal{G}_\delta\left((|\xi|-\delta-1)_+\right) - \mathcal{G}_\delta\left((|\eta|-\delta-1)_+\right)\right|^2 \leq c_p \left|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)\right|^2 \tag{2.10}$$

for any $\xi, \eta \in \mathbb{R}^n$.

Proof If $|\xi| < 1+\delta$ and $|\eta| < 1+\delta$ there is nothing to prove. So we will assume that $|\xi| > 1+\delta$, and without loss of generality we may suppose that $|\eta| \leq |\xi|$. Since $\mathcal{G}_\delta(t)$ is increasing, we have

$$\begin{aligned}
&\left|\mathcal{G}_\delta(|\xi|-1-\delta) - \mathcal{G}_\delta\left((|\eta|-1-\delta)_+\right)\right| \\
&= \mathcal{G}_\delta(|\xi|-1-\delta) - \mathcal{G}_\delta\left((|\eta|-1-\delta)_+\right) \\
&= \int_{(|\eta|-1-\delta)_+}^{|\xi|-1-\delta} \frac{s(s+\delta)^{\frac{p-2}{2}}}{\sqrt{1+\delta+s^2}}\, ds \\
&\leq \int_{(|\eta|-1-\delta)_+}^{|\xi|-1-\delta} (s+\delta)^{\frac{p-2}{2}}\, ds \\
&= \frac{2}{p}\left[(|\xi|-1)^{\frac{p}{2}} - \left[(|\eta|-\delta-1)_+ + \delta\right]^{\frac{p}{2}}\right].
\end{aligned}$$

Now, it can be easily checked that

$$\begin{aligned}
&(|\xi|-1)^{\frac{p}{2}} - \left[(|\eta|-\delta-1)_+ + \delta\right]^{\frac{p}{2}} \\
&= \begin{cases} (|\xi|-1)^{\frac{p}{2}} - \delta^{\frac{p}{2}} & \text{if } \quad |\xi| > \delta+1 \text{ and } |\eta| \leq \delta+1 \\ (|\xi|-1)^{\frac{p}{2}} - (|\eta|-1)^{\frac{p}{2}} & \text{if } \quad |\xi| > \delta+1 \text{ and } |\eta| > \delta+1. \end{cases}
\end{aligned}$$

In the first case, we have

$$\begin{aligned}
\left|(|\xi|-1)^{\frac{p}{2}} - \delta^{\frac{p}{2}}\right| &= (|\xi|-1)^{\frac{p}{2}} - \delta^{\frac{p}{2}} \leq (|\xi|-1)^{\frac{p}{2}} - (|\eta|-1)_+^{\frac{p}{2}} \\
&= \left|H_{\frac{p}{2}}(\xi)\right| - \left|H_{\frac{p}{2}}(\eta)\right| \leq \left|H_{\frac{p}{2}}(\eta) - H_{\frac{p}{2}}(\xi)\right|,
\end{aligned}$$

while, in the second,

$$(|\xi|-1)^{\frac{p}{2}} - \left[(|\eta|-\delta-1)_+ + \delta\right]^{\frac{p}{2}} = \left|H_{\frac{p}{2}}(\xi)\right| - \left|H_{\frac{p}{2}}(\eta)\right| \leq \left|H_{\frac{p}{2}}(\eta) - H_{\frac{p}{2}}(\xi)\right|.$$

Therefore,

$$\left|\mathcal{G}_\delta\left((|\xi|-\delta-1)_+\right) - \mathcal{G}_\delta\left((|\eta|-\delta-1)_+\right)\right|^2 \leq c_p \left|H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)\right|^2$$

for every $\xi, \eta \in \mathbb{R}^n$, which is (2.10). □

Arguing as in [21, Lemma 2.1], we prove the following.

Lemma 2.12 *Let* $0 < \delta \leq 1$ *and* $p \geq 2$*. Then the following inequalities hold*

$$c_{p,\delta}(t+\delta)^{\frac{p}{2}} - \tilde{c}_{p,\delta} \leq \mathcal{G}_\delta(t) \leq \frac{2}{p}(t+\delta)^{\frac{p}{2}}$$

with constants $\tilde{c}_{p,\delta}$ *and* $c_{p,\delta} < \dfrac{2}{p}$ *depending on* p *and* δ.

Proof If $p = 2$, one can explicitely calculate

$$\mathcal{G}_\delta(t) = \int_0^t \frac{s}{\sqrt{1+\delta+s^2}}\,ds = \left[\sqrt{1+\delta+s^2}\right]_0^t = \sqrt{1+\delta+t^2} - \sqrt{1+\delta},$$

and using the inequality $\sqrt{a^2+b^2} \geq \frac{1}{\sqrt{2}}(a+b)$ with $a = 1+\delta$ and $b = t$, we easily get

$$\frac{1}{\sqrt{2}}(t+\delta) - \sqrt{1+\delta} \leq \mathcal{G}_\delta(t) \leq t+\delta.$$

Let $p > 2$. The right inequality is a simple consequence of the trivial bound $\dfrac{s}{\sqrt{1+\delta+s^2}} < 1$. For the left inequality we start observing that

$$\sqrt{1+\delta+s^2} \leq \sqrt{1+\delta} + s \quad \Longrightarrow \quad \mathcal{G}_\delta(t) \geq \int_0^t \frac{s\,(s+\delta)^{\frac{p-2}{2}}}{\sqrt{1+\delta}+s}\,ds.$$

Now, we calculate the integral in previous formula. By the change of variable $r = \sqrt{1+\delta}+s$, we get

$$\begin{aligned}
\int_0^t \frac{s\,(s+\delta)^{\frac{p-2}{2}}}{\sqrt{1+\delta}+s}\,ds &= \int_{\sqrt{1+\delta}}^{t+\sqrt{1+\delta}} \frac{\left(r-\sqrt{1+\delta}\right)\left(r-\sqrt{1+\delta}+\delta\right)^{\frac{p-2}{2}}}{r}\,dr \\
&= \int_{\sqrt{1+\delta}}^{t+\sqrt{1+\delta}} \left(r-\sqrt{1+\delta}+\delta\right)^{\frac{p-2}{2}} dr - \sqrt{1+\delta}\int_{\sqrt{1+\delta}}^{t+\sqrt{1+\delta}} \frac{\left(r-\sqrt{1+\delta}+\delta\right)^{\frac{p-2}{2}}}{r}\,dr \\
&\geq \frac{2}{p}\left[\left(r-\sqrt{1+\delta}+\delta\right)^{\frac{p}{2}}\right]_{\sqrt{1+\delta}}^{t+\sqrt{1+\delta}} - \sqrt{1+\delta}\int_{\sqrt{1+\delta}}^{t+\sqrt{1+\delta}} \left(r-\sqrt{1+\delta}+\delta\right)^{\frac{p}{2}-2} dr,
\end{aligned}$$

since $0 < \delta \leq 1$, we have $\delta \leq \sqrt{1+\delta}$ and therefore $r - \sqrt{1+\delta} + \delta \leq r$. Calculating the last integral in previous formula, we get

$$\begin{aligned}
&\int_0^t \frac{s(s+\delta)^{\frac{p-2}{2}}}{\sqrt{1+\delta}+s}\,ds \\
&\geq \frac{2}{p}\left[\left(r-\sqrt{1+\delta}+\delta\right)^{\frac{p}{2}}\right]_{\sqrt{1+\delta}}^{t+\sqrt{1+\delta}} - \frac{2\sqrt{1+\delta}}{p-2}\left[\left(r-\sqrt{1+\delta}+\delta\right)^{\frac{p}{2}-1}\right]_{\sqrt{1+\delta}}^{t+\sqrt{1+\delta}} \\
&= \frac{2}{p}\left[(t+\delta)^{\frac{p}{2}} - \delta^{\frac{p}{2}}\right] - \frac{2\sqrt{1+\delta}}{p-2}\left[(t+\delta)^{\frac{p}{2}-1} - \delta^{\frac{p}{2}-1}\right] \\
&= \frac{2}{p}(t+\delta)^{\frac{p}{2}} - \frac{2\sqrt{1+\delta}}{p-2}(t+\delta)^{\frac{p}{2}-1} + \frac{2\sqrt{1+\delta}}{p-2}\delta^{\frac{p}{2}-1} - \frac{2}{p}\delta^{\frac{p}{2}}.
\end{aligned}$$

Therefore the lemma will be proven if there exists a constant $c_{p,\delta} < \frac{2}{p}$ such that

$$c_{p,\delta}(t+\delta)^{\frac{p}{2}} \le \frac{2}{p}(t+\delta)^{\frac{p}{2}} - \frac{2\sqrt{1+\delta}}{p-2}(t+\delta)^{\frac{p}{2}-1} + \frac{2\sqrt{1+\delta}}{p-2}\delta^{\frac{p}{2}-1} - \frac{2}{p}\delta^{\frac{p}{2}}$$

which, setting

$$h(t) = \frac{2\sqrt{1+\delta}}{p-2}(t+\delta)^{\frac{p}{2}-1} + \left(c_{p,\delta} - \frac{2}{p}\right)(t+\delta)^{\frac{p}{2}},$$

is equivalent to prove that there exists $c_{p,\delta}$ such that

$$h(t) \le \frac{2\sqrt{1+\delta}}{p-2}\delta^{\frac{p}{2}-1} - \frac{2}{p}\delta^{\frac{p}{2}}.$$

It is easy to check that $h(t)$ attains his maximum for $t+\delta = \frac{2\sqrt{1+\delta}}{2-pc_{p,\delta}}$ and so

$$h(t) \le h\left(\frac{2\sqrt{1+\delta}}{2-pc_{p,\delta}} - \delta\right) = \left(2\sqrt{1+\delta}\right)^{\frac{p}{2}}\left(\frac{1}{2-pc_{p,\delta}}\right)^{\frac{p-2}{2}}\frac{2}{p\,(p-2)}.$$

Therefore, to complete the proof it's enough to solve the following equation

$$\left(2\sqrt{1+\delta}\right)^{\frac{p}{2}}\left(\frac{1}{2-pc_{p,\delta}}\right)^{\frac{p-2}{2}}\frac{2}{p\,(p-2)} = \frac{2\sqrt{1+\delta}}{p-2}\delta^{\frac{p}{2}-1} - \frac{2}{p}\delta^{\frac{p}{2}},$$

which is equivalent to

$$\frac{1}{2-pc_{p,\delta}} = \left(\frac{\delta}{2\sqrt{1+\delta}}\right)^{\frac{p}{p-2}}\left(\frac{p\left(\sqrt{1+\delta}-\delta\right)}{\delta}+2\right)^{\frac{2}{p-2}}$$

that, for $0 < \delta < 1$, admits a unique solution $c_{p,\delta} < \frac{2}{p}$. □

3 The regularization

Let us fix $z_0 = (x_0, t_0) \in \Omega_T$ and $R \in (0,1)$ such that $Q_R(z_0) \Subset \Omega_T$. For $\varepsilon > 0$, we introduce the sequence of operators

$$A_\varepsilon(\xi) := (|\xi|-1)_+^{p-1}\frac{\xi}{|\xi|} + \varepsilon\left(1+|\xi|^2\right)^{\frac{p-2}{2}}\xi \tag{3.1}$$

and by

$$u^\varepsilon \in C^0\left(t_0-R^2, t_0; L^2\left(B_R\right)\right) \cap L^p\left(t_0-R^2, t_0; u+W_0^{1,p}\left(B_R\right)\right)$$

we denote the unique solution to the corresponding problems

$$\begin{cases} u_t^\varepsilon - \operatorname{div}\left(A_\varepsilon\left(Du^\varepsilon\right)\right) = f^\varepsilon & \text{in } Q_R\left(z_0\right) \\ u^\varepsilon = u & \text{in } \partial_{\text{par}} Q_R\left(z_0\right) \end{cases} \tag{3.2}$$

where $f^\varepsilon = f * \rho_\varepsilon$ with ρ_ε the usual sequence of mollifiers. One can easily check that the operator A_ε satisfies p-growth and p-ellipticity conditions with constants depending on ε. Therefore, by the results in [19], we have

$$V_{1,p}\left(Du^\varepsilon\right) \in L^2_{\mathrm{loc}}\left(0,T; W^{1,2}_{\mathrm{loc}}\left(B_R\left(x_0\right),\mathbb{R}^n\right)\right) \qquad \text{and} \qquad \left|Du^\varepsilon\right| \in L^{p+\frac{4}{n}}_{\mathrm{loc}}\left(Q_R\right)$$

and, by the definition of $V_{1,p}(\xi)$, this yields

$$DV_{1,p}\left(Du^\varepsilon\right) \approx \left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{4}} D^2u^\varepsilon \in L^2_{\mathrm{loc}}\left(Q_R;\mathbb{R}^{n\times n}\right) \quad \Longrightarrow \quad \left|D^2u^\varepsilon\right| \in L^2_{\mathrm{loc}}\left(Q_R\right). \tag{3.3}$$

By virtue of [3, Theorem 1.1], we also have $H_{\frac{p}{2}}\left(Du^\varepsilon\right) \in L^2_{\mathrm{loc}}\left(0,T; W^{1,2}_{\mathrm{loc}}\left(\Omega,\mathbb{R}^n\right)\right)$ and, by the definition of $H_{\frac{p}{2}}(\xi)$, it follows

$$\left|DH_{\frac{p}{2}}\left(Du\right)\right| \le c_p\left(\left|Du^\varepsilon\right|-1\right)_+^{\frac{p-2}{2}}\left|D^2u^\varepsilon\right| \in L^2_{\mathrm{loc}}\left(Q_R;\mathbb{R}^{n\times n}\right). \tag{3.4}$$

3.1 Uniform a priori estimates

The first step in the proof of Theorem 1.1 is the following estimate for solutions to the regularized problem (3.2).

Lemma 3.1 *Let* $u^\varepsilon \in C^0\left(t_0-R^2,t_0;L^2\left(B_R\right)\right) \cap L^p\left(t_0-R^2,t_0;u+W^{1,p}_0\left(B_R\right)\right)$ *be the unique solution to* (3.2). *Then the following estimate*

$$\begin{aligned}
&\sup_{\tau\in(t_0-4\rho^2,t_0)} \int_{B_\rho}\left(\left|Du^\varepsilon(x,\tau)\right|^2-1-\delta\right)_+ dx \\
&\quad + \int_{Q_\rho}\left|D\left[\mathcal{G}_\delta\left(\left(\left|Du^\varepsilon\right|-\delta-1\right)_+\right)\right]\right|^2 dz \\
&\le \frac{c}{\rho^2\delta^2}\left[\int_{Q_{2\rho}}\left(1+\left|Du^\varepsilon\right|^p\right)dz + \frac{1}{\delta^p}\int_{Q_{2\rho}}\left|f^\varepsilon\right|^2 dz\right]
\end{aligned} \tag{3.5}$$

holds for any $\varepsilon \in (0,1]$ *and for every* $Q_\rho \Subset Q_{2\rho} \Subset Q_R$, *with a constant* $c = c(n,p)$ *independent of* ε.

Proof The weak formulation of (3.2) reads as

$$\int_{Q_R}\left(u^\varepsilon\cdot\partial_t\varphi - \left\langle A_\varepsilon\left(Du^\varepsilon\right), D\varphi\right\rangle\right) dz = -\int_{Q_R} f^\varepsilon\cdot\varphi\, dz$$

for any test function $\varphi \in C^\infty_0\left(Q_R\right)$. Recalling the notations introduced at (2.1) and (3.1), and replacing φ with $\Delta_{-h}\varphi = \dfrac{\tau_{-h}\varphi}{h}$ for a sufficiently small $h \in \mathbb{R}\setminus\{0\}$, by virtue of the properties of difference quotients, we have

$$\begin{aligned}
&\int_{Q_R}\left(\Delta_h u^\varepsilon\cdot\partial_t\varphi - \left\langle\Delta_h H_{p-1}\left(Du^\varepsilon\right), D\varphi\right\rangle - \varepsilon\left\langle\Delta_h\left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}}Du^\varepsilon\right), D\varphi\right\rangle\right) dz \\
&\quad = -\int_{Q_R} f^\varepsilon\cdot\Delta_{-h}\varphi\, dz.
\end{aligned} \tag{3.6}$$

Arguing as in [19, Lemma 5.1], from (3.6) we get

$$\int_{Q_R} \partial_t \Delta_h u^\varepsilon \cdot \varphi \, dz + \int_{Q_R} \left\langle \Delta_h H_{p-1}\left(Du^\varepsilon\right), D\varphi \right\rangle dz$$
$$+\varepsilon \int_{Q_R} \left\langle \Delta_h \left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}} Du^\varepsilon \right), D\varphi \right\rangle dz = \int_{Q_R} f^\varepsilon \cdot \Delta_{-h}\varphi \, dz.$$

For $\Phi \in W_0^{1,\infty}(Q_R)$ non-negative and $g \in W^{1,\infty}(\mathbb{R})$ non-negative and non-decreasing, we choose $\varphi = \Phi \cdot \Delta_h u^\varepsilon \cdot g\left(|\Delta_h u^\varepsilon|^2\right)$ in the previous identity, thus getting

$$\int_{Q_R} \partial_t \left(\Delta_h u^\varepsilon\right) \Delta_h u^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \Phi \, dz$$
$$+ \int_{Q_R} \left\langle \Delta_h H_{p-1}\left(Du^\varepsilon\right), D\left[\Phi \Delta_h u^\varepsilon g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right] \right\rangle dz$$
$$+\varepsilon \int_{Q_R} \left\langle \Delta_h \left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}} Du^\varepsilon \right), D\left[\Phi \Delta_h u^\varepsilon g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right] \right\rangle dz$$
$$= \int_{Q_R} f^\varepsilon \cdot \Delta_{-h}\left(\Phi \Delta_h u^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right) dz,$$

i.e.

$$\int_{Q_R} \partial_t \left(\Delta_h u^\varepsilon\right) \Delta_h u^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \Phi \, dz$$
$$+ \int_{Q_R} \Phi \left\langle \Delta_h H_{p-1}\left(Du^\varepsilon\right), \Delta_h Du^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \right\rangle dz$$
$$+\varepsilon \int_{Q_R} \Phi \left\langle \Delta_h \left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}} Du^\varepsilon \right), \Delta_h Du^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \right\rangle dz$$
$$+2 \int_{Q_R} \Phi \left\langle \Delta_h H_{p-1}\left(Du^\varepsilon\right), \left|\Delta_h u^\varepsilon\right|^2 \Delta_h Du^\varepsilon \cdot g'\left(\left|\Delta_h u^\varepsilon\right|^2\right) \right\rangle dz$$
$$+2\varepsilon \int_{Q_R} \Phi \left\langle \Delta_h \left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}} Du^\varepsilon \right), \left|\Delta_h u^\varepsilon\right|^2 \Delta_h Du^\varepsilon \cdot g'\left(\left|\Delta_h u^\varepsilon\right|^2\right) \right\rangle dz$$
$$= - \int_{Q_R} \left\langle \Delta_h H_{p-1}\left(Du^\varepsilon\right), D\Phi \cdot \Delta_h u^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \right\rangle dz$$
$$-\varepsilon \int_{Q_R} \left\langle \Delta_h \left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}} Du^\varepsilon \right), D\Phi \cdot \Delta_h u^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \right\rangle dz$$
$$+ \int_{Q_R} f^\varepsilon \cdot \Delta_{-h}\left(\Phi \Delta_h u^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right) dz, \tag{3.7}$$

that we rewrite as follows

$$J_{h,1} + J_{h,2} + J_{h,3} + J_{h,4} + J_{h,5} = -J_{h,6} - J_{h,7} + J_{h,8}.$$

Arguing as in [7, 8], the first integral in Eq. (3.7) can be expressed as follows

$$J_{h,1} = \int_{Q_R} \partial_t \left(\Delta_h u^\varepsilon\right) \Delta_h u^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \Phi \, dz = \frac{1}{2} \int_{Q_R} \partial_t \left(\left|\Delta_h u^\varepsilon\right|^2\right) \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \Phi \, dz$$

$$= \frac{1}{2} \int_{Q_R} \partial_t \left(\int_0^{|\Delta_h u^\varepsilon|^2} g(s)\, ds \right) \Phi\, dz = -\frac{1}{2} \int_{Q_R} \left(\int_0^{|\Delta_h u^\varepsilon|^2} g(s)\, ds \right) \partial_t \Phi\, dz.$$

Using Lemma 2.2, since Φ, g are non-negative, we have

$$J_{h,2} \geq c_p \int_{Q_R} \Phi \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \left|\Delta_h Du^\varepsilon\right|^2 \frac{(|Du^\varepsilon| - 1)^p}{|Du^\varepsilon|\left(|Du^\varepsilon| + |Du^\varepsilon(x+h)|\right)}\, dz.$$

The right inequality in the assertion of Lemma 2.4 yields

$$J_{h,3} \geq \varepsilon c_p \int_{Q_R} \Phi \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \left|\Delta_h V_{1,p}\left(Du^\varepsilon\right)\right|^2\, dz$$

Moreover, again by Lemmas 2.2 and 2.4 and the fact that $g'(s) \geq 0$, we infer

$$J_{h,4} + J_{h,5} \geq 0.$$

Therefore (3.7) implies

$$\begin{aligned}
&-\frac{1}{2} \int_{Q_R} \left(\int_0^{|\Delta_h u^\varepsilon|^2} g(s)\, ds \right) \partial_t \Phi\, dz \\
&\quad + \int_{Q_R} \Phi \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \left|\Delta_h Du^\varepsilon\right|^2 \frac{(|Du^\varepsilon| - 1)^p}{|Du^\varepsilon|\left(|Du^\varepsilon| + |Du^\varepsilon(x+h)|\right)}\, dz \\
&\quad + c_p \varepsilon \int_{Q_R} \Phi \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \left|\Delta_h V_{1,p}\left(Du^\varepsilon\right)\right|^2\, dz \\
&\leq \int_{Q_R} |D\Phi| \left|\Delta_h H_{p-1}\left(Du^\varepsilon\right)\right| \left|\Delta_h u^\varepsilon\right| \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) dz \\
&\quad + \varepsilon \int_{Q_R} |D\Phi| \left|\Delta_h \left(\left(1 + \left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}} Du^\varepsilon\right)\right| \left|\Delta_h u^\varepsilon\right| \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) dz \\
&\quad + \int_{Q_R} \left|f^\varepsilon\right| \left|\Delta_{-h}\left(\Phi \Delta_h u^\varepsilon \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right)\right| dz. \qquad (3.8)
\end{aligned}$$

Now let us consider a parabolic cylinder $Q_\rho(z_0) \Subset Q_{2\rho}(z_0) \Subset Q_R(z_0)$ with $\rho < 2\rho < R$ and $t_0 > 0$. For a fixed time $\tau \in \left(t_0 - 4\rho^2, t_0\right)$ and $\theta \in (0, t_0 - \tau)$, we choose $\Phi(x,t) = \eta^2(x)\chi(t)\tilde{\chi}(t)$ with $\eta \in C_0^\infty\left(B_{2\rho}(x_0)\right)$, $0 \leq \eta \leq 1$, $\chi \in W^{1,\infty}([0,T])$ with $\partial_t \chi \geq 0$ and $\tilde{\chi}$ a Lipschitz continuous function defined, for $0 < \tau < \tau + \theta < T$, as follows

$$\tilde{\chi}(t) = \begin{cases} 1 & \text{if} \quad t \leq \tau, \\ 1 - \dfrac{t-\tau}{\theta} & \text{if} \quad \tau < t \leq \tau + \theta, \\ 0 & \text{if} \quad \tau + \theta < t \leq T, \end{cases}$$

so that (3.8) yields

$$\begin{aligned}
I_{h,1} + I_{h,2} + I_{h,3} := &\frac{1}{2} \int_{B_{2\rho}} \eta^2 \chi(\tau) \left(\int_0^{|\Delta_h u^\varepsilon(x,\tau)|^2} g(s)\, ds \right) dx \\
&+ c_p \int_{Q^\tau} \eta^2 \chi(t) \cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right) \left|\Delta_h Du^\varepsilon\right|^2 \\
&\times \frac{(|Du^\varepsilon| - 1)_+^p}{|Du^\varepsilon|\left(|Du^\varepsilon| + |Du^\varepsilon(x+h)|\right)}\, dz
\end{aligned}$$

$$
\begin{aligned}
&+c_p\varepsilon\int_{Q^\tau}\eta^2\chi(t)g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\left|\Delta_h V_{1,p}\left(Du^\varepsilon\right)\right|^2\,dz\\
\leq\ & 2\int_{Q^\tau}\eta\chi(t)\left|D\eta\right|\left|\Delta_h H_{p-1}\left(Du^\varepsilon\right)\right|\left|\Delta_h u^\varepsilon\right|\cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right)dz\\
&+2\varepsilon\int_{Q^\tau}\eta\chi(t)\left|D\eta\right|\left|\Delta_h\left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}}Du^\varepsilon\right)\right|\left|\Delta_h u^\varepsilon\right|\\
&\cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right)dz\\
&+\int_{Q^\tau}\chi(t)\left|f^\varepsilon\right|\left|\Delta_{-h}\left(\eta^2\Delta_h u^\varepsilon\cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right)\right|dz\\
&+\frac{1}{2}\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{\left|\Delta_h u^\varepsilon\right|^2}g(s)\,ds\right)dz\\
=:\ & I_{h,4}+I_{h,5}+I_{h,6}+I_{h,7},
\end{aligned}
\tag{3.9}
$$

where we used the notation $Q^\tau=B_{2\rho}(x_0)\times\left(t_0-4\rho^2,\tau\right)$.
Since $g\in W^{1,\infty}([0,\infty))$, by (3.3), by the last assertion of Lemma 2.9 and by Fatou's Lemma, we have

$$
\begin{aligned}
&\liminf_{h\to 0}\left(I_{h,1}+I_{h,2}+I_{h,3}\right)\\
&\geq\frac{1}{2}\int_{B_{2\rho}}\eta^2\chi(\tau)\left(\int_0^{\left|Du^\varepsilon(x,\tau)\right|^2}g(s)\,ds\right)dx\\
&+c_p\int_{Q^\tau}\eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\left|D^2u^\varepsilon\right|^2\frac{\left(\left|Du^\varepsilon\right|-1\right)^p}{\left|Du^\varepsilon\right|^2}\,dz\\
&+c_p\varepsilon\int_{Q^\tau}\eta^2\chi(t)g\left(\left|Du^\varepsilon\right|^2\right)\left|DV_{1,p}\left(Du^\varepsilon\right)\right|^2\,dz.
\end{aligned}
\tag{3.10}
$$

and

$$
\lim_{h\to 0}I_{h,7}=\frac{1}{2}\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{\left|Du^\varepsilon\right|^2}g(s)\,ds\right)dz.
\tag{3.11}
$$

Now let us observe that

$$
\left|DH_{p-1}\left(Du^\varepsilon\right)\right|\leq c_p\left(\left|Du^\varepsilon\right|-1\right)_+^{p-2}\left|D^2u^\varepsilon\right|
\tag{3.12}
$$

and, using Hölder's inequality with exponents $\left(\frac{2(p-1)}{p-2},\frac{2(p-1)}{p}\right)$, we have

$$
\begin{aligned}
\int_{B_R}\left|DH_{p-1}\left(Du^\varepsilon\right)\right|^{\frac{p}{p-1}}dx&\leq c_p\int_{B_R}\left[\left(\left|Du^\varepsilon\right|-1\right)_+^{p-2}\left|D^2u^\varepsilon\right|\right]^{\frac{p}{p-1}}dx\\
&\leq c_p\left(\int_{B_R}\left(\left|Du^\varepsilon\right|-1\right)_+^p dx\right)^{\frac{p-2}{2(p-1)}}\\
&\quad\cdot\left(\int_{B_R}\left[\left(\left|Du^\varepsilon\right|-1\right)_+^{\frac{p-2}{2}}\left|D^2u^\varepsilon\right|\right]^2dx\right)^{\frac{p}{2(p-1)}},
\end{aligned}
$$

and since, by (3.4), the right hand side of previous inequality is finite, again by Lemma 2.9 we have

$$\Delta_h H_{p-1}\left(Du^\varepsilon\right) \to DH_{p-1}\left(Du^\varepsilon\right) \qquad \text{strongly in} \qquad L^2\left(0, T; L^{\frac{p}{p-1}}\left(B_R\right)\right) \qquad \text{as } h \to 0,$$

which, since $\Delta_h u^\varepsilon \to Du^\varepsilon$ strongly in $L^2\left(0, T; L^p\left(B_R\right)\right)$ as $h \to 0$, implies

$$\lim_{h\to 0} I_{h,4} = 2\int_{Q^\tau} \eta\chi(t)\left|D\eta\right|\left|DH_{p-1}\left(Du^\varepsilon\right)\right|\left|Du^\varepsilon\right| g\left(\left|Du^\varepsilon\right|^2\right) dz. \tag{3.13}$$

Using similar arguments, we can check that

$$\lim_{h\to 0} I_{h,5} = 2\varepsilon\int_{Q^\tau} \eta\chi(t)\left|D\eta\right|\left|D\left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}} Du^\varepsilon\right)\right|\left|Du^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right) dz. \tag{3.14}$$

Now, by Proposition 2.7 (c), it holds

$$\begin{aligned}\left|\Delta_{-h}\left(\eta^2\Delta_h u^\varepsilon\cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right)\right| \le{}& c\|D\eta\|_\infty\left|\Delta_h u^\varepsilon\right|\left|g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right| \\ &+c\left|\Delta_{-h}\left(\Delta_h u^\varepsilon\right)\right|\left|g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right| \\ &+c\left|\Delta_h u^\varepsilon\right|^2\left|g'\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right|\left|\Delta_h Du^\varepsilon\right|.\end{aligned}$$

and choosing g such that

$$sg'\left(s^2\right) \le M, \tag{3.15}$$

for a positive constant M and for every $s > 0$, we have

$$\begin{aligned}\left|\Delta_{-h}\left(\eta^2\Delta_h u^\varepsilon\cdot g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right)\right| \le{}& c\|D\eta\|_\infty\left|\Delta_h u^\varepsilon\right|\left|g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right| \\ &+c\left|\Delta_{-h}\left(\Delta_h u^\varepsilon\right)\right|\left|g\left(\left|\Delta_h u^\varepsilon\right|^2\right)\right| \\ &+c_M\left|\Delta_h u^\varepsilon\right|\left|\Delta_{-h} Du^\varepsilon\right|.\end{aligned} \tag{3.16}$$

Since $\Delta_h u^\varepsilon \to Du^\varepsilon$, $\Delta_{-h}\left(\Delta_h u^\varepsilon\right) \to D^2u^\varepsilon$, $\Delta_{-h}Du^\varepsilon \to D^2u^\varepsilon$ strongly in $L^2\left(0, T; L^2_{\mathrm{loc}}\right.$ $\left.(\Omega)\right)$ as $h \to 0$, and $f^\varepsilon \in C^\infty\left(\Omega_T\right)$, thanks to (3.16), we have

$$\lim_{h\to 0} I_{h,6} = \int_{Q^\tau} \chi(t)\left|f^\varepsilon\right|\left|D\left(\eta^2 Du^\varepsilon\cdot g\left(\left|Du^\varepsilon\right|^2\right)\right)\right| dz. \tag{3.17}$$

So, collecting (3.10), (3.11), (3.13), (3.14) and (3.17), we can pass to the limit as $h \to 0$ in (3.9), thus getting

$$\begin{aligned}&\frac{1}{2}\int_{B_{2\rho}} \eta^2\chi(\tau)\left(\int_0^{\left|Du^\varepsilon(x,\tau)\right|^2} g(s)\,ds\right) dx \\ &\quad+c_p\int_{Q^\tau} \eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\left|D^2u^\varepsilon\right|^2 \frac{\left(\left|Du^\varepsilon\right|-1\right)^p}{\left|Du^\varepsilon\right|^2}\, dz \\ &\quad+c_p\varepsilon\int_{Q^\tau} \eta^2\chi(t) g\left(\left|Du^\varepsilon\right|^2\right)\left|DV_{1,p}\left(Du^\varepsilon\right)\right|^2 dz \\ &\le 2\int_{Q^\tau} \eta\chi(t)\left|D\eta\right|\left|DH_{p-1}\left(Du^\varepsilon\right)\right|\left|Du^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right) dz\end{aligned}$$

$$
\begin{aligned}
&+2\varepsilon\int_{Q^\tau}\eta\chi(t)\,|D\eta|\left|D\left(\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}}Du^\varepsilon\right)\right|\left|Du^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+\int_{Q^\tau}\chi(t)\left|f^\varepsilon\right|\left|D\left(\eta^2Du^\varepsilon\cdot g\left(\left|Du^\varepsilon\right|^2\right)\right)\right|dz\\
&+\frac{1}{2}\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{|Du^\varepsilon|^2}g(s)\,ds\right)dz\\
=:&\ \tilde{I}_1+\tilde{I}_2+\tilde{I}_3+\tilde{I}_4, \qquad (3.18)
\end{aligned}
$$

for every $g\in W^{1,\infty}([0,+\infty))$ such that (3.15) holds true. Now, by (3.12) and by Young's inequality, we have

$$
\begin{aligned}
\tilde{I}_1+\tilde{I}_2\le\ & c_p\int_{Q^\tau}\eta\chi(t)\,|D\eta|\left(\left|Du^\varepsilon\right|-1\right)_+^{p-2}\left|D^2u^\varepsilon\right|\left|Du^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+c_p\cdot\varepsilon\int_{Q^\tau}\eta\chi(t)\,|D\eta|\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-1}{2}}\left|D^2u^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
\le\ & \sigma\int_{Q^\tau}\eta^2\chi(t)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+\sigma\varepsilon\int_{Q^\tau}\eta^2\chi(t)\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p-2}{2}}\left|D^2u^\varepsilon\right|^2\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+c_\sigma\int_{Q^\tau}\chi(t)\,|D\eta|^2\left(\left|Du^\varepsilon\right|-1\right)_+^{p-4}\left|Du^\varepsilon\right|^4\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+c_{p,\sigma}\cdot\varepsilon\int_{Q^\tau}\chi(t)\,|D\eta|^2\left(1+\left|Du^\varepsilon\right|^2\right)^{\frac{p}{2}}\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
\le\ & \sigma\int_{Q^\tau}\eta^2\chi(t)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+\sigma\varepsilon\int_{Q^\tau}\eta^2\chi(t)\left|DV_{1,p}\left(Du^\varepsilon\right)\right|^2\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+c_{\sigma,p}\,\|D\eta\|_{L^\infty}^2\,\|g\|_{L^\infty}\int_{Q^\tau}\chi(t)\left(1+\left|Du^\varepsilon\right|\right)^p\,dz, \qquad (3.19)
\end{aligned}
$$

where we used (3.3), and where $\sigma>0$ is a parameter that will be chosen later.
Now, using Young's Inequality, we estimate the term $\tilde{I}_3$ as follows

$$
\begin{aligned}
\tilde{I}_3\le\ & c\int_{Q^\tau}\chi(t)\left|f^\varepsilon\right|\eta\,|D\eta|\left|Du^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+c\int_{Q^\tau}\chi(t)\left|f^\varepsilon\right|\eta^2\left|D^2u^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&+c\int_{Q^\tau}\chi(t)\left|f^\varepsilon\right|\eta^2\left|Du^\varepsilon\right|^2\left|D^2u^\varepsilon\right|\cdot g'\left(\left|Du^\varepsilon\right|^2\right)dz\\
\le\ & c\,\|D\eta\|_\infty\,\|g\|_{L^\infty}\int_{Q^\tau}\eta\chi(t)\left|f^\varepsilon\right|^2dz\\
&+c\,\|D\eta\|_\infty\,\|g\|_{L^\infty}\int_{Q^\tau}\eta\chi(t)\left|Du^\varepsilon\right|^2dz
\end{aligned}
$$

$$+c\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|D^2u^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz$$
$$+c\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|Du^\varepsilon\right|^2\left|D^2u^\varepsilon\right|\cdot g'\left(\left|Du^\varepsilon\right|^2\right)dz. \qquad (3.20)$$

Plugging (3.19) and (3.20) into (3.18), we get

$$\begin{aligned}
&\frac{1}{2}\int_{B_{2\rho}}\eta^2\chi(\tau)\left(\int_0^{|Du^\varepsilon(x,\tau)|^2}g(s)\,ds\right)dx\\
&\quad+c_p\int_{Q^\tau}\eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2dz\\
&\quad+c_p\varepsilon\int_{Q^\tau}\eta^2\chi(t)g\left(\left|Du^\varepsilon\right|^2\right)\left|DV_{1,p}\left(Du^\varepsilon\right)\right|^2dz\\
&\le\sigma\int_{Q^\tau}\eta^2\chi(t)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+\sigma\varepsilon\int_{Q^\tau}\eta^2\chi(t)\left|DV_{1,p}\left(Du^\varepsilon\right)\right|^2\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+c_{p,\sigma}\left\|D\eta\right\|_\infty\left\|g\right\|_{L^\infty}\int_{Q^\tau}\eta\chi(t)\left|f^\varepsilon\right|^2dz\\
&\quad+c_{p,\sigma}\left\|D\eta\right\|_\infty\left\|g\right\|_{L^\infty}\int_{Q^\tau}\eta\chi(t)\left(1+|Du^\varepsilon|\right)^p dz\\
&\quad+c\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|D^2u^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+c\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|Du^\varepsilon\right|^2\left|D^2u^\varepsilon\right|\cdot g'\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+\frac{1}{2}\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{|Du^\varepsilon|^2}g(s)\,ds\right)dz,
\end{aligned}$$

which, for a sufficiently small σ, gives

$$\begin{aligned}
&\frac{1}{2}\int_{B_{2\rho}}\eta^2\chi(\tau)\left(\int_0^{|Du^\varepsilon(x,\tau)|^2}g(s)\,ds\right)dx\\
&\quad+c_p\int_{Q^\tau}\eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2dz\\
&\quad+c_p\varepsilon\int_{Q^\tau}\eta^2\chi(t)g\left(\left|Du^\varepsilon\right|^2\right)\left|DV_{1,p}\left(Du^\varepsilon\right)\right|^2dz\\
&\le c_p\left\|D\eta\right\|_\infty\left\|g\right\|_{L^\infty}\int_{Q^\tau}\eta\chi(t)\left|f^\varepsilon\right|^2dz\\
&\quad+c_p\left\|D\eta\right\|_\infty\left\|g\right\|_{L^\infty}\int_{Q^\tau}\eta\chi(t)\left(1+|Du^\varepsilon|\right)^p dz\\
&\quad+c\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|D^2u^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+c\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|Du^\varepsilon\right|^2\left|D^2u^\varepsilon\right|\cdot g'\left(\left|Du^\varepsilon\right|^2\right)dz
\end{aligned}$$

$$+\frac{1}{2}\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{|Du^\varepsilon|^2}g(s)\,ds\right)dz,$$

that, neglecting the third integral in the left hand side, implies

$$\begin{aligned}
&\frac{1}{2}\int_{B_{2\rho}}\eta^2\chi(\tau)\left(\int_0^{|Du^\varepsilon(x,\tau)|^2}g(s)\,ds\right)dx\\
&\quad+c_p\int_{Q^\tau}\eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2dz\\
&\le c_p\|D\eta\|_\infty\|g\|_{L^\infty}\int_{Q^\tau}\eta\chi(t)\left|f^\varepsilon\right|^2dz\\
&\quad+c_p\|D\eta\|_\infty\|g\|_{L^\infty}\int_{Q^\tau}\eta\chi(t)\left(1+|Du^\varepsilon|\right)^p dz\\
&\quad+c\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|D^2u^\varepsilon\right|\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+c\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|Du^\varepsilon\right|^2\left|D^2u^\varepsilon\right|\cdot g'\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+\frac{1}{2}\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{|Du^\varepsilon|^2}g(s)\,ds\right)dz.
\end{aligned}\tag{3.21}$$

Now, for $\delta\in(0,1)$, recalling the notation in (2.3), we choose

$$g(s)=g_{1+\delta}\left((s-1-\delta)_+\right)=\frac{(s-1-\delta)_+^2}{1+\delta+(s-1-\delta)_+^2},$$

that is legitimate since, with this choice, $g\in W^{1,\infty}([0,+\infty))$ and $g'(s)\ge 0$. Moreover, we have $g(s)\in[0,1]$, for every $s\ge 0$, and thanks to (2.5), there exists a constant $c_\delta>0$ such that

$$sg'\left(s^2\right)\le c_\delta\qquad\text{for every } s\ge 0,$$

so that (3.15) holds. Therefore, since $g(s)$ vanishes on the set where $s\le 1+\delta$ and $g(s)\le 1$ for every s, (3.21) becomes

$$\begin{aligned}
&\frac{1}{2}\int_{B_{2\rho}}\eta^2\chi(\tau)\left(\int_0^{|Du^\varepsilon(x,\tau)|^2}g(s)\,ds\right)dx\\
&\quad+c_p\int_{Q^\tau}\eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2dz\\
&\le c\int_{Q^\tau\cap\{|Du^\varepsilon|^2>1+\delta\}}\eta^2\chi(t)\left|f^\varepsilon\right|\left|D^2u^\varepsilon\right|\frac{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}{|Du^\varepsilon|}\frac{|Du^\varepsilon|}{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}\cdot g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+c\int_{Q^\tau\cap\{|Du^\varepsilon|^2>1+\delta\}}\eta^2\chi(t)\left|f^\varepsilon\right|\left|Du^\varepsilon\right|^2\frac{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}{|Du^\varepsilon|}\frac{|Du^\varepsilon|}{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}\left|D^2u^\varepsilon\right|g'\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+c_p\|D\eta\|_\infty\|\chi\|_{L^\infty}\int_{Q^\tau}\left(1+|Du^\varepsilon|^p+\left|f^\varepsilon\right|^2\right)dz+\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{|Du^\varepsilon|^2}g(s)\,ds\right)dz
\end{aligned}$$

$$\leq \frac{c_p}{\delta^{\frac{p}{2}}} \int_{Q^\tau} \eta^2 \chi(t) \left|f^\varepsilon\right| \left|D^2 u^\varepsilon\right| \frac{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}{|Du^\varepsilon|} \cdot g\left(\left|Du^\varepsilon\right|^2\right) dz$$
$$+\frac{c_p}{\delta^{\frac{p}{2}}} \int_{Q^\tau} \eta^2 \chi(t) \left|f^\varepsilon\right| \left|Du^\varepsilon\right|^2 \frac{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}{|Du^\varepsilon|} \left|D^2 u^\varepsilon\right| g'\left(\left|Du^\varepsilon\right|^2\right) dz$$
$$+c_p \|D\eta\|_\infty \|\chi\|_{L^\infty} \int_{Q^\tau} \left(1+|Du^\varepsilon|^p+\left|f^\varepsilon\right|^2\right) dz+\int_{Q^\tau} \eta^2 \partial_t \chi(t) \left(\int_0^{|Du^\varepsilon|^2} g(s)\, ds\right) dz,$$

where we used that

$$\sup_{x\in(\sqrt{1+\delta},+\infty)} \frac{x}{(x-1)^{\frac{p}{2}}}=\frac{\sqrt{1+\delta}}{\left(\sqrt{1+\delta}-1\right)^{\frac{p}{2}}}=\frac{\sqrt{1+\delta}\left(\sqrt{1+\delta}+1\right)^{\frac{p}{2}}}{\delta^{\frac{p}{2}}} \leq \frac{c_p}{\delta^{\frac{p}{2}}},$$

since $\delta<1$ and $p\geq 2$. Using Young's inequality in the first integral in the right hand, previous estimate yields

$$\frac{1}{2} \int_{B_{2\rho}} \eta^2 \chi(\tau) \left(\int_0^{|Du^\varepsilon(x,\tau)|^2} g(s)\, ds\right) dx$$
$$+c_p \int_{Q^\tau} \eta^2 \chi(t) \cdot g\left(\left|Du^\varepsilon\right|^2\right) \frac{(|Du^\varepsilon|-1)_+^{p}}{|Du^\varepsilon|^2} \left|D^2 u^\varepsilon\right|^2 dz$$
$$\leq \frac{c_p(\beta)}{\delta^p} \int_{Q^\tau} \eta^2 \chi(t) \left|f^\varepsilon\right|^2 \cdot g\left(\left|Du^\varepsilon\right|^2\right) dz$$
$$+\beta \int_{Q^\tau} \eta^2 \chi(t) \frac{(|Du^\varepsilon|-1)_+^{p}}{|Du^\varepsilon|^2} \left|D^2 u^\varepsilon\right|^2 \cdot g\left(\left|Du^\varepsilon\right|^2\right) dz$$
$$+\frac{c_p}{\delta^{\frac{p}{2}}} \int_{Q^\tau} \eta^2 \chi(t) \left|f^\varepsilon\right| \left|Du^\varepsilon\right|^2 \frac{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}{|Du^\varepsilon|} \left|D^2 u^\varepsilon\right| g'\left(\left|Du^\varepsilon\right|^2\right) dz$$
$$+c_p \|D\eta\|_\infty \|\chi\|_{L^\infty} \int_{Q^\tau} \left(1+|Du^\varepsilon|^p+\left|f^\varepsilon\right|^2\right) dz$$
$$+\int_{Q^\tau} \eta^2 \partial_t \chi(t) \left(\int_0^{|Du^\varepsilon|^2} g(s)\, ds\right) dz.$$

Choosing β sufficiently small, reabsorbing the second integral in the right hand side by the left hand side and using that $g(s)\leq 1$, we get

$$\int_{B_{2\rho}} \eta^2 \chi(\tau) \left(\int_0^{|Du^\varepsilon(x,\tau)|^2} g(s)\, ds\right) dx$$
$$+\int_{Q^\tau} \eta^2 \chi(t) \cdot g\left(\left|Du^\varepsilon\right|^2\right) \frac{(|Du^\varepsilon|-1)_+^{p}}{|Du^\varepsilon|^2} \left|D^2 u^\varepsilon\right|^2 dz$$
$$\leq \frac{c_p}{\delta^{\frac{p}{2}}} \int_{Q^\tau} \eta^2 \chi(t) \left|f^\varepsilon\right| \left|Du^\varepsilon\right|^2 \frac{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}{|Du^\varepsilon|} \left|D^2 u^\varepsilon\right| g'\left(\left|Du^\varepsilon\right|^2\right) dz$$
$$+\int_{Q^\tau} \eta^2 \partial_t \chi(t) \left(\int_0^{|Du^\varepsilon|^2} g(s)\, ds\right) dz$$
$$+c\, \|D\eta\|_\infty \|\chi\|_\infty \int_{Q^\tau} \left(1+\left|Du^\varepsilon\right|\right)^p dz$$

$$+c\,\|\chi\|_{L^\infty}\left(\frac{c_p}{\delta^p}+\|D\eta\|_{L^\infty}\right)\int_{Q^\tau}\left|f^\varepsilon\right|^2 dz. \tag{3.22}$$

We now estimate the first integral in the right hand side of previous inequality with the use of (2.4) with $s=|Du^\varepsilon|^2$, $A=\dfrac{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}{|Du^\varepsilon|}\left|D^2u^\varepsilon\right|$, $B=\dfrac{c_p}{\delta^{\frac{p}{2}}}\,|f^\varepsilon|$ and $k=1+\delta$, thus getting

$$\begin{aligned}
&\frac{c_p}{\delta^{\frac{p}{2}}}\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|\left|Du^\varepsilon\right|^2\frac{(|Du^\varepsilon|-1)_+^{\frac{p}{2}}}{|Du^\varepsilon|}\left|D^2u^\varepsilon\right|g'\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\le 2\alpha\int_{Q^\tau}\eta^2\chi(t)\frac{(|Du^\varepsilon|-1)_+^{p}}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2 g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+\frac{c_{\alpha,p}}{\delta^p}\int_{Q^\tau}\eta^2\chi(t)\left|f^\varepsilon\right|^2 dz,
\end{aligned} \tag{3.23}$$

where we also used the fact that $\delta<1$.

Inserting (3.23) in (3.22), we find

$$\begin{aligned}
&\int_{B_{2\rho}}\eta^2\chi(\tau)\left(\int_0^{|Du^\varepsilon(x,\tau)|^2}g(s)\,ds\right)dx\\
&\quad+\int_{Q^\tau}\eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\frac{(|Du^\varepsilon|-1)_+^{p}}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2 dz\\
&\le 2\alpha\int_{Q^\tau}\eta^2\chi(t)\frac{(|Du^\varepsilon|-1)_+^{p}}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2 g\left(\left|Du^\varepsilon\right|^2\right)dz\\
&\quad+\frac{c_{\alpha,p}}{\delta^p}\int_{Q^\tau}\eta^2\chi(t)|f^\varepsilon|^2\,dz\\
&\quad+\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{|Du^\varepsilon|^2}g(s)\,ds\right)dz\\
&\quad+c\,\|D\eta\|_\infty\,\|\chi\|_\infty\int_{Q^\tau}\left(1+|Du^\varepsilon|\right)^p dz\\
&\quad+c\,\|\chi\|_{L^\infty}\left(\frac{c_p}{\delta^p}+\|D\eta\|_{L^\infty}\right)\int_{Q^\tau}\left|f^\varepsilon\right|^2 dz.
\end{aligned}$$

Choosing $\alpha=\frac{1}{4}$, we can reabsorb the first integral in the right hand side by the left hand side, thus obtaining

$$\begin{aligned}
&\int_{B_{2\rho}}\eta^2\chi(\tau)\left(\int_0^{|Du^\varepsilon(x,\tau)|^2}g(s)\,ds\right)dx\\
&\quad+\int_{Q^\tau}\eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\frac{(|Du^\varepsilon|-1)_+^{p}}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2 dz\\
&\le c\,\|D\eta\|_\infty\,\|\chi\|_\infty\int_{Q^\tau}\left(1+\left|Du^\varepsilon\right|\right)^p dz\\
&\quad+\frac{c}{\delta^p}\,\|\chi\|_{L^\infty}\left(1+\|D\eta\|_{L^\infty}\right)\int_{Q^\tau}\left|f^\varepsilon\right|^2 dz
\end{aligned}$$

$$+c\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\int_0^{|Du^\varepsilon|^2}g(s)\,ds\right)dz. \tag{3.24}$$

By the definition of g, we have

$$\int_0^\zeta g(s)\,ds=\begin{cases}0 & \text{if} \quad 0<\zeta\le 1+\delta,\\ \displaystyle\int_{1+\delta}^{\zeta}\frac{(s-1-\delta)^2}{1+\delta+(s-1-\delta)^2}\,ds & \text{if} \quad \zeta>1+\delta,\end{cases}$$

and so it is easy to check that

$$\int_0^\zeta g(s)\,ds=\begin{cases}0 & \text{if} \quad 0<\zeta\le 1+\delta,\\ \zeta-1-\delta-\sqrt{1+\delta}\arctan\left(\dfrac{\zeta-1-\delta}{\sqrt{1+\delta}}\right) & \text{if} \quad \zeta>1+\delta,\end{cases}$$

that is

$$\int_0^\zeta g(s)\,ds=(\zeta-1-\delta)_+-\sqrt{1+\delta}\arctan\left[\frac{(\zeta-1-\delta)_+}{\sqrt{1+\delta}}\right].$$

Therefore, by the previous equality and the properties of χ and η, (3.24) implies

$$\begin{aligned}&\int_{B_{2\rho}}\eta^2\chi(\tau)\left(\left|Du^\varepsilon(x,\tau)\right|^2-1-\delta\right)_+dx\\&\quad+\int_{Q^\tau}\eta^2\chi(t)\cdot g\left(\left|Du^\varepsilon\right|^2\right)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2dz\\&\le c\,\|D\eta\|_\infty\,\|\chi\|_\infty\int_{Q^\tau}\left(1+\left|Du^\varepsilon\right|\right)^p dz\\&\quad+\frac{c}{\delta^p}\,\|\chi\|_{L^\infty}\left(1+\|D\eta\|_{L^\infty}\right)\int_{Q^\tau}\left|f^\varepsilon\right|^2dz\\&\quad+c\int_{Q^\tau}\eta^2\partial_t\chi(t)\left(\left|Du^\varepsilon\right|^2-1-\delta\right)_+dz\\&\quad+c\,\|\partial_t\chi\|_\infty\left|Q^\tau\right|+c\,\|\chi\|_\infty\,|B_R|,\end{aligned} \tag{3.25}$$

which holds for almost every $\tau\in\left(t_0-4\rho^2,t_0\right)$. We now choose a cut-off function $\eta\in C^\infty\left(B_{2\rho}(x_0)\right)$ with $\eta\equiv 1$ on $B_\rho(x_0)$ such that $0\le\eta\le 1$ and $|D\eta|\le\frac{c}{\rho}$. For the cut-off function in time, we choose $\chi\in W^{1,\infty}\left(t_0-R^2,t_0,[0,1]\right)$ such that $\chi\equiv 0$ on $\left(t_0-R^2,t_0-4\rho^2\right]$, $\chi\equiv 1$ on $\left[t_0-\rho^2,t_0\right)$ and $\partial_t\chi\le\frac{c}{\rho^2}$ on $\left(t_0-4\rho^2,t_0-\rho^2\right)$. With these choices, (3.25) gives

$$\begin{aligned}&\sup_{\tau\in\left(t_0-4\rho^2,t_0\right)}\int_{B_\rho}\chi(\tau)\left(\left|Du^\varepsilon(x,\tau)\right|^2-1-\delta\right)_+dx\\&\quad+\int_{Q_\rho}g\left(\left|Du^\varepsilon\right|^2\right)\frac{(|Du^\varepsilon|-1)_+^p}{|Du^\varepsilon|^2}\left|D^2u^\varepsilon\right|^2dz\\&\le\frac{c}{\rho^2}\int_{Q_{2\rho}}\left(1+\left|Du^\varepsilon\right|^p\right)dz+\frac{c}{\rho^2\delta^p}\int_{Q_{2\rho}}\left|f^\varepsilon\right|^2dz\\&\quad+\frac{c\left|Q_{2\rho}\right|}{\rho^2}+c\left|B_{2\rho}\right|,\end{aligned}$$

and, since $\rho < 2\rho < R < 1$ and $Q_{2\rho} = B_\rho \times \left(t_0 - 4\rho^2, t_0\right)$, we have

$$\sup_{\tau \in \left(t_0 - 4\rho^2, t_0\right)} \int_{B_\rho} \left(\left|Du^\varepsilon(x,\tau)\right|^2 - 1 - \delta\right)_+ dx + \int_{Q_\rho} g\left(\left|Du^\varepsilon\right|^2\right) \frac{(|Du^\varepsilon| - 1)_+^p}{|Du^\varepsilon|^2} \left|D^2 u^\varepsilon\right|^2 dz \leq \frac{c}{\rho^2} \int_{Q_{2\rho}} \left(1 + \left|Du^\varepsilon\right|^p\right) dz + \frac{c}{\rho^2 \delta^p} \int_{Q_{2\rho}} \left|f^\varepsilon\right|^2 dz. \tag{3.26}$$

Now, with $\mathcal{G}_\delta(t)$ defined at (2.8), recalling (2.9), we have

$$\begin{aligned} &\left|D\left[\mathcal{G}_\delta\left(\left(\left|Du^\varepsilon\right| - \delta - 1\right)_+\right)\right]\right|^2 \\ &\leq \frac{(|Du^\varepsilon| - \delta - 1)_+^2}{1 + \delta + (|Du^\varepsilon| - \delta - 1)_+^2} \left[\left(\left|Du^\varepsilon\right| - \delta - 1\right)_+ + \delta\right]^{p-2} \left|D^2 u^\varepsilon\right|^2 \\ &= g\left(\left|Du^\varepsilon\right|\right) \left[\left(\left|Du^\varepsilon\right| - \delta - 1\right)_+ + \delta\right]^{p-2} \left|D^2 u^\varepsilon\right|^2. \end{aligned}$$

One can easily check that $g(s) \leq g\left(s^2\right)$, and therefore

$$\begin{aligned} \left|D\left[\mathcal{G}_\delta\left(\left(\left|Du^\varepsilon\right| - \delta - 1\right)_+\right)\right]\right|^2 &\leq g\left(\left|Du^\varepsilon\right|^2\right) \left(\left|Du^\varepsilon\right| - 1\right)_+^{p-2} \left|D^2 u^\varepsilon\right|^2 \\ &\leq \frac{c_p}{\delta^2} g\left(\left|Du^\varepsilon\right|^2\right) \frac{(|Du^\varepsilon| - 1)_+^p}{|Du^\varepsilon|^2} \left|D^2 u^\varepsilon\right|^2, \end{aligned} \tag{3.27}$$

where we also used that $g(s) = 0$, for $0 < s \leq 1 + \delta$. Using (3.27) in the left hand side of (3.26), we obtain

$$\sup_{\tau \in \left(t_0 - 4\rho^2, t_0\right)} \int_{B_\rho} \left(\left|Du^\varepsilon(x,\tau)\right|^2 - 1 - \delta\right)_+ dx + \int_{Q_\rho} \left|D\left[\mathcal{G}_\delta\left(\left(\left|Du^\varepsilon\right| - \delta - 1\right)_+\right)\right]\right|^2 dz \leq \frac{c}{\rho^2 \delta^2} \left[\int_{Q_{2\rho}} \left(1 + \left|Du^\varepsilon\right|^p\right) dz + \frac{1}{\delta^p} \int_{Q_{2\rho}} \left|f^\varepsilon\right|^2 dz\right],$$

which is (3.5). □

Combining Lemmas 3.1 and 2.8, we have the following.

Corollary 3.2 *Let $u^\varepsilon \in C^0\left(t_0 - R^2, t_0; L^2\left(B_R\right)\right) \cap L^p\left(t_0 - R^2, t_0; u + W_0^{1,p}\left(B_R\right)\right)$ be the unique solution to (3.2). Then the following estimate*

$$\int_{Q_{\frac{\rho}{2}}} \left|\tau_h\left[\mathcal{G}_\delta\left(\left(\left|Du^\varepsilon\right| - \delta - 1\right)_+\right)\right]\right|^2 dz \leq \frac{c|h|^2}{\rho^2 \delta^2} \left[\int_{Q_{2\rho}} \left(1 + \left|Du^\varepsilon\right|^p\right) dz + \frac{1}{\delta^p} \int_{Q_{2\rho}} \left|f^\varepsilon\right|^2 dz\right] \tag{3.28}$$

holds for $|h| < \frac{\rho}{4}$, for any parabolic cylinder $Q_{2\rho} \Subset Q_R\left(z_0\right)$.

4 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, that will be divided in two steps.
In the first one we shall establish an estimate that will allow us to measure the L^2-distance between $H_{\frac{p}{2}}(Du)$ and $H_{\frac{p}{2}}(Du^{\varepsilon})$ in terms of the L^2-distance between f and f^{ε}.
In the second one, we conclude combining this comparison estimate with the one obtained for the difference quotient of the solution to the regularized problem at (3.28).

Proof of Theorem 1.1 *Step 1: the comparison estimate.*
We formally proceed by testing Eqs. (1.1) and (3.2) with the map $\varphi = k(t)\,(u^{\varepsilon} - u)$, where $k \in W^{1,\infty}(\mathbb{R})$ is chosen such that

$$k(t) = \begin{cases} 1 & \text{if} \quad t \le t_2, \\ -\dfrac{1}{\omega}(t - t_2 - \omega) & \text{if} \quad t_2 < t < t_2 + \omega, \\ 0 & \text{if} \quad t \ge t_2 + \omega, \end{cases}$$

with $t_0 - R^2 < t_2 < t_2 + \omega < t_0$, and then letting $\omega \to 0$. We observe that, at this stage, it is important that u^{ε} and u agree on the parabolic boundary $\partial_{\text{par}} Q_R(z_0)$.
Proceeding in a standard way (see for example [19]), for almost every $t_2 \in \left(t_0 - R^2, t_0\right)$, we find

$$\begin{aligned} &\frac{1}{2}\int_{B_R(x_0)} \left|u^{\varepsilon}(x, t_2) - u(x, t_2)\right|^2 dx \\ &\quad + \int_{Q_{R,t_2}} \left\langle H_{p-1}\left(Du^{\varepsilon}\right) - H_{p-1}(Du), Du^{\varepsilon} - Du \right\rangle dz \\ &\quad + \varepsilon \int_{Q_{R,t_2}} \left\langle \left(1 + \left|Du^{\varepsilon}\right|^2\right)^{\frac{p-2}{2}} Du^{\varepsilon}, Du^{\varepsilon} - Du \right\rangle dz \\ &= \int_{Q_{R,t_2}} \left(f - f^{\varepsilon}\right)\left(u^{\varepsilon} - u\right) dz, \end{aligned} \tag{4.1}$$

where we used the abbreviation $Q_{R,t_2} = B_R(x_0) \times \left(t_0 - R^2, t_2\right)$. Using Lemma 2.1, the Cauchy-Schwarz inequality as well as Young's inequality, from (4.1) we infer

$$\begin{aligned} &\lambda_p \sup_{t \in (t_0 - R^2, t_0)} \left\|u^{\varepsilon}(\cdot, t) - u(\cdot, t)\right\|^2_{L^2(B_R(x_0))} \\ &\quad + \lambda_p \int_{Q_R} \left|H_{\frac{p}{2}}\left(Du^{\varepsilon}\right) - H_{\frac{p}{2}}(Du)\right|^2 dz + \varepsilon \int_{Q_R(z_0)} \left|Du^{\varepsilon}\right|^p dz \\ &\le \int_{Q_R} \left|f - f^{\varepsilon}\right|\left|u^{\varepsilon} - u\right| dz + \varepsilon \int_{Q_R} \left|Du^{\varepsilon}\right|^{p-1} |Du|\, dz \\ &\le \int_{Q_R} \left|f - f^{\varepsilon}\right|\left|u^{\varepsilon} - u\right| dz + \varepsilon \cdot c_p \int_{Q_R} |Du|^p\, dz \\ &\quad + \frac{1}{2} \cdot \varepsilon \int_{Q_R} \left|Du^{\varepsilon}\right|^p dz, \end{aligned} \tag{4.2}$$

where we set $\lambda_p = \min\left\{\frac{1}{2}, \frac{4}{p^2}\right\}$. Reabsorbing the last integral in the right-hand side of (4.2) by the left-hand side, we arrive at

$$\begin{aligned}
&\sup_{t\in(t_0-R^2,t_0)} \left\|u^\varepsilon(\cdot,t)-u(\cdot,t)\right\|^2_{L^2(B_R(x_0))}\\
&\quad+\int_{Q_R}\left|H_{\frac{p}{2}}\left(Du^\varepsilon\right)-H_{\frac{p}{2}}\left(Du\right)\right|^2 dz+\frac{\varepsilon}{2\lambda_p}\int_{Q_R}\left|Du^\varepsilon\right|^p dz\\
&\quad\le \varepsilon\, c_p\int_{Q_R}|Du|^p\,dz+c_p\int_{Q_R}\left|f-f^\varepsilon\right|\left|u^\varepsilon-u\right|dz.
\end{aligned}\tag{4.3}$$

Using in turn Hölder's inequality and Lemma 2.5, we get

$$\begin{aligned}
\tilde{I} &:= \int_{Q_R}\left|f-f^\varepsilon\right|\left|u^\varepsilon-u\right|dz\\
&\le C(n,p,R)\left\|f-f^\varepsilon\right\|_{L^2(Q_R)}\cdot\left(\int_{Q_R}\left|u^\varepsilon-u\right|^{p+\frac{2p}{n}}dz\right)^{\frac{n}{p(n+2)}}\\
&\le c(n,p,R)\left\|f-f^\varepsilon\right\|_{L^2(Q_R)}\cdot\left(\int_{Q_R}\left|Du^\varepsilon-Du\right|^p dz\right)^{\frac{n}{p(n+2)}}\\
&\quad\cdot\left(\sup_{t\in(t_0-R^2,t_0)}\left\|u^\varepsilon(\cdot,t)-u(\cdot,t)\right\|^2_{L^2(B_R(x_0))}\right)^{\frac{1}{n+2}}.
\end{aligned}\tag{4.4}$$

Now, let us notice that

$$\begin{aligned}
&\int_{Q_R}\left|Du^\varepsilon-Du\right|^p dz\\
&=\int_{Q_R\cap\{|Du^\varepsilon|\ge1\}}\left(\left|Du^\varepsilon\right|-1+1\right)^p dz+\int_{Q_R\cap\{|Du^\varepsilon|<1\}}\left|Du^\varepsilon\right|^p dz+\int_{Q_R}|Du|^p\,dz\\
&\le c_p\int_{Q_R}\left[\left(\left|Du^\varepsilon\right|-1\right)_+^{p}\right]dz+\int_{Q_R}\left(|Du|^p+1\right)dz\\
&\le c_p\int_{Q_R}\left(\left|H_{\frac{p}{2}}\left(Du^\varepsilon\right)-H_{\frac{p}{2}}(Du)+H_{\frac{p}{2}}(Du)\right|^2\right)dz+c_p\int_{Q_R}\left(|Du|^p+1\right)dz\\
&\le c_p\int_{Q_R}\left|H_{\frac{p}{2}}\left(Du^\varepsilon\right)-H_{\frac{p}{2}}(Du)\right|^2 dz+c_p\int_{Q_R}\left(|Du|^p+1\right)dz.
\end{aligned}\tag{4.5}$$

Inserting (4.5) in (4.4), we get

$$\begin{aligned}
\tilde{I}&\le c(n,p,R)\left\|f-f^\varepsilon\right\|_{L^2(Q_R(z_0))}\\
&\quad\cdot\left(\int_{Q_R}\left|H_{\frac{p}{2}}\left(Du^\varepsilon\right)-H_{\frac{p}{2}}(Du)\right|^2 dz+\int_{Q_R}\left(|Du|^p+1\right)dz\right)^{\frac{n}{p(n+2)}}\\
&\quad\cdot\left(\sup_{t\in(t_0-R^2,t_0)}\left\|u^\varepsilon(\cdot,t)-u(\cdot,t)\right\|^2_{L^2(B_R(x_0))}\right)^{\frac{1}{n+2}}\\
&\le c(n,p,R)\left\|f-f^\varepsilon\right\|_{L^2(Q_R)}\cdot\left(\int_{Q_R}\left|H_{\frac{p}{2}}\left(Du^\varepsilon\right)-H_{\frac{p}{2}}(Du)\right|^2 dz\right)^{\frac{n}{p(n+2)}}
\end{aligned}$$

$$\cdot \left(\sup_{t\in(t_0-R^2,t_0)} \left\| u^\varepsilon(\cdot,t) - u(\cdot,t) \right\|^2_{L^2(B_R(x_0))} \right)^{\frac{1}{n+2}}$$

$$+c\,(n,p,R)\left\| f - f^\varepsilon \right\|_{L^2(Q_R)} \cdot \left(\int_{Q_R} \left(|Du|^p + 1 \right) dz \right)^{\frac{n}{p(n+2)}}$$

$$\cdot \left(\sup_{t\in(t_0-R^2,t_0)} \left\| u^\varepsilon(\cdot,t) - u(\cdot,t) \right\|^2_{L^2(B_R(x_0))} \right)^{\frac{1}{n+2}}$$

and, by Young's inequality, we get

$$\tilde{I} \le \beta \int_{Q_R} \left| H_{\frac{p}{2}}\left(Du^\varepsilon\right) - H_{\frac{p}{2}}\left(Du\right) \right|^2 dz + \beta \sup_{t\in(t_0-R^2,t_0)} \left\| u^\varepsilon(\cdot,t) - u(\cdot,t) \right\|^2_{L^2(B_R(x_0))}$$
$$+c\,(n,p,R,\beta)\left\| f - f^\varepsilon \right\|^{\frac{n+2}{n+1}}_{L^2(Q_R)} \cdot \left(\int_{Q_R} \left(|Du|^p + 1 \right) dz \right)^{\frac{n}{p(n+1)}}$$
$$+c\,(n,p,R,\beta)\left\| f - f^\varepsilon \right\|^{\frac{p(n+2)}{n(p-1)+p}}_{L^2(Q_R)}. \tag{4.6}$$

Inserting (4.6) in (4.3), we obtain

$$\sup_{t\in(t_0-R^2,t_0)} \left\| u^\varepsilon(\cdot,t) - u(\cdot,t) \right\|^2_{L^2(B_R(x_0))}$$
$$+ \int_{Q_R} \left| H_{\frac{p}{2}}\left(Du^\varepsilon\right) - H_{\frac{p}{2}}\left(Du\right) \right|^2 dz + \frac{\varepsilon}{2\lambda_p} \int_{Q_R} \left| Du^\varepsilon \right|^p dz$$
$$\le \beta \int_{Q_R} \left| H_{\frac{p}{2}}\left(Du^\varepsilon\right) - H_{\frac{p}{2}}\left(Du\right) \right|^2 dz + \beta \sup_{t\in(t_0-R^2,t_0)} \left\| u^\varepsilon(\cdot,t) - u(\cdot,t) \right\|^2_{L^2(B_R(x_0))}$$
$$+c\,(n,p,R,\beta)\left\| f - f^\varepsilon \right\|^{\frac{n+2}{n+1}}_{L^2(Q_R)} \cdot \left(\int_{Q_R} \left(|Du|^p + 1 \right) dz \right)^{\frac{n}{p(n+1)}}$$
$$+c\,(n,p,R,\beta)\left\| f - f^\varepsilon \right\|^{\frac{p(n+2)}{n(p-1)+p}}_{L^2(Q_R)} + \varepsilon\, c_p \int_{Q_R} |Du|^p \, dz. \tag{4.7}$$

Choosing $\beta = \frac{1}{2}$ and neglecting the third non-negative term in the left hand side of (4.7), we get

$$\sup_{t\in(t_0-R^2,t_0)} \left\| u^\varepsilon(\cdot,t) - u(\cdot,t) \right\|^2_{L^2(B_R(x_0))} + \int_{Q_R} \left| H_{\frac{p}{2}}\left(Du^\varepsilon\right) - H_{\frac{p}{2}}\left(Du\right) \right|^2 dz$$
$$\le c\,(n,p,R)\left\| f - f^\varepsilon \right\|^{\frac{n+2}{n+1}}_{L^2(Q_R)} \cdot \left(\int_{Q_R} \left(|Du|^p + 1 \right) dz \right)^{\frac{n}{p(n+1)}}$$
$$+c\,(n,p,R)\left\| f - f^\varepsilon \right\|^{\frac{p(n+2)}{n(p-1)+p}}_{L^2(Q_R)} + \varepsilon\, c_p \int_{Q_R} |Du|^p \, dz. \tag{4.8}$$

For further needs, we also record that, combining (4.5) and (4.8), we have

$$\int_{Q_R} \left| Du^\varepsilon \right|^p dz \le c\,(n,p,R)\left\| f - f^\varepsilon \right\|^{\frac{n+2}{n+1}}_{L^2(Q_R)} \cdot \left(\int_{Q_R} \left(|Du|^p + 1 \right) dz \right)^{\frac{n}{p(n+1)}}$$
$$+c\,(n,p,R)\left\| f - f^\varepsilon \right\|^{\frac{p(n+2)}{n(p-1)+p}}_{L^2(Q_R)} + \varepsilon\, c_p \int_{Q_R} |Du|^p \, dz$$

$$+c_p \int_{Q_R} \left(|Du|^p + 1\right) dz. \tag{4.9}$$

Step 2: The conclusion. Let us fix $\rho > 0$ such that $Q_{2\rho} \subset Q_R$. We start observing that

$$\begin{aligned}
&\int_{Q_{\frac{\rho}{2}}} \left|\tau_h \left[\mathcal{G}_\delta \left((|Du| - \delta - 1)_+\right)\right]\right|^2 dz \\
&\le c \int_{Q_{\frac{\rho}{2}}} \left|\tau_h \left[\mathcal{G}_\delta \left(\left(\left|Du^\varepsilon\right| - \delta - 1\right)_+\right)\right]\right|^2 dz \\
&\quad + c \int_{Q_\rho} \left|\mathcal{G}_\delta \left(\left(\left|Du^\varepsilon\right| - \delta - 1\right)_+\right) - \mathcal{G}_\delta \left((|Du| - \delta - 1)_+\right)\right|^2 dz.
\end{aligned}$$

We estimate the right hand side of previous inequality using (3.28) and (2.10), as follows

$$\begin{aligned}
&\int_{Q_{\frac{\rho}{2}}} \left|\tau_h \left[\mathcal{G}_\delta \left((|Du| - \delta - 1)_+\right)\right]\right|^2 dz \\
&\le \frac{c|h|^2}{\delta^2\rho^2} \left[\int_{Q_{2\rho}} \left(1 + \left|Du^\varepsilon\right|^p\right) dz + \frac{1}{\delta^p} \int_{Q_{2\rho}} \left|f^\varepsilon\right|^2 dz\right] \\
&\quad + c_p \int_{Q_{2\rho}} \left|H_{\frac{p}{2}}\left(Du^\varepsilon\right) - H_{\frac{p}{2}}(Du)\right|^2 dz
\end{aligned}$$

that, thanks to (4.8), implies

$$\begin{aligned}
&\int_{Q_{\frac{\rho}{2}}} \left|\tau_h \left[\mathcal{G}_\delta \left((|Du| - \delta - 1)_+\right)\right]\right|^2 dz \\
&\le \frac{c|h|^2}{\delta^2\rho^2} \left[\int_{Q_{2\rho}} \left(1 + \left|Du^\varepsilon\right|^p\right) dz + \frac{1}{\delta^p} \int_{Q_{2\rho}} \left|f^\varepsilon\right|^2 dz\right] \\
&\quad + c(n, p, R) \left\|f - f^\varepsilon\right\|_{L^2(Q_R)}^{\frac{n+2}{n+1}} \cdot \left(\int_{Q_R} \left(|Du|^p + 1\right) dz\right)^{\frac{n}{p(n+1)}} \\
&\quad + c(n, p, R) \left\|f - f^\varepsilon\right\|_{L^2(Q_R)}^{\frac{p(n+2)}{n(p-1)+p}} + \varepsilon\, c_p \int_{Q_R} |Du|^p\, dz.
\end{aligned} \tag{4.10}$$

Now, using (4.9), we get

$$\begin{aligned}
\int_{Q_{2\rho}} \left(1 + \left|Du^\varepsilon\right|^p\right) dz &\le c(n, p, R) \left\|f - f^\varepsilon\right\|_{L^2(Q_R)}^{\frac{n+2}{n+1}} \cdot \left(\int_{Q_R} \left(|Du|^p + 1\right) dz\right)^{\frac{n}{p(n+1)}} \\
&\quad + c(n, p, R) \left\|f - f^\varepsilon\right\|_{L^2(Q_R)}^{\frac{p(n+2)}{n(p-1)+p}} + \varepsilon\, c_p \int_{Q_R} |Du|^p\, dz \\
&\quad + c_p \int_{Q_R} \left(|Du|^p + 1\right) dz,
\end{aligned}$$

which, combined with (4.10), implies

$$\begin{aligned}
&\int_{Q_{\frac{\rho}{2}}} \left|\tau_h \left[\mathcal{G}_\delta \left((|Du| - \delta - 1)_+\right)\right]\right|^2 dz \\
&\le \frac{c(n, p)\, |h|^2}{\delta^2\rho^2} \left[c(R) \left\|f - f^\varepsilon\right\|_{L^2(Q_R)}^{\frac{n+2}{n+1}} \cdot \left(\int_{Q_R} \left(|Du|^p + 1\right) dz\right)^{\frac{n}{p(n+1)}}\right.
\end{aligned}$$

$$+c(R)\left\|f-f^{\varepsilon}\right\|_{L^2(Q_R)}^{\frac{p(n+2)}{n(p-1)+p}}+\varepsilon\int_{Q_R}|Du|^p\,dz$$
$$+\int_{Q_R}\left(|Du|^p+1\right)dz+\frac{1}{\delta^p}\int_{Q_R}\left|f^{\varepsilon}\right|^2dz\Bigg].$$

Taking the limit as $\varepsilon\to0$, and since $f^{\varepsilon}\to f$ strongly in $L^2\left(B_R\right)$, we obtain

$$\int_{Q_{\frac{\rho}{2}}}\left|\tau_h\left[\mathcal{G}_\delta\left((|Du|-\delta-1)_+\right)\right]\right|^2dz$$
$$\le\frac{c\,(n,p)\,|h|^2}{\delta^2\rho^2}\left[\int_{Q_R}\left(|Du|^p+1\right)dz+\frac{1}{\delta^p}\int_{Q_R}|f|^2\,dz\right],$$

and thanks to Lemma 2.9, we have $\mathcal{G}_\delta\left((|Du|-\delta-1)_+\right)\in L^2\left(t_0-\rho^2,t_0;W^{1,2}\left(B_\rho\right)\right)$ with the following estimate

$$\int_{Q_{\frac{\rho}{2}}}\left|D\left[\mathcal{G}_\delta\left((|Du|-\delta-1)_+\right)\right]\right|^2dz$$
$$\le\frac{c\,(n,p)}{\delta^2\rho^2}\left[\int_{Q_R}\left(|Du|^p+1\right)dz+\frac{1}{\delta^p}\int_{Q_R}|f|^2\,dz\right].$$

Since previous estimate holds true for any $\rho>0$ such that $2\rho<R$, we may choose $\rho=\dfrac{R}{4}$ thus getting (1.2). □

5 Proof of Theorem 1.2

The higher differentiability result of Theorem 1.1 allows us to argue as in [19, Lemma 5.3] and [24, Lemma 3.2] to obtain the proof of Theorem 1.2.

Proof of Theorem 1.2 We start observing that

$$\left|D\left(\left(\mathcal{G}_\delta\left(\left(\left|Du^{\varepsilon}\right|-1-\delta\right)_+\right)\right)^{\frac{4}{np}+1}\right)\right|$$
$$\le c\left|\mathcal{G}_\delta\left(\left(\left|Du^{\varepsilon}\right|-1-\delta\right)_+\right)\right|^{\frac{4}{np}}\left|D\left[\mathcal{G}_\delta\left(\left(\left|Du^{\varepsilon}\right|-1-\delta\right)_+\right)\right]\right|,\tag{5.1}$$

where $c\equiv c(n,p)>0$ and $\mathcal{G}_\delta(t)$ is the function defined at (2.8).
With the notation used in the previous sections, for $B_{2\rho}\left(x_0\right)\Subset B_R\left(x_0\right)$, let $\varphi\in C_0^{\infty}\left(B_\rho\left(x_0\right)\right)$ and $\chi\in W^{1,\infty}\left((0,T)\right)$ be two non-negative cut-off functions with $\chi(0)=0$ and $\partial_t\chi\ge0$. Now, we fix a time $t_0\in(0,T)$ and apply the Sobolev embedding theorem on the time slices $\Sigma_t:=B_\rho(x_0)\times\{t\}$ for almost every $t\in(0,t_0)$, to infer that

$$\int_{\Sigma_t}\varphi^2\left(\left(\mathcal{G}_\delta\left(\left(\left|Du^{\varepsilon}\right|-1-\delta\right)_+\right)\right)^{\frac{4}{np}+1}\right)^2dx$$
$$\le c\left(\int_{\Sigma_t}\left|D\left(\varphi\left(\mathcal{G}_\delta\left(\left(\left|Du^{\varepsilon}\right|-1-\delta\right)_+\right)\right)^{\frac{4}{np}+1}\right)\right|^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{n}}$$

$$\leq c\left(\int_{\Sigma_t}\left|\varphi\, D\left(\left(\mathcal{G}_\delta\left(\left(|Du^\varepsilon|-1-\delta\right)_+\right)\right)^{\frac{4}{np}+1}\right)\right|^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{n}}$$

$$+c\left(\int_{\Sigma_t}\left|\left|\mathcal{G}_\delta\left(\left(|Du^\varepsilon|-1-\delta\right)_+\right)\right|^{\frac{4}{np}+1}D\varphi\right|^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{n}}$$

$$=:\, c\,I_1(t)\,+\,c\,I_2(t),$$

where, in the second to the last line, we have applied Minkowski's and Young's inequalities one after the other. We estimate $I_1(t)$ and $I_2(t)$ separately. Let us first consider $I_1(t)$. Using (5.1), Lemma 2.12 and Hölder's inequality with exponents $\left(\frac{n+2}{n}, \frac{n+2}{2}\right)$, we deduce

$$I_1(t)\leq c\left(\int_{\Sigma_t}\varphi^{\frac{2n}{n+2}}\left[\left(|Du^\varepsilon|-1\right)_+^{\frac{2}{n}}\left|D\mathcal{G}_\delta\left(\left(|Du^\varepsilon|-1-\delta\right)_+\right)\right|\right]^{\frac{2n}{n+2}}dx\right)^{\frac{n+2}{n}}$$

$$\leq c\int_{\Sigma_t}\varphi^2\left|D\mathcal{G}_\delta\left(\left(|Du^\varepsilon|-1-\delta\right)_+\right)\right|^2dx\left(\int_{\mathrm{supp}(\varphi)}\left(|Du^\varepsilon|-1\right)_+^2\,dx\right)^{\frac{2}{n}}$$

$$\leq c\int_{\Sigma_t}\varphi^2\left|D\mathcal{G}_\delta\left(\left(|Du^\varepsilon|-1-\delta\right)_+\right)\right|^2dx\left(\int_{\mathrm{supp}(\varphi)}|Du^\varepsilon|^2\,dx\right)^{\frac{2}{n}}.$$

We now turn our attention to $I_2(t)$. Lemma 2.12 and Hölder's inequality yield

$$I_2(t)\leq c\left(\int_{\Sigma_t}\left(|Du^\varepsilon|-1\right)_+^{\frac{np+4}{n+2}}|D\varphi|^{\frac{2n}{n+2}}\,dx\right)^{\frac{n+2}{n}}$$

$$\leq c\left(\int_{\Sigma_t}\left[|D\varphi|^2\,|Du^\varepsilon|^p\right]^{\frac{n}{n+2}}|Du^\varepsilon|^{\frac{4}{n+2}}\,dx\right)^{\frac{n+2}{n}}$$

$$\leq c\int_{\Sigma_t}|D\varphi|^2\,|Du^\varepsilon|^p\,dx\left(\int_{\mathrm{supp}(\varphi)}|Du^\varepsilon|^2\,dx\right)^{\frac{2}{n}}.$$

Putting together the last three estimates, using Lemma 2.12 in the left hand side, and integrating with respect to time, we obtain

$$\begin{aligned}\int_{Q^{t_0}}\chi\varphi^2\left(|Du^\varepsilon|-1\right)_+^{p+\frac{4}{n}}dz &\leq c\int_0^{t_0}\chi\left[\int_{\mathrm{supp}(\varphi)}\left|Du^\varepsilon(x,t)\right|^2dx\right]^{\frac{2}{n}}\cdot\\ &\quad\cdot\left[\int_{\Sigma_t}\left(\varphi^2\left|D\mathcal{G}_\delta\left(\left(|Du^\varepsilon|-1-\delta\right)_+\right)\right|^2+|D\varphi|^2\,|Du^\varepsilon|^p\right)dx\right]dt\\ &\leq c\int_{Q^{t_0}}\chi\left(\varphi^2\left|D\mathcal{G}_\delta\left(\left(|Du^\varepsilon|-1-\delta\right)_+\right)\right|^2+|D\varphi|^2\,|Du^\varepsilon|^p\right)dz\\ &\quad\cdot\left[\sup_{0<t<t_0,\ \chi(t)\neq 0}\int_{\mathrm{supp}(\varphi)}\left|Du^\varepsilon(x,t)\right|^2dx\right]^{\frac{2}{n}},\end{aligned}\tag{5.2}$$

where we have used the abbreviation $Q^{t_0} := B_\rho(x_0)\times(0,t_0)$. Now we choose $\chi\in W^{1,\infty}((0,T))$ such that $\chi\equiv 0$ on $\left(0,t_0-\rho^2\right]$, $\chi\equiv 1$ on $\left[t_0-\left(\frac{\rho}{2}\right)^2, T\right)$ and $\partial_t\chi\geq 0$.

For $\varphi \in C_0^\infty\left(B_\rho\left(x_0\right)\right)$, we assume that $\varphi \equiv 1$ on $B_{\frac{\rho}{2}}\left(x_0\right)$, $0 \leq \varphi \leq 1$ and $|D\varphi| \leq \dfrac{C}{\rho}$. With these choices (5.2) turns into

$$\int_{Q_{\frac{\rho}{2}}} \left(\left|Du^\varepsilon\right|-1\right)_+^{p+\frac{4}{n}} dz \leq c(n,p) \int_{Q_\rho} \left(\left|D\mathcal{G}_\delta\left(\left(\left|Du^\varepsilon\right|-1-\delta\right)_+\right)\right|^2+\rho^{-2}\left|Du^\varepsilon\right|^p\right) dz \cdot\left[\sup_{t_0-\rho^2<t<t_0} \int_{B_\rho(x_0)} \left|Du^\varepsilon(x,t)\right|^2 dx\right]^{\frac{2}{n}}. \tag{5.3}$$

We now use (3.5), in order to estimate the integrals on the right-hand side of (5.3), thus getting

$$\int_{Q_{\frac{\rho}{2}}} \left(\left|Du^\varepsilon\right|-1\right)_+^{p+\frac{4}{n}} dz \leq \frac{c}{(\delta\rho)^{\frac{2(n+2)}{n}}}\left[\int_{Q_{2\rho}} \left(1+\left|Du^\varepsilon\right|^p\right) dz+\frac{1}{\delta^p} \int_{Q_{2\rho}} \left|f^\varepsilon\right|^2 dz\right]^{\frac{2}{n}+1}.$$

Estimate (4.9) implies

$$\begin{aligned}
&\int_{Q_{\frac{\rho}{2}}} \left(|Du^\varepsilon|-1\right)_+^{p+\frac{4}{n}} dz \\
&\leq \frac{c\,(n,p)}{(\delta\rho)^{\frac{2(n+2)}{n}}}\left[c(\rho)\left\|f-f^\varepsilon\right\|_{L^2(Q_{2\rho}(z_0))}^{\frac{n+2}{n+1}} \cdot\left(\int_{Q_{2\rho}} \left(|Du|^p+1\right) dz\right)^{\frac{n}{p(n+1)}}\right]^{\frac{2}{n}+1} \\
&+\frac{c\,(n,p)}{(\delta\rho)^{\frac{2(n+2)}{n}}}\left[c(\rho)\left\|f-f^\varepsilon\right\|_{L^2(Q_{2\rho})}^{\frac{p(n+2)}{n(p-1)+p}}+\varepsilon\, c_p \int_{Q_{2\rho}} |Du|^p\, dz\right]^{\frac{2}{n}+1} \\
&+\frac{c\,(n,p)}{(\delta\rho)^{\frac{2(n+2)}{n}}}\left[\int_{Q_{2\rho}} \left(|Du|^p+1\right) dz+\frac{1}{\delta^p} \int_{Q_{2\rho}} \left|f^\varepsilon\right|^2 dz\right]^{\frac{2}{n}+1}.
\end{aligned} \tag{5.4}$$

Let us observe that estimate (4.8) in particular yields

$$\begin{aligned}
&\int_{Q_R} \left|H_{\frac{p}{2}}\left(Du^\varepsilon\right)-H_{\frac{p}{2}}(Du)\right|^2 dz \\
&\leq c\,(n,p,R)\left\|f-f^\varepsilon\right\|_{L^2(Q_R)}^{\frac{n+2}{n+1}} \cdot\left(\int_{Q_R} \left(|Du|^p+1\right) dz\right)^{\frac{n}{p(n+1)}} \\
&+c\,(n,p,R)\left\|f-f^\varepsilon\right\|_{L^2(Q_R)}^{\frac{p(n+2)}{n(p-1)+p}}+\varepsilon\, c_p \int_{Q_R} |Du|^p\, dz.
\end{aligned}$$

By the strong convergence of $f^\varepsilon \rightarrow f$ in $L^2\left(Q_R\right)$, passing to the limit as $\varepsilon \rightarrow 0$, from previous estimate we deduce

$$\lim_{\varepsilon \rightarrow 0} \int_{Q_R} \left|H_{\frac{p}{2}}\left(Du^\varepsilon\right)-H_{\frac{p}{2}}(Du)\right|^2 dz=0$$

that is $H_{\frac{p}{2}}\left(Du^\varepsilon\right) \rightarrow H_{\frac{p}{2}}(Du)$, strongly in $L^2\left(Q_R\right)$. Therefore, up to a not relabelled subsequence, we also have $H_{\frac{p}{2}}\left(Du^\varepsilon\right) \rightarrow H_{\frac{p}{2}}(Du)$, a.e. in $Q_R\left(z_0\right)$ and so

$$\left(\left|Du^\varepsilon\right|-1\right)_+ \rightarrow (|Du|-1)_+ \qquad \text{a.e. in } Q_R\left(z_0\right).$$

By Fatou's Lemma, taking the limit as $\varepsilon \to 0$ in both sides of (5.4)

$$\int_{Q_{\frac{\rho}{2}}} (|Du|-1)_+^{p+\frac{4}{n}}\, dz \leq \liminf_{\varepsilon\to 0} \int_{Q_{\frac{\rho}{2}}} (|Du^\varepsilon|-1)_+^{p+\frac{4}{n}}\, dz$$

$$\leq \frac{c\,(n,p)}{(\delta\rho)^{\frac{2(n+2)}{n}}} \left[\int_{Q_{2\rho}} \left(|Du|^p+1\right) dz + \frac{1}{\delta^p}\int_{Q_{2\rho}} |f|^2\, dz\right]^{\frac{2}{n}+1},$$

which holds for any $\delta \in (0,1)$, so fixing $\delta = \frac{1}{2}$, we get the conclusion (1.3). □

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