



Γ -convergence and stochastic homogenisation of singularly-perturbed elliptic functionals

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Abstract

We study the limit behaviour of singularly-perturbed elliptic functionals of the form

$$\mathcal{F}_k(u, v) = \int_A v^2 f_k(x, \nabla u) \, dx + \frac{1}{\varepsilon_k} \int_A g_k(x, v, \varepsilon_k \nabla v) \, dx,$$

where u is a vector-valued Sobolev function, $v \in [0, 1]$ a phase-field variable, and $\varepsilon_k > 0$ a singular-perturbation parameter; *i.e.*, $\varepsilon_k \rightarrow 0$, as $k \rightarrow +\infty$. Under mild assumptions on the integrands f_k and g_k , we show that if f_k grows superlinearly in the gradient-variable, then the functionals \mathcal{F}_k Γ -converge (up to subsequences) to a *brittle* energy-functional; *i.e.*, to a free-discontinuity functional whose surface integrand does *not* depend on the jump-amplitude of u . This result is achieved by providing explicit asymptotic formulas for the bulk and surface integrands which show, in particular, that volume and surface term in \mathcal{F}_k *decouple* in the limit. The abstract Γ -convergence analysis is complemented by a stochastic homogenisation result for *stationary random* integrands.

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1 Introduction

Since the seminal work of Modica and Mortola [48, 49] and of Ambrosio and Tortorelli [7, 8], singularly-perturbed elliptic functionals have proven to be an effective tool to approximate

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free-discontinuity problems in a variety of situations. Elliptic-functionals based approximations have been successfully used, e.g., for numerical simulations in imaging or brittle fracture (see, e.g., [18–20]), in cavitation problems [43, 44], and to define a notion of regular evolution in fracture mechanics [10, 41], just to mention few examples. Besides being an approximation tool, these kinds of functionals are also commonly employed to model a range of phenomena where “diffuse” interfaces appear (see, e.g., [15, 16, 30, 31, 37, 38, 51, 52, 55]), or as instances of gradient damage models (see, e.g., [34, 45, 53]).

In this paper we study the Γ -convergence, as $k \rightarrow +\infty$, of general elliptic functionals of the form

$$\mathcal{F}_k(u, v) = \int_A \psi(v) f_k(x, \nabla u) \, dx + \frac{1}{\varepsilon_k} \int_A g_k(x, v, \varepsilon_k \nabla v) \, dx, \tag{1.1}$$

where $\varepsilon_k \searrow 0$ is a singular-perturbation parameter. The set $A \subset \mathbb{R}^n$ is open bounded and with Lipschitz boundary, $u : A \rightarrow \mathbb{R}^m$ is a vectorial function, $v : A \rightarrow [0, 1]$ is a phase-field variable, and $\psi : [0, 1] \rightarrow [0, 1]$ is an increasing and continuous function satisfying $\psi(0) = 0$, $\psi(1) = 1$, and $\psi(s) > 0$ for $s > 0$. For every $k \in \mathbb{N}$, the integrands $f_k : \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$ and $g_k : \mathbb{R}^n \times [0, 1] \times \mathbb{R}^n \rightarrow [0, +\infty)$ belong to suitable classes of functions denoted by \mathcal{F} and \mathcal{G} , respectively (see Sect. 2.2 for their definition). The requirement $(f_k) \subset \mathcal{F}$ and $(g_k) \subset \mathcal{G}$ in particular ensures the existence of an exponent $p > 1$ such that for every $k \in \mathbb{N}$ and every $x \in \mathbb{R}^n$:

$$c_1 |\xi|^p \leq f_k(x, \xi) \leq c_2 |\xi|^p, \tag{1.2}$$

for every $\xi \in \mathbb{R}^{m \times n}$ and for some $0 < c_1 \leq c_2 < +\infty$, and

$$c_3 (|1 - v|^p + |w|^p) \leq g_k(x, v, w) \leq c_4 (|1 - v|^p + |w|^p), \tag{1.3}$$

for every $v \in [0, 1]$ and $w \in \mathbb{R}^n$, and for some $0 < c_3 \leq c_4 < +\infty$. As a consequence, the functionals \mathcal{F}_k are finite in $W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A)$ and are bounded both from below and from above by Ambrosio-Tortorelli functionals of the form

$$AT_k(u, v) = \int_A \psi(v) |\nabla u|^p \, dx + \int_A \left(\frac{(1 - v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) \, dx. \tag{1.4}$$

Therefore if $(u_k, v_k) \subset W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A)$ is a pair satisfying $\sup_k \mathcal{F}_k(u_k, v_k) < +\infty$, the lower bound on \mathcal{F}_k immediately yields $v_k \rightarrow 1$ in $L^p(A)$, as $k \rightarrow +\infty$. On the other hand, $|\nabla u_k|$ can blow up in the regions where v_k is asymptotically small, so that one expects a limit u which may develop discontinuities. In [7, 8] Ambrosio and Tortorelli showed that functionals of type (1.4) provide a variational approximation, in the sense of Γ -convergence, of the free-discontinuity functional of Mumford-Shah type given by

$$\int_A |\nabla u|^p \, dx + c_p \mathcal{H}^{n-1}(S_u), \tag{1.5}$$

where now the variable u belongs to the space of *generalised special functions of bounded variation* $GSBV^p(A; \mathbb{R}^m)$. As in the case of the Modica-Mortola approximation of the perimeter-functional [48, 49], the effect of the singular perturbation, $\varepsilon_k^{p-1} |\nabla v|^p$, in (1.4) is that of producing a transition layer around the discontinuity set of u , denoted by S_u . Similarly,

the pre-factor $c_p > 0$ is related to the cost of an optimal transition of the phase-field variable, now taking place between the value zero, where ψ vanishes, and the value one. Another similarity shared by the Modica-Mortola and the Ambrosio-Tortorelli approximation is that they are essentially one-dimensional, that is, in both cases the n -dimensional analysis can be carried out by resorting to an integral-geometric argument, the slicing procedure, which allows to reduce the general situation to the one-dimensional case.

A relevant feature of the Ambrosio-Tortorelli approximation is that the “regularised” bulk and surface terms in (1.4) separately converge to their sharp-interface counterparts in (1.5). This kind of volume-surface decoupling can be also observed in a number of variants of (1.4). Indeed, this is the case, e.g., of the anisotropic functionals analysed in [36], of the phase-field approximation of brittle fracture in linearised elasticity in [29], of the second-order variants proposed in [11, 26], and of the finite-difference discretisation of the Ambrosio-Tortorelli functional on periodic [12] and on stochastic grids [13]. More specifically, in [11, 26, 29, 36] the volume-surface decoupling is obtained by means of general integral-geometric arguments which can be employed thanks to the specific form of the approximating functionals. On the other hand, in [12, 13] the interplay between the singular perturbation and the discretisation parameters makes for a subtle problem for which an *ad hoc* proof is needed. In [12] this proof relies on an explicit geometric construction which, however, is feasible only in dimension $n = 2$. In fact, more refined arguments are necessary to deal with the case $n \geq 3$, as shown in [13]. Namely, in [13] the limit volume-surface decoupling is achieved by resorting to a weighted co-area formula, which is reminiscent of a technique introduced by Ambrosio [5] (see also [24, 27, 42, 54]). This procedure allows to identify an asymptotically small region where the phase-field variable v_k can be modified and set equal to zero while the corresponding u_k makes a steep transition between two constant values. In this way, a pair (u_k, v_k) is obtained whose bulk energy vanishes while the surface energy does not essentially increase.

In the present paper we show that the volume-surface decoupling illustrated above takes place also for the general functionals \mathcal{F}_k , whose integrands f_k and g_k combine both a k and an x dependence, and satisfy (1.2) and (1.3). Moreover, being the dependence on x only measurable, the case of homogenisation is covered by our analysis as well, as shown in this paper. We remark that the generality of the functionals does not allow us to use either the slicing or the blow-up method to establish a Γ -convergence result for \mathcal{F}_k , as it is instead customary for phase-field functionals of Ambrosio-Tortorelli type. Our approach is close in spirit to that of [27] and combines the localisation method of Γ -convergence [23, 32] with a careful local analysis which eventually allows us to completely characterise the integrands of the Γ -limit thus proving, in particular, that volume and surface term do not interact in the limit.

The volume-surface decoupling has been extensively analysed in the case of free-discontinuity functionals, starting with the seminal work [5]. It has then been proven that a decoupling takes place in the case of free-discontinuity functionals with periodically oscillating integrands [24], for scale-dependent scalar brittle energies both in the continuous [42] and in the discrete case [54], for general vectorial scale-dependent free-discontinuity functionals [27, 28], and, more recently, also in the setting of linearised elasticity [40]. The clear advantage of having such a decoupling is that limit volume and surface integrands can be determined independently from one another, by means of asymptotic formulas which are then easier to handle, e.g., computationally. Moreover in [42] it is shown that the noninteraction between volume and surface is crucial to prove the stability of unilateral minimality properties in the study of crack-propagation in composite materials. The same applies to the case of the evolution considered in [41], where this feature plays a central role in proving that

the regular quasistatic evolution for the Ambrosio-Tortorelli functional converges to a quasistatic evolution for brittle fracture in the sense of [39]. These considerations also motivate the analysis carried out in the present paper.

The main result of this paper is contained in Theorem 3.1 and is a Γ -convergence and an integral representation result for the Γ -limit. Namely, in Theorem 3.1 we show that if f_k and g_k satisfy rather mild assumptions, (see Sect. 2.2 for the complete list of hypotheses) together with (1.2) and (1.3), then (up to subsequences) the functionals \mathcal{F}_k Γ -converge to a free-discontinuity functional of the form

$$\mathcal{F}(u) = \int_A f_\infty(x, \nabla u) \, dx + \int_{S_u \cap A} g_\infty(x, \nu_u) \, d\mathcal{H}^{n-1}, \tag{1.6}$$

where now $u \in GSBVP(A; \mathbb{R}^m)$ and ν_u denotes the generalised normal to S_u . We observe that the surface term in \mathcal{F} is both inhomogeneous and anisotropic, however it does *not* depend on the jump-opening $[u] = u^+ - u^-$; in other words, \mathcal{F} is a so-called *brittle energy*. The form of the surface term in (1.6) is one of the effects of the volume-surface limit decoupling mentioned above, which is apparent from the asymptotic formulas defining f_∞ and g_∞ . In fact, in Theorem 3.1 we also provide formulas for f_∞ and g_∞ . Namely, we prove that

$$f_\infty(x, \xi) = \limsup_{\rho \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{\rho^n} \inf \int_{Q_\rho(x)} f_k(y, \nabla u) \, dy \tag{1.7}$$

where the infimum in (1.7) is taken over all functions $u \in W^{1,p}(Q_\rho(x); \mathbb{R}^m)$ with $u(y) = \xi y$ near $\partial Q_\rho(x)$. The surface energy density is given instead by

$$g_\infty(x, \nu) = \limsup_{\rho \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{\rho^{n-1}} \inf \frac{1}{\varepsilon_k} \int_{Q_\rho^v(x)} g_k(y, \nu, \varepsilon_k \nabla v) \, dy \tag{1.8}$$

where the cube $Q_\rho^v(x)$ is a suitable rotation of $Q_\rho(x)$ and the infimum in v is taken in a u -dependent class of functions. More precisely, the infimum in (1.8) is taken among all $v \in W^{1,p}(Q_\rho^v(x))$, with $0 \leq v \leq 1$, for which there exists $u \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m)$ such that $v \nabla u = 0$ a.e. in $Q_\rho^v(x)$ and $(u, v) = (u_x^v, 1)$ in $U \cap \{|(y - x) \cdot \nu| > \varepsilon_k\}$ where U is a neighbourhood of $\partial Q_\rho^v(x)$, and u_x^v is the jump function given by

$$u_x^v(y) = \begin{cases} e_1 & \text{if } (y - x) \cdot \nu \geq 0, \\ 0 & \text{if } (y - x) \cdot \nu < 0. \end{cases}$$

In (1.8) the boundary datum $(u_x^v, 1)$ cannot be prescribed in the vicinity of $\{y \in \mathbb{R}^n : (y - x) \cdot \nu = 0\}$ due to the discontinuity of u_x^v and to the fact that v must be equal to zero (and not to one) where u jumps. However, this mixed Dirichlet-Neumann boundary condition can be replaced by a Dirichlet boundary condition prescribed on the whole boundary of $Q_\rho^v(x)$, up to replacing u_x^v with a regularised counterpart defined, e.g., as in (I), Sect. 2.1.

In view of the growth conditions (1.2) satisfied by f_k and the properties of ψ , the constraint $v \nabla u = 0$ satisfied a.e. in $Q_\rho^v(x)$ is equivalent to

$$\int_{Q_\rho^v(x)} \psi(v) f_k(y, \nabla u) \, dy = 0,$$

which makes apparent why the bulk term in \mathcal{F}_k does not contribute to g_∞ . We notice, however, that due to the nature of the problem, the variable u must enter in the definition of g_∞ , so that in this case a decoupling is not to be intended as in the case of free-discontinuity functionals [27].

To derive the formula for f_∞ we follow a similar strategy as in [21] and use the co-area formula in the Modica-Mortola term in (1.1) to show that, in the set where v is bounded away from zero, \mathcal{F}_k behaves like a sequence of free-discontinuity functionals whose volume integrand is f_k . Then, we conclude by invoking the decoupling result for free-discontinuity functionals proven in [27]. In fact, we notice that (1.7) coincides with the asymptotic formula for the limit volume integrand provided in [27]. The proof of (1.8) is more subtle and is substantially different, e.g., from that in [13]. Namely, to prove (1.8) we need to modify a sequence (u_k, v_k) with bounded energy in the cube $Q_\rho^v(x)$ to get a new sequence with zero volume energy which can be used as a test in (1.8), hence, in particular, the modification to (u_k, v_k) shall not increase the surface energy. In the case of the discretised Ambrosio-Tortorelli functional considered in [13], the discrete nature of the problem allows for a construction which is not feasible in a continuous setting. In our case, instead, we follow an argument which is close in spirit to a construction in [41]. This argument amounts to partition the set where $\nabla u_k \neq 0$ and to use the bound on the energy to single out a set of the partition with small measure and small volume energy. Then in this set the function v_k is modified by suitably interpolating between the value zero and two functions explicitly depending on u_k . The advantage of this interpolation is that it allows to easily estimate the increment in surface energy in terms of the volume energy and at the same time to define a test pair for (1.8). Eventually, to prove that the increment in surface energy is asymptotically negligible we need to use that $p > 1$. We notice that the assumption $p > 1$ is optimal in the sense that if f_k is linear in the gradient variable; i.e., (1.2) holds with $p = 1$, then it is well known [2, 4] that the corresponding Ambrosio-Tortorelli functional Γ -converges to a free-discontinuity functional whose surface energy explicitly depends on $[u]$, this dependence being the result of a nontrivial limit volume-surface interaction.

Our general analysis is then applied to study the homogenisation of damage models; i.e., to deal with the case of integrands f_k and g_k of type

$$f_k(x, \xi) = f\left(\frac{x}{\varepsilon_k}, \xi\right) \text{ and } g_k(x, v, w) = g\left(\frac{x}{\varepsilon_k}, v, w\right), \tag{1.9}$$

for some $f \in \mathcal{F}$ and $g \in \mathcal{G}$. More specifically, in Theorem 3.5 we prove a homogenisation result for \mathcal{F}_k , with f_k and g_k as in (1.9), *without* requiring any spatial periodicity of the integrands, but rather assuming the existence and spatial homogeneity of the limit of certain scaled minimisation problems (cf. (3.9) and (3.10)). Eventually, we show that the assumptions of Theorem 3.5 are satisfied, almost surely, in the case where the integrands f and g are *stationary random variables* and derive the corresponding stochastic homogenisation result, Theorem 8.4. Thanks to the decoupling result, Theorem 3.1, the stochastic homogenisation of the bulk term readily follows from [35]. On the other hand, the treatment of the regularised surface term requires a new *ad hoc* analysis which shares some similarities with that developed for random surface functionals [3, 25, 28]. We also mention here the recent paper [46, 50] where the stochastic homogenisation of Modica-Mortola functionals with a stationary and ergodic gradient-perturbation is studied.

To conclude we notice that our analysis also allows to deduce a Γ -convergence result for functionals with oscillating integrands of type (1.9) when the heterogeneity scale does not necessarily coincide with the Ambrosio-Tortorelli parameter ε_k , but is rather given by a different infinitesimal scale $\delta_k > 0$. In this case, though, the asymptotic formulas provided by Theorem 3.1 would fully characterise the homogenised volume energy but not the surface energy. In fact, in this case a full characterisation of the homogenised surface integrand requires a further investigation which, in particular, shall distinguish between the regimes

$\varepsilon_k \ll \delta_k$ and $\varepsilon_k \gg \delta_k$. A complete analysis of this type, in the spirit of [9], goes beyond the purpose of the present paper and is instead the object of the ongoing work [14].

Outline of the paper This paper is organised as follows. In Sect. 2 we collect some notation used throughout, introduce the mathematical setting and the functionals we are going to analyse, moreover we prove some preliminary results. In Sect. 3 we state the main results of the paper, namely, the Γ -convergence and integral representation result (Theorem 3.1), a convergence result for some associated minimisation problems (Theorem 3.4), and a homogenisation result without periodicity assumptions (Theorem 3.5). In Sect. 4 we prove some properties satisfied by the limit volume and surface integrands (Proposition 4.1 and Proposition 4.5). In Sect. 5 we implement the localisation method of Γ -convergence proving, in particular, a fundamental estimate for the functionals \mathcal{F}_k (Proposition 5.1) and a compactness and integral representation result for the Γ -limit of \mathcal{F}_k (Theorem 5.2). In Sect. 6 we characterise the volume integrand of the Γ -limit (Proposition 6.1) and in Sect. 7 the surface integrand (Proposition 7.4), thus fully achieving the proof of the main result, Theorem 3.1. In Sect. 8 we prove a stochastic homogenisation result for stationary random integrands (Theorem 8.4). Eventually in the Appendix we prove two technical lemmas which are used in Sect. 8.

2 Setting of the problem and preliminaries

In this section we collect some notation, introduce the functionals we are going to study, and prove some preliminary results.

2.1 Notation

The present subsection is devoted to the notation we employ throughout.

- (a) $m \geq 1$ and $n \geq 2$ are fixed positive integers; we set $\mathbb{R}_0^m := \mathbb{R}^m \setminus \{0\}$;
- (b) $\mathbb{S}^{n-1} := \{v = (v_1, \dots, v_n) \in \mathbb{R}^n : v_1^2 + \dots + v_n^2 = 1\}$ and $\widehat{\mathbb{S}}_{\pm}^{n-1} := \{v \in \mathbb{S}^{n-1} : \pm v_{i(v)} > 0\}$, where $i(v) := \max\{i \in \{1, \dots, n\} : v_i \neq 0\}$;
- (c) \mathcal{L}^n and \mathcal{H}^{n-1} denote the Lebesgue measure and the $(n - 1)$ -dimensional Hausdorff measure on \mathbb{R}^n , respectively;
- (d) \mathcal{A} denotes the collection of all open and bounded subsets of \mathbb{R}^n with Lipschitz boundary. If $A, B \in \mathcal{A}$ by $A \subset\subset B$ we mean that A is relatively compact in B ;
- (e) Q denotes the open unit cube in \mathbb{R}^n with sides parallel to the coordinate axis, centred at the origin; for $x \in \mathbb{R}^n$ and $r > 0$ we set $Q_r(x) := rQ + x$. Moreover, Q' denotes the open unit cube in \mathbb{R}^{n-1} with sides parallel to the coordinate axis, centred at the origin, for every $r > 0$ we set $Q'_r := rQ'$;
- (f) for every $v \in \mathbb{S}^{n-1}$ let R_v denote an orthogonal $(n \times n)$ -matrix such that $R_v e_n = v$; we also assume that $R_{-v}Q = R_vQ$ for every $v \in \mathbb{S}^{n-1}$, $R_v \in \mathbb{Q}^{n \times n}$ if $v \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$, and that the restrictions of the map $v \mapsto R_v$ to $\widehat{\mathbb{S}}_{\pm}^{n-1}$ are continuous. For an explicit example of a map $v \mapsto R_v$ satisfying all these properties we refer the reader, e.g., to [27, Example A.1];
- (g) for $x \in \mathbb{R}^n$, $r > 0$, and $v \in \mathbb{S}^{n-1}$, we define $Q_r^v(x) := R_v Q_r(0) + x$.
- (h) for $\xi \in \mathbb{R}^{m \times n}$ we let u_ξ be the linear function whose gradient is equal to ξ ; i.e., $u_\xi(x) := \xi x$, for every $x \in \mathbb{R}^n$;
- (i) for $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, and $v \in \mathbb{S}^{n-1}$ we denote with $u_{x,\zeta}^v$ the piecewise constant function taking values 0, ζ and jumping across the hyperplane $\Pi^v(x) := \{y \in \mathbb{R}^n : (y-x) \cdot v = 0\}$;

i.e.,

$$u_{x,\zeta}^v(y) := \begin{cases} \zeta & \text{if } (y-x) \cdot v \geq 0, \\ 0 & \text{if } (y-x) \cdot v < 0, \end{cases}$$

when $\zeta = e_1$ we simply write u_x^v in place of u_{x,e_1}^v ;

(j) let $u \in C^1(\mathbb{R})$, $v \in C^1(\mathbb{R})$, with $0 \leq v \leq 1$, be one-dimensional functions satisfying the following two properties:

- i. $vu' \equiv 0$ in \mathbb{R} ;
- ii. $(u(t), v(t)) = (\chi_{(0,+\infty)}(t), 1)$ for $|t| > 1$.

(k) for $x \in \mathbb{R}^n$ and $v \in \mathbb{S}^{n-1}$ we set

$$\bar{u}_x^v(y) := u((y-x) \cdot v)e_1, \quad \bar{v}_x^v(y) := v((y-x) \cdot v);$$

(l) for $x \in \mathbb{R}^n$, $v \in \mathbb{S}^{n-1}$, $\zeta \in \mathbb{R}_0^m$ and $\varepsilon > 0$ we set

$$\bar{u}_{x,\zeta,\varepsilon}^v(y) := u\left(\frac{1}{\varepsilon}(y-x) \cdot v\right)\zeta, \quad \bar{v}_{x,\varepsilon}^v(y) := v\left(\frac{1}{\varepsilon}(y-x) \cdot v\right).$$

When $\zeta = e_1$ we simply write $\bar{u}_{x,\varepsilon}^v$ in place of $\bar{u}_{x,e_1,\varepsilon}^v$. We notice that in particular, $\bar{u}_{x,1}^v = \bar{u}_x^v$, $\bar{v}_{x,1}^v = \bar{v}_x^v$;

(m) for a given topological space X , $\mathcal{B}(X)$ denotes the Borel σ - algebra on X . If $X = \mathbb{R}^d$, with $d \in \mathbb{N}$, $d \geq 1$ we simply write \mathcal{B}^d in place of $\mathcal{B}(\mathbb{R}^d)$. For $d = 1$ we write \mathcal{B} .

For every \mathcal{L}^n -measurable set $A \subset \mathbb{R}^n$ we define $L^0(A; \mathbb{R}^m)$ as the space of all \mathbb{R}^m -valued Lebesgue measurable functions. We endow $L^0(A; \mathbb{R}^m)$ with the topology of convergence in measure on bounded subsets of A and recall that this topology is both metrisable and separable.

Let $A \in \mathcal{A}$; in this paper we deal with the functional space $SBV(A; \mathbb{R}^m)$ (resp. $GSBV(A; \mathbb{R}^m)$) of special functions of bounded variation (resp. of generalised special functions of bounded variation) on A , for which we refer the reader to the monograph [6]. Here we only recall that if $u \in SBV(A; \mathbb{R}^m)$ then its distributional derivative can be represented as

$$Du(B) = \int_B \nabla u \, dx + \int_{B \cap S_u} [u] \otimes \nu_u \, d\mathcal{H}^{n-1}, \tag{2.1}$$

for every $B \in \mathcal{B}^n$. In (2.1) ∇u denotes the approximate gradient of u (which makes sense also for $u \in GSBV$), S_u the set of approximate discontinuity points of u , $[u] := u^+ - u^-$ where u^\pm are the one-sided approximate limit points of u at S_u , and ν_u is the measure theoretic normal to S_u .

Let $p > 1$; we also consider

$$SBV^p(A; \mathbb{R}^m) := \{u \in SBV(A; \mathbb{R}^m) : \nabla u \in L^p(A; \mathbb{R}^{m \times n}) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\},$$

and

$$GSBV^p(A; \mathbb{R}^m) := \{u \in GSBV(A; \mathbb{R}^m) : \nabla u \in L^p(A; \mathbb{R}^{m \times n}) \text{ and } \mathcal{H}^{n-1}(S_u) < +\infty\}.$$

We recall that $GSBV^p(A; \mathbb{R}^m)$ is a vector space; moreover, if $u \in GSBV^p(A; \mathbb{R}^m)$ then we have that $\phi(u) \in SBV^p(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$, for every $\phi \in C_c^1(\mathbb{R}^m; \mathbb{R}^m)$ (see [33]).

Throughout the paper C denotes a strictly positive constant which may vary from line to line and within the same expression.

2.2 Setting of the problem

Let $p \in (1, +\infty)$; let $c_1, c_2, c_3, c_4, L_1, L_2$ be given constants such that $0 < c_1 \leq c_2 < +\infty$, $0 < c_3 \leq c_4 < +\infty$, $0 < L_1, L_2 < +\infty$. Let $\mathcal{F} := \mathcal{F}(p, c_1, c_2, L_1)$ denote the collection of all functions $f: \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$ satisfying the following conditions:

- (f1) (measurability) f is Borel measurable on $\mathbb{R}^n \times \mathbb{R}^{m \times n}$;
- (f2) (lower bound) for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m \times n}$

$$c_1 |\xi|^p \leq f(x, \xi);$$

- (f3) (upper bound) for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m \times n}$

$$f(x, \xi) \leq c_2 |\xi|^p;$$

- (f4) (continuity in ξ) for every $x \in \mathbb{R}^n$ we have

$$|f(x, \xi_1) - f(x, \xi_2)| \leq L_1 (1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}) |\xi_1 - \xi_2|,$$

for every $\xi_1, \xi_2 \in \mathbb{R}^{m \times n}$;

Moreover, $\mathcal{G} := \mathcal{G}(p, c_3, c_4, L_2)$ denotes the collection of all functions $g: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow [0, +\infty)$ satisfying the following conditions:

- (g1) (measurability) g is Borel measurable on $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$;
- (g2) (lower bound) for every $x \in \mathbb{R}^n$, every $v \in \mathbb{R}$, and every $w \in \mathbb{R}^n$

$$c_3 (|1 - v|^p + |w|^p) \leq g(x, v, w);$$

- (g3) (upper bound) for every $x \in \mathbb{R}^n$, every $v \in \mathbb{R}$, and every $w \in \mathbb{R}^n$

$$g(x, v, w) \leq c_4 (|1 - v|^p + |w|^p).$$

- (g4) (continuity in v and w) for every $x \in \mathbb{R}^n$ we have

$$|g(x, v_1, w_1) - g(x, v_2, w_2)| \leq L_2 \left((1 + |v_1|^{p-1} + |v_2|^{p-1}) |v_1 - v_2| + (1 + |w_1|^{p-1} + |w_2|^{p-1}) |w_1 - w_2| \right)$$

for every $v_1, v_2 \in \mathbb{R}$ and every $w_1, w_2 \in \mathbb{R}^n$;

- (g5) (monotonicity in v) for every $x \in \mathbb{R}^n$ and every $w \in \mathbb{R}^n$, $g(x, \cdot, w)$ is decreasing on $(-\infty, 1)$ and increasing on $[1, +\infty)$;
- (g6) (minimum in w) for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{R}$ it holds

$$g(x, v, 0) \leq g(x, v, w)$$

for every $w \in \mathbb{R}^n$.

Remark 2.1 Let $x \in \mathbb{R}^n$; we notice that gathering (f2) and (f3) readily implies that

$$f(x, \xi) = 0 \text{ if and only if } \xi = 0. \tag{2.2}$$

Moreover, from (g2) and (g3) we deduce that

$$g(x, v, w) = 0 \text{ if and only if } (v, w) = (1, 0). \tag{2.3}$$

Let $\psi : \mathbb{R} \rightarrow [0, +\infty)$ be continuous, decreasing on $(-\infty, 0]$, increasing on $(0, +\infty)$, such that $\psi(1) = 1$, and $\psi(v) = 0$ iff $v = 0$.

For $k \in \mathbb{N}$ let $(f_k) \subset \mathcal{F}$ and $(g_k) \subset \mathcal{G}$ and let (ε_k) be a decreasing sequence of strictly positive real numbers converging to zero, as $k \rightarrow +\infty$.

We consider the sequence of elliptic functionals $\mathcal{F}_k : L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ defined by

$$\mathcal{F}_k(u, v, A) := \begin{cases} \int_A \psi(v) f_k(x, \nabla u) \, dx \\ \quad + \frac{1}{\varepsilon_k} \int_A g_k(x, v, \varepsilon_k \nabla v) \, dx & \text{if } (u, v) \in W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A), \\ \quad 0 \leq v \leq 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Remark 2.2 Assumptions (f2)–(f3) and (g2)–(g3) imply that for every $A \in \mathcal{A}$ and every $(u, v) \in W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A)$, $0 \leq v \leq 1$ it holds

$$\begin{aligned} c_1 \int_A \psi(v) |\nabla u|^p \, dx + c_2 \int_A \left(\frac{(1-v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) \, dx &\leq \mathcal{F}_k(u, v, A) \\ &\leq c_3 \int_A \psi(v) |\nabla u|^p \, dx + c_4 \int_A \left(\frac{(1-v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) \, dx; \end{aligned} \tag{2.5}$$

that is, up to a multiplicative constant, the functionals \mathcal{F}_k are bounded from below and from above by the Ambrosio-Tortorelli functionals

$$AT_k(u, v) := \int_A \psi(v) |\nabla u|^p \, dx + \int_A \left(\frac{(1-v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) \, dx.$$

Remark 2.3 For later use, we notice that the assumptions on ψ , f_k and g_k ensure that for every $A \in \mathcal{A}$ the functionals $\mathcal{F}_k(\cdot, \cdot, A)$ are continuous in the strong $W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A)$ topology.

For every $A \in \mathcal{A}$, $u \in L^0(\mathbb{R}^n; \mathbb{R}^m)$ and $v \in L^0(\mathbb{R}^n)$ it is also convenient to write

$$\mathcal{F}_k(u, v, A) = \mathcal{F}_k^b(u, v, A) + \mathcal{F}_k^s(v, A),$$

where $\mathcal{F}_k^b : L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ and $\mathcal{F}_k^s : L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ denote the bulk and the surface part of \mathcal{F}_k , respectively; *i.e.*,

$$\mathcal{F}_k^b(u, v, A) := \begin{cases} \int_A \psi(v) f_k(x, \nabla u) \, dx & \text{if } (u, v) \in W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A), \, 0 \leq v \leq 1, \\ +\infty & \text{otherwise} \end{cases} \tag{2.6}$$

and

$$\mathcal{F}_k^s(v, A) := \begin{cases} \frac{1}{\varepsilon_k} \int_A g_k(x, v, \varepsilon_k \nabla v) \, dx & \text{if } v \in W^{1,p}(A), \, 0 \leq v \leq 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

For $\rho > 2\varepsilon_k$, $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$, and $v \in \mathbb{S}^{n-1}$ we consider the following two minimisation problems

$$\mathbf{m}_k^b(u_\xi, Q_\rho(x)) := \inf \{ \mathcal{F}_k^b(u, 1, Q_\rho(x)) : u \in W^{1,p}(Q_\rho(x); \mathbb{R}^m), u = u_\xi \text{ near } \partial Q_\rho(x) \} \tag{2.8}$$

and

$$\mathbf{m}_{k,N}^s(u_x^v, Q_\rho^v(x)) := \inf \{ \mathcal{F}_k^s(v, Q_\rho^v(x)) : v \in \mathcal{A}_{\varepsilon_k, \rho}(x, v) \}, \tag{2.9}$$

where

$$\begin{aligned} \mathcal{A}_{\varepsilon_k, \rho}(x, v) &:= \{ v \in W^{1,p}(Q_\rho^v(x)), 0 \leq v \leq 1 : \exists u \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m) \\ &\text{with } v \nabla u = 0 \text{ a.e. in } Q_\rho^v(x) \\ &\text{and } (u, v) = (u_x^v, 1) \text{ near } \partial Q_\rho^v(x) \text{ in } \{|(y-x) \cdot v| > \varepsilon_k\} \}. \end{aligned} \tag{2.10}$$

We observe that in (2.8) by “ $u = u_\xi$ near $\partial Q_\rho(x)$ ” we mean that the boundary datum is attained in a neighbourhood of $\partial Q_\rho(x)$. Whereas in (2.10) the boundary datum is prescribed only in

$$U \cap \{y \in \mathbb{R}^n : |(y-x) \cdot v| > \varepsilon_k\},$$

for some neighbourhood U of $\partial Q_\rho^v(x)$.

Remark 2.4 Clearly, the class of competitors $\mathcal{A}_{\varepsilon_k, \rho}(x, v)$ is nonempty. Indeed, the pair $(\bar{u}_{x, \varepsilon_k}^v, \bar{v}_{x, \varepsilon_k}^v)$ defined as in (I), with $\varepsilon = \varepsilon_k$, satisfies both

$$(\bar{u}_{x, \varepsilon_k}^v, \bar{v}_{x, \varepsilon_k}^v) = (u_x^v, 1) \text{ in } \{|(y-x) \cdot v| > \varepsilon_k\} \text{ and } \bar{v}_{x, \varepsilon_k}^v \nabla \bar{u}_{x, \varepsilon_k}^v \equiv 0.$$

Thus the restriction of $\bar{v}_{x, \varepsilon_k}^v$ to $Q_\rho^v(x)$ belongs to $\mathcal{A}_{\varepsilon_k, \rho}(x, v)$.

In view of (2.2) and of the properties satisfied by ψ , we also observe that in (2.10) the constraint

$$v \nabla u = 0 \text{ a.e. in } Q_\rho^v(x)$$

can be equivalently replaced by

$$\mathcal{F}_k^b(u, v, Q_\rho^v(x)) = 0.$$

Hence, in particular, for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, $v \in \mathbb{S}^{n-1}$, $\rho > 2\varepsilon_k$, and $k \in \mathbb{N}$ we have

$$\mathcal{F}_k(\bar{u}_{x, \zeta, \varepsilon_k}^v, \bar{v}_{x, \varepsilon_k}^v, Q_\rho^v(x)) = \mathcal{F}_k^s(\bar{v}_{x, \varepsilon_k}^v, Q_\rho^v(x)). \tag{2.11}$$

Finally, for every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m \times n}$ we define

$$f'(x, \xi) := \limsup_{\rho \rightarrow 0} \frac{1}{\rho^n} \liminf_{k \rightarrow +\infty} \mathbf{m}_k^b(u_\xi, Q_\rho(x)), \tag{2.12}$$

$$f''(x, \xi) := \limsup_{\rho \rightarrow 0} \frac{1}{\rho^n} \limsup_{k \rightarrow +\infty} \mathbf{m}_k^b(u_\xi, Q_\rho(x)), \tag{2.13}$$

while for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$ we set

$$g'(x, v) := \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \liminf_{k \rightarrow +\infty} \mathbf{m}_{k,N}^s(u_x^v, Q_\rho^v(x)), \tag{2.14}$$

$$g''(x, v) := \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \limsup_{k \rightarrow +\infty} \mathbf{m}_{k,N}^s(u_x^v, Q_\rho^v(x)). \tag{2.15}$$

2.3 Equivalent formulas for g' and g''

For later use, in Proposition 2.6 below, we prove that g' and g'' can be equivalently defined by replacing the boundary conditions in (2.14)–(2.15) with suitable Dirichlet boundary conditions on the whole boundary of $Q_\rho^v(x)$. More precisely, we consider the minimum values defined as follows: For every $x \in \mathbb{R}^n$, $v \in \mathbb{S}^{n-1}$, and $A \in \mathcal{A}$ we set

$$\mathbf{m}_k^s(\bar{u}_{x,\varepsilon_k}^v, A) := \inf\{\mathcal{F}_k^s(v, A) : v \in \mathcal{A}(\bar{u}_{x,\varepsilon_k}^v, A)\}, \tag{2.16}$$

where

$$\begin{aligned} \mathcal{A}(\bar{u}_{x,\varepsilon_k}^v, A) &:= \{v \in W^{1,p}(A), 0 \leq v \leq 1 : \exists u \in W^{1,p}(A; \mathbb{R}^m) \\ &\text{with } v \nabla u = 0 \text{ a.e. in } A \\ &\text{and } (u, v) = (\bar{u}_{x,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v) \text{ near } \partial A\}, \end{aligned} \tag{2.17}$$

with $(\bar{u}_{x,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v)$ as in (I).

Remark 2.5 Let $A \in \mathcal{A}$ be such that $A = A' \times I$ with $A' \subset \mathbb{R}^{n-1}$ open and bounded and $I \subset \mathbb{R}$ open interval. Let $v \in \mathbb{S}^{n-1}$ and set $A_v := R_v A$, with R_v as in (f). For every $k \in \mathbb{N}$ we have

$$\begin{aligned} \int_{A_v} \left(\frac{(1 - \bar{v}_{x,\varepsilon_k}^v(y))^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla \bar{v}_{x,\varepsilon_k}^v(y)|^p \right) dy &\leq \int_{A'} \int_{\mathbb{R}} ((1 - v(t))^p + |v'(t)|^p) dt dy' \\ &\leq C_v \mathcal{L}^{n-1}(A'), \end{aligned} \tag{2.18}$$

where

$$C_v := \int_{\mathbb{R}} ((1 - v(t))^p + |v'(t)|^p) dt = \int_{-1}^1 ((1 - v(t))^p + |v'(t)|^p) dt < +\infty.$$

In particular from (g3), Remark 2.4, and (2.18) we infer

$$\mathbf{m}_k^s(\bar{u}_{x,\varepsilon_k}^v, A_v) \leq \mathcal{F}_k^s(\bar{v}_{x,\varepsilon_k}^v, A_v) \leq c_4 C_v \mathcal{L}^{n-1}(A'). \tag{2.19}$$

We are now in a position to prove the following equivalent formulation for g' and g'' . We observe that the most delicate part in the proof of this result is to show that a suitable Dirichlet boundary datum can be prescribed on the whole $\partial Q_\rho^v(x)$ while keeping the nonconvex constraint $v \nabla u = 0$ a.e. in $Q_\rho^v(x)$.

Proposition 2.6 For every $\rho > 2\varepsilon_k$, $x \in \mathbb{R}^n$, and $v \in \mathbb{S}^{n-1}$ let $\mathbf{m}_k^s(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x))$ be as in (2.16) with $A = Q_\rho^v(x)$. Then we have

$$\begin{aligned} g'(x, v) &= \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \liminf_{k \rightarrow +\infty} \mathbf{m}_k^s(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)), \\ g''(x, v) &= \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \limsup_{k \rightarrow +\infty} \mathbf{m}_k^s(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)), \end{aligned}$$

where g', g'' are as in (2.14) and (2.15), respectively.

Proof We only prove the equality for g' , the proof of the equality for g'' being analogous. Let $x \in \mathbb{R}^n$ and $v \in \mathbb{S}^{n-1}$ be fixed and set

$$\underline{g}'(x, v) := \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \liminf_{k \rightarrow +\infty} \mathbf{m}_k^s(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)).$$

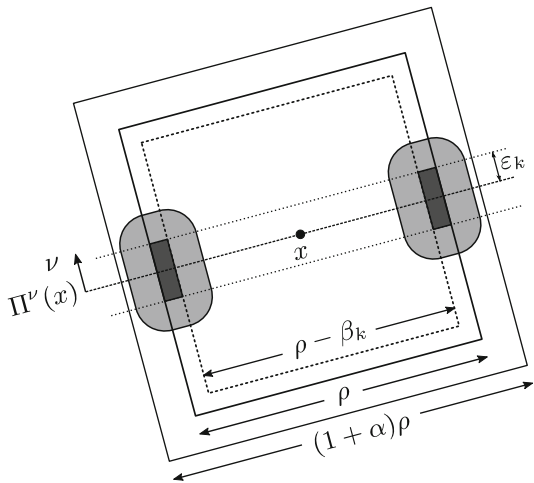


Fig. 1 The cubes $Q_{\rho-\beta_k}^v(x)$, $Q_\rho^v(x)$, $Q_{(1+\alpha)\rho}^v(x)$ and in gray the sets R_k (dark gray) and $\{d_k < 1\}$ (light gray)

In view of Remark 2.4 we readily have

$$g'(x, v) \leq \underline{g}'(x, v), \tag{2.20}$$

for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$. To prove the opposite inequality, let $\rho > 0$ and $\alpha \in (0, 1)$ be fixed and let $\bar{k} \in \mathbb{N}$ be such that $\varepsilon_k < \frac{\alpha\rho}{2}$, for every $k \geq \bar{k}$. Let moreover $v_k \in \mathcal{A}_{\varepsilon_k, \rho}(x, v)$ be such that

$$\mathcal{F}_k^s(v_k, Q_\rho^v(x)) \leq \mathbf{m}_{k, N}^s(u_x^v, Q_\rho^v(x)) + \rho^n. \tag{2.21}$$

Then, there exists $u_k \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m)$ satisfying $v_k \nabla u_k = 0$ a.e. in $Q_\rho^v(x)$ and

$$(u_k, v_k) = (u_x^v, 1) = (\bar{u}_{x, \varepsilon_k}^v, \bar{v}_{x, \varepsilon_k}^v) \text{ in } U_k \cap \{|(y-x) \cdot v| > \varepsilon_k\}, \tag{2.22}$$

where U_k is a neighbourhood of $\partial Q_\rho^v(x)$.

Starting from $(u_k, v_k) \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m) \times W^{1,p}(Q_\rho^v(x))$ we are now going to define a new pair $(\tilde{u}_k, \tilde{v}_k) \in W^{1,p}(Q_{(1+\alpha)\rho}^v(x); \mathbb{R}^m) \times W^{1,p}(Q_{(1+\alpha)\rho}^v(x))$ with $\tilde{v}_k \in \mathcal{A}(\bar{u}_{x, \varepsilon_k}^v, Q_{(1+\alpha)\rho}^v(x))$. Moreover, we will do this in a way such that $\mathcal{F}_k^s(\tilde{v}_k, Q_{(1+\alpha)\rho}^v(x))$ will be bounded from above by $\mathcal{F}_k^s(v_k, Q_\rho^v(x))$ and hence, thanks to (2.21), by $\mathbf{m}_{k, N}^s(u_x^v, Q_\rho^v(x))$.

To this end, let $\beta_k \in (0, 2\varepsilon_k) \subset (0, \rho)$ be such that $Q_\rho^v(x) \setminus \bar{Q}_{\rho-\beta_k}^v(x) \subset U_k$ and set

$$R_k := R_v \left((Q_\rho^v \setminus \bar{Q}_{\rho-\beta_k}^v) \times (-\varepsilon_k, \varepsilon_k) \right) + x,$$

where R_v is as in (f). By construction we have that

$$(Q_\rho^v(x) \setminus \bar{Q}_{\rho-\beta_k}^v(x)) \setminus \bar{R}_k \subset U_k \cap \{|(y-x) \cdot v| > \varepsilon_k\} \tag{2.23}$$

(see Fig. 1). Now let $\varphi_k \in C_c^1(Q_\rho^v)$ be a cut-off function between $Q_{\rho-\beta_k}^v$ and Q_ρ^v ; i.e., $0 \leq \varphi_k \leq 1$, and $\varphi_k \equiv 1$ on $Q_{\rho-\beta_k}^v$. Eventually, for $y = (y', y_n) \in Q_{(1+\alpha)\rho}^v(x)$ we define the pair $(\tilde{u}_k, \tilde{v}_k)$ by setting

$$\tilde{u}_k(y) := \varphi_k((R_v^T(y-x))') u_k(y) + (1 - \varphi_k((R_v^T(y-x))')) \bar{u}_{x, \varepsilon_k}^v(y)$$

and

$$\tilde{v}_k(y) := \begin{cases} \min\{v_k(y), d_k(y)\} & \text{in } Q_\rho^v(x), \\ \min\{\bar{v}_{x,\varepsilon_k}^v(y), d_k(y)\} & \text{in } Q_{(1+\alpha)\rho}^v(x) \setminus \bar{Q}_\rho^v(x), \end{cases}$$

where $d_k(y) := \frac{1}{\varepsilon_k} \text{dist}(y, R_k)$.

Clearly $\tilde{u}_k \in W^{1,p}(Q_{(1+\alpha)\rho}^v(x); \mathbb{R}^m)$, moreover, thanks to (2.22) the function \tilde{v}_k belongs to $W^{1,p}(Q_{(1+\alpha)\rho}^v(x))$. Furthermore, it holds $\tilde{v}_k \nabla \tilde{u}_k = 0$ a.e. in $Q_{(1+\alpha)\rho}^v(x)$. Indeed, $\tilde{v}_k |\nabla \tilde{u}_k| \leq v_k |\nabla u_k| = 0$ a.e. in $Q_{\rho-\beta_k}^v(x)$; similarly in $Q_{(1+\alpha)\rho}^v(x) \setminus \bar{Q}_\rho^v(x)$ there holds $\tilde{v}_k |\nabla \tilde{u}_k| \leq \bar{v}_{x,\varepsilon_k}^v |\nabla \bar{u}_{x,\varepsilon_k}^v| = 0$. Finally, thanks to (2.22) and (2.23), in $Q_\rho^v(x) \setminus \bar{Q}_{\rho-\beta_k}^v(x) \setminus \bar{R}_k$ we have $\nabla \tilde{u}_k = \nabla \bar{u}_{x,\varepsilon_k}^v = 0$, while, by definition, $\tilde{v}_k = 0$ in R_k .

Since moreover $d_k > 1$ in $Q_{(1+\alpha)\rho}^v(x) \setminus \bar{Q}_{\rho+2\varepsilon_k}^v(x)$, we also have $(\tilde{u}_k, \tilde{v}_k) = (\bar{u}_{x,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v)$ in $Q_{(1+\alpha)\rho}^v(x) \setminus \bar{Q}_{\rho+2\varepsilon_k}^v(x)$, and hence $\tilde{v}_k \in \mathcal{A}(\bar{u}_{x,\varepsilon_k}^v, Q_{(1+\alpha)\rho}^v(x))$ for $k \geq \bar{k}$.

Then, to conclude it only remains to estimate $\mathcal{F}_k^s(\tilde{v}_k, Q_{(1+\alpha)\rho}^v(x))$. To this end, we start noticing that $\tilde{v}_k = v_k$ in $Q_\rho^v(x) \cap \{d_k > 1\}$, $\tilde{v}_k = \bar{v}_{x,\varepsilon_k}^v$ in $(Q_{(1+\alpha)\rho}^v(x) \setminus \bar{Q}_\rho^v(x)) \cap \{d_k > 1\}$, while in $Q_\rho^v(x) \cap \{d_k < 1\}$ we have

$$\begin{aligned} \frac{1}{\varepsilon_k} g_k(y, \tilde{v}_k, \varepsilon_k \nabla \tilde{v}_k) &\leq \frac{1}{\varepsilon_k} g_k(y, v_k, \varepsilon_k \nabla v_k) + \frac{1}{\varepsilon_k} g_k(x, d_k, \varepsilon_k \nabla d_k) \\ &\leq \frac{1}{\varepsilon_k} g_k(y, v_k, \varepsilon_k \nabla v_k) + \frac{2c_4}{\varepsilon_k}, \end{aligned}$$

where the last estimate follows from (g3). Similarly, in $(Q_{(1+\alpha)\rho}^v(x) \setminus \bar{Q}_\rho^v(x)) \cap \{d_k < 1\}$ we have

$$\frac{1}{\varepsilon_k} g_k(y, \tilde{v}_k, \varepsilon_k \nabla \tilde{v}_k) \leq \frac{1}{\varepsilon_k} g_k(y, \bar{v}_{x,\varepsilon_k}^v, \varepsilon_k \nabla \bar{v}_{x,\varepsilon_k}^v) + \frac{2c_4}{\varepsilon_k}.$$

Thus, from (2.3) and (2.19) we infer

$$\begin{aligned} \mathcal{F}_k^s(\tilde{v}_k, Q_{(1+\alpha)\rho}^v(x)) &\leq \mathcal{F}_k^s(v_k, Q_\rho^v(x)) + \mathcal{F}_k^s(\bar{v}_{x,\varepsilon_k}^v, Q_{(1+\alpha)\rho}^v(x) \setminus \bar{Q}_\rho^v(x)) \\ &\quad + \frac{2c_4}{\varepsilon_k} \mathcal{L}^n(\{d_k < 1\}) \\ &\leq \mathcal{F}_k^s(v_k, Q_\rho^v(x)) + c_4 C_v \mathcal{L}^{n-1}(Q'_{(1+\alpha)\rho} \setminus \bar{Q}'_\rho) + C(\beta_k + \varepsilon_k) \rho^{n-2}. \end{aligned} \tag{2.24}$$

Then, thanks to (2.21), dividing both sides of (2.24) by $((1 + \alpha)\rho)^{n-1}$, since $\beta_k < 2\varepsilon_k$ we obtain

$$\begin{aligned} \frac{\mathbf{m}_k^s(\bar{u}_{x,\varepsilon_k}^v, Q_{(1+\alpha)\rho}^v(x))}{((1 + \alpha)\rho)^{n-1}} &\leq \frac{1}{(1 + \alpha)^{n-1}} \left(\frac{\mathbf{m}_{k,N}^s(u_x^v, Q_\rho^v(x))}{\rho^{n-1}} \right. \\ &\quad \left. + \rho + c_4 C_v ((1 + \alpha)^{n-1} - 1) + \frac{C\varepsilon_k}{\rho} \right). \end{aligned} \tag{2.25}$$

Eventually, from (2.25) passing first to the liminf as $k \rightarrow +\infty$ and then to the limsup as $\rho \rightarrow 0$ we deduce

$$(1 + \alpha)^{n-1} \underline{g}'(x, v) \leq g'(x, v) + c_4 C_v ((1 + \alpha)^{n-1} - 1),$$

which, together with (2.20), thanks to the arbitrariness of $\alpha > 0$ yields $\underline{g}'(x, v) = g'(x, v)$.

□

3 Statements of the main results

In this section we state the main results of this paper, namely, a Γ -convergence and integral representation result (Theorem 3.1), a converge result for some associated minimisation problems (Theorem 3.4), and a homogenisation result without periodicity assumptions (Theorem 3.5).

3.1 Γ -convergence

The following result asserts that, up to subsequences, the functionals \mathcal{F}_k Γ -converge to an integral functional of free-discontinuity type. Furthermore, it provides asymptotic cell formulas for the volume and surface limit integrands. These asymptotic cell formulas show, in particular, that volume and surface term decouple in the limit.

Theorem 3.1 (Γ -convergence) *Let $(f_k) \subset \mathcal{F}$, $(g_k) \subset \mathcal{G}$ and let \mathcal{F}_k be the functionals as in (2.4). Then there exists a subsequence, not relabelled, such that for every $A \in \mathcal{A}$ the functionals $\mathcal{F}_k(\cdot, \cdot, A)$ Γ -converge in $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ to $\mathcal{F}(\cdot, \cdot, A)$ with $\mathcal{F}: L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ given by*

$$\mathcal{F}(u, v, A) := \begin{cases} \int_A f_\infty(x, \nabla u) \, dx + \int_{S_u \cap A} g_\infty(x, \nu_u) \, d\mathcal{H}^{n-1} & \text{if } u \in GSBVP(A; \mathbb{R}^m), \\ +\infty & \text{if } v \neq 1 \text{ a.e. in } A, \\ +\infty & \text{otherwise,} \end{cases}$$

where $f_\infty: \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$ and $g_\infty: \mathbb{R}^n \times \mathbb{S}^{n-1} \rightarrow [0, +\infty)$ are Borel functions. Moreover, it holds:

i. for a.e. $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^{m \times n}$

$$f_\infty(x, \xi) = f'(x, \xi) = f''(x, \xi),$$

with f', f'' as in (2.12) and (2.13), respectively;

ii. for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$

$$g_\infty(x, v) = g'(x, v) = g''(x, v),$$

with g', g'' as in (2.14) and (2.15), respectively.

Remark 3.2 The choice of considering functionals \mathcal{F}_k which are finite when the variable v satisfies the bounds $0 \leq v \leq 1$ is a choice of convenience and it is not restrictive. Indeed, in view of (g5) and (g6) and of the properties of ψ , the functionals \mathcal{F}_k decrease under the transformation $v \rightarrow \min\{\max\{v, 0\}, 1\}$. Hence a Γ -convergence result for functionals \mathcal{F}_k defined on functions v with values in \mathbb{R} can be easily deduced from Theorem 3.1.

The proof of Theorem 3.1 will be achieved in four main steps which are addressed in Sects. 4, 5, 6, and 7, respectively. Firstly, we show that the functions $f', f'', g',$ and g'' satisfy a number of properties and, in particular, they are Borel measurable (see Proposition 4.1 and Proposition 4.5). In the second step, we prove the existence of a sequence (k_j) , with $k_j \rightarrow +\infty$ as $j \rightarrow +\infty$, such that for every $A \in \mathcal{A}$ the corresponding functionals $\mathcal{F}_{k_j}(\cdot, \cdot, A)$ Γ -converge to a free-discontinuity functional which is finite in $GSBVP(A; \mathbb{R}^m) \times \{1\}$ and is of the form

$$\int_A \hat{f}(x, \nabla u) \, dx + \int_{S_u \cap A} \hat{g}(x, [u], \nu_u) \, d\mathcal{H}^{n-1}, \tag{3.1}$$

for some Borel functions \hat{f} and \hat{g} (see Theorem 5.2).

In the third step we identify \hat{f} by showing that it is equal both to f' and f'' (see Proposition 6.1). Eventually, in the final step we identify \hat{g} by proving that it coincides with both g' and g'' (see Proposition 7.4). The representation result for \hat{g} implies, in particular, that the surface term in (3.1) does not depend on $[u]$.

The following result is an immediate consequence of Theorem 3.1 and of the Urysohn property of Γ -convergence [32, Proposition 8.3].

Corollary 3.3 *Let $(f_k) \subset \mathcal{F}$, $(g_k) \subset \mathcal{G}$ and let \mathcal{F}_k be the functionals as in (2.4). Let f', f'' be as in (2.12) and (2.13), respectively, and g', g'' be as in (2.14) and (2.15), respectively. Assume that*

$$f'(x, \xi) = f''(x, \xi) =: f_\infty(x, \xi), \text{ for a.e. } x \in \mathbb{R}^n \text{ and for every } \xi \in \mathbb{R}^{m \times n}$$

and

$$g'(x, \nu) = g''(x, \nu) =: g_\infty(x, \nu), \text{ for every } x \in \mathbb{R}^n \text{ and every } \nu \in \mathbb{S}^{n-1},$$

for some Borel functions $f_\infty: \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$ and $g_\infty: \mathbb{R}^n \times \mathbb{S}^{n-1} \rightarrow [0, +\infty)$. Then, for every $A \in \mathcal{A}$ the functionals $\mathcal{F}_k(\cdot, \cdot, A)$ Γ -converge in $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ to $\mathcal{F}(\cdot, \cdot, A)$ with $\mathcal{F}: L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ given by

$$\mathcal{F}(u, \nu, A) := \begin{cases} \int_A f_\infty(x, \nabla u) \, dx + \int_{S_u \cap A} g_\infty(x, \nu_u) \, d\mathcal{H}^{n-1} & \text{if } u \in GSBVP(A; \mathbb{R}^m), \\ +\infty & \text{if } \nu = 1 \text{ a.e. in } A, \\ & \text{otherwise.} \end{cases}$$

3.2 Convergence of minimisation problems

In view of Theorem 3.1 and Corollary 3.3 we are in a position to prove the following result on the convergence of certain minimisation problems associated with \mathcal{F}_k . Other minimisation problems can be treated similarly.

Theorem 3.4 (Convergence of minimisation problems) *Assume that the hypotheses of Corollary 3.3 are satisfied. Let $q \geq 1$, let $h \in L^q(A; \mathbb{R}^m)$, and set*

$$M_k := \inf \left\{ \mathcal{F}_k(u, \nu, A) + \int_A |u - h|^q \, dx : (u, \nu) \in L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \right\}.$$

Then $M_k \rightarrow M$ as $k \rightarrow +\infty$ where

$$M := \min \left\{ \mathcal{F}(u, 1, A) + \int_A |u - h|^q \, dx : u \in GSBVP(A; \mathbb{R}^m) \cap L^q(A; \mathbb{R}^m) \right\}. \tag{3.2}$$

Moreover if $(u_k, \nu_k) \subset L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ is such that

$$\lim_{k \rightarrow +\infty} \left(\mathcal{F}_k(u_k, \nu_k, A) + \int_A |u_k - h|^q \, dx - M_k \right) = 0, \tag{3.3}$$

then $\nu_k \rightarrow 1$ in $L^p(A)$ and there exists a subsequence of (u_k) which converges in $L^q(A; \mathbb{R}^m)$ to a solution of (3.2).

Proof Let $(u_k, v_k) \subset L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ be as in (3.3). Then the convergence $v_k \rightarrow 1$ in $L^p(A)$ readily follows by (g2), whereas the lower bound in (2.5) on the functionals \mathcal{F}_k together with [36, Lemma 4.1] give the existence of a subsequence $(u_{k_j}) \subset (u_k)$ with $u_{k_j} \rightarrow u$ in $L^0(A; \mathbb{R}^m)$, for some $u \in GSBVP(A; \mathbb{R}^m)$. Eventually, the convergence $M_k \rightarrow M$, the improved convergence $u_{k_j} \rightarrow u$ in $L^q(A; \mathbb{R}^m)$, and the fact that u is a solution to (3.2) follow arguing as in [34, Theorem 7.1], now invoking Corollary 3.3. \square

3.3 Homogenisation

In this subsection we prove a general homogenisation theorem without assuming any spatial periodicity of the integrands f_k and g_k . This theorem will be crucial to prove the stochastic homogenisation result Theorem 8.4.

In order to state the homogenisation result, we need to introduce some further notation.

For $f \in \mathcal{F}$, $g \in \mathcal{G}$, $A \in \mathcal{A}$, and $u \in W^{1,p}(A; \mathbb{R}^m)$ set

$$\mathcal{F}^b(u, A) := \int_A f(x, \nabla u) \, dx, \tag{3.4}$$

while for $v \in W^{1,p}(A)$ with $0 \leq v \leq 1$ set

$$\mathcal{F}^s(v, A) := \int_A g(x, v, \nabla v) \, dx. \tag{3.5}$$

Let $A \in \mathcal{A}$; for $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^{m \times n}$ we define

$$\mathbf{m}^b(u_\xi, A) := \inf\{\mathcal{F}^b(u, A) : u \in W^{1,p}(A; \mathbb{R}^m), u = u_\xi \text{ near } \partial A\}, \tag{3.6}$$

while for $z \in \mathbb{R}^n$, $v \in \mathbb{S}^{n-1}$, and \bar{u}_z^v as in (k) (with $x = z$) we set

$$\mathbf{m}^s(\bar{u}_z^v, A) := \inf\{\mathcal{F}^s(v, A) : v \in \mathcal{A}(\bar{u}_z^v, A)\}, \tag{3.7}$$

where $\mathcal{A}(\bar{u}_z^v, A)$ is as in (2.17), with $\bar{u}_{x,\varepsilon_k}^v$ and $\bar{v}_{x,\varepsilon_k}^v$ replaced, respectively, by \bar{u}_z^v and \bar{v}_z^v (that is, $x = z$ and $\varepsilon_k = 1$).

We are now ready to state the homogenisation result; the latter corresponds to the choice

$$f_k(x, \xi) := f\left(\frac{x}{\varepsilon_k}, \xi\right) \quad \text{and} \quad g_k(x, v, w) := g\left(\frac{x}{\varepsilon_k}, v, w\right), \tag{3.8}$$

with $f \in \mathcal{F}$ and $g \in \mathcal{G}$. We stress again that we will not assume any spatial periodicity of f and g .

Theorem 3.5 (Deterministic homogenisation) *Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Let also $\mathbf{m}^b(u_\xi, Q_r(rx))$ be as in (3.6) with $A = Q_r(rx)$ and $\mathbf{m}^s(\bar{u}_{rx}^v, Q_r^v(rx))$ be as in (3.7) with $z = rx$ and $A = Q_r^v(rx)$. Assume that for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$, $v \in \mathbb{S}^{n-1}$ the two following limits*

$$\lim_{r \rightarrow +\infty} \frac{\mathbf{m}^b(u_\xi, Q_r(rx))}{r^n} =: f_{\text{hom}}(\xi), \tag{3.9}$$

$$\lim_{r \rightarrow +\infty} \frac{\mathbf{m}^s(\bar{u}_{rx}^v, Q_r^v(rx))}{r^{n-1}} =: g_{\text{hom}}(v), \tag{3.10}$$

exist and are independent of x . Then, for every $A \in \mathcal{A}$ the functionals $\mathcal{F}_k(\cdot, \cdot, A)$ defined in (2.4) with f_k and g_k as in (3.8) Γ -converge in $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ to the functional

$\mathcal{F}_{\text{hom}}(\cdot, \cdot, A)$, with $\mathcal{F}_{\text{hom}}: L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ given by

$$\mathcal{F}_{\text{hom}}(u, v, A) := \begin{cases} \int_A f_{\text{hom}}(\nabla u) \, dx + \int_{S_u \cap A} g_{\text{hom}}(v_u) \, d\mathcal{H}^{n-1} & \text{if } u \in GSBVP(A; \mathbb{R}^m), \\ +\infty & \text{if } v \neq 1 \text{ a.e. in } A, \\ & \text{otherwise.} \end{cases}$$

Proof Let f', f'' be as in (2.12), (2.13), respectively, and g', g'' be as in (2.14), (2.15), respectively. By virtue of Corollary 3.3 it suffices to show that

$$f_{\text{hom}}(\xi) = f'(x, \xi) = f''(x, \xi), \quad g_{\text{hom}}(v) = g'(x, v) = g''(x, v), \quad (3.11)$$

for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$, and $v \in \mathbb{S}^{n-1}$.

We start by proving the first two equalities in (3.11). To this end, fix $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$, $\rho > 0$ and $k \in \mathbb{N}$. Take $u \in W^{1,p}(Q_\rho(x); \mathbb{R}^m)$ and let $u_k \in W^{1,p}(Q_{\frac{\rho}{\varepsilon_k}}(\frac{x}{\varepsilon_k}); \mathbb{R}^m)$ be defined as $u_k(y) := \frac{1}{\varepsilon_k} u(\varepsilon_k y)$. We notice that $u = u_\xi$ near $\partial Q_\rho(x)$ if and only if $u_k = u_\xi$ near $\partial Q_{\frac{\rho}{\varepsilon_k}}(\frac{x}{\varepsilon_k})$. Moreover, a change of variables gives $\mathcal{F}_k^b(u, 1, Q_\rho(x)) = \varepsilon_k^n \mathcal{F}^b(u_k, Q_{\frac{\rho}{\varepsilon_k}}(\frac{x}{\varepsilon_k}))$, from which, setting $r_k := \frac{\rho}{\varepsilon_k}$ we immediately deduce

$$\mathbf{m}_k^b(u_\xi, Q_\rho(x)) = \varepsilon_k^n \mathbf{m}^b(u_\xi, Q_{\frac{\rho}{\varepsilon_k}}(\frac{x}{\varepsilon_k})) = \frac{\rho^n}{r_k^n} \mathbf{m}^b(u_\xi, Q_{r_k}(\frac{x}{\rho})),$$

and thus (3.9) applied with x replaced by x/ρ yields

$$\lim_{k \rightarrow +\infty} \frac{\mathbf{m}_k^b(u_\xi, Q_\rho(x))}{\rho^n} = \lim_{k \rightarrow +\infty} \frac{\mathbf{m}^b(u_\xi, Q_{r_k}(\frac{x}{\rho}))}{r_k^n} = f_{\text{hom}}(\xi).$$

Eventually, by (2.12) and (2.13) we get $f'(x, \xi) = f''(x, \xi) = f_{\text{hom}}(\xi)$, for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$.

We now prove the second two equalities in (3.11). To this end, for fixed $x \in \mathbb{R}^n$, $v \in \mathbb{S}^{n-1}$, $\rho > 0$ and $k \in \mathbb{N}$, let $v \in \mathcal{A}(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x))$; then there exists $u \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m)$ such that

$$v \nabla u = 0 \text{ a.e. in } Q_\rho^v(x) \text{ and } (u, v) = (\bar{u}_{x, \varepsilon_k}^v, \bar{v}_{x, \varepsilon_k}^v) \text{ near } \partial Q_\rho^v(x). \quad (3.12)$$

Define $(u_k, v_k) \in W^{1,p}(Q_{\frac{\rho}{\varepsilon_k}}^v(\frac{x}{\varepsilon_k}); \mathbb{R}^m) \times W^{1,p}(Q_{\frac{\rho}{\varepsilon_k}}^v(\frac{x}{\varepsilon_k}))$ as $u_k(y) := u_k(\varepsilon_k y)$, $v_k(y) := v(\varepsilon_k y)$. Then (3.12) is equivalent to

$$v_k \nabla u_k = 0 \text{ a.e. in } Q_{\frac{\rho}{\varepsilon_k}}^v(\frac{x}{\varepsilon_k}) \text{ and } (u_k, v_k) = (\bar{u}_{\frac{x}{\varepsilon_k}}^v, \bar{v}_{\frac{x}{\varepsilon_k}}^v) \text{ near } \partial Q_{\frac{\rho}{\varepsilon_k}}^v(\frac{x}{\varepsilon_k}),$$

that is, $v_k \in \mathcal{A}(\bar{u}_{\frac{x}{\varepsilon_k}}^v, Q_{\frac{\rho}{\varepsilon_k}}^v(\frac{x}{\varepsilon_k}))$. Further, by a change of variables we immediately obtain that $\mathcal{F}_k^s(v, Q_\rho^v(x)) = \varepsilon_k^{n-1} \mathcal{F}^s(v_k, Q_{\frac{\rho}{\varepsilon_k}}^v(\frac{x}{\varepsilon_k}))$. Hence, again setting $r_k := \frac{\rho}{\varepsilon_k}$, we infer

$$\mathbf{m}_k^s(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x)) = \varepsilon_k^{n-1} \mathbf{m}^s(\bar{u}_{\frac{x}{\varepsilon_k}}^v, Q_{\frac{\rho}{\varepsilon_k}}^v(\frac{x}{\varepsilon_k})) = \frac{\rho^{n-1}}{r_k^{n-1}} \mathbf{m}^s(\bar{u}_{r_k \frac{x}{\rho}}^v, Q_{r_k}(\frac{x}{\rho})).$$

Hence invoking (3.10) applied with x replaced by x/ρ we get

$$\lim_{k \rightarrow +\infty} \frac{\mathbf{m}_k^s(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x))}{\rho^{n-1}} = \lim_{k \rightarrow +\infty} \frac{\mathbf{m}^s(\bar{u}_{r_k \frac{x}{\rho}}^v, Q_{r_k}(\frac{x}{\rho}))}{r_k^{n-1}} = g_{\text{hom}}(v).$$

Eventually, Proposition 2.6 gives $g'(x, \nu) = g''(x, \nu) = g_{\text{hom}}(\nu)$, for every $x \in \mathbb{R}^n, \nu \in \mathbb{S}^{n-1}$ and thus the claim. \square

4 Properties of f', f'', g', g''

This section is devoted to prove some properties satisfied by the functions $f', f'', g',$ and g'' defined in (2.12), (2.13), (2.14), and (2.15), respectively.

Proposition 4.1 *Let $(f_k) \subset \mathcal{F}$; then the functions f' and f'' defined, respectively, as in (2.12) and (2.13) satisfy properties 2.2-(f3). Moreover they also satisfy (f4), albeit with a different constant $L_1 > 0$.*

Proof The proof readily follows from [27, Lemma A.6]. \square

Remark 4.2 As shown in [27, Lemma A.6], hypothesis (f4) in the definition of the class \mathcal{F} can be weakened to obtain a larger class of volume integrands $\tilde{\mathcal{F}}$ which is closed, in the sense that if $(f_k) \subset \tilde{\mathcal{F}}$ then $f', f'' \in \tilde{\mathcal{F}}$.

Before proving the corresponding result for g', g'' we need to prove the two following technical lemmas.

Lemma 4.3 *Let $\rho, \delta, \varepsilon_k > 0$ with $\rho > \delta > 2\varepsilon_k$. Set*

$$\mathbf{m}_k^{s,\delta}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)) := \inf\{\mathcal{F}_k^s(v, Q_\rho^v(x)) : v \in \mathcal{A}^\delta(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x))\},$$

where

$$\begin{aligned} \mathcal{A}^\delta(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)) &:= \{v \in W^{1,p}(Q_\rho^v(x)), \\ &0 \leq v \leq 1 : \exists u \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m) \text{ with } v \nabla u = 0 \end{aligned}$$

$$\text{a.e. in } Q_\rho^v(x) \text{ and } (u, v) = (\bar{u}_{x,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v) \text{ in } Q_\rho^v(x) \setminus \bar{Q}_{\rho-\delta}^v(x)\},$$

and $(\bar{u}_{x,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v)$ are as in (1). Moreover, let $g'_\rho, g''_\rho : \mathbb{R}^n \times \mathbb{S}^{n-1} \rightarrow [0, +\infty]$ be the functions defined as

$$\begin{aligned} g'_\rho(x, \nu) &:= \inf_{\delta > 0} \liminf_{k \rightarrow +\infty} \mathbf{m}_k^{s,\delta}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)) = \lim_{\delta \rightarrow 0} \liminf_{k \rightarrow +\infty} \mathbf{m}_k^{s,\delta}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)), \\ g''_\rho(x, \nu) &:= \inf_{\delta > 0} \limsup_{k \rightarrow +\infty} \mathbf{m}_k^{s,\delta}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)) = \lim_{\delta \rightarrow 0} \limsup_{k \rightarrow +\infty} \mathbf{m}_k^{s,\delta}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)). \end{aligned} \tag{4.1}$$

Then the restrictions of g'_ρ, g''_ρ to the sets $\mathbb{R}^n \times \widehat{\mathbb{S}}_+^{n-1}$ and $\mathbb{R}^n \times \widehat{\mathbb{S}}_-^{n-1}$ are upper semicontinuous.

Proof We only show that the restriction of the function g'_ρ to the set $\mathbb{R}^n \times \widehat{\mathbb{S}}_+^{n-1}$ is upper semicontinuous, the other proofs being analogous.

Let $\rho > 0, x \in \mathbb{R}^n,$ and $\nu \in \widehat{\mathbb{S}}_+^{n-1}$ be fixed. Let $\delta' > \delta,$ since $\mathcal{A}^{\delta'}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)) \subset \mathcal{A}^\delta(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)),$ we immediately obtain that in (4.1) the infimum in $\delta > 0$ coincides with the limit as $\delta \rightarrow 0,$ which in particular exists. Thus, for fixed $\eta > 0$ there exists $\delta_\eta > 0$ such that

$$\liminf_{k \rightarrow +\infty} \mathbf{m}_k^{s,\delta}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)) \leq g'_\rho(x, \nu) + \eta, \tag{4.2}$$

for every $\delta \in (0, \delta_\eta).$ Fix $\delta_0 \in (0, \frac{\delta_\eta}{3}),$ hence by (4.2) we get

$$\liminf_{k \rightarrow +\infty} \mathbf{m}_k^{s,3\delta_0}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)) \leq g'_\rho(x, \nu) + \eta. \tag{4.3}$$

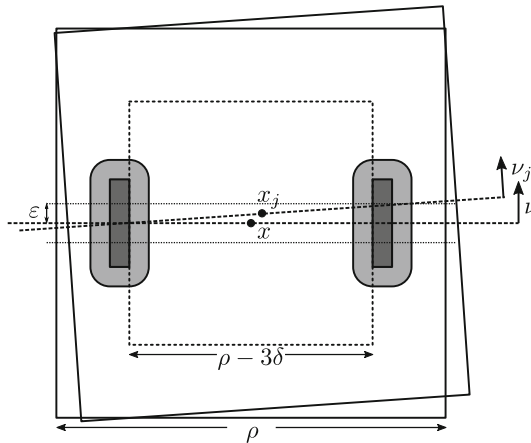


Fig. 2 The cubes $Q_\rho^v(x)$, $\bar{Q}_{\rho-3\delta_0}^v(x)$, $Q_\rho^{v_j}(x_j)$, and in gray the sets $R_{k,j}$ (dark gray) and $\{d_k < 1\}$ (light gray)

Now let $v_k \in \mathcal{A}^{3\delta_0}(\bar{u}_{x,\varepsilon_k}^v, Q_\rho^v(x))$ be such that

$$\mathcal{F}_k^s(v_k, Q_\rho^v(x)) \leq \mathbf{m}_k^{s,3\delta_0}(u_x^v, Q_\rho^v(x)) + \eta, \tag{4.4}$$

hence, in particular, $v_k \nabla u_k = 0$ a.e. in $Q_\rho^v(x)$, for some $u_k \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m)$; further

$$(u_k, v_k) = (\bar{u}_{x,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v) \text{ in } (Q_\rho^v(x) \setminus \bar{Q}_{\rho-3\delta_0}^v(x)). \tag{4.5}$$

Now let $(x_j, v_j) \subset \mathbb{R}^n \times \widehat{\mathbb{S}}_+^{n-1}$ be a sequence converging to (x, v) . Thanks to the continuity of the map $v \mapsto R_v$ there exists $\hat{j} = \hat{j}(\delta_0) \in \mathbb{N}$ such that for every $j \geq \hat{j}$ we have

$$Q_{\rho-2\delta_0}^v(x) \subset Q_{\rho-\delta_0}^{v_j}(x_j) \text{ and } Q_{\rho-5\delta_0}^{v_j}(x_j) \subset Q_{\rho-4\delta_0}^v(x). \tag{4.6}$$

We now modify (u_k, v_k) to obtain a new pair $(\tilde{u}_k, \tilde{v}_k) \in W^{1,p}(Q_\rho^{v_j}(x_j); \mathbb{R}^m) \times W^{1,p}(Q_\rho^{v_j}(x_j))$ with $\tilde{v}_k \in \mathcal{A}^\delta(\bar{u}_{x_j,\varepsilon_k}^{v_j}, Q_\rho^{v_j}(x_j))$ for any $j \geq \hat{j}$ and any $\delta \in (0, \delta_0)$. To this end, set

$$h_{k,j} := \varepsilon_k + |x - x_j| + \frac{\sqrt{n}}{2} \rho |v - v_j|$$

and

$$R_{k,j} := R_v \left((Q'_{\rho-3\delta_0+2\varepsilon_k} \setminus \bar{Q}'_{\rho-3\delta_0}) \times (-h_{k,j}, h_{k,j}) \right) + x,$$

(see Fig. 2).

Upon possibly increasing \hat{j} , we can assume that $R_{k,j} \subset Q'_{\rho-3\delta_0+2\varepsilon_k}(x) \setminus \bar{Q}'_{\rho-3\delta_0}(x)$, for every $j \geq \hat{j}$. Thus, setting $d_k(y) := \frac{1}{\varepsilon_k} \text{dist}(y, R_{k,j})$ and using (4.6), for $\varepsilon_k < \frac{\delta_0}{4}$ we also have

$$\{d_k < 1\} \subset Q_{\rho-2\delta_0}^v(x) \setminus \bar{Q}_{\rho-4\delta_0}^v(x) \subset Q_{\rho-\delta_0}^{v_j}(x_j) \setminus \bar{Q}_{\rho-5\delta_0}^{v_j}(x_j). \tag{4.7}$$

Now let $\varphi_k \in C_c^\infty(Q'_{\rho-3\delta_0+2\varepsilon_k})$ be a cut-off between $Q'_{\rho-3\delta_0}$ and $Q'_{\rho-3\delta_0+2\varepsilon_k}$; i.e., $0 \leq \varphi_k \leq 1$ and $\varphi_k = 1$ in $Q'_{\rho-3\delta_0}$. For $y \in Q_{\rho-3\delta_0}^{v_j}(x_j)$ we set

$$\tilde{u}_k(y) := \varphi_k \left((R_v^T(y-x))' \right) u_k(y) + \left(1 - \varphi_k \left(R_v^T(y-x)' \right) \right) \bar{u}_{x_j,\varepsilon_k}^{v_j}(y)$$

and

$$\tilde{v}_k(y) := \begin{cases} \min \{v_k(y), d_k(y)\} & \text{in } Q_{\rho-3\delta_0}^v(x), \\ \min \{\tilde{v}_{x_j, \varepsilon_k}^{vj}(y), d_k(y)\} & \text{in } Q_{\rho}^{vj}(x_j) \setminus Q_{\rho-3\delta_0}^v(x). \end{cases}$$

By construction, we have $\tilde{u}_k = u_k$, $\tilde{v}_k \leq v_k$ in $Q_{\rho-3\delta_0}^v(x)$ and $\tilde{u}_k = \tilde{u}_{x_j, \varepsilon_k}^{vj}$, $\tilde{v}_k \leq \tilde{v}_{x_j, \varepsilon_k}^{vj}$ in $Q_{\rho}^{vj}(x_j) \setminus Q_{\rho-3\delta_0+2\varepsilon_k}^v(x)$; therefore, in particular, it holds

$$\begin{aligned} 0 \leq \tilde{v}_k |\nabla \tilde{u}_k| &\leq v_k |\nabla u_k| = 0 \text{ a.e. in } Q_{\rho-3\delta_0}^v(x) \\ 0 \leq \tilde{v}_k |\nabla \tilde{u}_k| &\leq \tilde{v}_{x_j, \varepsilon_k}^{vj} |\nabla \tilde{u}_{x_j, \varepsilon_k}^{vj}| = 0 \text{ a.e. in } Q_{\rho}^{vj}(x_j) \setminus Q_{\rho-3\delta_0+2\varepsilon_k}^v(x). \end{aligned} \tag{4.8}$$

Moreover, for $y \in (Q_{\rho-3\delta_0+2\varepsilon_k}^v(x) \setminus \overline{Q}_{\rho-3\delta_0}^v(x)) \setminus \overline{R}_{k,j}$ we have

$$|(y-x) \cdot v| = |R_v^T(y-x) \cdot e_n| > h_{k,j} > \varepsilon_k. \tag{4.9}$$

Then, by applying the triangular inequality twice, also noticing that $Q_{\rho}^v(x) \subset B_{\frac{\sqrt{n}}{2}\rho}(x)$, we obtain

$$|(y-x_j) \cdot v_j| > h_{k,j} - |x-x_j| - |y-x||v-v_j| \geq \varepsilon_k. \tag{4.10}$$

In view of (4.5), gathering (4.9) and (4.10) we infer that

$$u_k = \tilde{u}_{x, \varepsilon_k}^v = \tilde{u}_{x_j, \varepsilon_k}^{vj}, \quad v_k = \tilde{v}_{x, \varepsilon_k}^v = \tilde{v}_{x_j, \varepsilon_k}^{vj} \text{ on } (Q_{\rho-3\delta_0+2\varepsilon_k}^v(x) \setminus \overline{Q}_{\rho-3\delta_0}^v(x)) \setminus \overline{R}_{k,j}.$$

Hence $\tilde{u}_k \in W^{1,p}(Q_{\rho}^{vj}(x_j))$ and $\nabla \tilde{u}_k = 0$ in $(Q_{\rho-3\delta_0+2\varepsilon_k}^v(x) \setminus \overline{Q}_{\rho-3\delta_0}^v(x)) \setminus \overline{R}_{k,j}$. Since moreover $\tilde{v}_k = 0$ in $\overline{R}_{k,j}$, we immediately obtain that $\tilde{v}_k \nabla \tilde{u}_k = 0$ in $Q_{\rho-3\delta_0+2\varepsilon_k}^v(x) \setminus \overline{Q}_{\rho-3\delta_0}^v(x)$, which in view of (4.8) yields

$$\tilde{v}_k \nabla \tilde{u}_k = 0 \text{ a.e. in } Q_{\rho}^{vj}(x_j).$$

Eventually, since in $Q_{\rho}^{vj}(x_j) \setminus Q_{\rho-3\delta_0}^v(x)$ we have $\tilde{v}_k = \tilde{v}_{x_j, \varepsilon_k}^{vj}$ if $d_k \geq 1$, from (4.7) we deduce that $(\tilde{u}_k, \tilde{v}_k) = (\tilde{u}_{x_j, \varepsilon_k}^{vj}, \tilde{v}_{x_j, \varepsilon_k}^{vj})$ in $Q_{\rho}^{vj}(x_j) \setminus \overline{Q}_{\rho-\delta_0}^{vj}(x_j)$, hence also in $Q_{\rho}^{vj}(x_j) \setminus \overline{Q}_{\rho-\delta}^{vj}(x_j)$ for every $\delta \in (0, \delta_0)$. In particular, $\tilde{v}_k \in \mathcal{A}^\delta(\tilde{u}_{x_j, \varepsilon_k}^{vj}, Q_{\rho}^{vj}(x_j))$ for any $\delta \in (0, \delta_0)$. Now it only remains to estimate $\mathcal{F}_k^s(\tilde{v}_k, Q_{\rho}^{vj}(x_j))$. Arguing as in Proposition 2.6, we get

$$\begin{aligned} \mathcal{F}_k^s(\tilde{v}_k, Q_{\rho}^{vj}(x_j)) &\leq \mathcal{F}_k^s(v_k, Q_{\rho}^v(x)) + \mathcal{F}_k^s(\tilde{v}_{x_j, \varepsilon_k}^{vj}, Q_{\rho}^{vj}(x_j) \setminus \overline{Q}_{\rho-3\delta_0}^v(x)) \\ &\quad + \frac{2c_4}{\varepsilon_k} \mathcal{L}^n(\{d_k < 1\}). \end{aligned} \tag{4.11}$$

Moreover, the second inclusion in (4.6) together with (2.3) and (2.19) implies that

$$\begin{aligned} \mathcal{F}_k^s(\tilde{v}_{x_j, \varepsilon_k}^{vj}, Q_{\rho}^{vj}(x_j) \setminus \overline{Q}_{\rho-3\delta_0}^v(x)) &\leq \mathcal{F}_k^s(\tilde{v}_{x_j, \varepsilon_k}^{vj}, Q_{\rho}^{vj}(x_j) \setminus \overline{Q}_{\rho-5\delta_0}^{vj}(x_j)) \\ &\leq c_4 C_v \mathcal{L}^{n-1}(Q'_{\rho} \setminus \overline{Q}'_{\rho-5\delta_0}) \leq C \delta_0 \rho^{n-2}, \end{aligned} \tag{4.12}$$

also

$$\begin{aligned} \frac{2c_4}{\varepsilon_k} \mathcal{L}^n(\{d_k < 1\}) &\leq \frac{4c_4}{\varepsilon_k} \mathcal{L}^{n-1}(Q'_{\rho-3\delta_0+4\varepsilon_k} \setminus \overline{Q}'_{\rho-3\delta_0-2\varepsilon_k})(h_{k,j} + \varepsilon_k) \\ &\leq C h_{k,j} (\rho - 3\delta_0)^{n-2} + \mathcal{O}(\varepsilon_k), \end{aligned} \tag{4.13}$$

as $k \rightarrow +\infty$.

Since $\tilde{v}_k \in \mathcal{A}^\delta(\bar{u}_{x_j, \varepsilon_k}^{v_j}, Q_\rho^{v_j}(x_j))$ for any $\delta \in (0, \delta_0)$, by combining (4.4) and (4.11)–(4.13), we get

$$\mathbf{m}_k^{s, \delta}(\bar{u}_{x_j, \varepsilon_k}^{v_j}, Q_\rho^{v_j}(x_j)) \leq \mathbf{m}_k^{s, 3\delta_0}(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x)) + \eta + C(\delta_0 + h_{k, j})\rho^{n-2} + \mathcal{O}(\varepsilon_k), \tag{4.14}$$

as $k \rightarrow +\infty$. Using (4.3) and taking in (4.14) first the liminf as $k \rightarrow +\infty$, then the limit as $\delta \rightarrow 0$, and finally the limsup as $j \rightarrow +\infty$ gives

$$\limsup_{j \rightarrow +\infty} g'_\rho(x_j, v_j) \leq g'(x, v) + 2\eta + C\delta_0\rho^{n-2},$$

since $\lim_j \lim_k h_{k, j} = 0$. Therefore, letting $\delta_0 \rightarrow 0$ the upper semicontinuity of g'_ρ restricted to $\widehat{\mathbb{S}}_+^{n-1} \times \mathbb{R}^n$ follows by the arbitrariness of $\eta > 0$. \square

Lemma 4.4 *Let g' and g'' be as in (2.14) and (2.15), respectively, and let g'_ρ, g''_ρ be as in (4.1). Then for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$ it holds*

$$g'(x, v) = \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} g'_\rho(x, v) \quad \text{and} \quad g''(x, v) = \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} g''_\rho(x, v).$$

Proof We prove the statement only for g' , the proof for g'' being analogous. We notice that since $\mathcal{A}^\delta(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x)) \subset \mathcal{A}(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x))$ for every $\delta \in (0, \rho)$, we clearly have

$$g'(x, v) \leq \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} g'_\rho(x, v),$$

therefore to conclude we just need to prove the opposite inequality. This can be done by means of an easy extension argument as follows. For fixed $\rho > 0$, $x \in \mathbb{R}^n$, and $v \in \mathbb{S}^{n-1}$ and for every $k \in \mathbb{N}$ such that $\varepsilon_k \in (0, \frac{\rho}{2})$ let $v_k \in \mathcal{A}(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x))$ satisfy

$$\mathcal{F}_k^s(v_k, Q_\rho^v(x)) \leq \mathbf{m}_k^s(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x)) + \rho^n, \tag{4.15}$$

and let $u_k \in W^{1, p}(Q_\rho^v(x); \mathbb{R}^m)$ be the corresponding u -variable. Let $\alpha > 0$ be arbitrary; thanks to the boundary conditions satisfied by (u_k, v_k) we can extend the pair (u_k, v_k) to $Q_{(1+\alpha)\rho}^v(x)$ by setting $(u_k, v_k) := (\bar{u}_{x, \varepsilon_k}^v, \bar{v}_{x, \varepsilon_k}^v)$ in $Q_{(1+\alpha)\rho}^v(x) \setminus Q_\rho^v(x)$. Then (2.3) and (2.19) yield

$$\begin{aligned} \mathcal{F}_k^s(v_k, Q_{(1+\alpha)\rho}^v(x)) &\leq \mathcal{F}_k^s(v_k, Q_\rho^v(x)) + \mathcal{F}_k^s(\bar{v}_{x, \varepsilon_k}^v, (Q_{(1+\alpha)\rho}^v(x) \setminus Q_\rho^v(x))) \\ &\leq \mathcal{F}_k^s(v_k, Q_\rho^v(x)) + c_4 C_v((1 + \alpha)^{n-1} - 1)\rho^{n-1}. \end{aligned} \tag{4.16}$$

Moreover, for $\delta \in (0, \alpha\rho)$ we have $v_k \in \mathcal{A}^\delta(\bar{u}_{x, \varepsilon_k}^v, Q_{(1+\alpha)\rho}^v(x))$, thus (4.15) and (4.16) give

$$\inf_{\delta > 0} \liminf_{k \rightarrow +\infty} \mathbf{m}_k^{s, \delta}(\bar{u}_{x, \varepsilon_k}^v, Q_{(1+\alpha)\rho}^v(x)) \leq \mathbf{m}_k^s(\bar{u}_{x, \varepsilon_k}^v, Q_\rho^v(x)) + \rho^n + c_4 C_v((1 + \alpha)^{n-1} - 1)\rho^{n-1}.$$

Hence, dividing the above inequality by $((1 + \alpha)\rho)^{n-1}$ and taking the limsup as $\rho \rightarrow 0$, thanks to Proposition 2.6 we obtain

$$(1 + \alpha)^{n-1} \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} g'_\rho(x, v) \leq g'(x, v) + c_4 C_v((1 + \alpha)^{n-1} - 1),$$

thus we conclude by the arbitrariness of $\alpha > 0$. \square

We are now ready to state and prove the following proposition which establishes the properties satisfied by g' and g'' .

Proposition 4.5 *Let $(g_k) \subset \mathcal{G}$; then the functions g' and g'' defined, respectively, as in (2.14) and (2.15) are Borel measurable and satisfy the following two properties:*

(1) (symmetry) for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$ it holds

$$g'(x, v) = g'(x, -v), \quad g''(x, v) = g''(x, -v); \tag{4.17}$$

(2) (boundedness) for every $x \in \mathbb{R}^n$ and every $v \in \mathbb{S}^{n-1}$ it holds

$$c_3c_p \leq g'(x, v) \leq c_4c_p, \quad c_3c_p \leq g''(x, v) \leq c_4c_p, \tag{4.18}$$

where $c_p := 2(p - 1)^{\frac{1-p}{p}}$.

Proof We prove the statement only for g' , the proof for g'' being analogous.

We divide the proof into three steps.

Step 1: g' is Borel measurable. Let $\rho > 0$ and let g'_ρ be the function defined in (4.1). Arguing as in the proof of Lemma 4.4 we deduce that the function $\rho \rightarrow g'_\rho(x, v) - c_4C_v\rho^{n-1}$ is nonincreasing on $(0, +\infty)$. From this it follows that

$$\lim_{\rho' \rightarrow \rho^-} g'_{\rho'}(x, v) \geq g'_\rho(x, v) \geq \lim_{\rho' \rightarrow \rho^+} g'_{\rho'}(x, v),$$

for every $x \in \mathbb{R}^n$, $v \in \mathbb{S}^{n-1}$, and every $\rho > 0$. Thus, if D is a countable dense subset of $(0, +\infty)$ we have

$$\limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} g'_\rho(x, v) = \limsup_{\substack{\rho \rightarrow 0 \\ \rho \in D}} \frac{1}{\rho^{n-1}} g'_\rho(x, v)$$

and hence by Lemma 4.4 we get

$$g'(x, v) = \limsup_{\substack{\rho \rightarrow 0 \\ \rho \in D}} \frac{1}{\rho^{n-1}} g'_\rho(x, v).$$

Therefore the Borel measurability of g' follows by Lemma 4.3 which guarantees, in particular, that the function $(x, v) \mapsto g'_\rho(x, v)$ is Borel measurable for every $\rho > 0$.

Step 2: g' is symmetric in v . Property (4.17) immediately follows from the definition of g' and from the fact that $u_x^v = -u_x^{-v} + e_1$ a.e. and $Q_\rho^v(x) = Q_\rho^{-v}(x)$ (see (f)), which implies in particular that $v \in \mathcal{A}_{\varepsilon_k, \rho}(x, v)$ with corresponding $u \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m)$ if and only if $v \in \mathcal{A}_{\varepsilon_k, \rho}(x, -v)$ with corresponding $w := -u + e_1 \in W^{1,p}(Q_\rho^{-v}(x); \mathbb{R}^m)$.

Step 3: g' is bounded. To prove that g' satisfies the bounds in (4.18) we start by observing that thanks to (g2) and (g3) we have

$$\begin{aligned} c_3 \int_{Q_\rho^v(x)} \left(\frac{(1-v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) dx &\leq \mathcal{F}_k^S(v, Q_\rho^v(x)) \\ &\leq c_4 \int_{Q_\rho^v(x)} \left(\frac{(1-v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) dx, \end{aligned}$$

for every $v \in W^{1,p}(Q_\rho^v(x))$ with $0 \leq v \leq 1$ a.e. in $Q_\rho^v(x)$. Therefore to establish (4.18) it is enough to show that

$$\lim_{k \rightarrow +\infty} \mathbf{m}_{k, \rho}(x, v) = c_p \rho^{n-1}, \tag{4.19}$$

for every $x \in \mathbb{R}^n$ and $\rho > 0$, where

$$\mathbf{m}_{k,\rho}(x, v) := \min \left\{ \int_{Q_\rho^v(x)} \left(\frac{(1-v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) dy : v \in \mathcal{A}_{\varepsilon_k, \rho}(x, v) \right\}.$$

Let $x \in \mathbb{R}^n$, $v \in \mathbb{S}^{n-1}$, $\rho > 0$, and let $k \in \mathbb{N}$ be such that $2\varepsilon_k < \rho$. By the homogeneity and rotation invariance of the Ambrosio-Tortorelli functional we have

$$\mathbf{m}_{k,\rho}(x, v) = \mathbf{m}_{\varepsilon_k, \rho}(0, e_n),$$

Let $\eta > 0$ be arbitrary; reasoning as in the construction of a recovery sequence for the Ambrosio-Tortorelli functional we find a sequence $(u_k, v_k) \subset W^{1,p}(Q_\rho(0); \mathbb{R}^m) \times W^{1,p}(Q_\rho(0))$ satisfying $v_k \nabla u_k = 0$ a.e. in $Q_\rho(0)$, $(u_k, v_k) = (u_0^{e_n}, 1)$ in $\{|y_n| > \varepsilon_k T_\eta\}$ for some $T_\eta > 0$ and

$$\limsup_{k \rightarrow +\infty} \int_{Q_\rho(0)} \left(\frac{(1-v_k)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v_k|^p \right) dx = (c_\rho + \eta) \rho^{n-1}. \tag{4.20}$$

Then, using a similar argument as in the proof of Proposition 2.6, we can modify v_k to obtain a function $\tilde{v}_k \in W^{1,p}(Q_\rho(0))$ satisfying $\tilde{v}_k = 1$ in $Q_\rho(0) \setminus \overline{Q_{\rho-\varepsilon_k}(0)} \cap \{|y_n| > \varepsilon_k\}$ and

$$\begin{aligned} & \int_{Q_\rho(0)} \left(\frac{(1-\tilde{v}_k)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla \tilde{v}_k|^p \right) dx \\ & \leq \int_{Q_\rho(0)} \left(\frac{(1-v_k)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v_k|^p \right) dx + C \varepsilon_k \rho^{n-2}. \end{aligned} \tag{4.21}$$

In particular, since $\tilde{v}_k \in \mathcal{A}_{\varepsilon_k, \rho}(0, e_n)$, gathering (4.20)–(4.21), by the arbitrariness of $\eta > 0$ we conclude that

$$\limsup_{k \rightarrow +\infty} \mathbf{m}_{k,\rho}(0, e_n) \leq c_\rho \rho^{n-1}. \tag{4.22}$$

We now turn to the proof of the lower bound. By the Fubini Theorem we have

$$\begin{aligned} & \int_{Q_\rho(0)} \left(\frac{(1-v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) dx \\ & = \int_{Q'_\rho} \int_{-\frac{\rho}{2}}^{\frac{\rho}{2}} \left(\frac{(1-v(x', x_n))^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v(x', x_n)|^p \right) dx_n dx' \\ & \geq \int_{Q'_\rho} \int_{-\frac{\rho}{2}}^{\frac{\rho}{2}} \left(\frac{(1-v(x', x_n))^p}{\varepsilon_k} + \varepsilon_k^{p-1} \left| \frac{\partial v(x', x_n)}{\partial x_n} \right|^p \right) dx_n dx'. \end{aligned} \tag{4.23}$$

Then, if $v \in \mathcal{A}_{\varepsilon_k, \rho}(0, e_n)$ the corresponding u coincides with $u_0^{e_n}$ in a neighbourhood of

$$\partial^\pm Q_\rho(0) := \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x' \in \overline{Q'_\rho}, x_n = \pm \frac{\rho}{2} \right\}.$$

Since it must hold that $v \nabla u = 0$ a.e. in $Q_\rho(0)$, then almost every straight line intersecting $\partial^\pm Q_\rho(0)$ and parallel to e_n also intersects the level set $\{v = 0\}$. Indeed, for \mathcal{L}^{n-1} -a.e. $x' \in Q'_\rho$ the pair $(u_{x'}(t), v_{x'}(t)) := (u(x', t) \cdot e_1, v(x', t))$ belongs to $W^{1,p}(-\frac{\rho}{2}, \frac{\rho}{2}) \times W^{1,p}(-\frac{\rho}{2}, \frac{\rho}{2})$ and satisfies

$$v_{x'}(t) u'_{x'}(t) = v(x', t) \frac{\partial u(x', t)}{\partial x_n} \cdot e_1 = 0 \text{ for } \mathcal{L}^1\text{-a.e. } t \in \left(-\frac{\rho}{2}, \frac{\rho}{2} \right), \tag{4.24}$$

as well as $v_{x'}(\pm \frac{\rho}{2}) = 1$, $u_{x'}(-\frac{\rho}{2}) = 0$, and $u_{x'}(\frac{\rho}{2}) = 1$. Since $u_{x'} \in W^{1,p}(-\frac{\rho}{2}, \frac{\rho}{2})$, the boundary conditions satisfied by $u_{x'}$ imply the existence of a subset of $(-\frac{\rho}{2}, \frac{\rho}{2})$ with positive \mathcal{L}^1 -measure on which $u'_{x'} \neq 0$, hence $v_{x'} = 0$ in view of (4.24). In particular, for \mathcal{L}^{n-1} -a.e. $x' \in Q'_\rho$ there exists $s \in (-\frac{\rho}{2}, \frac{\rho}{2})$ such that $v(x', s) = 0$.

Therefore, the Young Inequality

$$\frac{(1 - v(x', x_n))^p}{\varepsilon_k} + \varepsilon_k^{p-1} \left| \frac{\partial v(x', x_n)}{\partial x_n} \right|^p \geq \left(\frac{p}{p-1} \right)^{\frac{p-1}{p}} p^{\frac{1}{p}} (1 - v(x', x_n))^{p-1} \left| \frac{\partial v(x', x_n)}{\partial x_n} \right|,$$

applied for \mathcal{L}^{n-1} -a.e. $x' \in Q'_\rho$ together with the integration on $(-\frac{\rho}{2}, \frac{\rho}{2})$ give

$$\begin{aligned} & \int_{-\frac{\rho}{2}}^{\frac{\rho}{2}} \left(\frac{(1 - v(x', x_n))^p}{\varepsilon_k} + \varepsilon_k^{p-1} \left| \frac{\partial v(x', x_n)}{\partial x_n} \right|^p \right) dx_n \\ &= \int_{-\frac{\rho}{2}}^s \left(\frac{(1 - v(x', x_n))^p}{\varepsilon_k} + \varepsilon_k^{p-1} \left| \frac{\partial v(x', x_n)}{\partial x_n} \right|^p \right) dx_n \\ & \quad + \int_s^{\frac{\rho}{2}} \left(\frac{(1 - v(x', x_n))^p}{\varepsilon_k} + \varepsilon_k^{p-1} \left| \frac{\partial v(x', x_n)}{\partial x_n} \right|^p \right) dx_n \\ & \geq 2 \left(\frac{p}{p-1} \right)^{\frac{p-1}{p}} p^{\frac{1}{p}} \int_0^1 (1-t)^{p-1} dt = c_p, \end{aligned} \tag{4.25}$$

for \mathcal{L}^{n-1} -a.e. $x' \in Q'_\rho$. Thus, gathering (4.23) and (4.25) we get

$$\int_{Q_\rho(0)} \left(\frac{(1-v)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla v|^p \right) dx \geq c_p \rho^{n-1},$$

for every $v \in \mathcal{A}_{\varepsilon_k, \rho}(0, e_n)$. Passing to the infimum on v and to the liminf as $k \rightarrow +\infty$ we get

$$\liminf_{k \rightarrow +\infty} \mathbf{m}_{k, \rho}(0, e_n) \geq c_p \rho^{n-1}, \tag{4.26}$$

for every $\rho > 0$. Eventually, by combining (4.22) and (4.26) we get (4.19), and hence (4.18). \square

5 Γ -convergence and integral representation

In this section we show that, up to subsequences, the functionals \mathcal{F}_k Γ -converge in $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ to an integral functional of free-discontinuity type. This result is achieved by following a standard procedure which combines the localisation method of Γ -convergence (see e.g., [32, Chapters 14-18] or [23, Chapters 10, 11]) together with an integral-representation result in *SBV* [17, Theorem 1]. Though rather technical, this procedure is by now classical. For this reason here we only detail the adaptations of the theory to our specific setting, while we refer the reader to the literature for the more standard aspects.

We start by showing that the functionals \mathcal{F}_k satisfy the so-called fundamental estimate, uniformly in k .

Proposition 5.1 (Fundamental estimate) *Let \mathcal{F}_k be as in (2.4). Then, for every $\eta > 0$ and for every $A, A', B \in \mathcal{A}$ with $A \subset\subset A'$ there exists a constant $M_\eta > 0$ (also depending on A, A', B) satisfying the following property: For every $k \in \mathbb{N}$ and for every $(u, v) \in W^{1,p}(A'; \mathbb{R}^m) \times W^{1,p}(A')$, $(\tilde{u}, \tilde{v}) \in W^{1,p}(B; \mathbb{R}^m) \times W^{1,p}(B)$, $0 \leq v, \tilde{v} \leq 1$, there exists a*

pair $(\hat{u}, \hat{v}) \in W^{1,p}(A \cup B; \mathbb{R}^m) \times W^{1,p}(A \cup B)$ with $0 \leq \hat{v} \leq 1$ such that $(\hat{u}, \hat{v}) = (u, v)$ a.e. in A , $(\hat{u}, \hat{v}) = (\tilde{u}, \tilde{v})$ a.e. in $B \setminus \bar{A}'$ and

$$\begin{aligned} \mathcal{F}_k(\hat{u}, \hat{v}, A \cup B) &\leq (1 + \eta)(\mathcal{F}_k(u, v, A') + \mathcal{F}_k(\tilde{u}, \tilde{v}, B)) \\ &\quad + M_\eta(\|u - \tilde{u}\|_{L^p(S; \mathbb{R}^m)}^p + \varepsilon_k^{p-1}), \end{aligned} \tag{5.1}$$

where $S := (A' \setminus A) \cap B$.

Proof Fix $k \in \mathbb{N}$, $\eta > 0$ and $A, A', B \in \mathcal{A}$ with $A \subset\subset A'$. Let $N \in \mathbb{N}$ and $A_1, \dots, A_{N+1} \in \mathcal{A}$ with

$$A \subset\subset A_1 \subset\subset \dots \subset\subset A_{N+1} \subset\subset A'.$$

For each $i = 2, \dots, N + 1$ let φ_i be a smooth cut-off function between A_{i-1} and A_i and let

$$M := \max_{2 \leq i \leq N+1} \|\nabla \varphi_i\|_\infty.$$

Let (u, v) and (\tilde{u}, \tilde{v}) be as in the statement and consider the function $w \in L^0(\mathbb{R}^n)$ defined by

$$w := \min\{v, \tilde{v}\},$$

clearly $0 \leq w \leq 1$. For $i = 3, \dots, N$ we define $(\hat{u}^i, \hat{v}^i) \in W^{1,p}(A \cup B; \mathbb{R}^m) \times W^{1,p}(A \cup B)$ as follows

$$\hat{u}^i := \varphi_i u + (1 - \varphi_i) \tilde{u} \quad \text{and} \quad \hat{v}^i := \begin{cases} \varphi_{i-1} v + (1 - \varphi_{i-1}) w & \text{in } A_{i-1}, \\ w & \text{in } A_i \setminus A_{i-1}, \\ \varphi_{i+1} w + (1 - \varphi_{i+1}) \tilde{v} & \text{in } \mathbb{R}^n \setminus A_i. \end{cases}$$

Then, setting $S_i := A_i \setminus \bar{A}_{i-1}$ and taking into account the definition of (\hat{u}^i, \hat{v}^i) we have

$$\begin{aligned} \mathcal{F}_k(\hat{u}^i, \hat{v}^i, A \cup B) &\leq \mathcal{F}_k(u, v, A_{i-2}) + \mathcal{F}_k(u, \hat{v}^i, S_{i-1} \cap B) + \mathcal{F}_k(\hat{u}^i, w, S_i \cap B) \\ &\quad + \mathcal{F}_k(\tilde{u}, \hat{v}^i, S_{i+1} \cap B) + \mathcal{F}_k(\tilde{u}, \tilde{v}, B \setminus \bar{A}_{i+1}). \end{aligned} \tag{5.2}$$

We now come to estimate the three terms in (5.2) involving the sets S_{i-1} , S_i , and S_{i+1} . We start observing that thanks to (f3) and (f2), exploiting the definition of w and the fact that ψ is increasing, we have

$$\mathcal{F}_k^b(u, \hat{v}^i, S_{i-1} \cap B) \leq c_2 \int_{S_{i-1} \cap B} \psi(v) |\nabla u|^p \, dx \leq \frac{c_2}{c_1} \mathcal{F}_k^b(u, v, S_{i-1} \cap B). \tag{5.3}$$

Analogously, there holds

$$\mathcal{F}_k^b(\tilde{u}, \hat{v}^i, S_{i+1} \cap B) \leq \frac{c_2}{c_1} \mathcal{F}_k^b(\tilde{u}, \tilde{v}, S_{i+1} \cap B). \tag{5.4}$$

We complete the estimate of the bulk part of the energy by noticing that on $S_i \cap B$ we have

$$\begin{aligned} |\nabla \hat{u}^i|^p &\leq 3^{p-1} (|\nabla \varphi_i|^p |u - \tilde{u}|^p + |\nabla u|^p + |\nabla \tilde{u}|^p) \\ &\leq 3^{p-1} (M^p |u - \tilde{u}|^p + |\nabla u|^p + |\nabla \tilde{u}|^p). \end{aligned}$$

Integrating over $S_i \cap B$, using (f3) and (f2), the definition of w , and the monotonicity of ψ , we infer

$$\begin{aligned} \mathcal{F}_k^b(\hat{u}^i, w, S_i \cap B) &\leq c_2 \int_{S_i \cap B} \psi(w) |\nabla \hat{u}^i|^p \, dx \\ &\leq c_2 3^{p-1} \int_{S_i \cap B} (\psi(v) |\nabla u|^p + \psi(\tilde{v}) |\nabla \tilde{u}|^p) \, dx + 3^{p-1} M^p c_2 \int_{S_i \cap B} |u - \tilde{u}|^p \, dx \quad (5.5) \\ &\leq \frac{3^{p-1} c_2}{c_1} (\mathcal{F}_k^b(u, v, S_i \cap B) + \mathcal{F}_k^b(\tilde{u}, \tilde{v}, S_i \cap B)) + 3^{p-1} M^p c_2 \int_{S_i \cap B} |u - \tilde{u}|^p \, dx. \end{aligned}$$

It remains to estimate the surface term in \mathcal{F}_k . Thanks to (g3) it holds

$$\mathcal{F}_k^s(w, S_i \cap B) \leq c_4 \int_{S_i \cap B} \left(\frac{(1-w)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla w|^p \right) \, dx. \quad (5.6)$$

We now want to bound the right-hand side of (5.6) in terms of $\mathcal{F}_k^s(v, S_i \cap B) + \mathcal{F}_k^s(\tilde{v}, S_i \cap B)$. To this end we first observe that by definition of w we have

$$|\nabla w|^p \leq |\nabla v|^p + |\nabla \tilde{v}|^p \quad \text{and} \quad (1-w)^p \leq (1-v)^p + (1-\tilde{v})^p. \quad (5.7)$$

Thus, thanks to (g2), (5.6) becomes

$$\mathcal{F}_k^s(w, S_i \cap B) \leq \frac{c_4}{c_3} (\mathcal{F}_k^s(v, S_i \cap B) + \mathcal{F}_k^s(\tilde{v}, S_i \cap B)). \quad (5.8)$$

Moreover, it holds

$$\mathcal{F}_k^s(\hat{v}^i, (S_{i-1} \cup S_{i+1}) \cap B) \leq c_4 \int_{(S_{i-1} \cup S_{i+1}) \cap B} \left(\frac{(1-\hat{v}^i)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla \hat{v}^i|^p \right) \, dx. \quad (5.9)$$

By the definition of \hat{v}^i and by the convexity of $z \mapsto (1-z)^p$ for $z \in [0, 1]$, on $S_{i-1} \cap B$ we have

$$\begin{aligned} (1-\hat{v}^i)^p &= (\varphi_{i-1}(1-v) + (1-\varphi_{i-1})(1-w))^p \\ &\leq (1-v)^p + (1-w)^p \leq 2((1-v)^p + (1-\tilde{v})^p), \end{aligned}$$

where in the last step we used again (5.7). Similarly, there holds

$$\begin{aligned} |\nabla \hat{v}^i|^p &\leq 3^{p-1} (|\nabla \varphi_i|^p |v-w|^p + |\nabla v|^p + |\nabla w|^p) \\ &\leq 3^{p-1} (M^p |v-\tilde{v}|^p + 2(|\nabla v|^p + |\nabla \tilde{v}|^p)). \end{aligned}$$

Since analogous arguments hold on $S_{i+1} \cap B$, from (5.9) and (g2) we deduce

$$\begin{aligned} \mathcal{F}_k^s(\hat{v}^i, (S_{i-1} \cup S_{i+1}) \cap B) &\leq 3^{p-1} \frac{2c_4}{c_3} (\mathcal{F}_k^s(v, (S_{i-1} \cup S_{i+1}) \cap B) \\ &\quad + \mathcal{F}_k^s(\tilde{v}, (S_{i-1} \cup S_{i+1}) \cap B)) \\ &\quad + 3^{p-1} M^p c_4 \int_{(S_{i-1} \cup S_{i+1}) \cap B} \varepsilon_k^{p-1} |v-\tilde{v}|^p \, dx. \end{aligned} \quad (5.10)$$

Now set $\widehat{M} := \left(\frac{(1+3^{p-1})c_2}{c_1} + \frac{(1+3^{p-2})c_4}{c_3} \right)$; then, summing up in (5.2) over all i , gathering (5.3)–(5.5), (5.8), and (5.10), by averaging we find an index $i^* \in \{3, \dots, N\}$ such that

$$\mathcal{F}_k(\hat{u}^{i^*}, \hat{v}^{i^*}, A \cup B) \leq \frac{1}{N-2} \sum_{i=3}^N \mathcal{F}_k(\hat{u}^i, \hat{v}^i, A \cup B)$$

$$\begin{aligned} &\leq \left(1 + \frac{\widehat{M}}{N-2}\right) (\mathcal{F}_k(u, v, A') + \mathcal{F}_k(\tilde{u}, \tilde{v}, B)) \\ &\quad + \frac{3^{p-1}M^p c_2}{N-2} \int_S |u - \tilde{u}|^p \, dx + \frac{3^{p-1}M^p c_4}{N-2} \int_S \varepsilon_k^{p-1} |v - \tilde{v}|^p \, dx . \end{aligned}$$

Thus, upon choosing N large enough so that $\frac{\widehat{M}}{N-2} < \eta$, since $0 \leq v, \tilde{v} \leq 1$ we obtain (5.1) by setting $(\hat{v}, \hat{v}) := (\hat{u}^{i^*}, \hat{v}^{i^*})$ and $M_\eta := \frac{3^{p-1}M^p}{N-2} (c_2 + 2c_4 \mathcal{L}^n(S))$. \square

On account of the fundamental estimate, Proposition 5.1, we are now in a position to prove the following Γ -convergence result.

Theorem 5.2 *Let \mathcal{F}_k be as in (2.4). Then there exist a subsequence (\mathcal{F}_{k_j}) of (\mathcal{F}_k) and a functional $\mathcal{F} : L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ such that for every $A \in \mathcal{A}$ the functionals $\mathcal{F}_{k_j}(\cdot, \cdot, A)$ Γ -converge in $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ to $\mathcal{F}(\cdot, \cdot, A)$. Moreover, \mathcal{F} is given by*

$$\mathcal{F}(u, v, A) := \begin{cases} \int_A \hat{f}(x, \nabla u) \, dx + \int_{S_u \cap A} \hat{g}(x, [u], v_u) \, d\mathcal{H}^{n-1} & \text{if } u \in GSBVP(A; \mathbb{R}^m), v = 1 \text{ a.e. in } A, \\ +\infty & \text{otherwise,} \end{cases}$$

with $\hat{f} : \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$, $\hat{g} : \mathbb{R}^n \times \mathbb{R}_0^m \times \mathbb{S}^{n-1} \rightarrow [0, +\infty)$ given by

$$\hat{f}(x, \xi) := \limsup_{\rho \rightarrow 0} \frac{1}{\rho^n} \mathbf{m}(u_\xi, \mathcal{Q}_\rho(x)), \tag{5.11}$$

$$\hat{g}(x, \zeta, v) := \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \mathbf{m}(u_{x,\zeta}^v, \mathcal{Q}_\rho^v(x)), \tag{5.12}$$

for every $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$, $\zeta \in \mathbb{R}_0^m$, and $v \in \mathbb{S}^{n-1}$, where for $A \in \mathcal{A}$ and $\bar{u} \in SBVP(A; \mathbb{R}^m)$

$$\mathbf{m}(\bar{u}, A) := \inf \{ \mathcal{F}(u, 1, A) : u \in SBVP(A; \mathbb{R}^m), u = \bar{u} \text{ near } \partial A \}.$$

Proof Let $\mathcal{F}', \mathcal{F}'' : L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ be the functionals defined as

$$\mathcal{F}'(\cdot, \cdot, A) := \Gamma\text{-}\liminf_{k \rightarrow +\infty} \mathcal{F}_k(\cdot, \cdot, A) \quad \text{and} \quad \mathcal{F}''(\cdot, \cdot, A) := \Gamma\text{-}\limsup_{k \rightarrow +\infty} \mathcal{F}_k(\cdot, \cdot, A).$$

In view of Remark 2.2 we can invoke [36, Theorem 3.1] to deduce the existence of a constant $C > 0$ such that

$$\begin{aligned} \frac{1}{C} \left(\int_A |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u \cap A) \right) &\leq \mathcal{F}'(u, 1, A) \\ &\leq \mathcal{F}''(u, 1, A) \\ &\leq C \left(\int_A |\nabla u|^p \, dx + \mathcal{H}^{n-1}(S_u \cap A) \right), \end{aligned} \tag{5.13}$$

for every $A \in \mathcal{A}$ and every $u \in GSBVP(A; \mathbb{R}^m)$; moreover

$$\mathcal{F}'(u, v, A) = \mathcal{F}''(u, v, A) = +\infty \quad \text{if either } u \notin GSBVP(A; \mathbb{R}^m) \text{ or } v \neq 1. \tag{5.14}$$

By the general properties of Γ -convergence we know that for every $A \in \mathcal{A}$ fixed the functionals $\mathcal{F}'(\cdot, \cdot, A)$ and $\mathcal{F}''(\cdot, \cdot, A)$ are $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ lower semicontinuous [32, Proposition 6.8] and local [32, Proposition 16.15]. Further, the set functions $\mathcal{F}'(u, v, \cdot)$ and

$\mathcal{F}''(u, v, \cdot)$ are increasing [32, Proposition 6.7] and $\mathcal{F}'(u, v, \cdot)$ is superadditive [32, Proposition 16.12].

Invoking [32, Theorem 16.9] we can deduce the existence of a subsequence (k_j) , with $k_j \rightarrow +\infty$ as $j \rightarrow +\infty$, such that the corresponding \mathcal{F}' and \mathcal{F}'' also satisfy

$$\sup_{A' \subset \subset A, A' \in \mathcal{A}} \mathcal{F}'(u, v, A') = \sup_{A' \subset \subset A, A' \in \mathcal{A}} \mathcal{F}''(u, v, A') =: \mathcal{F}(u, v, A), \tag{5.15}$$

for every $(u, v) \in L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ and for every $A \in \mathcal{A}$. We notice that the set function $\mathcal{F}(u, v, \cdot)$ given by (5.15) is inner regular by definition. Moreover \mathcal{F} satisfies the following properties: the functional $\mathcal{F}(\cdot, \cdot, A)$ is $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ lower semicontinuous [32, Remark 15.10] and local [32, Remark 15.25], while the set function $\mathcal{F}(u, v, \cdot)$ is increasing and superadditive [32, Remark 15.10].

Thanks to the fundamental estimate Proposition 5.1 we can appeal to [32, Proposition 18.4] to deduce that $\mathcal{F}(u, v, \cdot)$ is also a subadditive set function. Here the only difference with a standard situation is that the remainder in (5.1) is infinitesimal with respect to the $L^p(\mathbb{R}^n; \mathbb{R}^m)$ convergence in u while we are considering the Γ -convergence of \mathcal{F}_{k_j} in $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$. However, this issue can be easily overcome by resorting to a truncation argument together with a sequential characterisation of \mathcal{F} (see e.g., [32, Proposition 16.4 and Remark 16.5]), which holds true on $SBVP(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$. Hence, we can now invoke the measure-property criterion of De Giorgi and Letta (see e.g., [32, Theorem 14.23]) to deduce that for every $(u, v) \in L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ the set function $\mathcal{F}(u, v, \cdot)$ is the restriction to \mathcal{A} of a Borel measure.

Furthermore, (5.13) together with [32, Proposition 18.6] and Proposition 5.1 imply that

$$\mathcal{F}(u, 1, A) = \mathcal{F}'(u, 1, A) = \mathcal{F}''(u, 1, A) \text{ if } u \in GSBVP(A; \mathbb{R}^m),$$

while, gathering (5.13) and (5.14) we may also deduce that

$$\mathcal{F}(u, v, A) = \mathcal{F}'(u, v, A) = \mathcal{F}''(u, v, A) = +\infty \text{ if either } u \notin GSBVP(A; \mathbb{R}^m) \text{ or } v \neq 1.$$

As a consequence, $\mathcal{F}(\cdot, \cdot, A)$ coincides with the Γ -limit of $\mathcal{F}_{k_j}(\cdot, \cdot, A)$ on $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$, for every $A \in \mathcal{A}$.

By [17, Theorem 1] and a standard perturbation and truncation argument (see e.g., [27, Theorem 4.3]), for every $A \in \mathcal{A}$ and $u \in GSBVP(A; \mathbb{R}^m)$ we can represent the Γ -limit \mathcal{F} in an integral form as

$$\mathcal{F}(u, 1, A) = \int_A \hat{f}(x, \nabla u) \, dx + \int_{S_u \cap A} \hat{g}(x, [u], \nu_u) \, d\mathcal{H}^{n-1},$$

for some Borel functions \hat{f} and \hat{g} . Eventually, thanks to (5.13), it can be easily shown that \hat{f} and \hat{g} are given by the same derivation formulas provided by [17, Theorem 1], that is, they coincide with (5.11) and (5.12), respectively. \square

6 Identification of the volume integrand

In this section we identify the volume integrand \hat{f} . Namely, we prove that \hat{f} coincides with both f' and f'' , given by (2.12) and (2.13), respectively. This shows, in particular, that the limit volume integrand \hat{f} depends only on f_k , and hence only on \mathcal{F}_k^b .

Proposition 6.1 *Let $(f_k) \subset \mathcal{F}$ and $(g_k) \subset \mathcal{G}$. Let (k_j) and \hat{f} be as in Theorem 5.2. Then it holds*

$$\hat{f}(x, \xi) = f'(x, \xi) = f''(x, \xi),$$

for a.e. $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^{m \times n}$, where f' and f'' are, respectively, as in (2.12) and (2.13) with k replaced by k_j .

Proof For notational simplicity, in what follows we still let k denote the index of the sequence provided by Theorem 5.2.

By definition $f' \leq f''$, hence to prove the claim it suffices to show that

$$f''(x, \xi) \leq \hat{f}(x, \xi) \leq f'(x, \xi), \tag{6.1}$$

for a.e. $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^{m \times n}$. We divide the proof of (6.1) into two steps.

Step 1: In this step we show that $\hat{f}(x, \xi) \geq f''(x, \xi)$ for a.e. $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^{m \times n}$.

By Theorem 5.2 we have that

$$\int_A \hat{f}(x, \xi) \, dx = \mathcal{F}(u_\xi, 1, A), \tag{6.2}$$

for every $A \in \mathcal{A}$ and for every $\xi \in \mathbb{R}^n$.

Now let $x \in \mathbb{R}^n$ be arbitrary, let $A \in \mathcal{A}$ be such that $x \in A$, and let $\rho > 0$ be so small that $Q_\rho(x) \subset A$. By Γ -convergence we can find $(u_k, v_k) \subset L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ which is a recovery sequence for $\mathcal{F}(u_\xi, 1, A)$; i.e., $(u_k, v_k) \subset W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A)$, $0 \leq v \leq 1$, (u_k, v_k) converges to $(u_\xi, 1)$ in measure on bounded sets and

$$\lim_{k \rightarrow +\infty} \mathcal{F}_k(u_k, v_k, A) = \mathcal{F}(u_\xi, 1, A). \tag{6.3}$$

Moreover, by (g2) we also have $v_k \rightarrow 1$ in $L^p(A)$.

We notice that (u_k, v_k) also satisfies

$$\lim_{k \rightarrow +\infty} \mathcal{F}_k(u_k, v_k, Q_\rho(x)) = \mathcal{F}(u_\xi, 1, Q_\rho(x)). \tag{6.4}$$

Indeed, thanks to (6.2) we have

$$\mathcal{F}(u_\xi, 1, A) = \mathcal{F}(u_\xi, 1, Q_\rho(x)) + \mathcal{F}(u_\xi, 1, A \setminus \overline{Q_\rho(x)}).$$

Therefore, again by Γ -convergence we get

$$\liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, v_k, Q_\rho(x)) \geq \mathcal{F}(u_\xi, 1, Q_\rho(x))$$

and

$$\liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, v_k, A \setminus \overline{Q_\rho(x)}) \geq \mathcal{F}(u_\xi, 1, A \setminus \overline{Q_\rho(x)}).$$

Hence (6.4) follows by (6.3).

We now estimate separately the surface and bulk term in \mathcal{F}_k . We notice that by Young's Inequality we have

$$\frac{p-1}{p} \frac{(1-v_k)^p}{\varepsilon_k} + \frac{1}{p} \varepsilon_k^{p-1} |\nabla v_k|^p \geq (1-v_k)^{p-1} |\nabla v_k|.$$

Hence, thanks to (g2), by using the co-area formula we get that

$$\mathcal{F}_k^s(v_k, Q_\rho(x)) \geq c_3 \int_{Q_\rho(x)} (1 - v_k)^{p-1} |\nabla v_k| \, dy = c_3 \int_0^1 (1 - t)^{p-1} \mathcal{H}^{n-1}(\partial^* E_{k,\rho}^t) \, dt,$$

where $E_{k,\rho}^t := \{y \in Q_\rho(x) : v_k(y) < t\}$. Now let $\eta \in (0, 1)$ be fixed and $\eta' \in (\eta, 1)$; then by the mean-value Theorem we deduce the existence of $\bar{t} = \bar{t}(k, \rho, \eta, \eta')$, $\bar{t} \in (\eta, \eta')$ such that

$$\mathcal{F}_k^s(v_k, Q_\rho(x)) \geq c_3 \int_\eta^{\eta'} (1 - t)^{p-1} \, dt \mathcal{H}^{n-1}(\partial^* E_{k,\rho}^{\bar{t}}).$$

Set $w_k := u_k \chi_{\mathbb{R}^n \setminus E_{k,\rho}^{\bar{t}}}$; since $u_k \in W^{1,p}(A; \mathbb{R}^m)$ and $E_{k,\rho}^{\bar{t}}$ is a set of finite perimeter, we have that $w_k \in GSBVP(A; \mathbb{R}^m)$ and $\mathcal{H}^{n-1}(S_{w_k}) \leq \mathcal{H}^{n-1}(\partial^* E_{k,\rho}^{\bar{t}})$. Moreover, since $v_k \rightarrow 1$ in $L^p(A)$ we have that $\mathcal{L}^n(E_{k,\rho}^{\bar{t}}) \rightarrow 0$ as $k \rightarrow +\infty$ so that $w_k \rightarrow u_\xi$ in measure on bounded sets. Eventually, since \mathcal{F}_k is increasing as set function, using the fact that ψ is increasing we obtain

$$\begin{aligned} \mathcal{F}_k(u_k, v_k, Q_\rho(x)) &= \mathcal{F}_k^b(u_k, v_k, Q_\rho(x)) + \mathcal{F}_k^s(v_k, Q_\rho(x)) \\ &\geq \psi(\eta) \int_{Q_\rho(x) \setminus E_{k,\rho}^{\bar{t}}} f_k(y, \nabla u_k) \, dy + c_3 \int_\eta^{\eta'} (1 - t)^{p-1} \, dt \mathcal{H}^{n-1}(\partial^* E_{k,\rho}^{\bar{t}}) \\ &\geq \psi(\eta) \int_{Q_\rho(x)} f_k(y, \nabla w_k) \, dy + c_3 \int_\eta^{\eta'} (1 - t)^{p-1} \, dt \mathcal{H}^{n-1}(S_{w_k}), \end{aligned}$$

where the last inequality follows from the definition of w_k .

In view of 2.2–(f4) the right-hand side belongs to the class of functionals considered in [27]. Then, thanks to [27, Theorem 3.5 and Theorem 5.2 (b)] we have

$$\lim_{k \rightarrow +\infty} \mathcal{F}_k(u_k, v_k, Q_\rho(x)) \geq \psi(\eta) \int_{Q_\rho(x)} f''(y, \xi) \, dy,$$

where f'' is given by (2.13).

Hence appealing to (6.4) gives

$$\mathcal{F}(u_\xi, 1, Q_\rho(x)) \geq \psi(\eta) \int_{Q_\rho(x)} f''(y, \xi) \, dy.$$

Then, by dividing both terms in the above inequality by ρ^n and using (6.2), we obtain

$$\frac{1}{\rho^n} \int_{Q_\rho(x)} \hat{f}(y, \xi) \, dy \geq \psi(\eta) \frac{1}{\rho^n} \int_{Q_\rho(x)} f''(y, \xi) \, dy.$$

Thus invoking the Lebesgue differentiation Theorem together with the continuity in ξ of \hat{f} and f'' (see [17] and Proposition 4.1) we deduce that

$$\hat{f}(x, \xi) \geq \psi(\eta) f''(x, \xi),$$

for a.e. $x \in \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{m \times n}$, and for every $\eta \in (0, 1)$. Eventually, since ψ is continuous and $\psi(1) = 1$, the claim follows by letting $\eta, \eta' \rightarrow 1$.

Step 2: In this step we show that $\hat{f}(x, \xi) \leq f'(x, \xi)$ for every $x \in \mathbb{R}^n$ and for every $\xi \in \mathbb{R}^{m \times n}$. The proof is similar to that of [27, Theorem 5.2]. However, we repeat it here for the readers' convenience.

Let $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m \times n}$, $\rho > 0$, and $\eta > 0$ be fixed. By (2.8) for every $k \in \mathbb{N}$ fixed we can find $u_k \in W^{1,p}(Q_\rho(x); \mathbb{R}^m)$ with $u_k = u_\xi$ near $\partial Q_\rho(x)$ such that

$$\mathcal{F}_k^b(u_k, 1, Q_\rho(x)) \leq \mathbf{m}_k^b(u_\xi, Q_\rho(x)) + \eta\rho^n. \tag{6.5}$$

Combining (6.5) with (f2) and (f3) yields

$$c_1 \|\nabla u_k\|_{L^p(Q_\rho(x); \mathbb{R}^{m \times n})}^p \leq \mathcal{F}_k^b(u_k, 1, Q_\rho(x)) \leq \rho^n (c_2 |\xi|^p + \eta), \tag{6.6}$$

where the second inequality follows by taking u_ξ as a test in the definition of $\mathbf{m}_k^b(u_\xi, Q_\rho(x))$. Let now (k_j) be a diverging sequence such that

$$\lim_{j \rightarrow +\infty} \mathcal{F}_{k_j}^b(u_{k_j}, 1, Q_\rho(x)) = \liminf_{k \rightarrow +\infty} \mathcal{F}_k^b(u_k, 1, Q_\rho(x)).$$

Since $u_{k_j} - u_\xi \in W_0^{1,p}(Q_\rho(x); \mathbb{R}^m)$, the uniform bound (6.6) together with the Poincaré Inequality provide us with a further subsequence (not relabelled) and a function $u \in W^{1,p}(Q_\rho(x); \mathbb{R}^m)$ such that $u_{k_j} \rightharpoonup u$ weakly in $W^{1,p}(Q_\rho(x); \mathbb{R}^m)$. Then, by the Rellich Theorem $u_{k_j} \rightarrow u$ in $L^p(Q_\rho(x); \mathbb{R}^m)$. We now extend u and u_k to functions $w, w_k \in W_{loc}^{1,p}(\mathbb{R}^n; \mathbb{R}^m)$ by setting

$$w := \begin{cases} u & \text{in } Q_\rho(x), \\ u_\xi & \text{in } \mathbb{R}^n \setminus Q_\rho(x), \end{cases} \quad w_k := \begin{cases} u_k & \text{in } Q_\rho(x), \\ u_\xi & \text{in } \mathbb{R}^n \setminus Q_\rho(x), \end{cases}$$

respectively; clearly, $w = u_\xi$ in a neighbourhood of $\partial Q_{(1+\eta)\rho}(x)$ and $w_{k_j} \rightarrow w$ in $L_{loc}^p(\mathbb{R}^n; \mathbb{R}^m)$. Hence by Γ -convergence, by (2.3) and (6.5) we get

$$\begin{aligned} \mathbf{m}(u_\xi, Q_{(1+\eta)\rho}(x)) &\leq \mathcal{F}(w, 1, Q_{(1+\eta)\rho}(x)) \leq \lim_{j \rightarrow +\infty} \mathcal{F}_{k_j}(w_{k_j}, 1, Q_{(1+\eta)\rho}(x)) \\ &\leq \lim_{j \rightarrow +\infty} \mathcal{F}_{k_j}^b(u_{k_j}, Q_\rho(x)) + c_2 |\xi|^p ((1+\eta)^n - 1) \rho^n \\ &\leq \liminf_{k \rightarrow +\infty} \mathbf{m}_k^b(u_\xi, Q_\rho(x)) + \eta\rho^n + c_2 |\xi|^p ((1+\eta)^n - 1) \rho^n. \end{aligned}$$

Eventually, dividing by ρ^n , passing to the limsup as $\rho \rightarrow 0$, and recalling the definition of \hat{f} and f' we get

$$(1+\eta)^n \hat{f}(x, \xi) \leq f'(x, \xi) + \eta + c_2 |\xi|^p ((1+\eta)^n - 1),$$

and hence the claim follows by the arbitrariness of $\eta > 0$. □

7 Identification of the surface integrand

In this section we identify the surface integrand \hat{g} . Namely, we show that \hat{g} coincides with both g' and g'' , given by (2.14) and (2.15), respectively. This shows, in particular, that the limit surface integrand \hat{g} is obtained by minimising only the surface term \mathcal{F}_k^s . We notice, however, that in this case the presence the bulk term \mathcal{F}_k^b affects the class of test functions over which the minimisation is performed (cf. (2.9)–(2.10)).

We start by proving some preliminary lemmas. The first lemma concerns the approximation of a minimisation problem involving the Γ -limit \mathcal{F} .

Lemma 7.1 (Approximation of minimum values) *Let $(f_k) \subset \mathcal{F}$ and $(g_k) \subset \mathcal{G}$. Let $\rho > 0$; for $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, $v \in \mathbb{S}^{n-1}$, and $k \in \mathbb{N}$ such that $2\varepsilon_k < \rho$ set*

$$\mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) := \inf\{\mathcal{F}_k(u, v, Q_\rho^v(x)) : (u, v) \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m) \times W^{1,p}(Q_\rho^v(x)), 0 \leq v \leq 1, (u, v) = (\bar{u}_{x,\zeta,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v) \text{ near } \partial Q_\rho^v(x)\}.$$

Let (k_j) be as in Theorem 5.2 and \hat{g} be as in (5.12). Then for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, and $v \in \mathbb{S}^{n-1}$ it holds

$$\begin{aligned} \hat{g}(x, \zeta, v) &= \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \liminf_{j \rightarrow +\infty} \mathbf{m}_{k_j}(\bar{u}_{x,\zeta,\varepsilon_{k_j}}^v, Q_\rho^v(x)) \\ &= \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \limsup_{j \rightarrow +\infty} \mathbf{m}_{k_j}(\bar{u}_{x,\zeta,\varepsilon_{k_j}}^v, Q_\rho^v(x)). \end{aligned} \tag{7.1}$$

Proof For notational simplicity, in what follows we still denote with k the index of the (sub)sequence provided by Theorem 5.2.

We divide the proof into two steps.

Step I: In this step we show that

$$\hat{g}(x, \zeta, v) \leq \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \liminf_{k \rightarrow +\infty} \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)), \tag{7.2}$$

for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, and $v \in \mathbb{S}^{n-1}$.

Let $\rho > 0$ and $\eta > 0$ be fixed; by definition of $\mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x))$ there exists $(u_k, v_k) \subset W^{1,p}(Q_\rho^v(x); \mathbb{R}^m) \times W^{1,p}(Q_\rho^v(x))$ such that $(u_k, v_k) = (\bar{u}_{x,\zeta,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v)$ in a neighbourhood of $\partial Q_\rho^v(x)$, and

$$\mathcal{F}_k(u_k, v_k, Q_\rho^v(x)) \leq \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) + \eta\rho^{n-1}. \tag{7.3}$$

Since the pair $(\bar{u}_{x,\zeta,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v)$ is admissible for $\mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x))$, then (2.11) and (2.19) readily give

$$\mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) \leq \mathcal{F}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v, Q_\rho^v(x)) = \mathcal{F}_k^s(\bar{v}_{x,\varepsilon_k}^v, Q_\rho^v(x)) \leq c_4 C_v \rho^{n-1}. \tag{7.4}$$

By a truncation argument (see e.g., [23, Lemma 3.5] or [27, Lemma 4.1]) it is not restrictive to assume that $\sup_k \|u_k\|_{L^\infty(Q_\rho^v(x); \mathbb{R}^m)} < +\infty$. We now extend u_k to a $W_{\text{loc}}^{1,p}(\mathbb{R}^n; \mathbb{R}^m)$ -function by setting

$$w_k := \begin{cases} u_k & \text{in } Q_\rho^v(x), \\ \bar{u}_{x,\zeta,\varepsilon_k}^v & \text{in } \mathbb{R}^n \setminus Q_\rho^v(x), \end{cases}$$

then, clearly, $\sup_k \|w_k\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^m)} < +\infty$. Now let (k_j) be such that

$$\lim_{j \rightarrow +\infty} \mathcal{F}_{k_j}(u_{k_j}, v_{k_j}, Q_\rho^v(x)) = \liminf_{k \rightarrow +\infty} \mathcal{F}_k(u_k, v_k, Q_\rho^v(x)).$$

In view of (7.4), (2.5), and the uniform $L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ -bound on w_{k_j} we can invoke [36, Lemma 4.1] to deduce the existence of a subsequence (not relabelled) such that

$$(w_{k_j}, v_{k_j}) \rightarrow (u, 1) \text{ in } L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m) \times L_{\text{loc}}^p(\mathbb{R}^n),$$

for some $u \in L_{\text{loc}}^p(\mathbb{R}^n; \mathbb{R}^m)$ also belonging to $SBV^p(Q_{(1+\eta)\rho}^v(x); \mathbb{R}^m)$. Moreover, we also have $u = u_{x,\zeta}^v$ in a neighbourhood of $\partial Q_{(1+\eta)\rho}^v(x)$, so that

$$\mathbf{m}(u_{x,\zeta}^v, Q_{(1+\eta)\rho}^v(x)) \leq \mathcal{F}(u, 1, Q_{(1+\eta)\rho}^v(x)). \tag{7.5}$$

Eventually, by Γ -convergence together with (7.3) we obtain

$$\begin{aligned} \mathcal{F}(u, 1, Q_{(1+\eta)\rho}^v(x)) &\leq \liminf_{j \rightarrow +\infty} \mathcal{F}_{k_j}(w_{k_j}, v_{k_j}, Q_{(1+\eta)\rho}^v(x)) \\ &\leq \lim_{j \rightarrow +\infty} \mathcal{F}_{k_j}(u_{k_j}, v_{k_j}, Q_\rho^v(x)) + c_4 C_v ((1 + \eta)^{n-1} - 1) \rho^{n-1} \\ &\leq \liminf_{k \rightarrow +\infty} \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) + \eta \rho^{n-1} + c_4 C_v ((1 + \eta)^{n-1} - 1) \rho^{n-1}. \end{aligned}$$

Thus, using (7.5), dividing the above inequality by ρ^{n-1} and passing to the limsup as $\rho \rightarrow 0$ we obtain

$$(1 + \eta)^{n-1} \hat{g}(x, \zeta, v) \leq \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \liminf_{k \rightarrow +\infty} \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) + \eta + c_4 C_v ((1 + \eta)^{n-1} - 1),$$

hence (7.2) follows by the arbitrariness of $\eta > 0$.

Step 2: In this step we show that

$$\limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \limsup_{k \rightarrow +\infty} \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) \leq \hat{g}(x, \zeta, v), \tag{7.6}$$

for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, $v \in \mathbb{S}^{n-1}$, and $\rho > 0$. To this end, we fix $\eta > 0$ and we choose $u \in SBV^p(Q_\rho^v(x); \mathbb{R}^m)$ with $u = u_{x,\zeta}^v$ near $\partial Q_\rho^v(x)$ and

$$\mathcal{F}(u, 1, Q_\rho^v(x)) \leq \mathbf{m}(u_{x,\zeta}^v, Q_\rho^v(x)) + \eta. \tag{7.7}$$

We extend u to the whole \mathbb{R}^n by setting $u = u_{x,\zeta}^v$ in $\mathbb{R}^n \setminus Q_\rho^v(x)$. Then, by Γ -convergence there exists a sequence (u_k, v_k) converging to $(u, 1)$ in measure on bounded sets such that

$$\lim_{k \rightarrow +\infty} \mathcal{F}_k(u_k, v_k, Q_\rho^v(x)) = \mathcal{F}(u, 1, Q_\rho^v(x)). \tag{7.8}$$

We notice, moreover, that thanks to a truncation argument (both on u and u_k) and to the bound (g2), it is not restrictive to assume that (u_k, v_k) converges to $(u, 1)$ in $L^p_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \times L^p_{\text{loc}}(\mathbb{R}^n)$. We now modify the sequence (u_k, v_k) in such a way that it satisfies the boundary conditions required in the definition of $\mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x))$. This will be done by resorting to the fundamental estimate Proposition 5.1. Namely, we choose $0 < \rho'' < \rho' < \rho$ such that $u = u_{x,\zeta}^v$ on $Q_{\rho'}^v(x) \setminus \bar{Q}_{\rho''}^v(x)$ and we apply Proposition 5.1 with $A = Q_{\rho''}^v(x)$, $A' = Q_{\rho'}^v(x)$, $B = Q_\rho^v(x) \setminus \bar{Q}_{\rho''}^v(x)$. In this way, we obtain a sequence $(\hat{u}_k, \hat{v}_k) \subset W^{1,p}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \times W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ converging to (u, v) in $L^p(Q_\rho^v(x); \mathbb{R}^m) \times L^p(Q_\rho^v(x))$ such that $(\hat{u}_k, \hat{v}_k) = (u_k, v_k)$ in $Q_{\rho''}^v(x)$, $(\hat{u}_k, \hat{v}_k) = (\bar{u}_{x,\zeta,\varepsilon_k}^v, \bar{v}_{x,\varepsilon_k}^v)$ in $Q_\rho^v(x) \setminus \bar{Q}_{\rho'}^v(x)$, and

$$\begin{aligned} \limsup_{k \rightarrow +\infty} \mathcal{F}_k(\hat{u}_k, \hat{v}_k, Q_\rho^v(x)) &\leq (1 + \eta) \limsup_{k \rightarrow +\infty} \left(\mathcal{F}_k(u_k, v_k, Q_{\rho'}^v(x)) \right. \\ &\quad \left. + \mathcal{F}_k^s(\bar{v}_{x,\varepsilon_k}^v, Q_\rho^v(x) \setminus \bar{Q}_{\rho'}^v(x)) \right) \\ &\leq (1 + \eta) \mathcal{F}(u, 1, Q_\rho^v(x)) + c_4 C_v \mathcal{L}^{n-1}(Q_\rho^v \setminus \bar{Q}_{\rho'}^v), \end{aligned} \tag{7.9}$$

where the second inequality follows from (7.8) and (2.19) together with (2.3), respectively.

Eventually, since (\hat{u}_k, \hat{v}_k) is admissible in the definition of $\mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x))$, gathering (7.7) and (7.9) we deduce that

$$\limsup_{k \rightarrow +\infty} \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) \leq (1 + \eta) (\mathbf{m}(u_{x,\zeta}^v, Q_\rho^v(x)) + \eta) + C(\rho^{n-1} - (\rho'')^{n-1}).$$

Then letting $\rho'' \rightarrow \rho$, by the arbitrariness of $\eta > 0$ we get

$$\limsup_{k \rightarrow +\infty} \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) \leq \mathbf{m}(u_{x,\zeta}^v, Q_\rho^v(x)), \tag{7.10}$$

therefore (7.6) follows by dividing both sides of (7.10) by ρ^{n-1} and passing to the limsup as $\rho \rightarrow 0$. \square

Remark 7.2 Thanks to the $(W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A))$ -continuity of $\mathcal{F}_k(\cdot, \cdot, A)$ (cf. Remark 2.3), a standard convolution argument shows that (7.1) holds also true if the minimisation in \mathbf{m}_k is carried over $C^1(Q_\rho^v(x); \mathbb{R}^m) \times C^1(Q_\rho^v(x))$.

The following lemma shows that if v is “small” in some region, then it can be replaced by a function which is equal to zero in that region, without essentially increasing \mathcal{F}_k^s .

Lemma 7.3 *Let $(g_k) \subset \mathcal{G}$, $A \in \mathcal{A}$, $v \in W^{1,p}(A)$, and $\eta \in (0, 1)$ be fixed. Let $v^\eta \in W^{1,p}(A)$ be defined as*

$$v^\eta := \min\{((1 + \sqrt{\eta})v - \eta)^+, v\}. \tag{7.11}$$

Then for a.e. $x \in A$ we have

$$v^\eta(x) = 0 \text{ iff } v(x) \leq \frac{\eta}{1 + \sqrt{\eta}} \quad \text{and} \quad v^\eta(x) = v(x) \text{ iff } v(x) \geq \sqrt{\eta}. \tag{7.12}$$

Moreover, v^η satisfies

$$\mathcal{F}_k^s(v^\eta, A) \leq (1 + c_\eta)\mathcal{F}_k^s(v, A), \tag{7.13}$$

where $c_\eta > 0$ is independent of v and A and such that $c_\eta \rightarrow 0$ as $\eta \rightarrow 0$.

Proof A direct computation shows that $((1 + \sqrt{\eta})v - \eta)^+ \leq v$ if and only if $v \leq \sqrt{\eta}$ and that $((1 + \sqrt{\eta})v - \eta)^+ = 0$ if and only if $v \leq \frac{\eta}{1 + \sqrt{\eta}}$; i.e., (7.12) is satisfied. Thus it remains to show that (7.13) holds true. To this end we introduce the sets

$$A^\eta := \left\{x \in A : v(x) \leq \frac{\eta}{1 + \sqrt{\eta}}\right\} \quad \text{and} \quad B^\eta := \left\{x \in A : \frac{\eta}{1 + \sqrt{\eta}} < v(x) < \sqrt{\eta}\right\},$$

so that $v^\eta = v$ in $A \setminus (A^\eta \cup B^\eta)$; we have

$$\begin{aligned} \mathcal{F}_k^s(v^\eta, A) &= \frac{1}{\varepsilon_k} \int_{A^\eta} g_k(x, 0, 0) \, dx + \frac{1}{\varepsilon_k} \int_{B^\eta} g_k(x, v^\eta, \varepsilon_k \nabla v^\eta) \, dx \\ &\quad + \frac{1}{\varepsilon_k} \int_{A \setminus (A^\eta \cup B^\eta)} g_k(x, v, \varepsilon_k \nabla v) \, dx. \end{aligned} \tag{7.14}$$

We start by estimating the first term on the right-hand side of (7.14). Since $v \leq \frac{\eta}{1 + \sqrt{\eta}} < \eta < 1$ in A^η , using (g4) and (g6) we get

$$\begin{aligned} g_k(x, 0, 0) &\leq g_k(x, v, 0) + L_2(1 + v^{p-1})v \\ &\leq g_k(x, v, \varepsilon_k \nabla v) + 2L_2\eta \leq g_k(x, v, \varepsilon_k \nabla v) + \frac{2L_2\eta}{(1 - \eta)^p}(1 - v)^p, \end{aligned}$$

in A^η , which together with (g2) yields

$$\frac{1}{\varepsilon_k} \int_{A^\eta} g_k(x, 0, 0) \, dx \leq \left(1 + \frac{2L_2\eta}{c_3(1 - \eta)^p}\right) \frac{1}{\varepsilon_k} \int_{A^\eta} g_k(x, v, \varepsilon_k \nabla v) \, dx. \tag{7.15}$$

We now come to estimate the second term on the right-hand side of (7.14). By definition, we have $v^\eta = (1 + \sqrt{\eta})v - \eta < v < \sqrt{\eta}$ in B^η , thus from (g4) we deduce that

$$g_k(x, v^\eta, \varepsilon_k \nabla v^\eta) \leq g_k(x, v, \varepsilon_k \nabla v) + L_2(1 + 2\eta^{\frac{p-1}{2}})(\eta - \sqrt{\eta}v) + L_2(1 + \varepsilon_k^{p-1}|\nabla v|^{p-1} + \varepsilon_k^{p-1}|\nabla v^\eta|^{p-1})\varepsilon_k|\nabla v - \nabla v^\eta|, \tag{7.16}$$

in B^η . Furthermore, since $v < \sqrt{\eta} < 1$ in B^η , it also holds

$$(1 + 2\eta^{\frac{p-1}{2}})(\eta - \sqrt{\eta}v) \leq 3\eta \leq \frac{3\eta}{(1 - \sqrt{\eta})^p}(1 - v)^p, \tag{7.17}$$

and similarly

$$|\nabla v - \nabla v^\eta| = \sqrt{\eta}|\nabla v| \leq \frac{\sqrt{\eta}}{(1 - \sqrt{\eta})^{p-1}}(1 - v)^{p-1}|\nabla v| \leq \frac{\sqrt{\eta}}{(1 - \sqrt{\eta})^{p-1}}\left(\frac{(1 - v)^p}{\varepsilon_k} + \varepsilon_k^{p-1}|\nabla v|^p\right), \tag{7.18}$$

where the last estimate follows by Young’s inequality. Eventually, we also have

$$\varepsilon_k^p(|\nabla v|^{p-1} + |\nabla v^\eta|^{p-1})|\nabla v - \nabla v^\eta| = \varepsilon_k^p(1 + (1 + \sqrt{\eta})^{p-1})\sqrt{\eta}|\nabla v|^p \leq \varepsilon_k^p(1 + 2^{p-1})\sqrt{\eta}|\nabla v|^p. \tag{7.19}$$

Gathering (7.16)–(7.19), dividing by ε_k and using (g2) give

$$\frac{1}{\varepsilon_k} \int_{B^\eta} g_k(x, v^\eta, \varepsilon_k \nabla v^\eta) \, dx \leq \frac{1}{\varepsilon_k} \int_{B^\eta} g_k(x, v, \varepsilon_k \nabla v) \, dx + c_\eta \frac{1}{\varepsilon_k} \int_{B^\eta} g_k(x, v, \varepsilon_k \nabla v) \, dx, \tag{7.20}$$

where $c_\eta := \frac{L_2}{\varepsilon_3} \max\{\frac{3\eta}{(1-\sqrt{\eta})^p} + (1+2^{p-1})\sqrt{\eta}, \frac{\sqrt{\eta}}{(1-\sqrt{\eta})^{p-1}}\}$. Then, since $(1 - \eta)^p > (1 - \sqrt{\eta})^p$ for $\eta \in (0, 1)$, gathering (7.14), (7.15), and (7.20) we obtain

$$\mathcal{F}_k^s(v^\eta, A) \leq (1 + c_\eta)\mathcal{F}_k^s(v, A).$$

Since $c_\eta \rightarrow 0$ as $\eta \rightarrow 0$ this concludes the proof. □

With the help of Lemma 7.1 and Lemma 7.3 we are now in a position to identify the surface integrand \hat{g} .

Proposition 7.4 *Let $(f_k) \subset \mathcal{F}$ and $(g_k) \subset \mathcal{G}$; let (k_j) and \hat{g} be as in Theorem 5.2. Then, it holds*

$$\hat{g}(x, \zeta, \nu) = g'(x, \nu) = g''(x, \nu),$$

for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, and $\nu \in \mathbb{S}^{n-1}$, where g' and g'' are, respectively, as in (2.14) and (2.15), with k replaced by k_j .

Proof For notational simplicity, in what follows we still denote with k the index of the sequence provided by Theorem 5.2.

By definition we have $g' \leq g''$; hence to prove the claim it suffices to show that

$$g''(x, \nu) \leq \hat{g}(x, \zeta, \nu) \leq g'(x, \nu), \tag{7.21}$$

for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, and $\nu \in \mathbb{S}^{n-1}$. We divide the proof of (7.21) into two steps.

Step 1: In this step we show that $\hat{g}(x, \zeta, \nu) \geq g''(x, \nu)$, for every $x \in \mathbb{R}^n$, $\zeta \in \mathbb{R}_0^m$, and $\nu \in \mathbb{S}^{n-1}$.

In view of Lemma 7.1 we have

$$\hat{g}(x, \zeta, \nu) = \limsup_{\rho \rightarrow 0} \frac{1}{\rho^{n-1}} \limsup_{k \rightarrow +\infty} \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^\nu, Q_\rho^\nu(x)).$$

Thanks to Remark 7.2 the minimisation in the definition of $\mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^\nu, Q_\rho^\nu(x))$ can be carried over C^1 -pairs (u_k, v_k) . Now let $\rho > 0$ and $\eta \in (0, 1)$ be fixed and for every k such that $2\varepsilon_k < \rho$ let $(u_k, v_k) \subset C^1(Q_\rho^\nu(x); \mathbb{R}^m) \times C^1(Q_\rho^\nu(x))$ satisfy

$$(u_k, v_k) = (\bar{u}_{x,\zeta,\varepsilon_k}^\nu, \bar{v}_{x,\varepsilon_k}^\nu) \text{ in } U_k, \tag{7.22}$$

where U_k is a neighbourhood of $\partial Q_\rho^\nu(x)$ and

$$\mathcal{F}_k(u_k, v_k, Q_\rho^\nu(x)) \leq \mathbf{m}_k(\bar{u}_{x,\zeta,\varepsilon_k}^\nu, Q_\rho^\nu(x)) + \eta\rho^{n-1}, \tag{7.23}$$

then, (7.4) readily gives

$$\mathcal{F}_k(u_k, v_k, Q_\rho^\nu(x)) \leq C\rho^{n-1}. \tag{7.24}$$

We now modify v_k in order to obtain a new function \tilde{v}_k for which there exists a corresponding \tilde{u}_k such that the pair $(\tilde{u}_k, \tilde{v}_k)$ satisfies both

$$(\tilde{u}_k, \tilde{v}_k) = (u_k^\nu, 1) \text{ in } \tilde{U}_k \cap \{|(y-x) \cdot \nu| > \varepsilon_k\}, \tag{7.25}$$

where \tilde{U}_k is a neighbourhood of $\partial Q_\rho^\nu(x)$, and the constraint

$$\tilde{v}_k \nabla \tilde{u}_k = 0 \text{ a.e. in } Q_\rho^\nu(x). \tag{7.26}$$

In this way we have $\mathcal{F}_k(\tilde{u}_k, \tilde{v}_k, Q_\rho^\nu(x)) = \mathcal{F}_k^s(\tilde{v}_k, Q_\rho^\nu(x))$ with $\tilde{v}_k \in \mathcal{A}_{\varepsilon_k, \rho}(x, \nu)$.

The modification as above shall be performed without essentially increasing the energy \mathcal{F}_k .

To this end, set $u_k := (u_k^1, \dots, u_k^m)$ and $\zeta := (\zeta^1, \dots, \zeta^m)$. Since $\zeta \in \mathbb{R}_0^m$ we can find $i \in \{1, \dots, m\}$ so that $\zeta^i \neq 0$; without loss of generality we assume that $\zeta^i > 0$. We now consider the open set

$$S_k^\rho := \left\{ y \in Q_\rho^\nu(x) : 0 < u_k^i(y) < \zeta^i \right\}.$$

Moreover, let $\sigma \in (0, 1)$ be fixed and consider the following partition of S_k^ρ :

$$S_k^\rho = \bigcup_{\ell=0}^{h_k^\sigma - 1} S_{k,\ell}^\rho$$

with

$$S_{k,\ell}^\rho := \left\{ y \in Q_\rho^\nu(x) : \ell \frac{\zeta^i}{h_k^\sigma} < u_k^i(y) \leq (\ell + 1) \frac{\zeta^i}{h_k^\sigma} \right\} \quad \ell = 0, \dots, h_k^\sigma - 2,$$

$$S_{k,h_k^\sigma - 1}^\rho := \left\{ y \in Q_\rho^\nu(x) : (h_k^\sigma - 1) \frac{\zeta^i}{h_k^\sigma} < u_k^i(y) < \zeta^i \right\}$$

and $h_k^\sigma \in \mathbb{N}$ to be chosen later.

Then, there exists $\bar{\ell} = \bar{\ell}(k, \rho, \sigma) \in \{0, \dots, h_k^\sigma - 1\}$ such that

$$\int_{\tilde{S}_k^\rho} \left(\sigma \psi(v_k) |\nabla u_k^i|^p + (1 - \sigma) \right) dy \leq \frac{1}{h_k^\sigma} \int_{S_k^\rho} \left(\sigma \psi(v_k) |\nabla u_k^i|^p + (1 - \sigma) \right) dy, \tag{7.27}$$

where $\tilde{S}_k^\rho := S_{k, \bar{\ell}}^\rho$.

Therefore, gathering (f2), (7.24), and (7.27) yields

$$\int_{\tilde{S}_k^\rho} \sigma \psi(v_k) |\nabla u_k^i|^p dy + (1 - \sigma) \mathcal{L}^n(\tilde{S}_k^\rho) \leq \frac{\rho^{n-1}}{h_k^\sigma} C(\sigma + (1 - \sigma)\rho). \tag{7.28}$$

In view of (7.22) we have that

$$\tilde{S}_k^\rho \cap U_k \subset \{|(y - x) \cdot v| \leq \varepsilon_k\} \cap U_k. \tag{7.29}$$

In this way any modification to v_k performed in the set \tilde{S}_k^ρ will not affect the boundary conditions. In order to modify v_k in \tilde{S}_k^ρ , we introduce an auxiliary function \hat{v}_k which interpolates in a suitable way between the values 0 and 1 in \tilde{S}_k^ρ . To this end, we set $\gamma_k := (\bar{\ell} + \frac{1}{2}) \frac{\xi_k^i}{h_k^\sigma}$ and $\tau_k := \frac{\xi_k^i}{4h_k^\sigma}$ and we define $\hat{v}_k \in W^{1,p}(Q_\rho^v(x))$ as follows:

$$\hat{v}_k := \begin{cases} \min\left\{\frac{u_k^i - \gamma_k - \tau_k}{\tau_k}, 1\right\} & \text{in } \{u_k^i \geq \gamma_k + \tau_k\}, \\ 0 & \text{in } \{\gamma_k - \tau_k < u_k^i < \gamma_k + \tau_k\}, \\ \min\left\{\frac{\gamma_k - \tau_k - u_k^i}{\tau_k}, 1\right\} & \text{in } \{u_k^i \leq \gamma_k - \tau_k\}, \end{cases}$$

Eventually, we let $v_k^\eta \in W^{1,p}(Q_\rho^v(x))$ be the function defined in (7.11) in Lemma 7.3, with v_k in place of v and we define $\tilde{v}_k \in W^{1,p}(Q_\rho^v(x))$ as

$$\tilde{v}_k := \min\{v_k^\eta, \hat{v}_k\}.$$

We notice that by definition

$$\hat{v}_k \equiv 1 \quad \text{in } Q_\rho^v(x) \setminus \tilde{S}_k^\rho,$$

so that in particular $\tilde{v}_k \equiv v_k^\eta$ in $Q_\rho^v(x) \setminus \tilde{S}_k^\rho$; moreover,

$$\tilde{v}_k \equiv 0 \quad \text{in } \{\gamma_k - \tau_k < u_k^i < \gamma_k + \tau_k\}.$$

By the regularity of u_k^i we can find $\xi_k > 0$ (possibly small) such that

$$\{y \in Q_\rho^v(x) : \text{dist}(y, \{u_k^i > \gamma_k\}) < \xi_k\} \subset \{y \in Q_\rho^v(x) : u_k^i(y) > \gamma_k - \tau_k\}.$$

Then, we define $\tilde{u}_k \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m)$ as

$$\tilde{u}_k(y) := \begin{cases} \left(1 - \frac{\text{dist}(y, \{u_k^i > \gamma_k\})}{\xi_k}\right) e_1 & \text{if } \text{dist}(y, \{u_k^i > \gamma_k\}) < \xi_k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the pair $(\tilde{u}_k, \tilde{v}_k)$ belongs to $W^{1,p}(Q_\rho^v(x); \mathbb{R}^m) \times W^{1,p}(Q_\rho^v(x))$ and by construction satisfies (7.26). Moreover, (7.29) together with (7.22) ensures that (7.25) is satisfied. Eventually, we have

$$\frac{1}{\rho^{n-1}} \mathbf{m}_{k,N}^s(u_k^v, Q_\rho^v(x)) \leq \frac{1}{\rho^{n-1}} \mathcal{F}_k^s(\tilde{v}_k, Q_\rho^v(x)). \tag{7.30}$$

To conclude the proof it only remains to show that, up to a small error, $\mathcal{F}_k^s(\tilde{v}_k, Q_\rho^v(x))$ is a lower bound for $\mathcal{F}_k(u_k, v_k, Q_\rho^v(x))$. To this end we consider the following partition of $Q_\rho^v(x)$:

$$S_k^1 := \{y \in Q_\rho^v(x) : v_k^\eta(y) \leq \hat{v}_k(y)\}, \quad S_k^2 := \{y \in Q_\rho^v(x) : v_k^\eta(y) > \hat{v}_k(y)\};$$

then, appealing to (7.13) in Lemma 7.3, we deduce

$$\begin{aligned} \mathcal{F}_k^s(\tilde{v}_k, Q_\rho^v(x)) &= \frac{1}{\varepsilon_k} \int_{S_k^1} g_k(y, v_k^\eta, \varepsilon_k \nabla v_k^\eta) \, dy + \frac{1}{\varepsilon_k} \int_{S_k^2} g_k(y, \hat{v}_k, \varepsilon_k \nabla \hat{v}_k) \, dy \\ &\leq (1 + c_\eta) \mathcal{F}_k^s(v_k, Q_\rho^v(x)) + \frac{1}{\varepsilon_k} \int_{S_k^2} g_k(y, \hat{v}_k, \varepsilon_k \nabla \hat{v}_k) \, dy, \end{aligned} \tag{7.31}$$

where $c_\eta \rightarrow 0$ as $\eta \rightarrow 0$. Hence, it remains to estimate the second term on the right-hand side of (7.31).

To this end, we start noticing that

$$S_k^2 \subset \left\{ v_k > \frac{\eta}{1 + \sqrt{\eta}} \right\}. \tag{7.32}$$

Indeed, since $v_k^\eta > 0$ a.e. in S_k^2 , by definition of v_k^η we readily get (7.32).

Therefore, by (g3), using that $\hat{v}_k \equiv 1$ in $Q_\rho^v(x) \setminus \tilde{S}_k^\rho$ we obtain

$$\begin{aligned} \frac{1}{\varepsilon_k} \int_{S_k^2} g_k(y, \hat{v}_k, \varepsilon_k \nabla \hat{v}_k) \, dy &\leq c_4 \int_{S_k^2} \left(\frac{(1 - \hat{v}_k)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla \hat{v}_k|^p \right) \, dy \\ &= c_4 \int_{S_k^2 \cap \tilde{S}_k^\rho} \left(\frac{(1 - \hat{v}_k)^p}{\varepsilon_k} + \varepsilon_k^{p-1} |\nabla \hat{v}_k|^p \right) \, dy \\ &\leq c_4 \left(\frac{\mathcal{L}^n(\tilde{S}_k^\rho)}{\varepsilon_k} + \frac{\varepsilon_k^{p-1}}{\psi(\frac{\eta}{1+\sqrt{\eta}})} \left(\frac{4h_k^\sigma}{\zeta^i} \right)^p \int_{S_k^2 \cap \tilde{S}_k^\rho} \psi(v_k) |\nabla u_k|^p \, dy \right), \end{aligned} \tag{7.33}$$

where the last inequality follows by (7.32), the definition of \hat{v}_k , and the monotonicity of ψ .

From (7.28) we deduce both that

$$\mathcal{L}^n(\tilde{S}_k^\rho) \leq C \frac{\rho^{n-1}}{h_k^\sigma} \left(\frac{\sigma}{1 - \sigma} + \rho \right) \tag{7.34}$$

and

$$\int_{S_k^2 \cap \tilde{S}_k^\rho} \psi(v_k) |\nabla u_k|^p \, dx \leq C \frac{\rho^{n-1}}{h_k^\sigma} \left(1 + \frac{1 - \sigma}{\sigma} \rho \right). \tag{7.35}$$

Hence, gathering (7.33), (7.34), and (7.35) we obtain

$$\begin{aligned} \frac{1}{\varepsilon_k} \int_{S_k^2} g_k(y, \hat{v}_k, \varepsilon_k \nabla \hat{v}_k) \, dy &\leq C c_4 \rho^{n-1} \left(\frac{1}{\varepsilon_k h_k^\sigma} \left(\frac{\sigma}{1 - \sigma} + \rho \right) \right. \\ &\quad \left. + K_\eta (\varepsilon_k h_k^\sigma)^{p-1} \left(1 + \frac{1 - \sigma}{\sigma} \rho \right) \right), \end{aligned} \tag{7.36}$$

with $K_\eta := 4^p (\psi(\frac{\eta}{1+\sqrt{\eta}})(\zeta^i)^p)^{-1}$. Now set $h_k^\sigma := \lfloor \tilde{h}_k^\sigma \rfloor$ where

$$\tilde{h}_k^\sigma := \frac{1}{\varepsilon_k} \left(\frac{\sigma}{1 - \sigma} \right)^{\frac{1}{p}}. \tag{7.37}$$

Using that $h_k^\sigma \leq \tilde{h}_k^\sigma \leq h_k^\sigma + 1$ we infer

$$\frac{1}{\varepsilon_k} \int_{S_k^2} g_k(y, \hat{v}_k, \varepsilon_k \nabla \hat{v}_k) \, dy \leq C c_4 \rho^{n-1} \left(\frac{\tilde{h}_k^\sigma}{h_k^\sigma - 1} + K_\eta \right) \left(\left(\frac{\sigma}{1 - \sigma} \right)^{\frac{p-1}{p}} + \rho \left(\frac{\sigma}{1 - \sigma} \right)^{-\frac{1}{p}} \right).$$

Plugging (7.36) into (7.31) gives

$$\begin{aligned} \frac{1}{\rho^{n-1}} \mathcal{F}_k^s(\tilde{v}_k, Q_\rho^v(x)) &\leq (1 + c_\eta) \frac{1}{\rho^{n-1}} \mathcal{F}_k(u_k, v_k, Q_\rho^v(x)) \\ &\quad + \widehat{K}_\eta \left(\left(\frac{\sigma}{1 - \sigma} \right)^{\frac{p-1}{p}} + \rho \left(\frac{\sigma}{1 - \sigma} \right)^{-\frac{1}{p}} \right), \end{aligned} \tag{7.38}$$

where $\widehat{K}_\eta := C c_4(1 + K_\eta)$. We notice that $c_\eta \rightarrow 0$ as $\eta \rightarrow 0$, while $\widehat{K}_\eta \rightarrow +\infty$ as $\eta \rightarrow 0$. Finally, by combining (7.23), (7.30), and (7.38) we get

$$\begin{aligned} \frac{1}{\rho^{n-1}} \mathbf{m}_{k,N}^s(u_x^v, Q_\rho^v(x)) &\leq (1 + c_\eta) \left(\frac{1}{\rho^{n-1}} \mathbf{m}_k(\tilde{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) + \eta \right) \\ &\quad + \widehat{K}_\eta \left(\left(\frac{\sigma}{1 - \sigma} \right)^{\frac{p-1}{p}} + \rho \left(\frac{\sigma}{1 - \sigma} \right)^{-\frac{1}{p}} \right). \end{aligned} \tag{7.39}$$

Eventually, the claim follows by passing to the limit in (7.39) in the following order: first as $k \rightarrow +\infty$, then as $\rho \rightarrow 0, \sigma \rightarrow 0$, and finally as $\eta \rightarrow 0$.

Step 2: In this step we show that

$$\hat{g}(x, \zeta, v) \leq g'(x, v),$$

for every $x \in \mathbb{R}^n, \zeta \in \mathbb{R}_0^m$ and $v \in \mathbb{S}^{n-1}$. Thanks to (5.12), in view of Lemma 7.1 and Proposition 2.6 it suffices to show that

$$\mathbf{m}_k(\tilde{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) \leq \mathbf{m}_k^s(\tilde{u}_{x,\varepsilon_k}^v, Q_\rho^v(x)), \tag{7.40}$$

for every $\varepsilon > 0$, where $\mathbf{m}_k^s(\tilde{u}_{x,\varepsilon_k}^v, Q_\rho^v(x))$ is defined in (2.16). To prove (7.40) let $v \in \mathcal{A}(\tilde{u}_{x,\varepsilon_k}^v, Q_\rho^v(x))$ with corresponding $u \in W^{1,p}(Q_\rho^v(x); \mathbb{R}^m)$ and notice that the pair (\tilde{u}, v) with $\tilde{u} := (u \cdot e_1)\zeta$ is an admissible competitor for $\mathbf{m}_k(\tilde{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x))$. Therefore, we obtain

$$\mathbf{m}_k(\tilde{u}_{x,\zeta,\varepsilon_k}^v, Q_\rho^v(x)) \leq \mathcal{F}_k(\tilde{u}, v, Q_\rho^v(x)) = \mathcal{F}_k^s(v, Q_\rho^v(x)),$$

from which we deduce (7.40) by passing to the infimum in $v \in \mathcal{A}(\tilde{u}_{x,\varepsilon_k}^v, Q_\rho^v(x))$. \square

Remark 7.5 We observe that the second term in the right hand side of (7.39) is infinitesimal as $\rho, \sigma \rightarrow 0$ thanks to the fact that $p > 1$. We notice, moreover, that in (7.39) the presence of the exponent p in the reminder is a consequence of the p -growth of the volume integrand f_k . Indeed if g_k satisfied conditions (g2)–(g4) with p replaced by some exponent $q \in (1, p]$, then arguing as in the proof of Proposition 7.4, in place of (7.36) we would get

$$\begin{aligned} &\frac{1}{\varepsilon_k} \int_{S_k^2} g_k(y, \hat{v}_k, \varepsilon_k \nabla \hat{v}_k) \, dy \\ &\leq C c_4 \rho^{n-1} \left(\frac{1}{\varepsilon_k h_k^\sigma} \left(\frac{\sigma}{1 - \sigma} + \rho \right) + K_\eta (\varepsilon_k h_k^\sigma)^{q-1} \left(1 + \frac{1 - \sigma}{\sigma} \rho \right)^{\frac{q}{p}} \left(\frac{\sigma}{1 - \sigma} + \rho \right)^{\frac{p-q}{p}} \right), \end{aligned}$$

with $K_\eta := 4^q (\psi(\frac{\eta}{1+\sqrt{\eta}}))(\zeta^i)^q)^{-1}$. Then, an easy computation shows that choosing $h_k^\sigma = \lfloor \tilde{h}_k^\sigma \rfloor$, with \tilde{h}_k^σ given by (7.37), exactly yields (7.39).

We are now in a position to prove the main result of this paper, namely, Theorem 3.1.

Proof of Theorem 3.1 The proof follows by combining Proposition 4.1, Proposition 4.5, Theorem 5.2, Proposition 6.1, and Proposition 7.4. □

8 Stochastic homogenisation

In this section we study the Γ -convergence of the functionals \mathcal{F}_k when f_k and g_k are random integrands of type

$$f_k(\omega, x, \xi) = f\left(\omega, \frac{x}{\varepsilon_k}, \xi\right), \quad g_k(\omega, x, v, w) = g\left(\omega, \frac{x}{\varepsilon_k}, v, w\right),$$

where ω belongs to the sample space Ω of a probability space (Ω, \mathcal{T}, P) .

Before stating the main result of this section, we need to recall some useful definitions.

Definition 8.1 (*Group of P -preserving transformations*) Let $d \in \mathbb{N}$, $d \geq 1$. A group of P -preserving transformations on (Ω, \mathcal{T}, P) is a family $(\tau_z)_{z \in \mathbb{Z}^d}$ of mappings $\tau_z : \Omega \rightarrow \Omega$ satisfying the following properties:

- (1) (measurability) τ_z is \mathcal{T} -measurable for every $z \in \mathbb{Z}^d$;
- (2) (invariance) $P(\tau_z(E)) = P(E)$, for every $E \in \mathcal{T}$ and every $z \in \mathbb{Z}^d$;
- (3) (group property) $\tau_0 = \text{id}_\Omega$ and $\tau_{z+z'} = \tau_z \circ \tau_{z'}$ for every $z, z' \in \mathbb{Z}^d$.

If, in addition, every $(\tau_z)_{z \in \mathbb{Z}^d}$ -invariant set (i.e., every $E \in \mathcal{T}$ with $\tau_z(E) = E$ for every $z \in \mathbb{Z}^d$) has probability 0 or 1, then $(\tau_z)_{z \in \mathbb{Z}^d}$ is called ergodic.

Let $a := (a_1, \dots, a_d)$, $b := (b_1, \dots, b_d) \in \mathbb{Z}^d$ with $a_i < b_i$ for all $i \in \{1, \dots, d\}$; we define the d -dimensional interval

$$[a, b) := \{x \in \mathbb{Z}^d : a_i \leq x_i < b_i \text{ for } i = 1, \dots, d\}$$

and we set

$$\mathcal{I}_d := \{[a, b) : a, b \in \mathbb{Z}^d, a_i < b_i \text{ for } i = 1, \dots, d\}.$$

Definition 8.2 (*Subadditive process*) A discrete subadditive process with respect to a group $(\tau_z)_{z \in \mathbb{Z}^d}$ of P -preserving transformations on (Ω, \mathcal{T}, P) is a function $\mu : \Omega \times \mathcal{I}_d \rightarrow \mathbb{R}$ satisfying the following properties:

- (1) (measurability) for every $A \in \mathcal{I}_d$ the function $\omega \mapsto \mu(\omega, A)$ is \mathcal{T} -measurable;
- (2) (covariance) for every $\omega \in \Omega$, $A \in \mathcal{I}_d$, and $z \in \mathbb{Z}^d$ we have $\mu(\omega, A + z) = \mu(\tau_z(\omega), A)$;
- (3) (subadditivity) for every $A \in \mathcal{I}_d$ and for every finite family $(A_i)_{i \in I} \subset \mathcal{I}_d$ of pairwise disjoint sets such that $A = \cup_{i \in I} A_i$, we have

$$\mu(\omega, A) \leq \sum_{i \in I} \mu(\omega, A_i) \quad \text{for every } \omega \in \Omega;$$

- (4) (boundedness) there exists $c > 0$ such that $0 \leq \mu(\omega, A) \leq c\mathcal{L}^d(A)$ for every $\omega \in \Omega$ and $A \in \mathcal{I}_d$.

Definition 8.3 (*Stationarity*) Let $(\tau_z)_{z \in \mathbb{Z}^n}$ be a group of P -preserving transformations on (Ω, \mathcal{T}, P) . We say that $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$ is stationary with respect to $(\tau_z)_{z \in \mathbb{Z}^n}$ if

$$f(\omega, x + z, \xi) = f(\tau_z(\omega), x, \xi)$$

for every $\omega \in \Omega, x \in \mathbb{R}^n, z \in \mathbb{Z}^n$ and $\xi \in \mathbb{R}^{m \times n}$.

Analogously, we say that $g: \Omega \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow [0, +\infty)$ is stationary with respect to $(\tau_z)_{z \in \mathbb{Z}^n}$ if

$$g(\omega, x + z, v, w) = g(\tau_z(\omega), x, v, w)$$

for every $\omega \in \Omega, x \in \mathbb{R}^n, z \in \mathbb{Z}^n, v \in [0, 1]$ and $w \in \mathbb{R}^n$.

In all that follows we consider random integrands $f: \Omega \times \mathbb{R}^n \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$ satisfying

(F1) f is $(\mathcal{T} \otimes \mathcal{B}^n \otimes \mathcal{B}^{m \times n})$ -measurable;

(F2) $f(\omega, \cdot, \cdot) \in \mathcal{F}$ for every $\omega \in \Omega$;

and random integrands $g: \Omega \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow [0, +\infty)$ satisfying

(G1) g is $(\mathcal{T} \otimes \mathcal{B}^n \otimes \mathcal{B} \otimes \mathcal{B}^n)$ -measurable;

(G2) $g(\omega, \cdot, \cdot, \cdot, \cdot) \in \mathcal{G}$ for every $\omega \in \Omega$.

Let f and g be random integrands satisfying (F1)–(F2) and (G1)–(G2), respectively. We consider the sequence of random elliptic functionals $\mathcal{F}_k(\omega): L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ given by

$$\mathcal{F}_k(\omega)(u, v, A) := \int_A \psi(v) f\left(\omega, \frac{x}{\varepsilon_k}, \nabla u\right) dx + \frac{1}{\varepsilon_k} \int_A g\left(\omega, \frac{x}{\varepsilon_k}, v, \varepsilon_k \nabla v\right) dx, \tag{8.1}$$

if $(u, v) \in W^{1,p}(A; \mathbb{R}^m) \times W^{1,p}(A), 0 \leq v \leq 1$ and extended to $+\infty$ otherwise. We also let $\mathcal{F}^b(\omega)$ be as in (3.4) and $\mathcal{F}^s(\omega)$ as in (3.5), with $f(\cdot, \cdot)$ and $g(\cdot, \cdot, \cdot)$ replaced by $f(\omega, \cdot, \cdot)$ and $g(\omega, \cdot, \cdot, \cdot)$, respectively. Moreover, for $\omega \in \Omega$ and $A \in \mathcal{A}$ we set

$$\mathbf{m}_\omega^b(u_\xi, A) := \inf\{\mathcal{F}^b(\omega)(u, A) : u \in W^{1,p}(A; \mathbb{R}^m), u = u_\xi \text{ near } \partial A\} \tag{8.2}$$

and

$$\mathbf{m}_\omega^s(\bar{u}_x^v, A) := \inf\{\mathcal{F}^s(\omega)(v, A) : v \in \mathcal{A}(\bar{u}_x^v, A)\}, \tag{8.3}$$

where $\mathcal{A}(\bar{u}_x^v, A)$ is as in (2.17) with \bar{u}_x^v in place of $\bar{u}_{x,\varepsilon_k}^v$; i.e., with $\varepsilon_k = 1$.

Eventually, we extend the definition of $\mathbf{m}_\omega^b(u_\xi, \cdot), \mathbf{m}_\omega^s(\bar{u}_x^v, \cdot)$, and $\mathcal{A}(\bar{u}_0^v, \cdot)$ to any $A \subset \mathbb{R}^n$ with $\text{int } A \in \mathcal{A}$ by setting $\mathbf{m}_\omega^b(u_\xi, A) := \mathbf{m}_\omega^b(u_\xi, \text{int } A), \mathbf{m}_\omega^s(\bar{u}_x^v, A) := \mathbf{m}_\omega^s(\bar{u}_x^v, \text{int } A)$, and $\mathcal{A}(\bar{u}_0^v, A) := \mathcal{A}(\bar{u}_0^v, \text{int } A)$.

We are now ready to state the main result of this section.

Theorem 8.4 (Stochastic homogenisation) *Let f and g be random integrands satisfying (F1)–(F2) and (G1)–(G2), respectively. Assume moreover that f and g are stationary with respect to a group $(\tau_z)_{z \in \mathbb{Z}^n}$ of P -preserving transformations on (Ω, \mathcal{T}, P) . For every $\omega \in \Omega$ let $\mathcal{F}_k(\omega)$ be as in (8.1) and $\mathbf{m}_\omega^b, \mathbf{m}_\omega^s$ be as in (8.2) and (8.3), respectively. Then there exists $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$ such that for every $\omega \in \Omega', x \in \mathbb{R}^n, \xi \in \mathbb{R}^{m \times n}, v \in \mathbb{S}^{n-1}$ the limits*

$$\lim_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^b(u_\xi, Q_r(rx))}{r^n} = \lim_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^b(u_\xi, Q_r(0))}{r^n} =: f_{\text{hom}}(\omega, \xi), \tag{8.4}$$

$$\lim_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_{rx}^v, Q_r(rx))}{r^{n-1}} = \lim_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^v, Q_r(0))}{r^{n-1}} =: g_{\text{hom}}(\omega, v) \tag{8.5}$$

exist and are independent of x . The function $f_{\text{hom}}: \Omega \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$ is $(\mathcal{T} \otimes \mathcal{B}^{m \times n})$ -measurable and $g_{\text{hom}}: \Omega \times \mathbb{S}^{n-1} \rightarrow [0, +\infty)$ is $(\mathcal{T} \otimes \mathcal{B}(\mathbb{S}^{n-1}))$ -measurable.

Moreover, for every $\omega \in \Omega'$ and for every $A \in \mathcal{A}$ the functionals $\mathcal{F}_k(\omega)(\cdot, \cdot, A)$ Γ -converge in $L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n)$ to the functional $\mathcal{F}_{\text{hom}}(\omega)(\cdot, \cdot, A)$ with $\mathcal{F}_{\text{hom}}(\omega) : L^0(\mathbb{R}^n; \mathbb{R}^m) \times L^0(\mathbb{R}^n) \times \mathcal{A} \rightarrow [0, +\infty]$ given by

$$\mathcal{F}_{\text{hom}}(\omega)(u, v, A) := \begin{cases} \int_A f_{\text{hom}}(\omega, \nabla u) \, dx + \int_{S_u \cap A} g_{\text{hom}}(\omega, \nu_u) \, d\mathcal{H}^{n-1} & \text{if } u \in GSBVP(A; \mathbb{R}^m), \\ & v = 1 \text{ a.e. in } A, \\ +\infty & \text{otherwise.} \end{cases}$$

If, in addition, $(\tau_z)_{z \in \mathbb{Z}^n}$ is ergodic, then f_{hom} and g_{hom} are independent of ω and

$$f_{\text{hom}}(\xi) = \lim_{r \rightarrow +\infty} \frac{1}{r^n} \int_{\Omega} \mathbf{m}_{\omega}^b(u_{\xi}, Q_r(0)) \, dP(\omega),$$

$$g_{\text{hom}}(\nu) = \lim_{r \rightarrow +\infty} \frac{1}{r^{n-1}} \int_{\Omega} \mathbf{m}_{\omega}^s(\bar{u}_0^{\nu}, Q_r(0)) \, dP(\omega),$$

thus, \mathcal{F}_{hom} is deterministic.

The almost sure Γ -convergence result in Theorem 8.4 is an immediate consequence of Theorem 3.5 once we show the existence of a \mathcal{T} -measurable set $\Omega' \subset \Omega$, with $P(\Omega') = 1$, such that for every $\omega \in \Omega'$ the limits in (8.4)–(8.5) exist and are independent of x . Therefore, the rest of this section is devoted to prove the existence of such a set.

8.1 Homogenisation formulas

In this subsection we prove that conditions (F1), (F2), (G1), and (G2) together with the stationarity of the random integrands f and g ensure that the assumptions of Theorem 3.5 are satisfied almost surely.

The proof of the following result is based on the pointwise Subadditive Ergodic Theorem [1, Theorem 2.4] applied to the function $(\omega, I) \mapsto \mathbf{m}_{\omega}^b(u_{\xi}, I)$, which defines a subadditive process on $\Omega \times \mathcal{I}_n$, as shown in [47, Proposition 3.2].

Proposition 8.5 (Homogenised volume integrand) *Let f satisfy (F1)–(F2) and assume that it is stationary with respect to a group $(\tau_z)_{z \in \mathbb{Z}^n}$ of P -preserving transformations on (Ω, \mathcal{T}, P) . For $\omega \in \Omega$ let \mathbf{m}_{ω}^b be as in (8.2). Then there exists $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$ and a $(\mathcal{T} \otimes \mathcal{B}^{m \times n})$ -measurable function $f_{\text{hom}} : \Omega \times \mathbb{R}^{m \times n} \rightarrow [0, +\infty)$ such that*

$$\lim_{r \rightarrow +\infty} \frac{\mathbf{m}_{\omega}^b(u_{\xi}, Q_r(rx))}{r^n} = \lim_{r \rightarrow +\infty} \frac{\mathbf{m}_{\omega}^b(u_{\xi}, Q_r(0))}{r^n} = f_{\text{hom}}(\omega, \xi).$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^{m \times n}$. If, in addition, $(\tau_z)_{z \in \mathbb{Z}^n}$ is ergodic, then f_{hom} is independent of ω and given by

$$f_{\text{hom}}(\xi) = \lim_{r \rightarrow +\infty} \frac{1}{r^n} \int_{\Omega} \mathbf{m}_{\omega}^b(u_{\xi}, Q_r(0)) \, dP(\omega).$$

Proof The proof follows by [47, Proposition 3.2] arguing as in [35, Theorem 1] (see also [47, Corollary 3.3]). □

We now deal with the existence of the homogenised surface integrand g_{hom} . Unlike the case of f_{hom} , the existence and x -homogeneity of the limit in (8.5) cannot be deduced by a direct application of the Subadditive Ergodic Theorem [1, Theorem 2.4]. In fact, due to

the x -dependent boundary datum appearing in $\mathbf{m}_\omega^s(\bar{u}_{rx}^v, Q_r^v(rx))$ (cf. (8.3)), the proof of the x -homogeneity of g_{hom} is rather delicate and follows by *ad hoc* arguments, which are typical of surface functionals [3, 28].

The proof of the existence of g_{hom} will be carried out in several steps. In a first step we prove that when $x = 0$ the minimisation problem (8.3) defines a subadditive process on $\Omega \times \mathcal{I}_{n-1}$. To do so we follow the same procedure as in [28, Section 5] (see also [3, 22]). Namely, given $v \in \mathbb{S}^{n-1}$ we let R_v be an orthogonal matrix as in (f). Then $\{R_v e_i : i = 1, \dots, n-1\}$ is an orthonormal basis for Π^v , further $R_v \in \mathbb{Q}^{n \times n}$, if $v \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$. Now let $M_v > 2$ be an integer such that $M_v R_v \in \mathbb{Z}^{n \times n}$; therefore $M_v R_v(z', 0) \in \Pi^v \cap \mathbb{Z}^n$ for every $z' \in \mathbb{Z}^{n-1}$.

Let $I \in \mathcal{I}_{n-1}$; i.e., $I = [a, b]$ with $a, b \in \mathbb{Z}^{n-1}$. Starting from I we define the n -dimensional interval I_v as

$$I_v := M_v R_v(I \times [-c, c]) \quad \text{where} \quad c := \frac{1}{2} \max_{i=1, \dots, n-1} (b_i - a_i). \tag{8.6}$$

Correspondingly, for fixed $v \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ we define the function $\mu_v : \Omega \times \mathcal{I}_{n-1} \mapsto \mathbb{R}$ as

$$\mu_v(\omega, I) := \frac{1}{M_v^{n-1}} \mathbf{m}_\omega^s(\bar{u}_0^v, I_v), \tag{8.7}$$

where $\mathbf{m}_\omega^s(\bar{u}_0^v, I_v)$ is as in (8.3) with $x = 0$ and $A = I_v$.

The following result asserts that μ_v defines a subadditive process on $\Omega \times \mathcal{I}_{n-1}$.

Proposition 8.6 *Let g satisfy (G1)–(G2) and assume that it is stationary with respect to a group $(\tau_z)_{z \in \mathbb{Z}^n}$ of P -preserving transformations on (Ω, \mathcal{T}, P) . Let $v \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ and let $\mu_v : \Omega \times \mathcal{I}_{n-1} \mapsto \mathbb{R}$ be as in (8.7). Then there exists a group of P -preserving transformations $(\tau_{z'}^v)_{z' \in \mathbb{Z}^{n-1}}$ on (Ω, \mathcal{T}, P) such that μ_v is a subadditive process on (Ω, \mathcal{T}, P) with respect to $(\tau_{z'}^v)_{z' \in \mathbb{Z}^{n-1}}$. Moreover, it holds*

$$0 \leq \mu_v(\omega, I) \leq c_4 C_v \mathcal{L}^{n-1}(I), \tag{8.8}$$

for P -a.e. $\omega \in \Omega$ and for every $I \in \mathcal{I}_{n-1}$.

Proof Let $v \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ be fixed; below we show that μ_v satisfies conditions (1)–(4) in Definition 8.2, for some group of P -preserving transformations $(\tau_{z'}^v)_{z' \in \mathbb{Z}^{n-1}}$.

We divide the proof into four steps, each of them corresponding to one of the four conditions in Definition 8.2.

Step 1: measurability. Let $I \in \mathcal{I}_{n-1}$ and let $I_v \subset \mathbb{R}^n$ be as in (8.6). Let $v \in W^{1,p}(I_v)$ be fixed. In view of (G1) the Fubini Theorem ensures that the map $\omega \mapsto \mathcal{F}^s(\omega)(v, I_v)$ is \mathcal{T} -measurable. On the other hand the space $W^{1,p}(I_v)$ is separable, hence the set of functions

$$\begin{aligned} \mathcal{A}(\bar{u}_0^v, I_v) &= \{v \in W^{1,p}(I_v), 0 \leq v \leq 1, v = \bar{v}_0^v \text{ near } \partial I_v \text{ and } \exists u \in W^{1,p}(I_v; \mathbb{R}^m), \\ &u = \bar{u}_0^v \text{ near } \partial I_v, \text{ such that } v \nabla u = 0 \text{ a.e. in } I_v\} \end{aligned}$$

defines a separable metric space, when endowed with the distance induced by the $W^{1,p}(I_v)$ -norm. Then, by the continuity of $\mathcal{F}^s(\omega)(\cdot, I_v)$ with respect to the strong $W^{1,p}(I_v)$ -topology, the infimum in the definition of $\mathbf{m}_\omega^s(\bar{u}_0^v, I_v)$ can be equivalently expressed as an infimum on a countable subset of $\mathcal{A}(\bar{u}_0^v, I_v)$, thus ensuring the \mathcal{T} -measurability of $\omega \mapsto \mathbf{m}_\omega^s(\bar{u}_0^v, I_v)$ and consequently that of $\omega \mapsto \mu_v(\omega, I)$, as desired.

Step 2: covariance. Let $z' \in \mathbb{Z}^{n-1}$ be fixed. Let $I \in \mathcal{I}_{n-1}$; by (8.6) we have

$$(I + z')_v = I_v + M_v R_v(z', 0) = I_v + z'_v,$$

where $z'_v := M_v R_v(z', 0) \in \mathbb{Z}^n \cap \Pi^v$. Then, by (8.7) we get

$$\mu_v(\omega, I + z') = \frac{1}{M_v^{n-1}} \mathbf{m}_\omega^s(\bar{u}_0^v, I_v + z'_v). \tag{8.9}$$

Now let $v \in \mathcal{A}(\bar{u}_0^v, I_v + z'_v)$ with its corresponding $u \in W^{1,p}(I'_v + z'_v; \mathbb{R}^m)$. Setting $\tilde{v}(x) := v(x + z'_v)$ for $x \in I_v$, a change of variables together with the stationarity of g yield

$$\begin{aligned} \mathcal{F}^s(\omega)(v, \text{int}(I_v + z'_v)) &= \int_{I_v + z'_v} g(\omega, x, v, \nabla v) \, dx = \int_{I_v} g(\omega, x + z'_v, \tilde{v}, \nabla \tilde{v}) \, dx \\ &= \int_{I_v} g(\tau_{z'_v}(\omega), x, \tilde{v}, \nabla \tilde{v}) \, dx = \mathcal{F}^s(\tau_{z'_v}(\omega))(\tilde{v}, \text{int } I_v). \end{aligned}$$

Set $(\tau_{z'_v}^v)_{z' \in \mathbb{Z}^{n-1}} := (\tau_{z'_v})_{z' \in \mathbb{Z}^{n-1}}$; we notice that $(\tau_{z'_v}^v)_{z' \in \mathbb{Z}^{n-1}}$ is well defined since $z'_v \in \mathbb{Z}^n$ and it defines a group of P -preserving transformations on (Ω, P, \mathcal{T}) . Then, the equality above can be rewritten as

$$\mathcal{F}^s(\omega)(v, \text{int}(I_v + z'_v)) = \mathcal{F}^s(\tau_{z'_v}^v(\omega))(\tilde{v}, \text{int } I_v). \tag{8.10}$$

Moreover, if we set $\tilde{u}(x) := u(x + z'_v)$ for $x \in I_v$, then $\tilde{v} \nabla \tilde{u} = 0$ a.e. in I_v , while since $z'_v \in \Pi^v$, we also have

$$(\tilde{u}, \tilde{v}) = (\bar{u}_0^v(\cdot + z'_v), \bar{v}_0^v(\cdot + z'_v)) = (\bar{u}_0^v, \bar{v}_0^v)$$

near ∂I_v ; i.e., $\tilde{v} \in \mathcal{A}(\bar{u}_0^v, I_v)$. Thus, gathering (8.9) and (8.10), by the arbitrariness of v we infer

$$\mu_v(\omega, I + z') = \mu_v(\tau_{z'_v}^v(\omega), I),$$

and hence the covariance of μ_v with respect to $(\tau_{z'_v}^v)_{z' \in \mathbb{Z}^{n-1}}$.

Step 3: subadditivity. Let $\omega \in \Omega$, $I \in \mathcal{I}_{n-1}$, and let $\{I_1, \dots, I_N\} \subset \mathcal{I}_{n-1}$ be a finite family of pairwise disjoint sets such that $I = \bigcup_i I_i$. Let $\eta > 0$ be fixed; for every $i = 1, \dots, N$ let $v_i \in W^{1,p}((I_i)_v)$ be admissible for $\mathbf{m}_\omega^s(\bar{u}_0^v, (I_i)_v)$ and such that

$$\mathcal{F}^s(\omega)(v_i, \text{int}(I_i)_v) \leq \mathbf{m}_\omega^s(\bar{u}_0^v, (I_i)_v) + \eta. \tag{8.11}$$

Therefore for every $i = 1, \dots, N$ there exists a corresponding $u_i \in W^{1,p}((I_i)_v; \mathbb{R}^m)$ such that

$$v_i \nabla u_i = 0 \text{ a.e. in } (I_i)_v \text{ with } (u_i, v_i) = (\bar{u}_0^v, \bar{v}_0^v) \text{ near } \partial(I_i)_v.$$

We define

$$v := \begin{cases} v_i & \text{in } (I_i)_v, \ i = 1, \dots, N, \\ \bar{v}_0^v & \text{in } I_v \setminus \bigcup_i (I_i)_v, \end{cases} \quad u := \begin{cases} u_i & \text{in } (I_i)_v, \ i = 1, \dots, N, \\ \bar{u}_0^v & \text{in } I_v \setminus \bigcup_i (I_i)_v, \end{cases}$$

so that $(u, v) \in W^{1,p}(I_v) \times W^{1,p}(I_v; \mathbb{R}^m)$ and

$$v \nabla u = 0 \text{ a.e. in } I_v \text{ with } (u, v) = (\bar{u}_0^v, \bar{v}_0^v) \text{ near } \partial I_v.$$

Therefore $v \in \mathcal{A}(\bar{u}_0^v, I_v)$. Furthermore, we have

$$\mathcal{F}^s(\omega)(v, \text{int } I_v) = \sum_{i=1}^N \mathcal{F}^s(\omega)(v_i, \text{int}(I_i)_v) + \mathcal{F}^s(\omega)(\bar{v}_0^v, \text{int}(I_v \setminus \bigcup_{i=1}^N (I_i)_v)). \tag{8.12}$$

Since $M_\nu > 2$ and $c \geq \frac{1}{2}$ in (8.6), it follows that $\{y \in I_\nu : |y \cdot \nu| \leq 1\} \subset \bigcup_i (I_i)_\nu$. Thus, $\bar{v}_0^\nu \equiv 1$ in $I_\nu \setminus \bigcup_i (I_i)_\nu$ which, thanks to (2.3), gives

$$\mathcal{F}^s(\omega)(\bar{v}_0^\nu, \text{int}(I_\nu \setminus \bigcup_{i=1}^N (I_i)_\nu)) = 0.$$

Eventually, gathering (8.11)–(8.12) we obtain

$$\mathbf{m}_\omega^s(\bar{u}_0^\nu, I_\nu) \leq \mathcal{F}^s(\omega)(v, \text{int } I_\nu) = \sum_{i=1}^N \mathcal{F}^s(\omega)(v_i, \text{int}(I_i)_\nu) \leq \sum_{i=1}^N \mathbf{m}_\omega^s(\bar{u}_0^\nu, (I_i)_\nu) + N\eta,$$

thus the subadditivity of μ_ν follows from (8.7) and the arbitrariness of $\eta > 0$.

Step 4: boundedness. Let $\omega \in \Omega$ and $I \in \mathcal{I}_{n-1}$. Then (8.6) and (2.19) readily imply

$$0 \leq \mu_\nu(\omega, I) = \frac{1}{M_\nu^{n-1}} \mathbf{m}_\omega^s(\bar{u}_0^\nu, I_\nu) \leq c_4 C_\nu \mathcal{L}^{n-1}(I).$$

□

Having at hand Proposition 8.6, with the help of Lemma A.1 and Lemma A.2 we now prove the following result, which establishes the almost sure existence of the limit defining g_{hom} when $x = 0$.

Proposition 8.7 (Homogenised surface integrand for $x = 0$) *Let g satisfy (G1)–(G2) and assume that it is stationary with respect to a group $(\tau_z)_{z \in \mathbb{Z}^n}$ of P -preserving transformations on (Ω, \mathcal{T}, P) . For $\omega \in \Omega$ let \mathbf{m}_ω^s be as in (8.3). Then there exist $\tilde{\Omega} \in \mathcal{T}$ with $P(\tilde{\Omega}) = 1$ and a $(\mathcal{T} \otimes \mathcal{B}(\mathbb{S}^{n-1}))$ -measurable function $g_{\text{hom}} : \Omega \times \mathbb{S}^{n-1} \rightarrow [0, +\infty)$ such that*

$$\lim_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}} = g_{\text{hom}}(\omega, \nu) \tag{8.13}$$

for every $\omega \in \tilde{\Omega}$ and every $\nu \in \mathbb{S}^{n-1}$. Moreover, $\tilde{\Omega}$ and g_{hom} are $(\tau_z)_{z \in \mathbb{Z}^n}$ -translation invariant; i.e., $\tau_z(\tilde{\Omega}) = \tilde{\Omega}$ for every $z \in \mathbb{Z}^n$ and

$$g_{\text{hom}}(\tau_z(\omega), \nu) = g_{\text{hom}}(\omega, \nu), \tag{8.14}$$

for every $z \in \mathbb{Z}^n$, for every $\omega \in \tilde{\Omega}$, and every $\nu \in \mathbb{S}^{n-1}$. Therefore, if $(\tau_z)_{z \in \mathbb{Z}^n}$ is ergodic then g_{hom} is independent of ω and given by

$$g_{\text{hom}}(\nu) = \lim_{r \rightarrow +\infty} \frac{1}{r^{n-1}} \int_{\tilde{\Omega}} \mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_r^\nu(0)) \, dP(\omega). \tag{8.15}$$

Proof We divide the proof into three steps.

Step 1: existence of the limit for $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$. Let $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$ be fixed. Thanks to Proposition 8.6 we can apply the Subadditive Ergodic Theorem [1, Theorem 2.4] to the subadditive process μ_ν defined on (Ω, \mathcal{T}, P) by (8.7). Then choosing $I = [-1, 1]^{n-1}$ we get the existence of a set $\Omega_\nu \in \mathcal{T}$, with $P(\Omega_\nu) = 1$, and of a \mathcal{T} -measurable function $g_\nu : \Omega \rightarrow [0, +\infty)$ such that

$$\lim_{j \rightarrow +\infty} \frac{\mu_\nu(\omega, jI)}{(2j)^{n-1}} = g_\nu(\omega),$$

for every $\omega \in \Omega_\nu$. By (8.7), since $I_\nu = 2M_\nu Q^\nu(0)$ this yields

$$g_\nu(\omega) = \lim_{j \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, j2M_\nu Q^\nu(0))}{(j2M_\nu)^{n-1}}, \tag{8.16}$$

for every $\omega \in \Omega_\nu$. Let (r_j) be a sequence of strictly positive real numbers with $r_j \rightarrow +\infty$, as $j \rightarrow +\infty$, and consider the two sequences of integers defined as follows:

$$r_j^- := 2M_\nu \left(\left\lfloor \frac{r_j}{2M_\nu} \right\rfloor - 1 \right) \quad \text{and} \quad r_j^+ := 2M_\nu \left(\left\lfloor \frac{r_j}{2M_\nu} \right\rfloor + 2 \right).$$

Let $j \in N$ be such that $r_j > 4(1 + M_\nu)$, and thus $r_j^- > 4$. We clearly have

$$Q_{r_j^-+2}^\nu(0) \subset\subset Q_{r_j}^\nu(0) \subset\subset Q_{r_j+2}^\nu(0) \subset\subset Q_{r_j^+}^\nu(0),$$

hence we can apply Lemma A.1 with $x = \tilde{x} = 0$ first choosing $r = r_j^-$ and $\tilde{r} = r_j$ to get

$$\frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_{r_j^-}^\nu(0))}{r_j^{n-1}} \leq \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_{r_j^-}^\nu(0))}{(r_j^-)^{n-1}} + \frac{L(r_j - r_j^- + 1)}{r_j}, \tag{8.17}$$

and then $r = r_j$ and $\tilde{r} = r_j^+$ to obtain

$$\frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_{r_j}^\nu(0))}{r_j^{n-1}} \geq \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_{r_j^+}^\nu(0))}{(r_j^+)^{n-1}} - \frac{L(r_j^+ - r_j + 1)}{r_j}. \tag{8.18}$$

Clearly, $r_j^+ - r_j \leq 4M_\nu$ and $r_j - r_j^- \leq 4M_\nu$, thus, if $\omega \in \Omega_\nu$, thanks to (8.16), passing to the limsup in as $j \rightarrow +\infty$ in (8.17) gives

$$\limsup_{j \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_{r_j}^\nu(0))}{r_j^{n-1}} \leq g_\nu(\omega), \tag{8.19}$$

while passing to the liminf in as $j \rightarrow +\infty$ in (8.18) yields

$$\liminf_{j \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_{r_j}^\nu(0))}{r_j^{n-1}} \geq g_\nu(\omega). \tag{8.20}$$

Hence gathering (8.19) and (8.20) gives for every $\omega \in \Omega_\nu$ the existence of the limit along (r_j) together with the equality

$$\lim_{j \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_{r_j}^\nu(0))}{r_j^{n-1}} = g_\nu(\omega).$$

Therefore setting

$$\tilde{\Omega} := \bigcap_{\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n} \Omega_\nu,$$

clearly gives $\tilde{\Omega} \in \mathcal{T}$ as well as $P(\tilde{\Omega}) = 1$; moreover we have

$$g_\nu(\omega) = \lim_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}$$

for every $\omega \in \tilde{\Omega}$ and every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$.

Step 2: existence of the limit for $\nu \in \mathbb{S}^{n-1} \setminus \mathbb{Q}^n$. Consider the two functions $\underline{g}, \bar{g}: \tilde{\Omega} \times \mathbb{S}^{n-1} \rightarrow [0, +\infty]$ defined as

$$\underline{g}(\omega, \nu) := \liminf_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}, \quad \bar{g}(\omega, \nu) := \limsup_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\bar{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}.$$

By the previous step we have that $\underline{g}(\omega, \nu) = \bar{g}(\omega, \nu) = g_\nu(\omega)$, for every $\omega \in \tilde{\Omega}$ and for every $\nu \in \mathbb{S}^{n-1} \cap \mathbb{Q}^n$. Therefore, if we show that the restrictions of the functions $\nu \mapsto \underline{g}(\omega, \nu)$ and $\nu \mapsto \bar{g}(\omega, \nu)$ to the sets $\widehat{\mathbb{S}}_{\pm}^{n-1}$ are continuous, by the density of $\widehat{\mathbb{S}}_{\pm}^{n-1} \cap \mathbb{Q}^n$ in $\widehat{\mathbb{S}}_{\pm}^{n-1}$ we can readily deduce that $\underline{g}(\omega, \nu) = \bar{g}(\omega, \nu) = g_\nu(\omega)$ for every $\omega \in \tilde{\Omega}$ and every $\nu \in \mathbb{S}^{n-1}$, and thus the claim.

Then it remains to show that $\underline{g}(\omega, \cdot)$ and $\bar{g}(\omega, \cdot)$ are continuous in $\widehat{\mathbb{S}}_{\pm}^{n-1}$. We only prove that $\bar{g}(\omega, \cdot)$ is continuous in $\widehat{\mathbb{S}}_+^{n-1}$, the other proofs being analogous. To this end, let $\nu \in \widehat{\mathbb{S}}_+^{n-1}$, $(\nu_j) \subset \widehat{\mathbb{S}}_+^{n-1}$ be such that $\nu_j \rightarrow \nu$, as $j \rightarrow +\infty$. Then, for every $\alpha \in (0, \frac{1}{2})$ there exists $j_\alpha \in \mathbb{N}$ such that the assumptions of Lemma A.2 are satisfied for every $j \geq j_\alpha$. Thus, choosing $x = 0$ and $\tilde{\nu} = \nu_j$ in (A.7) we obtain

$$\mathbf{m}_\omega^s(\tilde{u}_0^{\nu_j}, Q_{(1+\alpha)r}^{\nu_j}(0)) - c_\alpha r^{n-1} \leq \mathbf{m}_\omega^s(\tilde{u}_0^\nu, Q_r^\nu(0)) \leq \mathbf{m}_\omega^s(\tilde{u}_0^{\nu_j}, Q_{(1-\alpha)r}^{\nu_j}(0)) + c_\alpha r^{n-1},$$

where $c_\alpha \rightarrow 0$, as $\alpha \rightarrow 0$. Therefore, by definition of \bar{g} passing to the limsup as $r \rightarrow +\infty$ we get the two following inequalities

$$(1 + \alpha)^{n-1} \bar{g}(\omega, \nu_j) \leq \bar{g}(\omega, \nu) + c_\alpha, \tag{8.21}$$

$$(1 - \alpha)^{n-1} \bar{g}(\omega, \nu_j) \geq \bar{g}(\omega, \nu) - c_\alpha. \tag{8.22}$$

Passing to the limsup as $j \rightarrow +\infty$ in (8.21) and to the liminf as $j \rightarrow +\infty$ in (8.22) and eventually letting $\alpha \rightarrow 0$ we infer

$$\limsup_{j \rightarrow +\infty} \bar{g}(\omega, \nu_j) \leq \bar{g}(\omega, \nu) \leq \liminf_{j \rightarrow +\infty} \bar{g}(\omega, \nu_j),$$

and hence the claim.

For every $\omega \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$ set

$$g_{\text{hom}}(\omega, \nu) := \begin{cases} \bar{g}(\omega, \nu) & \text{if } \omega \in \tilde{\Omega}, \\ c_2 c_P & \text{if } \omega \in \Omega \setminus \tilde{\Omega}, \end{cases} \tag{8.23}$$

then, for every $\omega \in \tilde{\Omega}$ and every $\nu \in \mathbb{S}^{n-1}$ we have

$$g_{\text{hom}}(\omega, \nu) = \lim_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\tilde{u}_0^\nu, Q_r^\nu(0))}{r^{n-1}}. \tag{8.24}$$

Moreover

$$\omega \mapsto \bar{g}(\omega, \nu) \text{ is } \mathcal{T}\text{-measurable in } \tilde{\Omega}, \text{ for every } \nu \in \mathbb{S}^{n-1}$$

and

$$\nu \mapsto \bar{g}(\omega, \nu) \text{ is continuous in } \widehat{\mathbb{S}}_{\pm}^{n-1}, \text{ for every } \omega \in \tilde{\Omega}.$$

Therefore the restriction of \bar{g} to $\tilde{\Omega} \times \widehat{\mathbb{S}}_{\pm}^{n-1}$ is measurable with respect to the σ -algebra induced in $\tilde{\Omega} \times \widehat{\mathbb{S}}_{\pm}^{n-1}$ by $\mathcal{T} \otimes \mathcal{B}(\mathbb{S}^{n-1})$ thus, finally, g_{hom} is $(\mathcal{T} \otimes \mathcal{B}(\mathbb{S}^{n-1}))$ -measurable on $\Omega \times \mathbb{S}^{n-1}$.

Step 3: $(\tau_z)_{z \in \mathbb{Z}^n}$ -translation invariance. Let $z \in \mathbb{Z}^n$, $\omega \in \tilde{\Omega}$, and $\nu \in \mathbb{S}^{n-1}$ be fixed. Let $r > 4$ and $\nu \in \mathcal{A}(\tilde{u}_0^\nu, Q_r^\nu(0))$ with

$$\mathcal{F}^s(\omega)(\nu, Q_r^\nu(0)) \leq \mathbf{m}_\omega^s(\tilde{u}_0^\nu, Q_r^\nu(0)) + 1. \tag{8.25}$$

Set $\tilde{\nu}(y) := \nu(y + z)$; by the stationarity of g we have

$$\mathcal{F}^s(\omega)(\nu, Q_r^\nu(0)) = \mathcal{F}^s(\tau_z(\omega))(\tilde{\nu}, Q_r^\nu(-z)),$$

further, since $\tilde{v} \in \mathcal{A}(\bar{u}_{-z}^v, Q_r^v(-z))$, by (8.25) we immediately get

$$\mathbf{m}_{\tau_z(\omega)}^s(\bar{u}_{-z}^v, Q_r^v(-z)) \leq \mathbf{m}_\omega^s(\bar{u}_0^v, Q_r^v(0)) + 1. \tag{8.26}$$

Now let r, \tilde{r} be such that $\tilde{r} > r$ and

$$Q_{r+2}^v(-z) \subset\subset Q_{\tilde{r}}^v(0) \text{ and } \text{dist}(0, \Pi^v(-z)) \leq \frac{r}{4},$$

therefore applying Lemma A.1 with $x = -z$ and $\tilde{x} = 0$ we may deduce the existence of a constant $L > 0$ such that

$$\mathbf{m}_{\tau_z(\omega)}^s(\bar{u}_0^v, Q_{\tilde{r}}^v(0)) \leq \mathbf{m}_{\tau_z(\omega)}^s(\bar{u}_{-z}^v, Q_r^v(-z)) + L(|z| + |r - \tilde{r}| + 1)(\tilde{r})^{n-2}. \tag{8.27}$$

Hence, gathering (8.26) and (8.27) we obtain

$$\frac{\mathbf{m}_{\tau_z(\omega)}^s(\bar{u}_0^v, Q_{\tilde{r}}^v(0))}{\tilde{r}^{n-1}} \leq \frac{\mathbf{m}_\omega^s(\bar{u}_0^v, Q_r^v(0)) + 1}{r^{n-1}} + \frac{L(|z| + |r - \tilde{r}| + 1)}{\tilde{r}}. \tag{8.28}$$

An analogous argument, now replacing ω with $\tau_z(\omega)$ and z with $-z$, yields

$$\frac{\mathbf{m}_\omega^s(\bar{u}_0^v, Q_{\tilde{r}}^v(0))}{\tilde{r}^{n-1}} \leq \frac{\mathbf{m}_{\tau_z(\omega)}^s(\bar{u}_0^v, Q_r^v(0)) + 1}{r^{n-1}} + \frac{L(|z| + |r - \tilde{r}| + 1)}{\tilde{r}}. \tag{8.29}$$

Taking in (8.28) the limsup as $\tilde{r} \rightarrow +\infty$ and the limit as $r \rightarrow +\infty$ gives

$$\limsup_{\tilde{r} \rightarrow +\infty} \frac{\mathbf{m}_{\tau_z(\omega)}^s(\bar{u}_0^v, Q_{\tilde{r}}^v(0))}{\tilde{r}^{n-1}} \leq g_{\text{hom}}(\omega, \nu), \tag{8.30}$$

while taking in (8.29) the limit as $\tilde{r} \rightarrow +\infty$ and the liminf as $r \rightarrow +\infty$ entails

$$g_{\text{hom}}(\omega, \nu) \leq \liminf_{r \rightarrow +\infty} \frac{\mathbf{m}_{\tau_z(\omega)}^s(\bar{u}_0^v, Q_r^v(0))}{r^{n-1}}. \tag{8.31}$$

Eventually, by combining (8.30) and (8.31) we both deduce that $\tau_z(\omega) \in \tilde{\Omega}$ and that

$$g_{\text{hom}}(\tau_z(\omega), \nu) = g_{\text{hom}}(\omega, \nu),$$

for every $z \in \mathbb{Z}$, $\omega \in \tilde{\Omega}$, and $\nu \in \mathbb{S}^{n-1}$. Then, we observe that thanks to the group properties of $(\tau_z)_{z \in \mathbb{Z}^n}$ we also have that $\omega \in \tau_z(\tilde{\Omega})$, for every $z \in \mathbb{Z}^n$. Indeed we have $\omega = \tau_z(\tau_{-z}(\omega))$ and $\tau_{-z}(\omega) \in \tilde{\Omega}$.

If $(\tau_z)_{z \in \mathbb{Z}^n}$ is ergodic, then the independence of ω of the function of g_{hom} is a direct consequence of (8.14) (cf. [28, Corollary 6.3]). Furthermore, (8.15) can be obtained by integrating (8.24) on Ω and using the Dominated Convergence Theorem, thanks to (8.8) (see also (8.7)). \square

The following result is of crucial importance in our analysis as it extends Proposition 8.7 to the case of an arbitrary $x \in \mathbb{R}^n$. More precisely, Proposition 8.8 below establishes the existence of the limit in (8.13) when $x = 0$ is replaced by any $x \in \mathbb{R}^n$; moreover it shows that this limit is x -independent, and hence it coincides with (8.13).

The proof of the following proposition can be obtained arguing exactly as in [28, Theorem 6.1] (see also [3, Theorem 5.5]), now appealing to Proposition 8.7, Lemma A.1, and Lemma A.2. For this reason we omit its proof here.

Proposition 8.8 (Homogenised surface integrand) *Let g satisfy (G1)–(G2) and assume that it is stationary with respect to a group $(\tau_z)_{z \in \mathbb{Z}^n}$ of P -preserving transformations on (Ω, \mathcal{T}, P) . For $\omega \in \Omega$ let \mathbf{m}_ω^s be as in (8.3). Then there exists $\Omega' \in \mathcal{T}$ with $P(\Omega') = 1$ such that*

$$\lim_{r \rightarrow +\infty} \frac{\mathbf{m}_\omega^s(\tilde{u}_{rx}^v, Q_r^v(rx))}{r^{n-1}} = g_{\text{hom}}(\omega, v) \tag{8.32}$$

for every $\omega \in \Omega'$, every $x \in \mathbb{R}^n$, and every $v \in \mathbb{S}^{n-1}$, where g_{hom} is given by (8.13). In particular, the limit in (8.32) is independent of x . Moreover, if $(\tau_z)_{z \in \mathbb{Z}^n}$ is ergodic, then g_{hom} is independent of ω and given by (8.15).

We conclude this section with the proof of Theorem 8.4.

Proof of Theorem 8.4 The proof follows by Theorem 3.5 now invoking Proposition 8.5 and Proposition 8.8. □

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Declarations

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Appendix

In this last section we state and prove two technical lemmas which are used in Sect. 8.1.

For $A \in \mathcal{A}$, $x \in \mathbb{R}^n$, and $v \in \mathbb{S}^{n-1}$, in what follows $\mathbf{m}^s(\tilde{u}_x^v, A)$ denotes the infimum value given by (3.7).

Lemma A.1 *Let $g \in \mathcal{G}$; let $v \in \mathbb{S}^{n-1}$, $x, \tilde{x} \in \mathbb{R}^n$, and $\tilde{r} > r > 4$ be such that*

$$(i) \quad Q_{r+2}^v(x) \subset\subset Q_{\tilde{r}}^v(\tilde{x}), \quad (ii) \quad \text{dist}(\tilde{x}, \Pi^v(x)) \leq \frac{r}{4}.$$

Then there exists a constant $L > 0$ (independent of $v, x, \tilde{x}, r, \tilde{r}$) such that

$$\mathbf{m}^s(\tilde{u}_{\tilde{x}}^v, Q_{\tilde{r}}^v(\tilde{x})) \leq \mathbf{m}^s(\tilde{u}_x^v, Q_r^v(x)) + L(|x - \tilde{x}| + |r - \tilde{r}| + 1)\tilde{r}^{n-2}. \tag{A.1}$$

Proof To prove (A.1) we are going to perform a construction which is similar to that used in Proposition 2.6.

Let $v \in S^{n-1}$, $\eta > 0$ be fixed and let $v \in \mathcal{A}(\bar{u}_x^v, Q_r^v(x))$ with

$$\mathcal{F}^s(v, Q_r^v(x)) \leq \mathbf{m}^s(\bar{u}_x^v, Q_r^v(x)) + \eta. \tag{A.2}$$

Let $u \in W^{1,p}(Q_r^v(x); \mathbb{R}^m)$ correspond to the v as above; i.e., $v \nabla u = 0$ a.e. in $Q_r^v(x)$ and $(u, v) = (\bar{u}_x^v, \bar{v}_x^v)$ a.e. in U , where U is a neighbourhood of $\partial Q_r^v(x)$. Let moreover $\beta \in (0, 1)$ be such that $Q_r^v(x) \setminus \bar{Q}_{r-\beta}^v(x) \subset U$. Let $\tilde{x} \in \mathbb{R}^n$ and $\tilde{r} > r > 4$ satisfy (i)–(ii). We set

$$R := R_v \left((Q_r^v \setminus \bar{Q}_{r-\beta}^v) \times \left(-1 - \frac{|(x - \tilde{x}) \cdot v|}{2}, 1 + \frac{|(x - \tilde{x}) \cdot v|}{2} \right) \right) + x + \frac{(\tilde{x} - x) \cdot v}{2} v,$$

where R_v be as in (f) (see Fig. 3a). Then let $\varphi \in C_c^\infty(Q_r^v)$ be a cut-off function between $Q_{r-\beta}^v$ and Q_r^v ; i.e., $0 \leq \varphi \leq 1$, and $\varphi \equiv 1$ on $Q_{r-\beta}^v$. Eventually, for $y = (y', y_n) \in Q_{\tilde{r}}^v(\tilde{x})$ we define the pair (\tilde{u}, \tilde{v}) by setting

$$\begin{aligned} \tilde{u}(y) &:= \varphi((R_v^T(y-x))')u(y) + (1 - \varphi((R_v^T(y-x))'))\bar{u}_x^v(y), \\ \tilde{v}(y) &:= \begin{cases} \min\{v(y), d(y)\} & \text{in } Q_r^v(x), \\ \min\{\bar{v}_x^v(y), d(y)\} & \text{in } Q_{\tilde{r}}^v(\tilde{x}) \setminus \bar{Q}_r^v(x), \end{cases} \end{aligned}$$

where $d(y) := \text{dist}(y, R)$. Clearly, $\tilde{u} \in W^{1,p}(Q_{\tilde{r}}^v(\tilde{x}); \mathbb{R}^m)$, moreover, by construction we have

$$|(y-x) \cdot v| > 1 \text{ and } |(y-\tilde{x}) \cdot v| > 1 \text{ for every } y \in (Q_r^v(x) \setminus \bar{Q}_{r-\beta}^v(x)) \setminus \bar{R}.$$

Hence, the boundary conditions satisfied by v imply that

$$v = \bar{v}_x^v = \bar{v}_x^v = 1 \text{ in } (Q_r^v(x) \setminus \bar{Q}_{r-\beta}^v(x)) \setminus \bar{R},$$

thus $\tilde{v} \in W^{1,p}(Q_{\tilde{r}}^v(\tilde{x}))$.

Thanks to (ii) and to the fact that $r > 4$ we have $R \subset Q_r^v(x)$ so that $\{d < 1\} \subset Q_{r+2}^v(x)$. Therefore, in view of (i) we get that $\tilde{v} = \bar{v}_x^v$ in a neighbourhood of $\partial Q_{\tilde{r}}^v(\tilde{x})$, further $\tilde{u} = \bar{u}_x^v$ in $Q_{\tilde{r}}^v(\tilde{x}) \setminus \bar{Q}_r^v(x)$. Then, to show that \tilde{v} is admissible for $\mathbf{m}^s(\bar{u}_x^v, Q_{\tilde{r}}^v(\tilde{x}))$ it only remains to check that $\tilde{v} \nabla \tilde{u} = 0$ a.e. in $Q_{\tilde{r}}^v(\tilde{x})$. Clearly, $\tilde{v} \nabla \tilde{u} = 0$ a.e. in R . On the other hand, in $Q_r^v(x) \setminus \bar{R}$ we have $\tilde{v} |\nabla \tilde{u}| \leq v |\nabla u| = 0$, while in $(Q_{\tilde{r}}^v(\tilde{x}) \setminus Q_r^v(x)) \setminus \bar{R}$ we get $\tilde{v} |\nabla \tilde{u}| \leq \bar{v}_x^v |\nabla \bar{u}_x^v| = 0$, and hence the claim.

Eventually, arguing as in Proposition 2.6 we obtain

$$\mathcal{F}^s(\tilde{v}, Q_{\tilde{r}}^v(\tilde{x})) \leq \mathcal{F}^s(v, Q_r^v(x)) + \mathcal{F}^s(\bar{v}_x^v, Q_{\tilde{r}}^v(\tilde{x}) \setminus \bar{Q}_r^v(x)) + 2c_4 \mathcal{L}^n(\{d < 1\}). \tag{A.3}$$

Thanks to (2.3) and (2.19) we have

$$\mathcal{F}^s(\bar{v}_x^v, Q_{\tilde{r}}^v(\tilde{x}) \setminus \bar{Q}_r^v(x)) \leq c_4 C_v \mathcal{L}^{n-1}(Q_r^v \setminus \bar{Q}_r^v) \leq C|r - \tilde{r}|r^{n-2}, \tag{A.4}$$

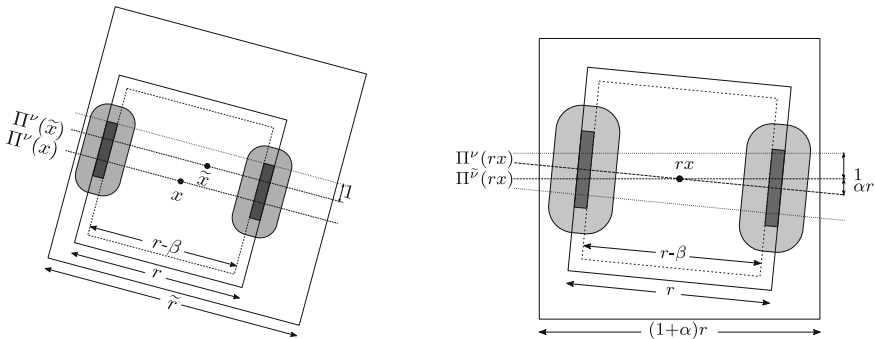
moreover

$$\mathcal{L}^n(\{d < 1\}) \leq C(|x - \tilde{x}| + 1)(\beta + 1)r^{n-2}. \tag{A.5}$$

We observe that both in (A.4) and (A.5) the positive constant $C > 0$ does not depend on any of the parameters and on v . Therefore, gathering (A.2) and (A.3) we finally obtain

$$\mathbf{m}^s(\bar{u}_x^v, Q_{\tilde{r}}^v(\tilde{x})) \leq \mathcal{F}^s(\tilde{v}, Q_{\tilde{r}}^v(\tilde{x})) \leq \mathbf{m}^s(\bar{u}_x^v, Q_r^v(x)) + L(|x - \tilde{x}| + |r - \tilde{r}| + 1)\tilde{r}^{n-2} + \eta,$$

for some $L > 0$ independent of $x, \tilde{x}, r, \tilde{r}, v$, thus (A.1) follows by the arbitrariness of $\eta > 0$. \square



(a) The sets $Q_{r-\beta}^\nu(x)$, $Q_r^\nu(x)$, $Q_r^\nu(\tilde{x})$ and in gray the sets R (dark gray) and $\{d < 1\}$ (light gray).

(b) The sets $Q_{r-\beta}^\nu(rx)$, $Q_r^\nu(rx)$, $Q_{(1+\alpha)r}^\nu(rx)$ and in gray the sets R (dark gray) and $\{d < 1\}$ (light gray).

Fig. 3 The sets used in the construction of (\tilde{u}, \tilde{v}) in Lemma A.1 and Lemma A.2

Lemma A.2 Let $g \in \mathcal{G}$; let $\alpha \in (0, \frac{1}{2})$ and $v, \tilde{v} \in \mathbb{S}^{n-1}$ be such that

$$\max_{1 \leq i \leq n-1} |R_\nu e_i - R_{\tilde{\nu}} e_i| + |v - \tilde{v}| < \frac{\alpha}{\sqrt{n}}, \tag{A.6}$$

where R_ν and $R_{\tilde{\nu}}$ are orthogonal $(n \times n)$ -matrices as in (f). Then there exists a constant $c_\alpha > 0$ (independent of v, \tilde{v}), with $c_\alpha \rightarrow 0$ as $\alpha \rightarrow 0$, such that for every $x \in \mathbb{R}^n$ and every $r > 2$ we have

$$\begin{aligned} \mathbf{m}^s(\tilde{u}_{rx}^{\tilde{\nu}}, Q_{(1+\alpha)r}^{\tilde{\nu}}(rx)) - c_\alpha r^{n-1} &\leq \mathbf{m}^s(\tilde{u}_{rx}^\nu, Q_r^\nu(rx)) \\ &\leq \mathbf{m}^s(\tilde{u}_{rx}^{\tilde{\nu}}, Q_{(1-\alpha)r}^{\tilde{\nu}}(rx)) + c_\alpha r^{n-1}. \end{aligned} \tag{A.7}$$

Proof We only prove that

$$\mathbf{m}^s(\tilde{u}_{rx}^{\tilde{\nu}}, Q_{(1+\alpha)r}^{\tilde{\nu}}(rx)) - c_\alpha r^{n-1} \leq \mathbf{m}^s(\tilde{u}_{rx}^\nu, Q_r^\nu(rx)), \tag{A.8}$$

for some $c_\alpha > 0$, with $c_\alpha \rightarrow 0$, as $\alpha \rightarrow 0$; the proof of the other inequality is analogous.

Let $x \in \mathbb{R}^n$, $r > 2$, and set $r_\alpha^\pm := (1 \pm \alpha)r$. We notice that condition (A.6) readily implies that

$$Q_{r_\alpha^-}^{\tilde{\nu}}(rx) \subset\subset Q_r^\nu(rx) \subset\subset Q_{r_\alpha^+}^{\tilde{\nu}}(rx). \tag{A.9}$$

Moreover, we let r be so large that $Q_{r+2}^\nu(rx) \subset\subset Q_{r_\alpha^+}^{\tilde{\nu}}(rx)$. This allows us to use a similar argument as in the proofs of Proposition 2.6 and Lemma A.1, which we repeat here for the readers' convenience.

Let $\eta > 0$ be fixed and let $v \in W^{1,p}(Q_r^\nu(rx))$ be a test function for $\mathbf{m}^s(\tilde{u}_{rx}^\nu, Q_r^\nu(rx))$ satisfying

$$\mathcal{F}^s(v, Q_r^\nu(rx)) \leq \mathbf{m}^s(\tilde{u}_{rx}^\nu, Q_r^\nu(rx)) + \eta. \tag{A.10}$$

Therefore we know that there exist $u \in W^{1,p}(Q_r^\nu(rx); \mathbb{R}^m)$ and a neighbourhood U of $\partial Q_r^\nu(rx)$ such that

$$(u, v) = (\tilde{u}_{rx}^\nu, \tilde{v}_{rx}^\nu) \text{ in } U \text{ and } v \nabla u = 0 \text{ a.e. in } Q_r^\nu(rx). \tag{A.11}$$

Now let $\beta \in (0, 1)$ be such that $Q_r^v \setminus \overline{Q}_{r-\beta}^v(rx) \subset U$ and set

$$R := R_v \left(Q_r' \setminus \overline{Q}_{r-\beta}' \times (-1 - \alpha r, 1 + \alpha r) \right) + rx,$$

where R_v is as in (f) (see Fig. 3b). Let $\varphi \in C_c^\infty(Q_r')$ be a cut-off function between $Q_{r-\beta}'$ and Q_r' ; we define

$$\begin{aligned} \tilde{u}(y) &:= \varphi((R_v^T(y - rx))')u(y) + \left(1 - \varphi((R_v^T(y - rx))')\right)\tilde{u}_{rx}^v(y), \\ \tilde{v}(y) &:= \begin{cases} \min\{v(y), d(y)\} & \text{in } Q_r^v(rx), \\ \min\{\tilde{v}_{rx}^v(y), d(y)\} & \text{in } Q_{r\alpha}^v(rx) \setminus \overline{Q}_r^v(rx), \end{cases} \end{aligned}$$

where $d(y) := \text{dist}(y, R)$. We now observe that in view of (A.6) and (A.11), by definition of R we get

$$v = \tilde{v}_{rx}^v = \tilde{v}_{rx}^v = 1 \text{ in } (Q_r^v(rx) \setminus \overline{Q}_{r-\beta}^v(rx)) \setminus \overline{R},$$

so that $\tilde{v} \in W^{1,p}(Q_{r\alpha}^v(rx))$. Since $Q_{r+2}^v(rx) \subset\subset Q_{r\alpha}^v(rx)$, arguing as in Lemma A.1 it is easy to check that the pair $(\tilde{u}, \tilde{v}) \in W^{1,p}(Q_{r\alpha}^v(rx); \mathbb{R}^m) \times W^{1,p}(Q_{r\alpha}^v(rx))$ also satisfies

$$(\tilde{u}, \tilde{v}) = (\tilde{u}_{rx}^v, \tilde{v}_{rx}^v) \text{ near } \partial Q_{r\alpha}^v(rx) \text{ and } \tilde{v} \nabla \tilde{u} = 0 \text{ a.e. in } Q_{r\alpha}^v(rx).$$

Therefore, \tilde{v} is admissible for $\mathbf{m}^s(\tilde{u}_{rx}^v, Q_{r\alpha}^v(rx))$. Then, using the same arguments as in Proposition 2.6 we deduce

$$\begin{aligned} \mathcal{F}^s(\tilde{v}, Q_{r\alpha}^v(rx)) &\leq \mathcal{F}^s(v, Q_r^v(rx)) + \mathcal{F}^s(\tilde{v}_{rx}^v, Q_{r\alpha}^v(rx) \setminus \overline{Q}_r^v(rx)) + 2c_4\mathcal{L}^n(\{d < 1\}) \\ &\leq \mathcal{F}^s(v, Q_r^v(rx)) + \mathcal{F}^s(\tilde{v}_{rx}^v, Q_{r\alpha}^v(rx) \setminus \overline{Q}_{r\alpha}^v(rx)) + 2c_4\mathcal{L}^n(\{d < 1\}), \end{aligned} \tag{A.12}$$

where to obtain the second inequality we used the first inclusion in (A.9). Furthermore by (2.3) and (2.19) we have

$$\begin{aligned} \mathcal{F}^s(\tilde{v}_{rx}^v, Q_{r\alpha}^v(rx) \setminus \overline{Q}_{r\alpha}^v(rx)) &\leq c_4C_v\mathcal{L}^{n-1}(Q_{r\alpha}'(rx) \setminus Q_{r\alpha}'(rx)) \\ &= c_4C_v((1 + \alpha)^{n-1} - (1 - \alpha)^{n-1})r^{n-1} \end{aligned} \tag{A.13}$$

whereas

$$\mathcal{L}^n(\{d < 1\}) \leq C(\beta + 1)r^{n-2}\alpha r \leq C\alpha r^{n-1}, \tag{A.14}$$

for some $C > 0$ independent of x, v , and r . Thus, gathering (A.10), (A.12), (A.13), and (A.14), thanks to (A.11) we infer

$$\mathbf{m}^s(\tilde{u}_{rx}^v, Q_{r\alpha}^v(rx)) \leq \mathbf{m}^s(\tilde{u}_{rx}^v, Q_r^v(rx)) + c_\alpha r^{n-1} + \eta,$$

where $c_\alpha := c_4C_v((1 + \alpha)^{n-1} - (1 - \alpha)^{n-1}) + C\alpha$. Eventually, (A.8) follows by the arbitrariness of $\eta > 0$. \square

References

1. Akcoglu, M.A., Krengel, U.: Ergodic theorems for superadditive processes. *J. Reine Angew. Math.* **323**, 53–67 (1981)

2. Alicandro, R., Braides, A., Shah, J.: Free-discontinuity problems via functionals involving the L^1 -norm of the gradient and their approximations. *Interfaces Free Bound.* **1**, 17–37 (1999)
3. Alicandro, R., Cicalese, M., Ruf, M.: Domain formation in magnetic polymer composites: an approach via stochastic homogenization. *Arch. Ration. Mech. Anal.* **218**, 945–984 (2015)
4. Alicandro, R., Focardi, M.: Variational approximation of free-discontinuity energies with linear growth. *Commun. Cont. Math.* **4**(4), 685–723 (2002)
5. Ambrosio, L.: On the lower semicontinuity of quasi-convex integrals in $SBV(\Omega; \mathbb{R}^k)$. *Nonlinear Anal.* **23**, 405–425 (1994)
6. Ambrosio, L., Fusco, N., Pallara, D.: *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York (2000)
7. Ambrosio, L., Tortorelli, V.M.: Approximation of functionals depending on jumps by elliptic functionals via Γ -convergence. *Commun. Pure Appl. Math.* **43**(8), 999–1036 (1990)
8. Ambrosio, L., Tortorelli, V.M.: On the approximation of free discontinuity problems. *Boll. Unione Mat. Ital.* **7**, 105–123 (1992)
9. Anisini, N., Braides, A., Chiadó Piat, V.: Gradient theory of phase transitions in composite media. *Proc. R. Soc. Edinb.* **133A**, 265–296 (2003)
10. Babadjian, J.-F., Millot, V.: Unilateral gradient flow of the Ambrosio–Tortorelli functional by minimizing movements. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31**(4), 779–822 (2014)
11. Bach, A.: Anisotropic free-discontinuity functionals as the Γ -limit of second-order elliptic functionals. *ESAIM Control Optim. Calc. Var.* **24**(3), 1107–1140 (2018)
12. Bach, A., Braides, A., Zeppieri, C.I.: Quantitative analysis of finite-difference approximations of free-discontinuity functionals. *Interfaces Free Bound.* **22**(3), 317–381 (2020)
13. Bach, A., Cicalese, M., Ruf, M.: Random finite-difference discretizations of the Ambrosio–Tortorelli functional with optimal mesh size. *SIAM J. Math. Anal.* **53**(3), 2275–2318 (2021)
14. Bach, A., Esposito, T., Marziani, R., Zeppieri, C.I.: Gradient damage models for heterogeneous materials. *SIAM J. Math. Anal.* (to appear)
15. Barroso, A.C., Fonseca, I.: Anisotropic singular perturbations—The vectorial case. *Proc. R. Soc. Edinb. Sect. A* **124**, 527–571 (1994)
16. Bouchitté, G.: Singular perturbations of variational problems arising from a two-phase transition model. *Appl. Math. Opt.* **21**, 289–314 (1990)
17. Bouchitté, G., Fonseca, I., Leoni, G., Mascarenhas, L.: A global method for relaxation in $W^{1,p}$ and SBV_p . *Arch. Ration. Mech. Anal.* **165**, 187–242 (2002)
18. Bourdin, B.: Image segmentation with a finite element method. *ESAIM Math. Model. Numer. Anal.* **33**(2), 229–244 (1999)
19. Bourdin, B., Francfort, G.A., Marigo, J.-J.: Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids.* **48**(4), 797–826 (2000)
20. Bourdin, B., Francfort, G.A., Marigo, J.-J.: The variational approach to fracture. *J. Elasticity* **9**, 5–148 (2008)
21. Braides, A., Chambolle, A., Solci, M.: A relaxation result for energies defined on pairs set-function and applications. *ESAIM Control Optim. Calc. Var.* **13**(4), 717–734 (2007)
22. Braides, A., Cicalese, M., Ruf, M.: Continuum limit and stochastic homogenization of discrete ferromagnetic thin films. *Anal. PDE* **11**, 499–553 (2018)
23. Braides, A., Defranceschi, A.: *Homogenization of Multiple Integrals*. Oxford University Press, New York (1998)
24. Braides, A., Defranceschi, A., Vitali, E.: Homogenization of free-discontinuity problems. *Arch. Ration. Mech. Anal.* **135**, 297–356 (1996)
25. Braides, A., Piatnitski, A.: Homogenization of surface and length energies for spin systems. *J. Funct. Anal.* **264**, 1296–1328 (2013)
26. Burger, M., Esposito, T., Zeppieri, C.I.: Second-order Edge-penalization in the Ambrosio–Tortorelli functional. *SIAM Multisc. Model. Simul.* **13**(4), 1354–1389 (2015)
27. Cagnetti, F., Dal Maso, G., Scardia, L., Zeppieri, C.I.: Γ -convergence of free-discontinuity problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire.* **36**, 1035–1079 (2019)
28. Cagnetti, F., Dal Maso, G., Scardia, L., Zeppieri, C.I.: Stochastic homogenisation of free discontinuity problems. *Arch. Ration. Mech. Anal.* **233**, 935–974 (2019)
29. Chambolle, A.: An approximation result for special functions with bounded deformation. *J. Math. Pures Appl.* **83**, 929–954 (2004)
30. Chermisi, M., Dal Maso, G., Fonseca, I., Leoni, G.: Singular perturbation models in phase transitions for second-order materials. *Indiana Univ. Math. J.* **60**, 367–409 (2011)
31. Cicalese, M., Spadaro, E., Zeppieri, C.I.: Asymptotic analysis of a second-order singular perturbation model for phase transitions. *Calc. Var. Part. Differ. Equ.* **41**, 127–150 (2011)

32. Dal Maso, G.: An introduction to Γ -convergence, Progress in Nonlinear Differential Equations and their Applications, vol. 8. Birkhäuser Boston Inc., Boston, MA (1993)
33. Dal Maso, G., Francfort, G., Toader, R.: Quasistatic crack growth in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **176**, 165–225 (2005)
34. Dal Maso, G., Iurlano, F.: Fracture models as Γ -limits of damage models. *Commun. Pure Appl. Anal.* **12**, 1657–1686 (2013)
35. Dal Maso, G., Modica, L.: Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.* **368**, 28–42 (1986)
36. Focardi, M.: On the variational approximation of free-discontinuity problems in the vectorial case. *Math. Models Methods App. Sci.* **11**, 663–684 (2001)
37. Fonseca, I., Mantegazza, C.: Second order singular perturbation models for phase transitions. *SIAM J. Math. Anal.* **31**(5), 1121–1143 (2000)
38. Fonseca, I., Tartar, L.: The gradient theory of phase transitions for systems with two potential wells. *Proc. Roy. Soc. Edinb. Sect. A* **111**, 89–102 (1989)
39. Francfort, G., Larsen, C.J.: Existence and convergence for quasistatic evolution in brittle fracture. *Commun. Pure Appl. Math.* **56**, 1465–1500 (2003)
40. Friedrich, M., Perugini, M., Solombrino, F.: Γ -convergence for free-discontinuity problems in linear elasticity: Homogenization and relaxation, *Indiana Univ. Math. J.* (to appear)
41. Giacomini, A.: Ambrosio–Tortorelli approximation of quasi-static evolution of brittle fractures. *Calc. Var. Part. Differ. Equ.* **22**, 129–172 (2005)
42. Giacomini, A., Ponsiglione, M.: A Γ -convergence approach to stability of unilateral minimality properties. *Arch. Ration. Mech. Anal.* **180**, 399–447 (2006)
43. Henao, D., Mora-Corral, C., Xu, X.: Γ -convergence approximation of fracture and cavitation in nonlinear elasticity. *Arch. Ration. Mech. Anal.* **216**, 813–879 (2015)
44. Henao, D., Mora-Corral, C., Xu, X.: A numerical study of void coalescence and fracture in nonlinear elasticity. *Comput. Methods Appl. Mech. Eng.* **303**, 163–184 (2016)
45. Iurlano, F.: Fracture and plastic models as Γ -limits of damage models under different regimes. *Adv. Calc. Var.* **6**, 165–189 (2013)
46. Marziani, R.: Γ -convergence and stochastic homogenisation of phase-transition functionals. *ESAIM Control Optim. Calc Var.* **29**, 44 (2023)
47. Messaoudi, K., Michaille, G.: Stochastic homogenization of nonconvex integral functionals. *Rairo Modél. Math. Anal. Numér.* **28**(3), 329–356 (1991)
48. Modica, L.: The gradient theory of phase transitions and the minimal interface criterion. *Arch. Ration. Mech. Anal.* **98**, 123–142 (1987)
49. Modica, L., Mortola, S.: Un esempio di Γ -convergenza. *Boll. Un. Mat. Ital.* **14–B**, 285–299 (1977)
50. Morfe, P.S.: Surface tension and Γ -convergence of Van der Waals–Cahn–Hilliard phase transitions in stationary ergodic media. *J. Stat. Phys.* **181**(6), 2225–2256 (2020)
51. Owen, N.C.: Nonconvex variational problems with general singular perturbations. *Trans. Am. Math. Soc.* **310**, 393–404 (1988)
52. Owen, N.C., Sternberg, P.: Nonconvex variational problems with anisotropic perturbations. *Nonlinear Anal.* **16**, 705–719 (1991)
53. Pham, K., Marigo, J.-J., Maurini, C.: The issues of the uniqueness and the stability of the homogeneous response in uniaxial tests with gradient damage models. *J. Mech. Phys. Solids* **59**(6), 1163–1190 (2011)
54. Ruf, M.: Discrete stochastic approximations of the Mumford–Shah functional. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36**(4), 887–937 (2019)
55. Sternberg, P.: The effect of a singular perturbation on nonconvex variational problems. *Arch. Ration. Mech. Anal.* **101**, 209–260 (1988)