



A vectorial problem with thin free boundary

Daniela De Silva¹ · Giorgio Tortone²

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Abstract

We consider the vectorial analogue of the thin free boundary problem introduced by Caffarelli et al. (J Eur math Soc 12:1151–1179, 2010) as a realization of a nonlocal version of the classical Bernoulli problem. We study optimal regularity, nondegeneracy, and density properties of local minimizers. Via a blow-up analysis based on a Weiss type monotonicity formula, we show that the free boundary is the union of a “regular” and a “singular” part. Finally we use a viscosity approach to prove $C^{1,\alpha}$ regularity of the regular part of the free boundary.

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1 Introduction

In this paper we are interested in the vectorial version of the minimization problem introduced in [6]. Precisely, given a regular open set $\Omega \subset \mathbb{R}^{n+1}$, we consider the vectorial free boundary problem

$$\min \left\{ \int_{\Omega} |\nabla G|^2 dX + \mathcal{L}_n(\Omega \cap \{|G| > 0\} \cap \mathbb{R}^n) : G \in H^1(\Omega, \mathbb{R}^m), G = \Phi \text{ on } \partial\Omega \right\}, \quad (1.1)$$

with boundary data $\Phi = (\varphi^1, \dots, \varphi^m)$. Since the free boundary lies on the lower dimensional subspace $\{x_{n+1} = 0\}$, such a problem is usually called a *thin* free boundary problem. With a slight abuse of notation, whenever it does not create confusion, we will denote with G both the local minimizer in \mathbb{R}^{n+1} and its trace on $\mathbb{R}^n \times \{0\}$. Since we are developing a local analysis, it is not restrictive to assume $\Omega = B_1 \subset \mathbb{R}^{n+1}$. Hence, given a ball $B \subset B_1 \subset \mathbb{R}^{n+1}$ and the following class of admissible competitors (say $\Phi \in H^1(\Omega)$),

$$\mathcal{K}(B) = \{G \in H^1(B, \mathbb{R}^m), G = \Phi \text{ on } \partial B\},$$

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✉ Giorgio Tortone
giorgio.tortone@dm.unipi.it

Daniela De Silva
desilva@math.columbia.edu

¹ Department of Mathematics, Barnard College, Columbia University, New York, NY 10027, USA

² Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

we say that G is a *local minimizer* of

$$\mathcal{J}(G, B_1) = \int_{B_1} |\nabla G|^2 \, dX + \mathcal{L}_n(B_1 \cap \{|G| > 0\} \cap \mathbb{R}^n), \tag{1.2}$$

in B_1 if it minimizes $\mathcal{J}(G, B)$ in the class of competitors $\mathcal{K}(B)$, for every ball $B \subset B_1$. Similarly G is a *global minimizer* in \mathbb{R}^{n+1} if G is a local minimizer on every ball $B \subset \mathbb{R}^{n+1}$.

The scalar one-phase case $G = g \geq 0$, was first investigated by Caffarelli, Roquejoffre and Sire [6] in relation with the theory of semi-permeable membranes and models where turbulence or long-range interactions occur, for example in flame propagation and also in the propagation of surfaces of discontinuities (see [6, 16] and references therein). Moreover, the authors considered this free boundary problem as the local realization of the classical one-phase problem (also called the Bernoulli type problem) for the fractional Laplacian.

They proved general properties (optimal regularity, non-degeneracy and classification of global solutions) for local minimizers, corresponding to those proved by Alt and Caffarelli in their pioneering paper [3] for the standard one-phase Bernoulli problem (see [9, 26] for a comprehensive survey of the results in this setting).

A major step toward understanding the regularity theory for thin free boundaries was then obtained in [10], where the first author and Roquejoffre introduced the notion of viscosity solution to the Euler-Lagrange equation associated to the minimization problem:

$$\begin{cases} \Delta g = 0, & \text{in } B_1^+(g) := B_1 \setminus \{(x, 0) : g(x, 0) = 0\}, \\ \frac{\partial g}{\partial t^{1/2}} = 1, & \text{on } F(g) := B_1 \cap \partial_{\mathbb{R}^n} \{(x, 0) : g(x, 0) > 0\}, \end{cases} \tag{1.3}$$

where

$$\frac{\partial}{\partial t^{1/2}} g(x_0) = \lim_{(t,z) \rightarrow (0,0)} \frac{g(x_0 + tv(x_0), x_{n+1})}{t^{1/2}}, \quad x_0 \in F(g),$$

with $v(x_0)$ the unit normal to the free boundary $F(g)$ at x_0 pointing toward $\{g > 0\}$.

In [10] it was proved that in any dimension if the free boundary $F(g)$ is sufficiently “flat” then it is $C^{1,\alpha}$. Afterwards, in a series of paper [11–13] the first author and Savin improved the known results by answering the question of whether Lipschitz free boundaries are smooth. By combining variational and nonvariational techniques, they also showed that local minimizers have smooth free boundary except possibly for a small singular set of Hausdorff dimension $n - 3$.

Recently in [17] the authors used the Rectifiable-Reifenberg and quantitative stratification framework of Naber-Valtorta to prove Hausdorff measure and structure results for the singular set. We remark that in [1] the authors removed the sign assumption by considering a two-phase problem with thin free boundary, in the same spirit of the classical work of Alt-Caffarelli-Friedman [2]. While in [2] it is proved the existence of a common free boundary between the two phases, in [1] the authors showed that positive and negative phases are always separated by nontrivial dead core.

On the other hand, in [7] the authors initiated the study of free boundary problems where several flows are involved, and interact whenever there is a phase transition. Indeed, this problem describes stationary thermal insulation, allowing a prescribed heat loss from the insulating layer. Similarly, this set-up arose in population dynamics where several species coexist, and overflow the patches. While in the case of competitive systems it is well known that competition gives rise to the so-called junction points, under cooperation the solutions tend to congregate and to show a smooth free boundary.

In [7, 20], the authors considered the classical Bernoulli problem,

$$\min \left\{ \int_{\Omega} |\nabla G|^2 \, dx + \mathcal{L}_{n+1}(\Omega \cap \{|G| > 0\}) : G \in H^1(\Omega, \mathbb{R}^m), G = \Phi \text{ on } \partial\Omega \right\}, \tag{1.4}$$

and applied a reduction method to reduce the problem to its scalar counterpart by assuming nonnegativity of the components. This assumption was successively removed in [21]. As expected, in this case the structure of the singular set changes and the set of branching points $\text{Sing}_2(F(G))$ arises, as natural in two-phase problems (see also the recent work [8]).

Recently, in [15] we developed a vectorial viscosity approach to understand the regularity of the free boundary in (1.4), which takes advantage of the fact that the norm $|G|$ is a viscosity subsolution to the scalar one-phase problem.

We also remark that as pointed out in [18–20], problem (1.4) is related to a class of shape optimization problems involving the eigenvalues of the Dirichlet Laplacian.

1.1 Main results and organization of the paper

In the first part of this paper, we consider the classical questions of optimal regularity, non-degeneracy, and density estimates for local minimizers to (1.2). Then, we derive a Weiss-type monotonicity formula, which allows us to use a blow-up analysis and characterize global blow-up limits, which in turn leads to the definition of the regular and singular part of the free boundary. In the second part of the paper, we prove that local minimizers are viscosity solutions (see Sect. 5 for the precise definition) of the vectorial thin one-phase problem ($A_0 > 0$ a precise dimensional constant):

$$\begin{cases} \Delta G = 0 & \text{in } B_1^+(|G|) := B_1 \cap \{(x, 0) : |G(x, 0)| > 0\}; \\ \frac{\partial}{\partial t} |G| = A_0 & \text{on } F(G) := B_1 \cap \partial_{\mathbb{R}^n} \{(x, 0) : |G(x, 0)| > 0\}. \end{cases} \tag{1.5}$$

Thus, the analysis of the regular part of the free boundary can be carried out with the viscosity methods developed in [10, 15]. Combining the two parts we obtain the following main theorem.

Theorem 1.1 *The problem (1.1) admits a solution $G \in H^1(B_1; \mathbb{R}^m)$. Moreover, any solution is locally $C^{0,1/2}$ -Hölder continuous in B_1 and the set $\{|G| > 0\} \cap \{x_{n+1} = 0\}$ has a locally finite perimeter in $B_1 \cap \{x_{n+1} = 0\}$. More precisely, the free boundary $F(G)$ is a disjoint union of a regular part $\text{Reg}(F(G))$ and one-phase singular set $\text{Sing}(F(G))$:*

1. $\text{Reg}(F(G))$ is an open subset of $F(G)$ and is locally the graph of a $C^{1,\alpha}$ function.
2. $\text{Sing}(F(G))$ consists only of points in which the Lebesgue density of $\{|G| > 0\} \cap \{x_{n+1} = 0\}$ is strictly between $1/2$ and 1 . Moreover, there is $n^* \geq 3$ such that:
 - if $n < n^*$, then $\text{Sing}(F(G))$ is empty;
 - if $n = n^*$, then $\text{Sing}(F(G))$ contains at most a finite number of isolated points;
 - if $n > n^*$, then the $(n - n^*)$ -dimensional Hausdorff measure of $\text{Sing}(F(G))$ is locally finite in $B_1 \cap \{x_{n+1} = 0\}$.

As already remarked, in the local analogue [21] the authors proved the existence of a closed set of locally finite $(n - 1)$ -Hausdorff measure of branching point $\text{Sing}_2(F(G))$. This set consists only of points in which the Lebesgue density of the positivity set $\{|G| > 0\}$ is 1 and the blow-up limits are linear functions: this blow-up analysis implies that cusps pointing inwards, might appear. While in the local case this feature is natural, in the thin case the picture changes. Indeed, as pointed out earlier in the thin two-phase problem the positive and

negative phases are always separated thus the problem reduces locally back to a one-phase problem.

The main theorem of the second part is independent of the minimization problem and it reads as follows.

Theorem 1.2 *Let G be a viscosity solution to (1.5) in B_1 . There exists a universal constant $\bar{\varepsilon} > 0$ such that if G is $\bar{\varepsilon}$ -flat in the (f, ν) -directions in B_1 , i.e. for some unit directions $f \in \mathbb{R}^m, \nu \in \mathbb{R}^n$*

$$|G(X) - U(\langle x, \nu \rangle, x_{n+1})f| \leq \bar{\varepsilon} \text{ in } B_1,$$

and

$$|G|(x, 0) \equiv 0 \text{ in } B_1 \cap \{\langle x, \nu \rangle < -\bar{\varepsilon}\},$$

then $F(G) \in C^{1,\alpha}$ in $B_{1/2}$.

In the viscosity setting, differently from the local case treated in [15], the reduction from the vector valued problem to the scalar one, is almost straightforward and hence most technical results leading to the proof of Theorem 1.2 follow from slight modifications of those in [10] (see Subsection 6.1).

Remark 1.3 As in the scalar case, in light of the extension facts related to the half-laplacian our theory applies, among others, to vectorial Bernoulli type problem involving non local energies, like, for instance the solutions to the following problem (when $s = 1/2$):

$$\min \left\{ \sum_{i=1}^m \int_{\mathbb{R}^{2n}} \frac{|g^i(x) - g^i(y)|^2}{|x - y|^{n+2s}} dx dy + \mathcal{L}_n(\mathcal{B}_1 \cap \{|G| > 0\}) : \begin{array}{l} g^i \equiv \varphi^i \text{ on } \mathbb{R}^n \setminus \mathcal{B}_1 \\ \text{for } i = 1, \dots, m \end{array} \right\},$$

where $\mathcal{B}_1 = B_1 \cap \{x_{n+1} = 0\}$, $\Phi = (\varphi^1, \dots, \varphi^m) \in H^{1/2}(\mathbb{R}^n; \mathbb{R}^m)$ is the boundary data in $\mathbb{R}^n \setminus \mathcal{B}_1$. In this regard, our results extend the theory of vectorial free boundary problem to the fractional case.

In the same spirit of [18–20], since our methodologies are quite robust, we believe that our results can be extended to the case of almost minimizer of (1.1) (see [14] for the theory of almost minimizers to the scalar thin-one phase problem) and suitably adapted to the nonlocal shape optimization problem

$$\min \left\{ \sum_{i=1}^m \lambda_i^s(A) : A \subset \Omega \text{ } s\text{-quasi open, } |A| \leq c \right\}, \text{ with } \lambda_i^s(A) = \min_{u \perp E_{i-1}} \frac{[u]_{H^s(\mathbb{R}^N)}^2}{\|u\|_{L^2(\mathbb{R}^N)}^2},$$

where $\Omega \subset \mathbb{R}^n$ open and bounded, $c < |\Omega|$ and $E_i \subset H_0^s(A)$ is the space spanned by the first i eigenfunctions (see [4, 25] for more details in this direction). This line of research will be treated in a subsequent work [24], in which the second author extend Theorem 1.2 to the case of almost minimizers of a vectorial problem with fractional diffusion $s \in (0, 1)$.

The paper is organized as follows: in Sect. 2 we study the local behavior of minimizers near the free boundary by answering the classical questions of optimal regularity, non-degeneracy and density estimates for local minimizers. Then, in Sect. 3 we derive a Weiss-type formula which will allow in Sect. 4 to characterize global blow-up limits. The blow-up analysis of Sect. 4 will lead to the definition of the regular $\text{Reg}(F(G))$ and singular part $\text{Sing}(F(G))$ of the free-boundary. Finally, in the remaining part of the paper we will use a viscosity approach to obtain $C^{1,\alpha}$ smoothness of the regular part $\text{Reg}(F(G))$. First, in Sect. 5 we introduce a vector valued analogue of the notion of viscosity solution of [10] and we prove that local

minimizers are viscosity solutions. In Sect. 6 we develop a vectorial Harnack inequality, which will be the basic tool for our analysis of the regular part of the free boundary. Finally, in Sect. 7 we prove the improvement of flatness result, from which the $C^{1,\alpha}$ regularity of a flat free boundary follows by standard arguments.

1.2 Notation

From now on, we denote by $\{e_i\}_{i=1,\dots,n}$ and $\{f^i\}_{i=1,\dots,m}$ canonical basis in \mathbb{R}^n and \mathbb{R}^m respectively. Unit directions in \mathbb{R}^n and \mathbb{R}^m will be typically denoted by e and f . The Euclidean norm in either space is denoted by $|\cdot|$ while the dot product is denoted by $\langle \cdot, \cdot \rangle$.

A point $X \in \mathbb{R}^{n+1}$ will be denoted by $X = (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ and we will use the notation $x = (x', x_n)$. Moreover, a ball in \mathbb{R}^{n+1} with radius $r > 0$ centered at X is denoted by $B_r(X)$ and for simplicity $B_r = B_r(0)$. Also, we use $\mathcal{B}_r = B_r \cap \{x_{n+1} = 0\}$ to denote the n -dimensional ball in $\mathbb{R}^n \times \{x_{n+1} = 0\}$.

We will often consider the following sets: let g be a continuous non-negative function in B_r , then

$$B_r^+(g) := B_r \setminus \{(x, 0) : g(x, 0) = 0\} \subset \mathbb{R}^{n+1}$$

$$\mathcal{B}_r^+(g) := B_r^+(g) \cap \{x_{n+1} = 0\} \subset \mathbb{R}^n.$$

By abuse of notation, if $G = (g^1, \dots, g^m)$ is a vector valued continuous function, we use $B_r^+(G), \mathcal{B}_r^+(G)$ in place of $B_r^+(|G|), \mathcal{B}_r^+(|G|)$ respectively. Also, we will denote with P and L respectively the half-hyperplane $P := \{X \in \mathbb{R}^{n+1} : x_n \leq 0, x_{n+1} = 0\}$ and $L := \{X \in \mathbb{R}^{n+1} : x_n = 0, x_{n+1} = 0\}$.

In regard to the problem (1.3), we remark that if $F(g)$ is C^2 then any function g which is harmonic in $B_1^+(g)$ has an asymptotic expansion at a point $x_0 \in F(g)$,

$$g(x, s) = \alpha(x_0)U((x - x_0) \cdot \nu(x_0), s) + o(|x - x_0|^{1/2} + s^{1/2}).$$

Here $U(t, s)$ is the real part of \sqrt{z} which in the polar coordinates

$$t = r \cos \theta, \quad s = r \sin \theta, \quad r \geq 0, \quad -\pi \leq \theta \leq \pi,$$

is given by

$$U(t, s) = r^{1/2} \cos \frac{\theta}{2}. \tag{1.6}$$

Then, the free boundary condition in (1.3) requires that $\alpha \equiv 1$ on $F(g)$.

The function U plays a fundamental role in our analysis.

Throughout the paper we will often use the invariance of a local minimizer of (1.1) with respect to translations and dilations. More precisely, fixed $X_0 \in F(G)$ we define as the blow-up sequence of G centered at X_0 the family

$$G_{X_0,r}(X) = \frac{1}{r^{1/2}} G(X_0 + rX) \tag{1.7}$$

with $r > 0$. Indeed, for every $R > 0$ we get

$$\mathcal{J}(G, B_R(X_0)) = r^n \mathcal{J}(G_{X_0,r}, B_{R/r}). \tag{1.8}$$

Consequently, G is a local minimizer of $\mathcal{J}(\cdot, B_R(X_0))$ if and only if $G_{X_0,r}$ is a local minimizer of $\mathcal{J}(\cdot, B_{R/r})$. As in [10] it is not restrictive to reduce the analysis to the case of minimizers

that are symmetric with respect to the x_{n+1} -variable. Indeed, if $G_e = (g_e^1, \dots, g_e^m)$ is the even part of G with respect to $\{x_{n+1} = 0\}$,

$$g_e^i(x, x_{n+1}) = \frac{g^i(x, x_{n+1}) + g^i(x, -x_{n+1})}{2}, \quad \text{for } i = 1, \dots, m$$

we get

$$\mathcal{J}(G, B_1) = \mathcal{J}(G_e, B) + \int_{B_1} |\nabla G_o|^2 dX,$$

with $G_o = G - G_e$ the odd part of G with respect to $\{x_{n+1} = 0\}$. By the minimality of G , for every $V \in H^1(B; \mathbb{R}^m)$, $V = G_e + G_o$ on ∂B we have

$$\mathcal{J}(G_e, B) \leq \mathcal{J}(V, B) - \int_{B_1} |\nabla G_o|^2 dX \leq \mathcal{J}(V - G_o, B),$$

which proves our claim once we noticed that $V - G_o = G_e$ on ∂B . Thus, throughout the paper this will be tacitly assumed.

2 Local behavior of solutions

Since the existence of an optimal vector for problem (1.1) is nowadays standard (see [6][Proposition 3.2]), we start by focusing on the properties of local minimizers and obtain in this section, optimal regularity, non-degeneracy, and density estimates. We use the notation from Subsection 1.1.

2.1 Optimal regularity

Our proof follows the lines of the scalar case in [6] and it is based on an harmonic replacement of each component of the minimizing vector $G = (g^1, \dots, g^m)$.

First, we make the following basic observation.

Lemma 2.1 *If G is a local minimizer in B_1 , then g^i is harmonic in $B_1 \setminus \{x_{n+1} = 0\}$.*

Proof Denote by $B_1^+ := B_1 \cap \{x_{n+1} > 0\}$, and similarly $B_1^- := B_1 \cap \{x_{n+1} < 0\}$. Let φ be in $C_0^\infty(B_1^+)$ and call $G_i^\pm := (g^1, \dots, g^i \pm \varepsilon\varphi, \dots, g^m)$, for any $i = 1, \dots, m$. Thus, since $G \equiv G_i^\pm$ on $\{x_{n+1} = 0\}$, the minimality of G ,

$$\mathcal{J}(G, B_1) \leq \mathcal{J}(G_i^\pm, B_1)$$

implies

$$\int_{B_1} |\nabla g^i|^2 dX \leq \int_{B_1} |\nabla(g^i \pm \varepsilon\varphi)|^2 dX$$

and hence

$$\int_{B_1} \langle \nabla g^i, \nabla \varphi \rangle dX = 0,$$

that is g^i is harmonic in B_1^+ . We argue similarly for $\varphi \in C_0^\infty(B_1^-)$. □

We now prove the optimal regularity.

Proposition 2.2 *Let G be a local minimizer, then $G \in C^{0,1/2}(K; \mathbb{R}^m)$, for every compact set $K \subset B_1$. Moreover,*

$$|g^i(X)| \leq C_1 \text{dist}(X, \partial\{|G| > 0\})^{1/2} \quad \text{in } B_{1/2}, \tag{2.1}$$

for every $i = 1, \dots, m$, with $C_1 > 0$ a universal constant.

Proof Let $K \subset B_1$ be a compact and $i = 1, \dots, m$. We claim that there exists an universal constant $C > 0$ such that, for any $X_0 \in K$ and $r \in (0, 1 - |X_0|)$

$$\frac{1}{r^n} \int_{B_r(X_0)} |\nabla g^i|^2 dX \leq C. \tag{2.2}$$

Then, by a Morrey type embedding, we deduce $g^i \in C^{0,1/2}(K)$ (see [22]). Moreover, since the constant $C > 0$ is universal, the inequality (2.1) is satisfied for a constant $C > 0$ independent of the local minimizer.

Since by Lemma 2.1, the components of G are harmonic in $B_1 \setminus \{x_{n+1} = 0\}$, it is not restrictive to suppose that $X_0 \in \{x_{n+1} = 0\}$. By the translation invariance of the problem, let us suppose $X_0 = 0$ and $r \in (0, 1)$. Inspired by the proof of [6][Theorem 1.1.], let $\tilde{g}_r^i : B_r \rightarrow \mathbb{R}$ be the harmonic replacement of g^i in B_r , i.e. be such that

$$\begin{cases} \Delta \tilde{g}_r^i = 0 & \text{in } B_r \\ \tilde{g}_r^i = g^i & \text{on } \partial B_r. \end{cases}$$

By an integration by parts, we easily deduce

$$\int_{B_r} \langle \nabla \tilde{g}_r^i, \nabla (g^i - \tilde{g}_r^i) \rangle dX = 0. \tag{2.3}$$

Consider now the competitor $\tilde{G}_i = (g^1, \dots, \tilde{g}_r^i, \dots, g^m)$. By the minimality of G we get $\mathcal{J}(G, B_r) \leq \mathcal{J}(\tilde{G}_i, B_r)$, which implies

$$\int_{B_r} |\nabla g^i|^2 dX \leq \int_{B_r} |\nabla \tilde{g}_r^i|^2 dX + \omega_n r^n.$$

Combining the ‘‘quasi-minimality’’ of g^i with (2.3), we get

$$\int_{B_r} |\nabla (g^i - \tilde{g}_r^i)|^2 dX \leq \omega_n r^n.$$

Thus, for $\rho \in (r, 1)$ we get

$$\begin{aligned} \int_{B_r} |\nabla g^i|^2 dX &\leq 2 \left(\int_{B_\rho} |\nabla (g^i - \tilde{g}_\rho^i)|^2 dX + \int_{B_r} |\nabla \tilde{g}_\rho^i|^2 dX \right) \\ &\leq C \rho^n + C \int_{B_r} |\nabla \tilde{g}_\rho^i|^2 dX \\ &\leq C \rho^n + C \left(\frac{r}{\rho}\right)^{n+1} \int_{B_\rho} |\nabla \tilde{g}_\rho^i|^2 dX \\ &\leq C \rho^n + C \left(\frac{r}{\rho}\right)^{n+1} \int_{B_\rho} |\nabla g^i|^2 dX, \end{aligned}$$

where in the third inequality we used that $|\nabla \tilde{g}_\rho^i|^2$ is subharmonic. Hence, fixed $\delta < 1/2$ such that $q = C\delta < 1$, if $\rho = \delta^{k-1}$, $r = \delta^k$ and $\mu = \delta^n$ we get

$$\int_{B_{\delta^k}} |\nabla g^i|^2 dX \leq C\mu^{k-1} + C\mu\delta \int_{B_{\delta^{k-1}}} |\nabla g^i|^2 dX$$

and iterating the previous estimate

$$\int_{B_{\delta^k}} |\nabla g^i|^2 dX \leq C\mu^{k-1} \sum_{i=0}^{k-1} q^i \leq C\mu^{k-1} \frac{1}{1-q}.$$

hence there exists a universal constant $\tilde{C} > 0$ such that

$$\int_{B_r} |\nabla g^i|^2 dX \leq \tilde{C}r^n,$$

for every $r \in (0, 1/2)$. By a covering argument we obtain the claimed inequality (2.2). \square

By the continuity, we immediately deduce the following corollary.

Corollary 2.3 *Let G be a local minimizer, then the sets*

$$\{|G| > 0\} \text{ and } \{g^i > 0\}, \{g^i < 0\} \text{ for } i = 1, \dots, m,$$

and their restrictions on $\{x_{n+1} = 0\}$ are respectively open in \mathbb{R}^{n+1} and \mathbb{R}^n .

With the continuity at hands, we can easily obtain the harmonicity of the components away from $\{|G| = 0\}$.

Lemma 2.4 *Let G be a local minimizer in B_1 . Then for every $i = 1, \dots, m$ we get*

$$\Delta g^i = 0 \text{ in } B_1^+(G),$$

and consequently

$$g_\pm^i \text{ is subharmonic in } B_1.$$

Moreover $\lambda_i = \Delta g^i$ is a signed Radon measure supported on $\partial\{|G| > 0\}$ with the total variation $|\Delta g^i|$ satisfying

$$\langle |\Delta g^i|, \chi_K \rangle \leq C(n, K) \int_B |\nabla g^i|^2 dX,$$

for every compact set $K \subset B_1$.

Proof As in [3], the first part of the result follows by computing the first variation of the functional $\mathcal{J}(\cdot, B_1)$ with respect to a direction ξe^i , with $\xi \in C^\infty(\{|G| > 0\})$.

More precisely, fixed $i = 1, \dots, m$, consider the competitor $G_\varepsilon = G + \varepsilon \xi e^i$, for some $\xi \in C_c^\infty(\{|G| > 0\})$ and $i = 1, \dots, m$. By the previous corollary, $\{|G| > 0\}$ is an open set, and passing through the first variation, we get

$$0 = \frac{1}{2} \frac{d}{d\varepsilon} \mathcal{J}(G_\varepsilon, K) \Big|_{\varepsilon=0} = \int_{\{|G|>0\} \cap K} \langle \nabla g^i, \nabla \xi \rangle dX,$$

for every compact $K \subset B_1$.

Now, since g_{\pm}^i are both nonnegative subharmonic in B_1 and g^i is harmonic in $\{|G| > 0\}$, then $\lambda_i = \Delta g^i$ is a signed Radon measure supported in $\partial\{|G| > 0\}$. Moreover, by a standard argument, let $\eta \in C_c^\infty(B_1)$ be such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on K . Then

$$\left\langle |\Delta g^i|, \chi_K \right\rangle \leq \left\langle |\Delta g^i|, \eta \right\rangle = \int_B \langle \nabla g^i, \nabla \eta \rangle dX \leq C(n, K) \int_B |\nabla g^i|^2 dX,$$

as we claimed. □

Remark 2.5 By explicit computation we now easily deduce

$$2|G| \Delta(|G|) = \Delta(|G|^2) - \frac{|\nabla |G|^2|^2}{2|G|^2} \geq 0 \quad \text{in } B_1^+(G),$$

and consequently that $|G|$ is subharmonic in $B_1^+(G)$.

As in the scalar case in [6], we can now detail the connection of global minimizer with the fractional analogue of the Bernoulli one-phase problem.

Corollary 2.6 *Let G be a global minimizer in \mathbb{R}^{n+1} and $0 \in F(G)$. Then, for every $i = 1, \dots, m$, the trace of g^i on $\{x_{n+1} = 0\}$ solves*

$$\begin{cases} (-\Delta)^{1/2} g^i(\cdot, 0) = 0 & \text{in } \{|G| > 0\} \\ g^i(\cdot, 0) = 0 & \text{in } \{|G| = 0\}. \end{cases}$$

2.2 Non-degeneracy

The non-degeneracy of solutions near the free boundary points allows us to obtain several results on the measure-theoretic structure of the free boundary via the blow-up analysis. We start by proving the following weak non-degeneracy condition.

Proposition 2.7 *Let G be a local minimizer. Then, there exists a universal constant $c_2 > 0$ such that*

$$|G|(X) \geq c_2 \text{dist}(X, \partial\{|G| > 0\})^{1/2} \quad \text{in } B_{1/2}^+(G). \tag{2.4}$$

Proof Up to translation and rescaling, it is enough to show that if G is a local minimizer in a large ball and

$$\text{dist}(0, \partial\{|G| > 0\}) = 1, \tag{2.5}$$

then

$$|G|(0) \geq c_2 > 0$$

for some c_2 small to be made precise later. Indeed, assume not, then $B_1 \subset \{|G| > 0\}$ and

$$g^i \text{ is harmonic in } B_1, \quad g^i(0) \leq c_2, \quad \text{for every } i = 1, \dots, m.$$

By the $C^{0,1/2}$ -regularity of minimizers we deduce that the g^i 's are uniformly bounded say in $B_{3/4}$ and hence, since they are harmonic

$$|g^i(X) - g^i(0)| \leq K|X|, \quad \text{in } B_{1/2},$$

for $K > 0$ universal. Thus,

$$g^i(X) \leq c_2 + K|X|, \quad \text{in } B_{1/2}.$$

Let

$$G_\delta(X) = \frac{1}{\delta^{1/2}}G(\delta X), \quad X \in B_1$$

with $\delta > 0$ universal to be chosen universal later. Then, for $c_2 \leq \delta$ we get

$$g_\delta^i \leq c_2\delta^{-1/2} + K\delta^{1/2} \leq C\delta^{1/2} \quad \text{in } B_1,$$

for every $i = 1, \dots, m$. Moreover, since the g_δ^i 's are harmonic in B_1 , the bound above implies

$$\|g_\delta^i\|_{L^\infty(B_1)}, \|\nabla g_\delta^i\|_{L^\infty(B_{1/2})} \leq C\delta^{1/2}.$$

Let $\varphi \in C_0^\infty(B_{1/2})$, $0 \leq \varphi \leq 1$ such that $\varphi \equiv 1$ in $B_{1/4}$, then

$$\int_{B_1} |\nabla g_\delta^i|^2 dX \geq \int_{B_1} |\nabla(g_\delta^i(1 - \varphi))|^2 dX - C\delta$$

and on the other hand

$$\mathcal{L}_n(\mathcal{B}_1^+(|G_\delta|)) \geq \mathcal{L}_n(\mathcal{B}_1^+(|G_\delta|(1 - \varphi))) + C_0.$$

In conclusion, by the minimality of G_δ

$$0 \geq -C\delta + C_0,$$

and we reach a contradiction for δ (hence c_2) sufficiently small. □

The following result improves the non-degeneracy property of Proposition 2.7, and it will be fundamental in the proof of existence of non trivial blow-up limits.

Proposition 2.8 *Let G be a local minimizer and $0 \in F(G)$. Then, for every $r \in (0, 1/2)$*

$$\sup_{B_r} |G| \geq cr^{1/2}, \tag{2.6}$$

for some universal constant $c > 0$.

In view of Proposition 2.2 and Remark 2.5, Proposition 2.8 follows immediately from the next lemma.

Lemma 2.9 *Let $v \geq 0$ be defined in B_1 and subharmonic in $B_1^+(v)$. Assume that there is a small constant $\eta > 0$ such that*

$$\|v\|_{C^{1/2}(B_1)} \leq \eta^{-1}, \tag{2.7}$$

and v satisfies the non-degeneracy condition on B_1 ,

$$v(X) \geq \eta \operatorname{dist}(X, \{v = 0\})^{1/2} \quad \text{for every } X \in B_1. \tag{2.8}$$

Then if $0 \in F(v)$, we get

$$\sup_{B_r} v \geq c(\eta) r^{1/2}, \quad \text{for } r \leq 1.$$

Proof The proof follows the lines of [5][Lemma 7] (see also [6][Proposition 3.3]).

Given a point $X_0 \in \mathcal{B}_1^+(v)$ (to be chosen close to the free boundary point $0 \in F(v)$) we construct a sequence of points $(X_k)_k \subset \mathcal{B}_1$ such that

$$v(X_{k+1}) \geq (1 + \delta)v(X_k), \quad |X_{k+1} - X_k| \leq C(\eta)\text{dist}(X_k, \{v = 0\}),$$

with δ small depending on η .

Then, using (2.8) and that $(v(X_k))_k$ grows geometrically, we find

$$\begin{aligned} |X_{k+1} - X_0| &\leq \sum_{i=0}^k |X_{i+1} - X_i| \leq C(\eta) \sum_{i=0}^k \text{dist}(X_i, \{v = 0\}) \\ &\leq \frac{C(\eta)}{\eta^2} \sum_{i=0}^k v^2(X_i) \leq c(\eta)v^2(X_{k+1}). \end{aligned}$$

Hence for a sequence of radii $r_k = \text{dist}(X_k, \{v = 0\})$, we have that

$$\sup_{\mathcal{B}_{r_k}(X_0)} v \geq cr_k^{1/2}$$

from which we obtain that

$$\sup_{\mathcal{B}_r(X_0)} v \geq cr^{1/2}, \quad \text{for all } r \geq |X_0|.$$

The conclusion follows by letting X_0 go to $0 \in F(v)$.

We now show that the sequence of X_k 's exists. After scaling, assume we constructed X_k such that

$$v(X_k) = 1.$$

Let us call with $Y_k \in F(v)$ the point where the distance from X_k to $\{v = 0\}$ is achieved. By (2.7) and (2.8), we get

$$c(\eta) \leq r_k = |X_k - Y_k| \leq C(\eta).$$

Assume by contradiction that we cannot find X_{k+1} in $\mathcal{B}_M(X_k)$ with M large to be specified later, such that

$$v(X_{k+1}) \geq 1 + \delta.$$

Then $v \leq 1 + \delta + w$ with w harmonic in $B_M^+(X_k)$ and such that

$$w = 0 \quad \text{on } \{x_{n+1} = 0\}, \quad w = v \quad \text{on } \partial B_M(X_k) \cap \{x_{n+1} > 0\}.$$

Thus, we have

$$w \leq C(n) \frac{x_n}{M} \sup_{B_M^+(X_k)} v \leq C\eta^{-1}x_nM^{-1/2} \leq \delta \quad \text{in } B := B_{r_k}(X_k),$$

for M sufficiently large depending on δ . Thus,

$$v \leq 1 + 2\delta \quad \text{in } B. \tag{2.9}$$

On the other hand, $v(Y_k) = 0$, $Y_k \in \partial B$. Thus from the Hölder continuity of v we find

$$v \leq \frac{1}{2}, \quad \text{in } B_{c(\eta)}(Y_k). \tag{2.10}$$

If δ is sufficiently small (2.9)-(2.10) contradict that

$$1 = v(X_k) \leq \int_B v.$$

□

The following lemma is on the convergence of sequences of minimizers.

Lemma 2.10 *Let $(G_k)_k$ be a sequence of local minimizer in B_1 uniformly bounded in $L^2(B_1)$. Then, up to a subsequence, there exists a limit function G_∞ such that*

- $G_\infty \in H^1_{loc}(B_1) \cap C^{0,1/2}(\overline{B_1})$;
- $G_k \rightarrow G_\infty$ in $C^{0,\alpha}_{loc}(\overline{B_1})$, for every $\alpha \in (0, 1/2)$;
- $G_k \rightharpoonup G_\infty$ weakly in $H^1_{loc}(B_1)$;
- G_∞ is a local minimizer in B_1 .

Proof By Proposition 2.2 we already know that $G_k \rightarrow G_\infty$ uniformly on every compact set of B_1 and in $C^{0,\alpha}_{loc}(\overline{B})$, for every $\alpha \in (0, 1/2)$. Moreover, by Ascoli-Arzelá theorem it follows that $G_\infty \in C^{0,1/2}(\overline{B})$. Now, let us prove that the sequence is uniformly bounded in $H^1_{loc}(B_1)$ in order to ensure the weak convergence of sequence G_k . Fixed $i = 1, \dots, m$ and $r \in (0, 1)$, consider the competitor $G_{k,\varepsilon} = G_k - \varepsilon g_{k,\pm}^i \eta^2 e^i$, with $\eta \in C^\infty_c(B_r)$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B_{r/2}, \quad |\nabla \eta| \leq \frac{C}{r}$$

and $\varepsilon > 0$ small enough. Note that $G_{k,\varepsilon} = G_k$ on ∂B_r and $\{|G_k| > 0\} = \{|G_{k,\varepsilon}| > 0\}$. Therefore, from the local minimality of G_k we get $\mathcal{J}(G_k, B_r) \leq \mathcal{J}(G_{k,\varepsilon}, B_r)$, which implies

$$\int_{B_r} \langle \nabla g_k^i, \nabla (g_{k,\pm}^i \eta^2) \rangle dX \leq \frac{\varepsilon}{2} \int_{B_r} |\nabla (g_{k,\pm}^i \eta^2)|^2 dX.$$

Finally, letting $\varepsilon \rightarrow 0$ and proceeding as in the proof of the standard Caccioppoli inequality, we deduce

$$\int_{B_{r/2}} |\nabla g_{k,\pm}^i|^2 dX \leq \frac{C}{r^2} \int_{B_r} (g_{k,\pm}^i)^2 dX, \tag{2.11}$$

with $C > 0$ universal constant and $r \in (0, 1)$. Thus, since the sequence $(G_k)_k$ is uniformly bounded in $L^2(B_1)$, by (2.11) we get that the sequence is uniformly bounded in $H^1_{loc}(B_1)$ and it weakly converges to some $G_\infty \in H^1(B_1)$.

In conclusion, let us show that for every $r \in (0, 1)$ we have

$$\mathcal{J}(G_\infty, B_r) \leq \mathcal{J}(G_\infty + \Psi, B_r), \quad \text{for every } \Psi = (\psi^1, \dots, \psi^m) \in H^1_0(B_r; \mathbb{R}^m).$$

Since we already know by Proposition 2.2 that there exists a local minimizer Hölder continuous of class $C^{0,1/2}$, we can assume that Ψ is continuous. Therefore, for every $k > 0$ let us consider the competitor

$$G_{k,\varepsilon} = \sum_{i=1}^m (g_k^i + \psi^i - \varepsilon \eta)_+ e^i - (g_k^i + \psi^i + \varepsilon \eta)_- e^i,$$

with $\eta \in C^\infty_c(B_{(1+r)/2})$ such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on a neighbourhood of $\overline{B_r}$. Hence, by the local minimality of G_k in $B_{(1+r)/2}$, namely $\mathcal{J}(G_k, B_{(1+r)/2}) \leq \mathcal{J}(G_{k,\varepsilon},$

$B_{(1+r)/2}$), we have

$$\begin{aligned} \mathcal{L}_n(B_{(1+r)/2} \cap \{|G_k| > 0\}) &\leq \sum_{i=1}^m \int_{B_{(1+r)/2}} |\nabla \psi^i|^2 + 2\langle \nabla \psi^i, \nabla g_k^i \rangle dX + \\ &+ \varepsilon \sum_{i=1}^m \int_{\text{supp} \eta \setminus B_r} \varepsilon |\nabla \eta|^2 + 2\langle \nabla \eta, \nabla (g_k^i + \psi) \rangle dX + \\ &+ \mathcal{L}_n(B_{(1+r)/2} \cap \{|G_{k,\varepsilon}| > 0\}). \end{aligned}$$

In particular, localizing the measure of the positivity set in B_r , we get

$$\begin{aligned} \mathcal{L}_n(B_r \cap \{|G_k| > 0\}) &\leq \sum_{i=1}^m \int_{B_{(1+r)/2}} |\nabla \psi^i|^2 + 2\langle \nabla \psi^i, \nabla g_k^i \rangle dX + C\varepsilon + \\ &+ \int_{B_{(1+r)/2}} \chi_{\{|G_{k,\varepsilon}| > 0\}} dx - \int_{B_{(1+r)/2} \setminus \overline{B_r}} \chi_{\{|G_k| > 0\}} dx, \end{aligned}$$

where we used that $(G_k)_k$ is uniformly bounded in $H^1(B_{(1+r)/2})$. Since

$$\begin{aligned} \{g_k^i - \varepsilon \eta > 0\} \setminus \overline{B_r} &\subseteq \{g_k^i > 0\} \setminus \overline{B_r} \\ \{g_k^i + \varepsilon \eta < 0\} \setminus \overline{B_r} &\subseteq \{g_k^i < 0\} \setminus \overline{B_r} \end{aligned}$$

and by the uniform convergence

$$\begin{aligned} \{g_k^i + \psi^i - \varepsilon > 0\} \cap \overline{B_r} &\subseteq \{g_\infty^i + \psi^i > 0\} \cap \overline{B_r} \\ \{g_k^i + \psi^i + \varepsilon < 0\} \cap \overline{B_r} &\subseteq \{g_\infty^i + \psi^i < 0\} \cap \overline{B_r}, \end{aligned}$$

we deduce

$$\begin{aligned} \mathcal{L}_n(B_r \cap \{|G_k| > 0\}) &\leq \int_{B_{(1+r)/2}} (|\nabla \Psi|^2 + 2\langle \nabla \Psi, \nabla G_k \rangle) dX \\ &+ \mathcal{L}_n(B_r \cap \{|G_\infty + \Psi| > 0\}) + C\varepsilon. \end{aligned}$$

Now, using that $G_k \rightharpoonup G_\infty$ weakly in $H^1_{\text{loc}}(B_1)$ and uniformly on $\overline{B_r}$, we obtain

$$\mathcal{J}(G_\infty, B_r) \leq \int_{B_r} |\nabla(G_\infty + \Psi)|^2 dX + \mathcal{L}_n(B_r \cap \{|G_\infty + \Psi| > 0\}) + C\varepsilon$$

for every $\varepsilon > 0$, which implies the desired inequality. □

Finally, we conclude the section by proving the first corollaries of the non-degeneracy results Proposition 2.8. These density estimates for the positivity set of $|G|$ are obtained by a straightforward combination of the non-degeneracy condition (2.6) and the optimal regularity of local minimizer.

Corollary 2.11 *Let G be a local minimizer in B_1 and $0 \in F(G)$. Then, for every $r \in (0, 1/2)$ there exists $X_r \in B_r$ be such that*

$$B_{C_0 r}(X_0) \subset B_r^+(G),$$

for some universal constant $C_0 > 0$. Equivalently, there exists $\varepsilon_0 > 0$ such that

$$\mathcal{L}_n(B_r \cap \{|G| > 0\}) \geq \varepsilon_0 \omega_n r^n.$$

Proof The proof of the interior corkscrew condition is a combination of Proposition 2.2 and Proposition 2.8. More precisely, on one hand for r small enough there exists $X_r \in \mathcal{B}_r^+(G)$ such that $|G|(X_r) \geq Cr^{1/2}$. On the other one, since $|G|$ is of class $C^{0,1/2}$, by setting

$$C_0 = \min \left\{ 1, \frac{C}{[|G|]_{C^{0,1/2}}} \right\},$$

we have that $|G| > 0$ in $\mathcal{B}_{C_0r}^+(|G|)$, which proves the claimed lower bound. □

Remark 2.12 The following estimate is a specific feature of the non-local attitude of the vectorial thin one-phase problem. Indeed, for the local case [21, Remark 2.2] the authors highlight that, unlike in [7, 20] where it was assumed at least one component g^i to be positive, they cannot hope to have a density estimate from above on the positivity set.

Instead, since in our case the traces are $(-\Delta)^{1/2}$ -harmonic in $\{|G| > 0\}$, the upper bound holds true thanks to the different local regularity of $(-\Delta)^{1/2}$ -harmonic functions near their zero set depending on whether or not they change sign.

Corollary 2.13 *Let G be a local minimizer in B_1 and $0 \in F(G)$. Then, for every $r \in (0, 1/2)$*

$$\mathcal{L}_n(\mathcal{B}_r^+(G)) \leq (1 - \varepsilon_0)\omega_n r^n, \tag{2.12}$$

for some universal constant $\varepsilon_0 > 0$.

Proof Since $|G|$ is non-negative, up to rescaling, condition 2.12 is equivalent to

$$\mathcal{L}_n(\mathcal{B}_1 \cap \{|G| = 0\}) \geq \varepsilon_0.$$

Thus, suppose there exists a sequence $(G_k)_k$ of local minimizers in B_1 such that $0 \in F(G_k)$ and

$$\lim_{k \rightarrow \infty} \mathcal{L}_n(\mathcal{B}_1 \cap \{|G_k| = 0\}) = 0.$$

By Proposition 2.2 and Lemma 2.10, we already know that $G_k \rightarrow G_\infty$ weakly in $H^1(B_{1/2})$ and uniformly on every compact set of $B_{1/2}$. Moreover, $G_\infty \in H_{loc}^1(B_{1/2}^+) \cap C_{loc}^{0,1/2}(\overline{B_{1/2}})$ is a local minimizer in $B_{1/2}$. Now, let $\tilde{g}_k^i : B_1 \rightarrow \mathbb{R}$ be the harmonic replacement of g_k^i in B_1 , i.e. be such that

$$\begin{cases} \Delta \tilde{g}_k^i = 0 & \text{in } B_1 \\ \tilde{g}_k^i = g_k^i & \text{on } \partial B_1. \end{cases}$$

By the minimality of G_k , given the competitor $\tilde{G}_k = (g_k^1, \dots, \tilde{g}_k^i, \dots, g_k^m)$, from (2.3) we deduce

$$\int_{B_1} \left| \nabla(g_k^i - \tilde{g}_k^i) \right|^2 dX \leq \mathcal{L}_n(\mathcal{B}_1 \cap \{|G_k| = 0\}) \rightarrow 0, \tag{2.13}$$

as $k \rightarrow \infty$. Thus, up to a subsequence, the sequence $(\tilde{G}_k)_k$ do converge uniformly on every compact set of $B_{1/2}$ to some function $\tilde{G}_\infty \in H_{loc}^1(B_{1/2}^+)$ which is harmonic in $B_{1/2}$. Finally, by applying Fatou's Lemma to (2.13), we get

$$\int_{B_{1/2}} \left| \nabla(g_\infty^i - \tilde{g}_\infty^i) \right|^2 dX = 0,$$

namely for every $i = 1, \dots, m$ we deduce that g_∞^i is harmonic in $B_{1/2}$ such that $0 \in F(G_\infty)$. Hence, we already know that $g_\infty^i \in C_{loc}^{0,\alpha}(\mathbb{R}^{n+1})$, for every $\alpha \in (0, 1)$, in contradiction with Proposition 2.8 for $\alpha > 1/2$. □

3 Weiss monotonicity formula

In this section we establish a Weiss type monotonicity formula in the spirit of [20, 21]. In the case $m = 1$, our result recovers the one in [1] for the scalar case. As it is well known in the literature, this result will imply the convergence of a blow-up sequence to an homogenous global minimizer.

For a vector-valued function $G \in H^1(B_1; \mathbb{R}^m)$, let us consider

$$W(X_0, G, r) = \frac{1}{r^n} \mathcal{J}(G, B_r(X_0)) - \frac{1}{2r^{n+1}} \int_{\partial^+ B_r(X_0)} |G|^2 \, d\sigma \tag{3.1}$$

The monotonicity of $r \mapsto W(X_0, g, r)$ is a fundamental tool for the classification of the blow-up limits.

Theorem 3.1 *Let G be a local minimizer of (1.2) and $X_0 \in F(G)$. Then, the Weiss type functional $r \mapsto W(X_0, G, r)$ is monotone non-decreasing for every $r \in (0, 1 - |X_0|)$. More precisely, we have*

$$\frac{d}{dr} W(X_0, G, r) \geq \frac{1}{r^{n+2}} \sum_{i=1}^m \int_{\partial B_r(X_0)} \left(\langle \nabla g^i, X - X_0 \rangle - \frac{1}{2} g^i \right)^2 \, d\sigma. \tag{3.2}$$

Moreover, $W(X_0, G, \cdot)$ is constant in $(0, +\infty)$ if and only if G is s -homogeneous with respect to X_0 .

Through the paper we will always denote with $W(X_0, G, 0^+)$ the limit of the Weiss monotonicity formula as $r \rightarrow 0^+$.

In order to simplify the notation, since the problem is invariant under translation, in the following computations we will assume $X_0 = 0$ and denote $W(r) = W(0, G, r)$.

Lemma 3.2 *Let G be a local minimizer of (1.2) and $0 \in F(G)$. Then, we get*

$$\begin{aligned} \int_{B_r} |\nabla G|^2 \, dX + \mathcal{L}_n(B_r^+(G)) &\leq \frac{1}{n} \int_{\partial B_r} \left(r |\nabla_{S^n} G|^2 + \frac{1}{4} \frac{|G|^2}{r} \right) \, d\sigma + \\ &+ \frac{r}{n} \mathcal{H}^{n-1}(\partial B_r \cap \{|G| > 0\}), \end{aligned}$$

for every $r \in (0, 1)$.

Proof Let us consider now the $1/2$ -homogeneous extension $\tilde{G} = (\tilde{g}_1, \dots, \tilde{g}_m)$ of the trace of G on ∂B_r , defined by

$$\tilde{G}(X) = \frac{|X|^{1/2}}{r^{1/2}} G \left(X \frac{r}{|X|} \right).$$

Then, for every $i = 1, \dots, m$ we get

$$|\nabla \tilde{g}_i|^2(X) = \frac{1}{4} \frac{1}{r|X|} g^i \left(X \frac{r}{|X|} \right)^2 + \frac{r}{|X|} |\nabla_{S^n} g^i|^2 \left(X \frac{r}{|X|} \right).$$

Integrating over B_r^+ and summing for $i = 1, \dots, m$, we obtain

$$\begin{aligned} \int_{B_r} |\nabla \tilde{G}|^2 \, dX &= \int_0^r \frac{1}{\rho} \int_{\partial B_\rho} \left(\frac{1}{4} \frac{1}{r} |G|^2 \left(X \frac{r}{\rho} \right) + r |\nabla_{S^n} G|^2 \left(X \frac{r}{\rho} \right) \right) \, d\sigma \, d\rho \\ &= \frac{r}{n} \int_{\partial B_r} \left(\frac{1}{4} \frac{|G|^2}{r} + r |\nabla_{S^n} G|^2 \right) \, d\sigma, \end{aligned}$$

while for the measure term we have that

$$\mathcal{L}_n(\mathcal{B}_r \cap \{|G| > 0\}) = \frac{r}{n} \mathcal{H}^{n-1}(\partial \mathcal{B}_r \cap \{|G| > 0\})$$

Finally, since $\tilde{G} = G$ on ∂B_r , the minimality assumption $\mathcal{J}(G, B_r) \leq \mathcal{J}(\tilde{G}, B_r)$ gives the claimed inequality. \square

Proof of Theorem 3.1 By the estimate of Lemma 3.2, we immediately get

$$\begin{aligned} W'(r) &= \frac{1}{r^n} \left(\int_{\partial B_r} |\nabla G|^2 \, dX + \mathcal{H}^{n-1}(\partial \mathcal{B}_r \cap \{|G| > 0\}) \right) + \\ &\quad - \frac{n}{r^{n+1}} \left(\int_{B_r} |\nabla G|^2 \, dX + \mathcal{L}_n(\mathcal{B}_r \cap \{|G| > 0\}) \right) + \\ &\quad - \frac{1}{r^{n+1}} \sum_{i=1}^m \int_{\partial B_r} g^i \partial_r g^i \, d\sigma + \frac{1}{2r^{n+2}} \int_{\partial B_r} |G|^2 \, d\sigma \\ &\geq \frac{1}{r^n} \sum_{i=1}^m \int_{\partial B_r} \left(|\partial_r g^i|^2 - g^i \partial_r u^i + \frac{1}{4r^2} |g^i|^2 \right) \, d\sigma \\ &= \frac{1}{r^n} \sum_{i=1}^m \int_{\partial B_r} \left(\partial_r g^i - \frac{1}{2r} g^i \right)^2 \, d\sigma. \end{aligned}$$

Finally, since the right hand side of (3.2) is non-negative, we deduce that $W'(r) \equiv 0$ for $r \in (0, +\infty)$ if and only if

$$\left\langle \nabla g^i(X), \frac{X}{|X|} \right\rangle = \frac{1}{2|X|} g^i(X) \quad \text{in } \mathbb{R}^{n+1},$$

i.e. the components g^i are 1/2-homogeneous in \mathbb{R}^{n+1} . \square

4 Compactness and convergence of blow-up sequences

This section is dedicated to the convergence of the blow-up sequences and the analysis of the blow-up limits, both being essential for determining the local behavior of the free boundary and for the characterization of the Regular and Singular sets.

Let us recall the notion of blow-up sequence associated to a local minimizer G in B_1 . Given $(X_k)_k \subset F(G)$ and $r_k \searrow 0^+$ such that $B_{r_k}(X_k) \subset B_1$, we define a blow-up sequence by

$$G_{X_k, r_k}(X) = \frac{1}{r_k^{1/2}} G(X_k + r_k X). \tag{4.1}$$

Then the sequence $(G_{X_k, r_k})_k$ is uniformly Hölder continuous in the class $C^{0,1/2}$ and locally uniformly bounded in \mathbb{R}^{n+1} . Thus, by Lemma 2.10, up to a subsequence, $(G_{X_k, r_k})_k$ converges locally uniformly on every compact set to a function $G_0 \in H_{\text{loc}}^1(B_1) \cap C_{\text{loc}}^{0,1/2}(\overline{B_1})$ such that, for every $R > 0$ the following properties hold

- $G_{X_k, r_k} \rightarrow G_0$ in $C_{\text{loc}}^{0,\alpha}(\overline{B_R})$, for every $\alpha \in (0, 1/2)$;
- $G_{X_k, r_k} \rightharpoonup G_0$ weakly in $H^1(B_R)$;
- G_0 is a local minimizer in B_R .

Moreover, by the non-degeneracy results of the previous section, we can guarantee the existence of a non-degenerate blow-up limit.

Proposition 4.1 *Let G be a local minimizer in B_1 . Given $(X_k)_k \subset F(G)$ and $r_k \searrow 0^+$ such that $B_{r_k}(X_k) \subset B_1$, for every $R > 0$ the following properties hold (up to extracting a subsequence):*

- $G_{X_k, r_k} \rightarrow G_0$ strongly in $H^1(B_R; \mathbb{R}^m)$;
- the sequence of the characteristic functions

$$\chi(\{|G_{X_k, r_k}| > 0\}) \rightarrow \chi(\{|G_0| > 0\})$$

strongly in $L^1(B_R)$;

- the sequence of the closed sets $\overline{\mathcal{B}_R^+(G_{X_k, r_k})}$ and its complement in \mathbb{R}^n , converge in the Hausdorff sense respectively to $\mathcal{B}_R^+(G_0)$ and $\mathbb{R}^n \setminus \mathcal{B}_R^+(G_0)$
- the blow-up limit G_0 is non-degenerate at zero, i.e. there exists a dimensional constant $c_0 > 0$ such that

$$\sup_{B_r} |G_0| \geq c_0 r^{1/2} \text{ for every } r > 0.$$

Proof For notational simplicity, we set $G_k = G_{X_k, r_k}$. Since $|G_k|$ converges locally uniformly to $|G_0|$, we get

$$\chi(\{|G_0| > 0\}) \leq \liminf_{k \rightarrow \infty} \chi(\{|G_k| > 0\})$$

Now, let us prove that G_k converges strongly in $H^1_{loc}(\mathbb{R}^{n+1}; \mathbb{R}^m)$ to G_0 and that the characteristic functions $\chi(\{|G_k| > 0\})$ converge to $\chi(\{|G_0| > 0\})$ in L^1 . Namely, fixed a radius $R > 0$, it is sufficient to prove that

$$\lim_{k \rightarrow \infty} \int_{B_R} |\nabla G_k|^2 dX + \mathcal{L}_n(\mathcal{B}_R^+(G_k)) = \int_{B_R} |\nabla G_0|^2 dX + \mathcal{L}_n(\mathcal{B}_R^+(G_0)).$$

Consider now $\eta \in C_c^\infty(\mathbb{R}^{n+1})$, $0 \leq \eta \leq 1$ such that $\eta \equiv 1$ on B_R , and the competitor $\tilde{G}_k \in H^1(B_1; \mathbb{R}^m)$ defined by

$$\tilde{G}_k = \eta G_0 + (1 - \eta)G_k.$$

For the sake of notational simplicity, let us set:

$$\Omega_k = \{|G_k| > 0\} \cap \mathbb{R}^n, \quad \tilde{\Omega}_k = \{|\tilde{G}_k| > 0\} \cap \mathbb{R}^n \quad \text{and} \quad \Omega_0 = \{|G_0| > 0\} \cap \mathbb{R}^n.$$

Since $\tilde{G}_k = G_k$ on $\{\eta = 0\}$, by the optimality of G_k we get

$$\begin{aligned} \int_{\{\eta > 0\}} |\nabla G_k|^2 dX + \mathcal{L}_n(\Omega_k \cap \{\eta > 0\}) &\leq \int_{\{\eta > 0\}} |\nabla \tilde{G}_k|^2 dX + |\tilde{\Omega}_k \cap \{\eta > 0\}| \\ &\leq \int_{\{\eta > 0\}} |\nabla \tilde{G}_k|^2 dX + \mathcal{L}_n(\Omega_0 \cap \{\eta = 1\}) + \mathcal{L}_n(\{0 < \eta < 1\}). \end{aligned} \tag{4.2}$$

On $\{\eta > 0\}$ we calculate

$$\begin{aligned} |\nabla G_k|^2 - |\nabla \tilde{G}_k|^2 &= |\nabla G_k|^2 - |\eta \nabla G_0 + (1 - \eta) \nabla G_k + (G_0 - G_k) \nabla \eta|^2 \\ &= (1 - (1 - \eta)^2) |\nabla G_k|^2 - \eta^2 |\nabla G_0|^2 - |G_0 - G_k|^2 |\nabla \eta|^2 + \\ &\quad - 2(G_0 - G_k) \langle \nabla \eta, \eta \nabla G_0 + (1 - \eta) \nabla G_k \rangle - 2\eta(1 - \eta) \langle \nabla G_0, \nabla G_k \rangle. \end{aligned}$$

Since G_k converges strongly in $L^2(B_R; \mathbb{R}^m)$ and weakly $H^1_{loc}(\mathbb{R}^{n+1}; \mathbb{R}^m)$ to G_0 , we can estimate

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \int_{\{\eta > 0\}} (|\nabla G_k|^2 - |\nabla \tilde{G}_k|^2) \, dX \\ &= \limsup_{k \rightarrow \infty} \int_{\{\eta > 0\}} \left((1 - (1 - \eta)^2) |\nabla G_k|^2 - \eta^2 |\nabla G_0|^2 - 2\eta(1 - \eta) \langle \nabla G_0, \nabla G_k \rangle \right) \, dX \\ &= \limsup_{k \rightarrow \infty} \int_{\{\eta > 0\}} (1 - (1 - \eta)^2) (|\nabla G_k|^2 - |\nabla G_0|^2) \, dX \\ &\geq \limsup_{k \rightarrow \infty} \int_{\{\eta = 1\}} (|\nabla G_k|^2 - |\nabla G_0|^2) \, dX, \end{aligned}$$

where in the last inequality we used that $|\nabla G_k|$ weakly converges in $L^2(\{0 < \eta < 1\})$ to $|\nabla G_0|$.

Combining this fact with inequality (4.2), we obtain

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left(\int_{\{\eta = 1\}} (|\nabla G_k|^2 - |\nabla G_0|^2) \, dX + \mathcal{L}_n(\Omega_k \cap \{\eta = 1\}) - \mathcal{L}_n(\Omega_0 \cap \{\eta = 1\}) \right) \\ &\leq \limsup_{k \rightarrow \infty} \left(\int_{\{\eta > 0\}} (|\nabla G_k|^2 - |\nabla \tilde{G}_k|^2) \, dX + \mathcal{L}_n(\Omega_k \cap \{\eta = 1\}) - \mathcal{L}_n(\Omega_0 \cap \{\eta = 1\}) \right) \\ &\leq \limsup_{k \rightarrow \infty} \mathcal{L}_n(\Omega_k \cap \{\eta = 1\}) - \mathcal{L}_n(\Omega_k \cap \{\eta > 0\}) + \mathcal{L}_n(\{0 < \eta < 1\}) \\ &\leq \mathcal{L}_n(\{0 < \eta < 1\}). \end{aligned}$$

Finally, since η is arbitrary outside B_R , the right hand side can be made arbitrarily small, and this implies the desired equality.

By Corollary 2.11 and Corollary 2.13, we already know that

$$\varepsilon_0 \omega_n r^n \leq \mathcal{L}_n(B_r \cap \{|G_k| > 0\}) \leq (1 - \varepsilon_0) \omega_n r^n, \quad \text{for } r < r_0/r_k, \tag{4.3}$$

and for every $k > 0$. Now, it is well-known that the convergence of the sequence of characteristic functions in the strong topology of L^1 , together with (4.3), implies the Hausdorff convergence of $\overline{\Omega_k \cap B_R}$ to $\overline{\Omega_0 \cap B_R}$ locally in \mathbb{R}^n . Obviously, the same result holds for the complements Ω_k^c .

Finally, the non-degeneracy of the blow-up limit is a straightforward combination of the uniform convergence and the non-degeneracy condition (2.6). Namely, by Proposition 2.8, for every $k > 0$ the rescaled function G_k is non-degenerate in the sense

$$\text{for every } y \in \overline{\Omega_k}, r \leq \frac{1}{2r_k} \sup_{B_r(y) \cap \mathbb{R}^n} |G_k| \geq c_0 r^{1/2}.$$

The previous inequality is obtained by applying (2.6) in $B_{r_k r}(y)$ for the local minimizer G . Finally, by the uniform convergence of G_k and the Hausdorff convergence of $\Omega_k \cap B_r$ in \mathbb{R}^n , for every $y \in \overline{\Omega_0}$ we get

$$\sup_{B_r(y) \cap \mathbb{R}^n} |G_k| \geq c_0 r^{1/2}, \quad \text{for every } r > 0.$$

□

The following is a straightforward application of the Weiss monotonicity formula to the blow-up limit.

Corollary 4.2 *Let G be a local minimizer and $X_0 \in F(G)$. Then every blow-up limit $G_0 = (g_0^i, \dots, g_0^m)$ of G at X_0 is 1/2-homogeneous in \mathbb{R}^{n+1} , i.e.*

$$\left\langle \nabla g_0^i(X), \frac{X}{|X|} \right\rangle = \frac{1}{2|X|} g_0^i(X) \text{ in } \mathbb{R}^{n+1},$$

for every $i = 1, \dots, m$. Moreover, the Lebesgue density of $F(G)$ exists finite at every $X_0 \in F(G)$ and it satisfies

$$\begin{aligned} \{|G| > 0\}^{(\gamma)} &= \left\{ X_0 \in F(G) : \lim_{r \rightarrow 0^+} \frac{\mathcal{L}_n(B_r \cap \{|G_k| > 0\})}{\mathcal{L}_n(B_r)} = \gamma \right\} \\ &= \{X_0 \in F(G) : W(X_0, G, 0^+) = \omega_n \gamma\}. \end{aligned} \tag{4.4}$$

Proof Let $X_0 \in F(G)$ and G_0 a blow-up limit of G at X_0 associated to a sequence $r_k \searrow 0^+$. By Lemma 2.10 and Proposition 4.1, we already know that G_0 is a global minimizer of the vectorial Bernoulli problem. On the other hand, by the definition of the Weiss formula, for every $\rho, r > 0$ we get

$$W(X_0, G, r\rho) = W(0, G_{X_0, r}, \rho).$$

Fixed $\rho > 0$, since up to a subsequence $G_{X_0, r_k} \rightarrow G_0$ uniformly and strongly in $H^1(B_\rho, \mathbb{R}^m)$, we deduce

$$W(0, G_0, \rho) = \lim_{k \rightarrow \infty} W(0, G_{X_0, r_k}, \rho) = \lim_{k \rightarrow \infty} W(X_0, G, \rho r_k) = \lim_{r \rightarrow 0^+} W(X_0, G, r),$$

where the last limit is unique and it does not depend on the sequence $(r_k)_k$ by the monotonicity result Theorem 3.1. Finally, since $W(0, G_0, \rho)$ is constant we get that the blow-up limit is 1/2-homogeneous.

Moreover, the homogeneity of the blow-up limits and the strong convergence of the blow-up sequences imply

$$\begin{aligned} \mathcal{L}_n(B_1 \cap \{|G_0| > 0\}) &= W(0, G_0, 1) = \lim_{r \rightarrow 0^+} W(X_0, G, r) \\ &= \lim_{r \rightarrow 0^+} \frac{\mathcal{L}_n(B_r(X_0) \cap \{|G| > 0\})}{r^n}. \end{aligned} \tag{4.5}$$

Hence, the density $W(X_0, G, 0^+)$ coincides, up to a multiplicative constant, with the Lebesgue density of the free boundary. □

Remark 4.3 By (4.5), we note that for every $X_0 \in F(G)$, the measure of the positivity set in B_1 of the blow-up limit does not depend on the blow-up limit itself.

Remark 4.4 In the classification of the blow-up limits, we will use some results related to eigenvalues of the Laplace-Beltrami operator

$$\Delta S^n = \operatorname{div}_{S^n}(\nabla S^n),$$

with div_{S^n} and ∇S^n respectively the tangential divergence and gradient on S^n and $x_{n+1} = r \sin(\theta_n)$. In particular, the following results hold true.

Let $\omega \subset S^{n-1} \times \{0\}$ be an open subset of the $(n - 1)$ -sphere and let $\Sigma_\omega = \{r\theta : \theta \in \omega, r > 0\} \times \{0\}$ be the cone generated by ω in $\{x_{n+1} = 0\}$. Then, g is a α -homogeneous solution of

$$\begin{cases} -\Delta g = 0 & \text{in } \mathbb{R}_+^{n+1} \\ \partial_{x_{n+1}} g = 0 & \text{on } \Sigma_\omega \\ g = 0 & \text{on } \{x_{n+1} = 0\} \setminus \Sigma_\omega, \end{cases}$$

if and only if its trace $\varphi = g|_{S^n}$ on the sphere satisfies

$$\begin{cases} -\Delta_{S^n} \varphi = \lambda(\alpha)\varphi & \text{in } S^n_+ \\ \partial_{\theta_n} \varphi = 0 & \text{on } \omega \\ \varphi = 0 & \text{on } (S^{n-1} \times \{0\}) \setminus \omega, \end{cases} \tag{4.6}$$

with $\lambda(\alpha) = \alpha(\alpha + n - 1)$ the characteristic eigenvalue associated to the section ω . Moreover, both the map $\omega \mapsto \alpha(\omega)$ and $\omega \mapsto \lambda(\alpha(\omega))$ are monotone with respect to the inclusion of spherical sets.

In particular, if $\alpha < 1$, then φ cannot change sign and it is indeed a multiple of the principle eigenvalue. Finally, for every spherical set $\omega \subset S^{n-1}$ such that $\mathcal{H}^{n-1}(S) \leq n\omega_n/2$ we have the inequality

$$\lambda_1 \geq \frac{1}{2} \left(n - \frac{1}{2} \right),$$

and the equality is achieved if and only if, up to a rotation, $\omega = S^{n-1} \cap \{x_n > 0\}$.

The proof of these claims uses the monotonicity of the eigenvalue with respect to the inclusion of spherical set and the Pólya-Szegő inequality for the Schwarz symmetrization applied to the eigenvalue problem (4.6) (see [23] for further details).

The following Lemma characterizes the structure of the blow-up limits. In particular, we can prove that the norm of every blow-up limit is a global minimizer of the scalar thin one-phase functional.

Proposition 4.5 *Let G be a local minimizer and $X_0 \in F(G)$. Then, every blow-up limit G_0 is of the form*

$$G_0(X) = \xi |G_0|(X) \text{ where } \xi \in \mathbb{R}^m, |\xi| = 1$$

and $|G_0|$ is a global minimizer of the scalar thin one-phase functional

$$\mathcal{J}(g, B_R) = \int_{B_R} |\nabla g|^2 dX + \mathcal{L}_n(B_R \cap \{g > 0\}), \text{ for } R > 0. \tag{4.7}$$

Moreover, there exists a dimensional constant $\delta \in (0, 1/2)$ such that one of the following possibilities holds:

1. The Lebesgue density of $\{|G| > 0\}$ at X_0 is $1/2$ and every blow-up limit G_0 is of the form

$$G_0(X) = \xi AU(\langle x, \nu \rangle, x_{n+1}) \text{ where } \xi \in \mathbb{R}^m, |\xi| = 1, \nu \in S^{n-1} \times \{0\} \tag{4.8}$$

and $A > 0$ a specific constant depending only on n .

2. The Lebesgue density of $\{|G| > 0\}$ at X_0 satisfies

$$\frac{1}{2} + \delta \leq \lim_{r \rightarrow 0} \frac{|\mathcal{B}_r(X_0) \cap \{|G| > 0\}|}{|\mathcal{B}_r|} \leq 1 - \delta, \tag{4.9}$$

and $|G_0|$ is a nonnegative global minimizer of (4.7) with singularity in zero.

Proof Let $X_0 \in F(G)$ and G_0 a blow-up limit of G at X_0 . By Corollary 4.2 the limit G_0 is an $1/2$ -homogeneous global minimizer such that

$$|\mathcal{B}_1^+(G_0)| = \gamma |\mathcal{B}_1|,$$

for some $\gamma \in (0, 1)$ (because of the density estimates). By Lemma 2.4 and Corollary 2.6, we have that the blow-up limit satisfies

$$\begin{cases} \Delta g_0^i = 0 & \text{in } \mathbb{R}_+^{n+1} \\ \partial_{x_{n+1}} g_0^i = 0 & \text{on } \{|G_0| > 0\} \cap \mathbb{R}^n \\ g_0^i = 0 & \text{on } \mathbb{R}^n \setminus \{|G_0| > 0\}. \end{cases}$$

Hence, in view of Remark 4.4 all the components are equal up to a multiplicative constant. Moreover by nondegeneracy, G_0 cannot be identically zero.

Thus, there exists $\xi \in \mathbb{R}^m$ such that $|\xi| = 1$ and $G_0 = \xi u$, where $|G_0| = g$ and g is a global minimizer of (4.7). Indeed, for every $R > 0$ let $\tilde{g} \in H_{loc}^1(\mathbb{R}^n)$ be such that $\text{supp}(g - \tilde{g}) \subseteq B_R$. Then, given the competitor $\tilde{G} = \xi \tilde{g}$, we easily get that $\mathcal{J}(G_0, B_R) \leq \mathcal{J}(\tilde{G}, B_R)$ is equivalent to

$$\int_{B_R} |\nabla g|^2 \, dX + \mathcal{L}_n(\mathcal{B}_R \cap \{g > 0\}) \leq \int_{B_R} |\nabla \tilde{g}|^2 \, dX + \mathcal{L}_n(\mathcal{B}_R \cap \{\tilde{g} > 0\}).$$

The desired claims now follow by the known results for the scalar case, see [12, Proposition 5.3]. □

The previous analysis allows us to extend the results from the scalar case to our vectorial counterpart. In particular, the problem of the existence of singular global minimizer for (1.1) coincides with its scalar counterpart. Indeed, by [12] we have

$$n^* = \inf\{k \in \mathbb{N} : \text{there exists an } 1/2 \text{ --homogeneous global minimizer with singularity in zero}\} \geq 3.$$

Corollary 4.6 *Let G_0 be a global minimizer in \mathbb{R}^{n+1} with $n < n^*$. Then G_0 is of the form (4.8).*

Finally, we introduce the notion of regular and singular part of $F(G)$. The rest of the paper will be devoted to analyzing the smoothness of the regular part of the free boundary.

Definition 4.7 Let $X_0 \in F(G)$. We say that

- X_0 is a regular point in $\text{Reg}(F(G))$, if the Lebesgue density of $\{|G| > 0\}$ at X_0 is $1/2$;
- X_0 is a singular point in $\text{Sing}(F(G))$, if $X_0 \notin \text{Reg}(F(G))$.

5 Viscosity formulation around $\text{Reg}(F(G))$

In this short section we recall some basic facts about the scalar thin one-phase free boundary problem, and we state the viscosity formulation of the vector valued analogue. We show that local minimizers are indeed viscosity solutions. Hence, the analysis of the regular part of the free boundary can be performed with the viscosity methods of [10, 15]. However, as pointed out in the introduction, differently from the local case, the reduction from the vector valued problem to the scalar one is now almost straightforward. For this reason, we start by recalling definitions and basic property for the scalar problem.

5.1 The scalar problem

In this subsection we collect basic definitions and results for the scalar thin one-phase free boundary problem

$$\begin{cases} \Delta g = 0, & \text{in } B_1^+(g) := B_1 \setminus \{(x, 0) : g(x, 0) = 0\}, \\ \frac{\partial g}{\partial t^{1/2}} = 1, & \text{on } F(g) := B_1 \cap \partial_{\mathbb{R}^n} \{(x, 0) : g(x, 0) > 0\}, \end{cases} \tag{5.1}$$

where

$$\frac{\partial g}{\partial t^{1/2}}(x_0) := \lim_{t \rightarrow 0^+} \frac{g(x_0 + t\nu(x_0), 0)}{\sqrt{t}}, \quad X_0 = (x_0, 0) \in F(g), \tag{5.2}$$

with $\nu(x_0)$ the unit normal to the free boundary $F(g)$ at x_0 pointing toward $B_1^+(g)$. For further details and proofs, we refer the reader to [6, 10–13].

First, we state the notion of viscosity solutions to (5.1), as introduced in [10].

Definition 5.1 Given g, v continuous, we say that v touches g by below (resp. above) at $X_0 \in B_1$ if $g(X_0) = v(X_0)$, and

$$g(X) \geq v(X) \quad (\text{resp. } g(X) \leq v(X)) \quad \text{in a neighbourhood } O \text{ of } X_0.$$

If this inequality is strict in $O \setminus \{X_0\}$, we say that v touches g strictly by below (resp. above).

Definition 5.2 We say that $v \in C(B_1)$ is a (strict) comparison subsolution to (5.1) if v is a non-negative function in B_1 which is even with respect to $x_{n+1} = 0$ and it satisfies

- (i) v is C^2 and $\Delta v \geq 0$ in $B_1^+(v)$;
- (ii) $F(v)$ is C^2 and if $x_0 \in F(v)$ we have

$$v(x_0 + t\nu(x_0), 0) = \alpha(x_0)\sqrt{t} + o(\sqrt{t}), \quad \text{as } t \rightarrow 0^+,$$

with

$$\alpha(x_0) \geq 1,$$

where $\nu(x_0)$ denotes the unit normal at x_0 to $F(v)$ pointing toward $B_1^+(v)$;

- (iii) Either v is not harmonic in $B_1^+(v)$ or $\alpha(x_0) > 1$ at all $x_0 \in F(v)$.

Similarly one can define a (strict) comparison supersolution.

Definition 5.3 We say that g is a viscosity solution to (5.1) if g is a continuous non-negative function in B_1 which is even with respect to $x_{n+1} = 0$ and it satisfies

1. $\Delta g = 0$ in $B_1^+(g)$;
2. Any (strict) comparison subsolution (resp. supersolution) cannot touch g by below (resp. by above) at a point $X_0 = (x_0, 0) \in F(g)$.

5.1.1 The function \tilde{g}

In this subsection we recall the notion of ε -domain variation from [10]. Via this transformation the problem (5.1) can be “linearized”, as long as an appropriate Harnack type inequality is established. This is the heart of the strategy developed in [10] and that we plan to adapt to the vectorial context.

Recall that we denote by P the half-hyperplane

$$P := \{X \in \mathbb{R}^{n+1} : x_n \leq 0, x_{n+1} = 0\}$$

and by

$$L := \{X \in \mathbb{R}^{n+1} : x_n = 0, x_{n+1} = 0\}.$$

Also, we call $U(X) := U(x_n, x_{n+1})$, where U is the function defined in (1.6).

Let g be a continuous non-negative function in \overline{B}_ρ . We define the multivalued map \tilde{g} which associate to each $X \in \mathbb{R}^{n+1} \setminus P$ the set $\tilde{g}(X) \subset \mathbb{R}$ via the formula

$$U(X) = g(X - we_n), \quad \forall w \in \tilde{g}(X). \tag{5.3}$$

We write $\tilde{g}(X)$ to denote any of the values in this set.

This change of variables has the same role as the partial Hodograph transform for the standard one-phase problem. Our free boundary problem becomes a problem with fixed boundary for \tilde{g} , and the limiting values of \tilde{g} on L give the free boundary of g as a graph in the e_n direction.

Recall that if g satisfies

$$U(X - \varepsilon e_n) \leq g(X) \leq U(X + \varepsilon e_n) \quad \text{in } B_\rho, \text{ for } \varepsilon > 0 \tag{5.4}$$

then $\tilde{g}(X) \neq \emptyset$ for $X \in B_{\rho-\varepsilon} \setminus P$ and $|\tilde{g}(X)| \leq \varepsilon$, hence we can associate to g a possibly multi-valued function \tilde{g} defined at least on $B_{\rho-\varepsilon} \setminus P$ and taking values in $[-\varepsilon, \varepsilon]$ which satisfies

$$U(X) = g(X - \tilde{g}(X)e_n). \tag{5.5}$$

Moreover if g is strictly monotone in the e_n -direction in $B_\rho^+(g)$, then \tilde{g} is single-valued. See [10, Section 3] for the basic properties of \tilde{g} .

5.2 The Vector Valued Case

We consider now the vector valued thin problem:

$$\begin{cases} \Delta G = 0 & \text{in } B_1^+(|G|); \\ \frac{\partial}{\partial t^{1/2}} |G| = 1 & \text{on } F(G). \end{cases} \tag{5.6}$$

Here and henceforth, for notational simplicity we use $B_1^+(G)$ in place of $B_1^+(|G|)$.

Definition 5.4 We say that $G = (g^1, \dots, g^m) \in C(B_1, \mathbb{R}^m)$ is a viscosity solution to (5.6) in B_1 if each g^i is even with respect to $\{x_{n+1} = 0\}$,

$$\Delta g^i = 0 \quad \text{in } B_1^+(G), \quad \forall i = 1, \dots, m, \tag{5.7}$$

and the free boundary condition is satisfied in the following sense. Given $X_0 \in F(G)$, and a continuous function φ in a neighborhood of X_0 , then

- (i) If φ is a strict comparison subsolution to (5.1), then for all unit directions f , $\langle G, f \rangle$ cannot be touched by below by φ at X_0 .
- (ii) If φ is a strict comparison supersolution to (5.1), then $|G|$ cannot be touched by above by φ at X_0 .

Remark 5.5 We remark that if G is a viscosity solution to (5.6) in B_λ , then

$$G_\lambda(X) = \lambda^{-1/2}G(\lambda X), \quad X \in B_1$$

is a viscosity solution to (5.6) in B_1 .

Remark 5.6 Notice that, if G is a viscosity solution to (5.6), then $|G|$ is a viscosity subsolution to the scalar thin one-phase problem (5.1). Indeed, by the free boundary condition in Definition 5.4, we easily deduce the validity of its scalar counterpart in Definition 5.3 (see also Remark 2.5).

In the next proposition we prove that local minimizers are indeed viscosity solutions.

Proposition 5.7 *Let G be a local minimizer in B_1 . Then, up to a scalar multiple, G is a viscosity solution of (5.6) in B_1 .*

Proof Since the constant $A > 0$ in (4.8) depends only on the dimension n , up to a scalar multiplication it is not restrictive to assume that $A = 1$ in Theorem 4.5.

By Lemma 2.4 we already know that (5.7) is satisfied. Hence, let φ be a strict comparison subsolution to (5.1), and suppose by contradiction that there exists a unit direction f in \mathbb{R}^m such that $\langle G, f \rangle$ is touched by below by φ at $Y_0 \in F(G)$.

Consider now the blow-up sequences centered in the touching point

$$G_k(X) = \frac{1}{r_k^{1/2}}G(Y_0 + r_k X) \quad \text{and} \quad \varphi_k(X) = \frac{1}{r_k^{1/2}}\varphi(Y_0 + r_k X),$$

for some sequence of radii $r_k \rightarrow 0^+$. Up to a subsequence, they converge respectively to some G_0 and φ_0 uniformly on every compact set of \mathbb{R}^{n+1} . By Definition 5.2, we get, up to rotation, that

$$\varphi_0(X) = \alpha U(x_n, x_{n+1}) \quad \text{with } \alpha > 1, \tag{5.8}$$

On the other side, by Proposition 4.5 the norm $|G_0|$ is a 1/2-homogeneous global minimizer of the scalar thin one-phase functional (4.7) such that $\{|G_0| > 0\} \cap \{x_{n+1} = 0\} \supset \{x_{n+1} = 0, x_n > 0\}$. By Remark 4.4, we deduce that $\{|G_0| = 0\} \cap \{x_{n+1} = 0\} = P$ and consequently

$$G_0(X) = \xi U(x_n, x_{n+1}) \quad \text{where } \xi \in \mathbb{R}^m, |\xi| = 1.$$

Hence, we immediately deduce that

$$\varphi_0(X) = \alpha U(x_n, x_{n+1}) \leq \langle G_0, f \rangle = \langle \xi, f \rangle U(x_n, x_{n+1}),$$

in contradiction with the hypothesis (5.8).

On the other hand, let φ be a comparison strict supersolution and let us assume that $|G|$ is touched by above by φ at some $Y_0 \in F(G)$. By the same blow-up procedure we get, up to rotation, that

$$\varphi_0(X) = \alpha U(x_n, x_{n+1}) \quad \text{with } \alpha < 1,$$

and that $|G_0|$ is a 1/2-homogeneous global minimizer of the scalar thin one-phase functional (4.7) such that $\{|G_0| = 0\} \cap \{x_{n+1} = 0\} \subset P$. As before, we get

$$|G_0|(X) = U(x_n, x_{n+1}).$$

Since $|G_0| \leq \varphi_0$, the absurd follows from the fact that $\alpha < 1$. □

6 Flat free boundaries: the Harnack inequality

In this section we develop the basic tools for our analysis of the regular part of the free boundary. In view of Definition 4.7, Proposition 5.7, and non-degeneracy, this boils down to understanding "flat" viscosity solutions defined below.

Definition 6.1 Let G be a viscosity solution to (5.6) in B_1 . We say that G is ε -flat in the (f, ν) -directions in B_1 , if for some unit directions $f \in \mathbb{R}^m, \nu \in \mathbb{R}^n$,

$$|G(X) - U(\langle x, \nu \rangle, x_{n+1})f| \leq \varepsilon \quad \text{in } B_1, \tag{6.1}$$

and

$$|G|(x, 0) \equiv 0 \quad \text{in } B_1 \cap \{\langle x, \nu \rangle < -\varepsilon\}. \tag{6.2}$$

6.1 Key lemmas.

Below is the key proposition that allows us to reduce our analysis to the scalar case. As already remarked, this is different from the approach we followed for the local vectorial one-phase problem in [15] where we did not reduce to the scalar counterpart. In this case such reduction just requires the construction of an appropriate barrier. In the local case a "component-wise" strategy has been used by the authors in [21], however it required delicate geometric measure theory tools.

Proposition 6.2 *Let G be a viscosity solution to (5.6) in B_1 . There exists $\varepsilon_0 > 0$ universal such that, if G is ε_0 -flat in the (f^1, e_n) -directions in B_1 , then*

$$g^1 > 0 \quad \text{in } B_1^+(G).$$

Proof Let $\varepsilon_0 > 0$ to be chosen later, by the flatness assumption (6.1) we deduce that

$$U(X) - \varepsilon_0 \leq g^1(X) \leq U(X) + \varepsilon_0 \quad \text{in } B_1.$$

For $\delta_0 > 0$ to be made precise later, $x_0 \in B_1$, set $\Phi_{x_0}(X) = \delta_0(\Lambda x_{n+1}^2 - |x - x_0|^2)$ with $\Lambda > 0$ universal such that $\Delta \Phi > 0$ in B_1 . We aim to show that $g^1 \geq \Phi_0$ on $\partial B_1^+(G)$, which implies by the comparison principle that $g^1 > 0$ on the x_{n+1} -axis minus the origin. Hence, by comparing g^1 with Φ_{x_0} and varying x_0 in B_1 we get that $g^1 > 0$ in $B_1 \setminus \{x_{n+1} = 0\}$. Our claim then follows by continuity.

Clearly, on the set $\{|G| \equiv 0\} \cap \{x_{n+1} = 0\}$ we have $g^1 = 0 \geq \Phi_0$. On the other hand on $\partial B_1 \setminus (\{|G| \equiv 0\} \cap \{x_{n+1} = 0\})$ we argue as follows: given $\Lambda' > \Lambda$ let

$$C = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : \Lambda' x_{n+1}^2 - |x|^2 > 0\} \supset \{\Phi > 0\}$$

be a slighter larger cone in \mathbb{R}^{n+1} . Since $g^1 \geq U - \varepsilon_0$, there exists an universal constant $c_0 > 0$ such that

$$\begin{cases} g^1 \geq c_0 > 0 & \text{on } \partial B_1 \cap C \\ g^1 \geq -\varepsilon_0 & \text{on } \partial B_1 \setminus C \end{cases}.$$

Finally, fixed

$$M_0 = \max_{\partial B_1 \cap C} \Phi, \quad m_0 = \max_{\partial B_1 \setminus C} |\Phi|,$$

let us choose $\delta_0 > 0$ so that

$$\delta_0 M_0 \leq c_0 \text{ on } \partial B_1 \cap C.$$

Then, up to choose $\varepsilon_0 > 0$ small enough, we get

$$-m_0 \delta_0 \leq -\varepsilon_0 \text{ on } \partial B_1 \setminus C.$$

Thus, $g^1 \geq \Phi_0$ on $\partial B_1^+(G)$ as desired. □

The following lemma allows us to translate the flatness assumption on the vector-valued function G into the property that one of its components is trapped between nearby translation of a one-plane solution, while the remaining ones are small.

Lemma 6.3 *Let G be a viscosity solution to (5.6) in B_1 . There exists $\varepsilon_0 > 0$ universal such that, if G is ε_0 -flat in the (f^1, e_n) -directions in B_1 , then*

1. for $i = 2, \dots, m$,

$$|g^i| \leq C\varepsilon_0 U(X + \varepsilon_0 e_n) \text{ in } B_{1/2}; \tag{6.3}$$

2.

$$U(X - C\varepsilon_0 e_n) \leq g^1 \leq |G| \leq U(X + C\varepsilon_0 e_n) \text{ in } B_{1/2}, \tag{6.4}$$

with $C > 0$ universal.

Proof For the bound (i), let v be the harmonic function in $B_1 \setminus \{X \in B_1 : x_n < -\varepsilon_0\}$ such that

$$v = \varepsilon_0 \text{ on } \partial B_1, \quad v = 0 \text{ on } \{X \in B_1 : x_n \leq -\varepsilon_0\}.$$

Since $|g^i|$ is subharmonic in B_1 and it satisfies

$$|g^i| \leq \varepsilon_0, \quad g^i \equiv 0 \text{ on } \{X \in B_1 : x_n \leq -\varepsilon_0\},$$

by comparison principle $|g^i| \leq v$ in B_1 . Then by the boundary Harnack inequality, say for $\bar{X} = \frac{1}{2}e_n$, we deduce

$$v(X) \leq \bar{C} \frac{v(\bar{X})}{U(\bar{X} + \varepsilon_0 e_n)} U(X + \varepsilon_0 e_n) \leq C\varepsilon_0 U(X + \varepsilon_0 e_n) \text{ in } B_{1/2},$$

with $C > 0$ universal.

For the bounds in (ii), let $\varepsilon_0 > 0$ be as in Proposition 6.2. Then, g^1 is strictly positive and harmonic in $B_1^+(g_1)$ and it satisfies

$$U(X) - \varepsilon_0 \leq g^1(X) \leq U(X) + \varepsilon_0 \text{ in } B_1,$$

and (for ε_0) possibly smaller,

$$\{X \in B_1 : x_n \leq -\varepsilon_0\} \subset \{X \in B_1 : g^1 = 0\} \subset \{X \in B_1 : x_n \leq \varepsilon_0\}$$

Thus, by [10][Lemma 5.3.], there exists $C > 0$ universal such that

$$U(X - C\varepsilon_0 e_n) \leq g^1(X) \leq U(X + C\varepsilon_0 e_n) \text{ in } B_{1/2}.$$

According to the proof of [10][Lemma 5.3.], since the norm $|G|$ is subharmonic in B_1 , and $|G| = 0$ on $\{x_n \leq -\varepsilon_0\}$, the claimed bound for $|G|$ also follows. Details are omitted as they apply verbatim. □

6.2 Harnack Inequality

In this subsection we state and prove a Harnack type inequality which is crucial for our method. As in the local case (see [15, Lemma 2.4]), in the proof we use the observation that $|G|$ is a subsolution for the scalar one phase problem in B_1 . The key difference here is that we also have that $g^1 > 0$, which means that the strategy of the scalar case applies straightforwardly in this context. Most details are omitted as the results of [10] can be applied directly, after observing that in their proofs it is enough for the function to be either a subsolution or a supersolution of (5.1) (depending on the desired bound), or simply a positive harmonic function away from its zero set on the plate $\{x_{n+1} = 0\}$.

Theorem 6.4 *There exists a universal constant $\bar{\varepsilon} > 0$ such that, if G solves (5.6) in B_1 and*

$$U(X + \varepsilon a_0 e_n) \leq g^1 \leq |G| \leq U(X + \varepsilon b_0 e_n) \text{ in } B_r(X_0) \subset B_1, \tag{6.5}$$

with

$$\varepsilon(b_0 - a_0) \leq \bar{\varepsilon}r,$$

and

$$|g^i| \leq r^{1/2} \left(\frac{b_0 - a_0}{r} \varepsilon \right)^{5/8} \text{ in } B_{1/2}(X_0), \quad i = 2, \dots, m, \tag{6.6}$$

then

$$U(X + \varepsilon a_1 e_n) \leq g^1 \leq |G| \leq U(X + \varepsilon b_1 e_n) \text{ in } B_{\eta r}(X_0), \tag{6.7}$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 = (1 - \eta)(b_0 - a_0),$$

for a small universal constant $\eta > 0$.

The following key corollary is immediately obtained. Here \tilde{g}_ε^1 and $|\widetilde{G}_\varepsilon|$ are the ε -domain variations associated to g^1 and $|G|$ respectively and

$$a_\varepsilon := \{(X, \tilde{g}_\varepsilon^1(X)) : X \in B_{1-\varepsilon} \setminus P\} \quad \text{and} \quad A_\varepsilon := \{(X, |\widetilde{G}_\varepsilon|(X)) : X \in B_{1-\varepsilon} \setminus P\}.$$

Since domain variations may be multivalued, we mean that given X all pairs $(X, \tilde{g}_\varepsilon^1(X))$ belong to a_ε for all possible values of $\tilde{g}_\varepsilon^1(X)$, and similarly for A_ε .

Corollary 6.5 *There exists a universal constant $\bar{\varepsilon} > 0$ such that, if G solves (5.6) in B_1 ,*

$$U(X - \varepsilon e_n) \leq g^1 \leq |G| \leq U(X + \varepsilon e_n) \text{ in } B_1,$$

and

$$|g^i| \leq \varepsilon^{3/4} \text{ in } B_{1/2}, \quad i = 2, \dots, m,$$

with $\varepsilon \leq \bar{\varepsilon}/2$ and $m_0 > 0$ such that (C universal)

$$4\varepsilon(1 - \eta)^{m_0} \eta^{-m_0} \leq \bar{\varepsilon}, \quad \varepsilon \leq C(1 - \eta)^{5m_0}, \tag{6.8}$$

then the sets $a_\varepsilon \cap (B_{1/2} \times [-1, 1])$ and $A_\varepsilon \cap (B_{1/2} \times [-1, 1])$ are trapped above the graph of a function $y = a_\varepsilon(X)$ and below the graph of a function $y = b_\varepsilon(X)$ with

$$b_\varepsilon - a_\varepsilon \leq 2(1 - \eta)^{m_0-1},$$

where $a_\varepsilon, b_\varepsilon$ have modulus of continuity bounded by the Hölder function αt^β , with α, β depending only on η .

Indeed, we can apply repeatedly the Harnack inequality for $m = 0, \dots, m_0$ (the second inequality in (6.8) guarantees that (6.6) is preserved), and obtain

$$U(X + \varepsilon a_m e_n) \leq g^1 \leq |G| \leq U(X + \varepsilon b_m e_n) \quad \text{in } B_{\eta^m} \tag{6.9}$$

with $b_m - a_m = 2(1 - \eta)^m$. Thus, by the properties of the ε -domain variations (see [10, Lemma 3.1]) we get

$$a_m \leq \tilde{g}_\varepsilon^1 \leq \widetilde{|G_\varepsilon|} \leq b_m \quad \text{in } B_{\eta^m - \varepsilon},$$

and

$$a_\varepsilon \cap (B_{\eta^m - \varepsilon} \times [-1, 1]) \subset B_{\eta^m - \varepsilon} \times [a_m, b_m], \quad A_\varepsilon \cap (B_{\eta^m - \varepsilon} \times [-1, 1]) \subset B_{\eta^m - \varepsilon} \times [a_m, b_m],$$

for $m = 0, \dots, m_0$.

We are left with the proof of the Harnack inequality, that follows easily from the next lemma.

Lemma 6.6 *There exists $\varepsilon_0 > 0$ universal such that if G is a solution to (1.5) in B_1 such that for $0 < \varepsilon \leq \varepsilon_0$,*

$$U(X) \leq g^1(X) \leq |G|(X) \quad \text{in } B_{1/2},$$

and at $\bar{X} \in B_{1/8}(\frac{1}{4}e_n)$ we have $U(\bar{X} + \varepsilon e_n) \leq g^1(\bar{X})$, then

$$U(X + \tau \varepsilon e_n) \leq g^1(X) \leq |G|(X) \quad \text{in } B_\delta,$$

for universal constants $\tau, \delta > 0$. Similarly, if

$$g^1(X) \leq |G|(X) \leq U(X) \quad \text{in } B_{1/2}$$

and

$$|g^i| \leq \varepsilon^{5/8} \quad \text{in } B_{1/2}, \quad i = 2, \dots, m, \tag{6.10}$$

then if $g^1(\bar{X}) \leq U(\bar{X} - \varepsilon e_n)$, we get

$$g^1(X) \leq |G|(X) \leq U(X - \tau \varepsilon e_n) \quad \text{in } B_\delta.$$

Proof of Lemma 6.6 The first statement follows immediately from the fact that g^1 is a supersolution to (5.1) hence we can apply [10, Lemma 6.3].

Now, let us consider the case

$$g^1(X) \leq |G|(X) \leq U(X) \quad \text{in } B_{1/2}$$

and

$$|g^i| \leq \varepsilon^{5/8} \quad \text{in } B_{1/2}, \quad i = 2, \dots, m.$$

Since $|G|$ is a subsolution, in order to apply again in [10, Lemma 6.3], we need to check that

$$|G|(\bar{X}) \leq U(\bar{X} - c\varepsilon e_n)$$

for some $c > 0$ universal. Since $g^1(\bar{X}) \leq U(\bar{X} - \varepsilon e_n)$, we get

$$g^1(\bar{X}) - U(\bar{X}) \leq U(\bar{X} - \varepsilon e_n) - U(\bar{X}) = -\partial_t U(\bar{X} - \lambda e_n) \varepsilon \leq -c\varepsilon, \quad \lambda \in (0, \varepsilon) \tag{6.11}$$

and

$$|G|(\bar{X}) - U(\bar{X}) \leq g^1(\bar{X}) + C\varepsilon^{5/4} - U(\bar{X}) \leq -\frac{c}{2}\varepsilon.$$

The desired bound follows arguing as in (6.11). □

We are now ready to sketch the proof of Theorem 6.4.

Proof of Theorem 6.4 Without loss of generality, let us assume $a_0 = -1$ and $b_0 = 1$. Also, up to rescaling, we can take $r = 1$ (hence $2\varepsilon \leq \bar{\varepsilon}$). Moreover, we denote with ε_0 and δ the universal constants in Lemma 6.6, and choose $\bar{\varepsilon} = \varepsilon_0$.

We distinguish two cases depending on the position of $B_r(X_0)$.

Case 1. If $\text{dist}(X_0, \{x_n = -\varepsilon, x_{n+1} = 0\}) \leq \delta/2$ we aim to apply Lemma 6.6. Assume that for $\bar{X} = 1/4e_n$ (the other case is analogous to the scalar counterpart [10, Theorem 6.1])

$$g^1(\bar{X}) \leq U(\bar{X}).$$

Since,

$$g^1 \leq |G| \leq U(X + \varepsilon e_n) \quad \text{in } B_{1/2}(-\varepsilon e_n) \subset B_1(X_0),$$

and for ε small enough, it holds $\bar{X} \in B_{1/8}((-\varepsilon + 1/4)e_n)$, by (6.6) we can apply Lemma 6.6 and conclude that

$$g^1 \leq |G| \leq U(X + (1 - \eta)\varepsilon e_n) \quad \text{in } B_\delta(-\varepsilon e_n).$$

Finally, the improvement follows by choosing $\eta < \delta/2$, which implies that $B_\eta(X_0) \subset B_\delta(-\varepsilon e_n)$.

Case 2. If $\text{dist}(X_0, \{x_n = -\varepsilon, x_{n+1} = 0\}) > \delta/2$, then we can apply directly [10, Theorem 6.1], as in this case we only use that g^1 is a positive harmonic function in $B_1^+(g^1)$, thus the conclusion

$$U(X + \varepsilon a_1 e_n) \leq g^1 \leq U(X + \varepsilon b_1 e_n) \quad \text{in } B_\eta(X_0), \tag{6.12}$$

does hold for η small. On the other hand, reasoning as in Lemma 6.3-(i) we have in the same ball,

$$|G| \leq U(X + \varepsilon b_1 e_n) + C\varepsilon^{5/8}U(X + \varepsilon e_n) \leq U(X + \bar{b}_1 \varepsilon e_n),$$

and our claim is proved. □

7 The improvement of flatness lemma

In this section we prove our main lemma, from which the $C^{1,\alpha}$ regularity of a flat free boundary follows by standard arguments (see for example [15]). In view of Lemma 6.3 the flatness can be expressed as in (7.1)-(7.2).

Lemma 7.1 [Improvement of flatness] *Let G be a viscosity solution to (5.6) in B_1 with $0 \in F(G)$, satisfying*

$$U(X - \varepsilon e_n) \leq g^1 \leq |G| \leq U(X + \varepsilon e_n) \quad \text{in } B_1, \tag{7.1}$$

and

$$|G - g^1 f^1| \leq \varepsilon^{3/4} \quad \text{in } B_1. \tag{7.2}$$

If $0 < \rho \leq \rho_0$ for a universal $\rho_0 > 0$ and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on ρ , then for unit vectors $v \in \mathbb{R}^n$ and $\bar{f}^1 \in \mathbb{R}^m$,

$$U\left(\langle x, v \rangle - \frac{\varepsilon}{2}\rho, x_{n+1}\right) \leq G \cdot \bar{f}^1 \leq |G| \leq U\left(\langle x, v \rangle + \frac{\varepsilon}{2}\rho, x_{n+1}\right) \quad \text{in } B_\rho, \tag{7.3}$$

and

$$|G - (G \cdot \bar{f}^1)\bar{f}^1| \leq \left(\frac{\varepsilon}{2}\right)^{3/4} \rho^{1/2} \quad \text{in } B_\rho, \tag{7.4}$$

with $|v - e_n|, |f^1 - \bar{f}^1| \leq C\varepsilon, i = 1, \dots, m$, for a universal constant $C > 0$.

Proof Following the strategies of [10, 15], we proceed by contradiction. Once the argument reduces to the scalar case, we omit the details and refer the reader to the corresponding steps in the proof of [10, Theorem 7.2].

Step 1 - Compactness and linearization. Fix $\rho \leq \rho_0$ to be made precise later. Let us suppose there exist $\varepsilon_k \rightarrow 0$ and a sequence of solutions $(G_k)_k$ of (5.6) such that $0 \in F(G_k)$ and (7.1) and (7.2) are satisfied for every k ,

$$U(X - \varepsilon_k e_n) \leq g_k^1 \leq |G_k| \leq U(X + \varepsilon_k e_n) \quad \text{in } B_1, \tag{7.5}$$

and

$$|G_k - g_k^1 f^1| \leq \varepsilon_k^{3/4} \quad \text{in } B_1, \tag{7.6}$$

but either of the conclusions (7.3) or (7.4) does not hold. Let \tilde{g}_k^1 and $|\widetilde{G}_k|$ be the ε_k -domain variations of g_k^1 and $|G_k|$ respectively. In view of (7.5)-(7.6), we can apply Corollary 6.5 and Ascoli-Arzelà to conclude that, up to a subsequence, the sets

$$a_k := \{(X, \tilde{g}_k^1(X)) : X \in B_{1-\varepsilon_k} \setminus P\} \quad \text{and} \quad A_k := \{(X, |\widetilde{G}_k|(X)) : X \in B_{1-\varepsilon_k} \setminus P\},$$

converge uniformly, with respect to the Hausdorff distance, in $B_{1/2} \setminus P$ to the graphs

$$a_\infty := \{(X, \tilde{g}_\infty^1(X)) : X \in B_{1/2} \setminus P\} \quad \text{and} \quad A_\infty := \{(X, |\widetilde{G}_\infty|(X)) : X \in B_{1/2} \setminus P\},$$

with \tilde{g}_∞^1 and $|\widetilde{G}_\infty|$ Hölder continuous functions in $B_{1/2}$. Moreover,

$$|\widetilde{G}_\infty| \equiv \tilde{g}_\infty^1 \quad \text{in } B_{1/2}. \tag{7.7}$$

Since g_k^1 is a sequence of supersolutions to the scalar thin one-phase problem (5.1), while $|G_k|$ is a sequence of subsolutions to the same problem, we conclude by the arguments in Step 2 of [10, Theorem 7.1] that $|\widetilde{G}_\infty| \equiv \tilde{g}_\infty^1$ satisfies (in the viscosity sense) the linearized problem

$$\begin{cases} \Delta(U_n w) = 0 & \text{in } B_1 \setminus P, \\ |\nabla_r w| = 0 & \text{on } B_1 \cap L, \end{cases} \tag{7.8}$$

where we recall that

$$|\nabla_r w|(X_0) = \lim_{(x_n, x_{n+1}) \rightarrow (0,0)} \frac{w(x'_0, x_n, z) - w(x'_0, 0, 0)}{r}, \quad r = |(x_n, x_{n+1})|.$$

In particular, since $(\tilde{g}_k^1)_k$ and $(|\widetilde{G}_k|)_k$ are uniformly bounded in B_1 , we get a uniform bound on $\tilde{g}_\infty^1 \equiv |\widetilde{G}_\infty|$, hence by [10, Lemma 4.2], since $\tilde{g}_\infty^1(0) = 0$, we deduce that for C_0 universal,

$$|\tilde{g}_\infty^1(X) - \langle \xi', x' \rangle| \leq C_0 \rho^{3/2} \quad \text{in } B_{2\rho}, \tag{7.9}$$

for some vector $\xi' \in \mathbb{R}^{n-1}$. Details are omitted as we reduced to the scalar case, hence the arguments of [10, Theorem 7] apply verbatim.

Step 2 - Improvement of flatness. In view of (7.9), for $\rho < 1/(8C_0)$ small enough, we get

$$\langle \xi', x' \rangle - \frac{1}{8}\rho \leq \bar{g}_\infty^1(X) \leq \langle \xi', x' \rangle + \frac{1}{8}\rho \quad \text{in } B_{2\rho}$$

and, for k sufficiently large, we deduce from the uniform convergence of a_k to a_∞ and of A_k to A_∞ that

$$\langle \xi', x' \rangle - \frac{1}{4}\rho \leq \bar{g}_k^1(X) \leq \widetilde{|G_k|}(X) \leq \langle \xi', x' \rangle + \frac{1}{4}\rho \quad \text{in } B_{2\rho} \setminus P. \tag{7.10}$$

The argument of Step 2 in [10, Theorem 7.1] then gives (again details are omitted):

$$U\left(\langle x, v \rangle - \frac{\varepsilon_k}{4}\rho, x_{n+1}\right) \leq g_k^1 \leq |G_k| \leq U\left(\langle x, v \rangle + \frac{\varepsilon_k}{4}\rho, x_{n+1}\right) \quad \text{in } B_{\frac{3}{2}\rho}, \tag{7.11}$$

for a unit vector v with $|v - e_n| \leq C\varepsilon_k$.

On the other hand, by (7.5)-(7.6) we conclude that, up to a subsequence, $g_k^i/\varepsilon_k^{3/4} \rightarrow g_*^i$ uniformly, with g_*^i harmonic in $B_{1/2} \setminus P$ and $g_*^i = 0$ on $L \cap B_{1/2}$, $i = 2, \dots, m$. Thus, for k large, $|M_i| \leq M$ universal,

$$|g_k^i - M_i U \varepsilon_k^{3/4}| \leq C \varepsilon_k^{3/4} \rho U \quad \text{in } B_{\frac{3}{2}\rho}.$$

From the properties of the function U and (7.5), we conclude that (C universal)

$$|g_k^i - M_i g_k^1 \varepsilon_k^{3/4}| \leq C \varepsilon_k^{3/4} (\rho^{3/2} + \varepsilon_k^{1/2}) \leq \left(\frac{\varepsilon_k}{8}\right)^{3/4} \rho^{1/2} \quad \text{in } B_{\frac{3}{2}\rho}, \tag{7.12}$$

by choosing $\rho \leq \rho_0$ small enough universal and then k large.

Now, set

$$\xi_k^1 := f^1 + \varepsilon_k^{3/4} \sum_{i \neq 1} M_i f^i, \quad \bar{f}_k^1 := \frac{\xi_k^1}{|\xi_k^1|}.$$

Notice that,

$$\bar{f}_k^1 = \xi_k^1 + O(\varepsilon_k^{3/2}). \tag{7.13}$$

We claim that

$$U\left(\langle x, v \rangle - \frac{\varepsilon_k}{2}\rho, x_{n+1}\right) \leq G_k \cdot \bar{f}_k^1 \leq |G_k| \leq U\left(\langle x, v \rangle + \frac{\varepsilon_k}{2}\rho, x_{n+1}\right) \quad \text{in } B_\rho, \tag{7.14}$$

and

$$|G_k - (G_k \cdot \bar{f}_k^1) \bar{f}_k^1| \leq \left(\frac{\varepsilon_k}{2}\right)^{3/4} \rho^{1/2} \quad \text{in } B_\rho, \tag{7.15}$$

thus reaching a contradiction. Indeed, the upper bound in (7.14) is also a straightforward consequence of (7.11). For the lower bound, we observe that by (7.5), (7.6) and (7.13),

$$|G_k - U \bar{f}_k^1| \rightarrow 0, \quad \text{as } k \rightarrow \infty,$$

while

$$|G_k| \equiv 0 \quad \text{in } \{x_n \leq -\varepsilon_k\}. \tag{7.16}$$

Proposition 6.2 then gives

$$G_k \cdot \bar{f}_k^1 > 0 \quad \text{in } B_{\frac{1}{2}}^+(G_k). \tag{7.17}$$

Moreover, by the definition of \bar{f}_k^1 , (7.13) and (7.17),

$$G_k \cdot \bar{f}_k^1 \geq (U \left(\langle x, \nu \rangle - \frac{\varepsilon_k}{4} \rho, x_{n+1} \right) - C\varepsilon_k^{3/2})^+. \tag{7.18}$$

Call

$$h(X) := (U \left(\langle x, \nu \rangle - \frac{\varepsilon_k}{4} \rho, x_{n+1} \right) - C\varepsilon_k^{3/2})^+.$$

Let V be the harmonic function in $B_{\frac{3}{2}\rho} \setminus \{ \langle x, \nu \rangle \leq \frac{\varepsilon_k}{4} \rho \}$ with

$$V = h \text{ on } \partial B_{\frac{3}{2}\rho}, \quad V = 0 \text{ on } \{ \langle x, \nu \rangle = \frac{\varepsilon_k}{4} \rho \}.$$

Then, by (7.11)-(7.17)-(7.18) and the comparison principle, we conclude that

$$G_k \cdot \bar{f}_k^1 \geq V \text{ in } B_{\frac{3}{2}\rho}.$$

On the other hand, by Boundary Harnack,

$$V \geq (1 - C\varepsilon_k^{3/2})U \left(\langle x, \nu \rangle - \frac{\varepsilon_k}{4} \rho, x_{n+1} \right) \text{ on } B_\rho,$$

for $C > 0$ universal, from which the required lower bound follows for k large.

We are left with the proof of (7.15). In view of (7.13), we need to show that

$$|G_k - (G_k \cdot \xi_k^1) \xi_k^1| \leq \left(\frac{\varepsilon_k}{4} \right)^{3/4} \rho^{1/2} \text{ in } B_\rho.$$

Call

$$\bar{G}_k := G_k - (G_k \cdot \xi_k^1) \xi_k^1.$$

Then,

$$|\bar{g}_k^1| = \varepsilon^{3/4} \left| \sum_{i \neq 1} M_i g^i \right| \leq C\varepsilon_k^{3/2},$$

in view of assumption (7.6). For the remaining components we use (7.12), hence

$$|\bar{g}_k^i| = |g_k^i - \varepsilon_k^{3/4} M_i g_k^1 - \varepsilon_k^{3/2} M_i \sum_{j \neq 1} M_j g_k^j| \leq \left(\frac{\varepsilon_k}{8} \right)^{3/4} \rho^{1/2} + C\varepsilon_k^{9/4},$$

and the desired bound follows for k large. □

The proof of our main result Theorem 1.1 now follows combining Proposition 4.7 (and its corollary), Definition 4.7, Proposition 5.7, and Theorem 1.2. The statements about n^* and the fact that $\{|G| > 0\} \cap \{x_{n+1} = 0\}$ has locally finite perimeter follow exactly as in the scalar case (see [12, Section 5] and [17, Theorem 1.2.]).

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References

- Allen, M., Petrosyan, A.: A two-phase problem with a lower-dimensional free boundary. *Interfaces Free Bound.* **14**(3), 307–342 (2012)
- Alt, H.W., Caffarelli, L., Friedman, A.: Variational problems with two phases and their free boundaries. *Trans. Amer. Math. Soc.* **282**(2), 431–461 (1984)
- Alt, H.W., Caffarelli, L.A.: Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325**, 105–144 (1981)
- Bonder, J.F., Ritorto, A., Salort, A.M.: A class of shape optimization problems for some nonlocal operators. *Adv. Calc. Var.* **11**(4), 373–386 (2018)
- Caffarelli, L.A.: A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on x . *Annali della Scuola Normale Superiore Pisa: Classe Sci. Ser.* **15**(4), 583–602 (1988)
- Caffarelli, L.A., Roquejoffre, J.-M., Sire, Y.: Variational problems for free boundaries for the fractional Laplacian. *J. Eur. Math. Soc. (JEMS)* **12**(5), 1151–1179 (2010)
- Caffarelli, L.A., Shahgholian, H., Yeressian, K.: A minimization problem with free boundary related to a cooperative system. *Duke Math. J.* **167**(10), 1825–1882 (2018)
- De Philippis, G., Spolaor, L., Velichkov, B.: Regularity of the free boundary for the two-phase Bernoulli problem. *Invent. Math.* **225**(2), 347–394 (2021)
- De Silva, D., Ferrari, F., Salsa, S.: On two phase free boundary problems governed by elliptic equations with distributed sources. *Discrete Contin. Dyn. Syst. Ser. 7*(4), 673–693 (2014)
- De Silva, D., Roquejoffre, JM: Regularity in a one-phase free boundary problem for the fractional Laplacian. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(3):335–367 (2012)
- De Silva, D., Savin, O.: $C^{2,\alpha}$ regularity of flat free boundaries for the thin one-phase problem. *J. Diff. Equ.* **253**(8), 2420–2459 (2012)
- De Silva, D., Savin, O.: Regularity of lipschitz free boundaries for the thin one-phase problem. *J. Eur. Math. Soc.* **017**(6), 1293–1326 (2015)
- De Silva, D., Savin, O.: C^∞ regularity of certain thin free boundaries. *Indiana Univ. Math. J.* **64**(5), 1575–1608 (2015)
- De Silva, D., Savin, O.: Thin one-phase almost minimizers. *Nonlinear Anal.* **193**, 111507 (2020)
- De Silva, D., Tortone, G.: Improvement of flatness for vector valued free boundary problems. *Math. Eng.* **2**, 598 (2020)
- Duvaut, G., Lions, J.L.: *Inequalities in Mechanics and Physics*. Springer, Berlin Heidelberg (1976)
- Engelstein, M., Kauranen, A., Prats, M., Sakellaris, G., Sire, Y.: Minimizers for the thin one-phase free boundary problem. *Commun. Pure Appl. Math.* **74**(9), 1971–2022 (2021)
- Kriventsov, D., Lin, F.-H.: Regularity for shape optimizers: the nondegenerate case. *Commun. Pure Appl. Math.* **71**(8), 1535–1596 (2018)
- Kriventsov, D., Lin, F.-H.: Regularity for shape optimizers: the degenerate case. *Commun. Pure Appl. Math.* **72**(8), 1678–1721 (2019)
- Mazzoleni, D., Terracini, S., Velichkov, B.: Regularity of the optimal sets for some spectral functionals. *Geom. Funct. Anal.* **27**(2), 373–426 (2017)
- Mazzoleni, D., Terracini, S., Velichkov, B.: Regularity of the free boundary for the vectorial Bernoulli problem. *Anal. PDE* **13**(3), 741–764 (2020)
- Morrey, C.B.: *Multiple Integrals in the Calculus of Variations*. Springer, Berlin Heidelberg (1966)

23. Terracini, S., Tortone, G., Vita, S.: On s -harmonic functions on cones. *Anal. PDE* **11**(7), 1653–1691 (2018)
24. Tortone, G.: Regularity of shape optimizers for some spectral fractional problems. *J. Funct. Anal.* **281**(12), 109271 (2021)
25. Tortone, G., Zilio, A.: Regularity results for segregated configurations involving fractional Laplacian. *Nonlinear Anal.* **193**, 111532 (2020)
26. Velichkov, B.: Regularity of the one-phase free boundaries. *Lect. Notes* (2023). <https://doi.org/10.1007/978-3-031-13238-4>

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