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Calculus of Variations

Exponential decay of the solutions to nonlinear Schrödinger systems

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Abstract

We show that the components of finite energy solutions to general nonlinear Schrödinger systems have exponential decay at infinity. Our results apply to positive or sign-changing components, and to cooperative, competitive, or mixed-interaction systems. As an application, we use the exponential decay to derive an upper bound for the least possible energy of a solution with a prescribed number of positive and nonradial sign-changing components.

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1 Introduction

Consider the nonlinear Schrödinger system

$$\begin{cases} -\Delta u_i + V_i(x)u_i = \sum_{j=1}^{\ell} \beta_{ij}|u_j|^p|u_i|^{p-2}u_i, \\ u_i \in H^1(\mathbb{R}^N), \qquad i = 1, \dots, \ell, \end{cases} \tag{1.1}$$

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where $N \geq 1$, $V_i \in L^\infty(\mathbb{R}^N)$, $\beta_{ij} \in \mathbb{R}$ and $1 < p < \frac{2^*}{2}$. Here 2^* is the usual critical Sobolev exponent, namely, $2^* := \frac{2N}{N-2}$ if $N \geq 3$ and $2^* := \infty$ for $N = 1, 2$.

Systems of this type occur as models for various natural phenomena. In physics, for example, they describe the behavior of standing waves for a mixture of Bose-Einstein condensates of different hyperfine states which overlap in space [13]. The coefficients β_{ij} determine the type of interaction between the states; if $\beta_{ij} > 0$, then there is an attractive force between u_i and u_j, similarly, if $\beta_{ij} < 0$, then the force is repulsive, and if $\beta_{ij} = 0$, then there is no direct interaction between these components. Whenever all the interaction coefficients are positive, we say that the system is cooperative. If $\beta_{ii} > 0$ and $\beta_{ij} < 0$ for all $i \neq j$, then the system is called competitive. And if some β_{ij} are positive and others are negative for $i \neq j$, then we say that the system has mixed couplings. All these regimes exhibit very different qualitative behaviors and have been studied extensively in recent years, see for instance [5, 6, 8–12, 17, 19–24, 26] and the references therein.

System (1.1) has a variational structure, and therefore a natural strategy is to find weak solutions by minimizing an associated energy functional on a suitable set, under additional assumptions on the matrix (β_{ij}) and on the potentials V_i. Using this approach, several kinds of solutions have been found in terms of their signs and their symmetries. However, there seems to be no information available about the decay of these solutions at infinity. In this paper, we show that finite energy solutions must decay exponentially at infinity, and a rate can be found in terms of the potentials V_i. Our main result is the following one.

Theorem 1.1 *Assume that, for every $i = 1, \ldots, \ell$,*

(V_1) *$V_i : \mathbb{R}^N \to \mathbb{R}$ is Hölder continuous and bounded,*
(V_2) *there exists $\rho \geq 0$ such that*

$$\sigma_i := \inf_{\mathbb{R}^N \smallsetminus B_\rho(0)} V_i > 0.$$

Let $(u_1, \ldots, u_\ell) \in \left(H^1(\mathbb{R}^N)\right)^\ell$ be a solution of (1.1) and let $\mu_i \in (0, \sqrt{\sigma_i})$. Then, there is $C > 0$ such that

$$|u_i(x)| \leq C\mathrm{e}^{-\mu_i |x|} \quad \text{for all } x \in \mathbb{R}^N \text{ and } i = 1, \ldots, \ell. \tag{1.2}$$

Furthermore, if $V_i \equiv 1$ for every $i = 1, \ldots, \ell$, then (1.2) holds true with $\mu_i = 1$.

We emphasize that each component may have a different decay depending on each potential V_i. The main obstacle to showing (1.2) is to handle the possibly sublinear term $|u_i|^{p-2}u_i$ for $p \in (1, 2)$ (which is always the case for $N \geq 4$). To explain this point in more detail, assume that $(u_1, \ldots, u_\ell)$ is a solution of (1.1) and write the i-th equation of the system as

$$-\Delta u_i + \left(a_i(x) - c_i(x)|u_i(x)|^{p-2}\right)u_i = 0, \qquad a_i := V_i - \beta_{ii}|u_i|^{2p-2}, \quad c_i := \sum_{j \neq i}^{\ell} \beta_{ij}|u_j|^p. \tag{1.3}$$

Since every $u_j \in H^1(\mathbb{R}^N) \cap \mathcal{C}^0(\mathbb{R}^N)$, we know that a_i and c_i are bounded in $\mathbb{R}^N$, but $|u_i|^{p-2} \to \infty$ as $|x| \to \infty$ and it is also singular at the nodal set of a sign-changing solution. As a consequence, one cannot use directly previously known results about exponential decay for scalar equations, such as those in [1, 3, 18]. In fact, one can easily construct a one-dimensional solution of a similar scalar equation that has a power-type decay. For instance, let $w \in \mathcal{C}^2(\mathbb{R})$ be a positive function such that $w(x) = |x|^{-2/3}$ for $|x| > 1$ and let

$$c(x) := \frac{-w''(x) + w(x)}{w(x)^{\frac{1}{2}}}, \qquad x \in \mathbb{R}.$$

Then, $w \in H^1(\mathbb{R})$ is a solution of $-w'' + w = c\, w^{\frac{1}{2}}$ in $\mathbb{R}$, $c(x) \to 0$ as $|x| \to \infty$, and w decays as a power at infinity.

This shows that the proof of the exponential estimate in Theorem 1.1 must rely on a careful study of the *system structure*. In other words, although the sublinear nonlinearity $|u_i|^{p-2}u_i$ appears in (1.1), the system is not sublinear. As a whole, it is always superlinear.

With this in mind, we adapt some of the arguments in [1, 18] preserving at each step the system structure of the problem. These arguments rely basically on elliptic regularity and comparison principles.

The exponential decay of solutions is a powerful tool in their qualitative study. As an application of Theorem 1.1, we derive *energy bounds* of solutions having prescribed positive and nonradial sign-changing components. For this, power type decay would not be enough.

To be more precise, we consider the autonomous system

$$\begin{cases} -\Delta u_i + u_i = \sum\limits_{j=1}^{\ell} \beta_{ij} |u_j|^p |u_i|^{p-2} u_i, \\ u_i \in H^1(\mathbb{R}^N), \qquad i = 1, \dots, \ell. \end{cases} \tag{1.4}$$

where the β_{ij}'s satisfy the following condition:

(B_1) The matrix (β_{ij}) is symmetric and admits a block decomposition as follows: For some $1 \le q \le \ell$ there exist $0 = \ell_0 < \ell_1 < \cdots < \ell_{q-1} < \ell_q = \ell$ such that, if we set

$$I_h := \{i \in \{1, \dots, \ell\} : \ell_{h-1} < i \le \ell_h\}, \qquad h \in \{1, \dots, q\},$$

then $\beta_{ii} > 0$, $\beta_{ij} \ge 0$ if $i, j \in I_h$, and $\beta_{ij} < 0$ if $i \in I_h$, $j \in I_k$ and $h \ne k$.

According to this decomposition, a solution $\boldsymbol{u} = (u_1, \dots, u_\ell)$ to (1.1) may be written in block-form as

$$\boldsymbol{u} = (\overline{u}_1, \dots, \overline{u}_q) \qquad \text{with } \overline{u}_h = (u_{\ell_{h-1}+1}, \dots, u_{\ell_h}), \quad h = 1, \dots, q.$$

We say that $\boldsymbol{u}$ is *fully nontrivial* if every component u_i is different from zero.

Set $Q := \{1, \dots, q\}$. Given a partition $Q = Q^+ \cup Q^-$ with $Q^+ \cap Q^- = \emptyset$ we look for solutions such that every component of $\overline{u}_h$ is positive if $h \in Q^+$ and every component of $\overline{u}_h$ is nonradial and changes sign if $h \in Q^-$. To this end, we use variational methods in a space having suitable symmetries. As shown in [11, Section 3], to guarantee that the solutions obtained are fully nontrivial we need to assume the following two conditions:

(B_2) For each $h \in Q$, the graph whose set of vertices is I_h and whose set of edges is $E_h := \{\{i, j\} : i, j \in I_h,\ i \ne j,\ \beta_{ij} > 0\}$ is connected.

(B_3) If $q \ge 2$ then, for every $h \in \{1, \dots, q\}$ such that $\ell_h - \ell_{h-1} \ge 2$, the inequality

$$\Big(\min_{\{i,j\} \in E_h} \beta_{ij}\Big) \left[\frac{\min\limits_{h=1,\dots,q} \max\limits_{i \in I_h} \beta_{ii}}{\sum\limits_{i,j \in I_h} \beta_{ij}} \right]^{\frac{p}{p-1}} > C_* \sum_{\substack{k=1 \\ k \ne h}}^{q} \sum_{\substack{i \in I_h \\ j \in I_k}} |\beta_{ij}|$$

holds true, where $C_* = C_*(N, p, q, Q^+) > 0$ is the explicit constant given in (3.7) below.

In [11] it is shown that, for any q, the system (1.1) has a fully nontrivial solution satisfying the sign requirements described above. Furthermore, an upper bound for its energy is exhibited, but only for systems with at most two blocks, i.e., for $q = 1, 2$. Here we use Theorem 1.1 to obtain an energy bound for *any* number of blocks.

For each $h = 1, \dots, q$, let $\mathbb{R}^{I_h} := \{\overline{s} = (s_{\ell_{h-1}+1}, \dots, s_{\ell_h}) : s_i \in \mathbb{R} \text{ for all } i \in I_h\}$ and define

$$\mu_h := \inf_{\substack{\overline{s}\in\mathbb{R}^{I_h} \\ \overline{s}\neq 0}} \left(\frac{\sum_{i\in I_h} s_i^2}{\left(\sum_{i,j\in I_h} \beta_{ij}|s_i|^p|s_j|^p\right)^{\frac{2}{2p}}} \right)^{\frac{p}{p-1}}. \tag{1.5}$$

For any $\ell \in \mathbb{N}$, we write $\|\boldsymbol{u}\|$ for the usual norm of $\boldsymbol{u} = (u_1, \dots, u_\ell)$ in $(H^1(\mathbb{R}^N))^\ell$, i.e.,

$$\|\boldsymbol{u}\|^2 := \sum_{i=1}^{\ell} \int_{\mathbb{R}^N} (|\nabla u_i|^2 + |u_i|^2).$$

We prove the following result.

Theorem 1.2 *Let $N = 4$ or $N \geq 6$, and let $Q = Q^+ \cup Q^-$ with $Q^+ \cap Q^- = \emptyset$. Assume (B_1), (B_2), and (B_3). Then, there exists a fully nontrivial solution $\boldsymbol{u} = (\overline{u}_1, \dots, \overline{u}_q)$ to the system (1.4) with the following properties:*

(*a*) *Every component of $\overline{u}_h$ is positive if $h \in Q^+$ and every component of $\overline{u}_h$ is nonradial and changes sign if $h \in Q^-$.*

(*b*) *If $q = 1$, then*

$$\|u\|^2 = \mu_1\|\omega\|^2 \text{ if } Q = Q^+ \qquad \text{and} \qquad \|u\|^2 < 10\,\mu_1\|\omega\|^2 \text{ if } Q = Q^-.$$

(*c*) *If $q \geq 2$ the following estimate holds true*

$$\|\boldsymbol{u}\|^2 < \left(\min_{k\in Q} \left(a_k\mu_k + \sum_{h\in Q\smallsetminus\{k\}} b_h\mu_h \right) \right) \|\omega\|^2, \tag{1.6}$$

where $a_k := 1$ if $k \in Q^+$, $a_k := 12$ if $k \in Q^-$, $b_h := 6$ if $h \in Q^+$, $b_h := 12$ if $h \in Q^-$, and ω is the unique positive radial solution to the equation

$$-\Delta w + w = |w|^{2p-2}w, \qquad w \in H^1(\mathbb{R}^N). \tag{1.7}$$

To prove Theorem 1.2, we follow the approach in [11] and impose on the variational setting some carefully constructed symmetries which admit finite orbits. This approach immediately gives energy estimates but it requires showing a quantitative compactness condition which needs precise knowledge about the asymptotic decay of the components of the system. Here is where we use Theorem 1.1.

The paper is organized as follows. Section 2 is devoted to the proof of the exponential decay stated in Theorem 1.1. The application of this result to derive energy bounds is contained in Section 3, where we also give some concrete examples.

2 Exponential decay

This section is devoted to the proof of Theorem 1.1. As a first step, we extend the argument in [2, Lemma 5.3] to systems. Let B_r denote the ball of radius r in $\mathbb{R}^N$ centered at zero. Let σ_i and β_{ij} as in (V_2) and (1.1), then we let $\boldsymbol{\sigma} := (\sigma_1, \dots, \sigma_\ell)$ and $\boldsymbol{\beta} := (\beta_{ij})_{i,j=1}^{\ell}$.

Lemma 2.1 *Let $V_i \in L^\infty(\mathbb{R}^N)$ satisfy (V_2) and let $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ be a solution of (1.1). Set*

$$\xi_i(r) := \int_{\mathbb{R}^N \smallsetminus B_r} \left(|\nabla u_i|^2 + |u_i|^2\right) \qquad \text{and} \qquad \boldsymbol{\xi}(r) := (\xi_1(r), \ldots, \xi_\ell(r)).$$

Then, there are positive constants $C = C(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\beta}, N, \rho, p)$ and $\vartheta = \vartheta(\boldsymbol{\sigma})$, with ρ and σ_i as in (V_2), such that

$$|\boldsymbol{\xi}(r)|_1 := \sum_{i=1}^{\ell} \xi_i(r) \leq C\mathrm{e}^{-\vartheta r} \qquad \text{for every } r \geq 0.$$

Proof Let $\chi : \mathbb{R}^N \to \mathbb{R}$ be given by $\chi(r) := 0$ if $r \leq 0$, $\chi(r) := r$ if $r \in (0, 1)$ and $\chi(r) := 1$ if $r \geq 1$. Let $u_i^r(x) := \chi(|x| - r)u_i(x)$ for $r \geq 0$, $x \in \mathbb{R}^N$, and $i = 1, \ldots, \ell$. Then $u_i^r \in H^1(\mathbb{R}^N)$ and

$$u_i^r(x) = (|x| - r)u_i(x), \qquad \nabla u_i^r(x) = (|x| - r)\nabla u_i(x) + \frac{x}{|x|}u_i(x), \qquad \text{if } x \in B_{r+1} \smallsetminus B_r.$$

Set $\delta := \min\{\sigma_1, \ldots, \sigma_\ell, 1\}$. Using that $|u_i \frac{x}{|x|} \cdot \nabla u_i| \leq \frac{1}{2}(|\nabla u_i|^2 + |u_i|^2)$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \left(\nabla u_i \cdot \nabla u_i^r + V_i\, u_i u_i^r\right) &\geq \delta\xi_i(r+1) + \int_{B_{r+1} \smallsetminus B_r} \left[(|x| - r)\left(|\nabla u_i|^2 + V_i\, u_i^2\right) + u_i \frac{x}{|x|} \cdot \nabla u_i\right] \\ &\geq \delta\xi_i(r+1) - \frac{1}{2}\int_{B_{r+1} \smallsetminus B_r} \left(|\nabla u_i|^2 + |u_i|^2\right) \\ &\geq (\delta + \tfrac{1}{2})\xi_i(r+1) - \tfrac{1}{2}\xi_i(r) \qquad \text{if } r+1 \geq \rho. \end{aligned} \tag{2.1}$$

As $\boldsymbol{u}$ solves (1.1) we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} \nabla u_i \cdot \nabla u_i^r + V_i\, u_i u_i^r\right| &= \left|\int_{\mathbb{R}^N} \sum_{j=1}^{\ell} \beta_{ij}|u_j|^p|u_i|^{p-2}u_i u_i^r\right| \\ &\leq \sum_{j=1}^{\ell} \int_{\mathbb{R}^N \smallsetminus B_r} |\beta_{ij}||u_j|^p|u_i|^{p-2}|u_i|^2 = \sum_{j=1}^{\ell} |\beta_{ij}| \int_{\mathbb{R}^N \smallsetminus B_r} |u_j|^p|u_i|^p\end{aligned}$$

and since $|u_m|^p \leq \left(\sum_{k=1}^{\ell} |u_k|^{2p}\right)^{1/2}$ for every $m = 1, \ldots, \ell$, we obtain

$$\left|\int_{\mathbb{R}^N} \nabla u_i \cdot \nabla u_i^r + V_i\, u_i u_i^r\right| \leq \left(\sum_{j=1}^{\ell} |\beta_{ij}|\right) \sum_{k=1}^{\ell} \int_{\mathbb{R}^N \smallsetminus B_r} |u_k|^{2p}.$$

Given that $u_k \in H^1(\mathbb{R}^N)$ for all $k = 1, \ldots, \ell$, Lemma A.1 implies the existence of a constant $C_1 = C_1(N, p) > 0$ such that

$$\left|\int_{\mathbb{R}^N} \nabla u_i \cdot \nabla u_i^r + V_i\, u_i u_i^r\right| \leq C_1 \left(\sum_{j=1}^{\ell} |\beta_{ij}|\right) \sum_{k=1}^{\ell} \left(\int_{\mathbb{R}^N \smallsetminus B_r} \left(|\nabla u_k|^2 + |u_k|^2\right)\right)^p \tag{2.2}$$

for every $r \geq 1$ and $i = 1, \ldots, \ell$. Set $C_2 := C_1 \sum_{i,j=1}^{\ell} |\beta_{ij}|$. From (2.1) and (2.2), assuming without loss of generality that $\rho \geq 2$ and adding over i, we get

$$\frac{2\delta+1}{2}\,|\boldsymbol{\xi}(r+1)|_1 - \frac{1}{2}|\boldsymbol{\xi}(r)|_1 \leq C_2 \sum_{k=1}^{\ell} |\xi_k(r)|^p =: C_2\,|\boldsymbol{\xi}(r)|_p^p \qquad \text{if } r+1 \geq \rho.$$

Therefore,

$$\frac{|\boldsymbol{\xi}(r+1)|_1}{|\boldsymbol{\xi}(r)|_1} \leq \frac{1}{2\delta+1}\left(1 + 2C_2\frac{|\boldsymbol{\xi}(r)|_p^p}{|\boldsymbol{\xi}(r)|_1}\right) \leq \frac{1}{2\delta+1}\left(1 + 2C_2|\boldsymbol{\xi}(r)|_1^{p-1}\right) =: \gamma(r) \quad \text{if } r+1 \geq \rho. \tag{2.3}$$

Since $|\boldsymbol{\xi}(r)|_1 \to 0$ as $r \to \infty$, there is $r_0 = r_0(\boldsymbol{u}, p, \boldsymbol{\beta}, \rho) \in \mathbb{N}$ such that $r_0 \geq \rho$ and $\gamma(r) \leq \gamma_0^{-1}$ for all $r \geq r_0$ with $\gamma_0 := \frac{2\delta+1}{\delta+1} > 1$. Then, for $r > r_0 + 1$,

$$|\boldsymbol{\xi}(r)|_1 \leq |\boldsymbol{\xi}(\lfloor r\rfloor)|_1 = |\boldsymbol{\xi}(r_0)|_1 \prod_{k=r_0}^{\lfloor r\rfloor -1} \frac{|\boldsymbol{\xi}(k+1)|_1}{|\boldsymbol{\xi}(k)|_1} \leq |\boldsymbol{\xi}(r_0)|_1 \gamma_0^{r_0-\lfloor r\rfloor} \leq \|\boldsymbol{u}\|^2 \gamma_0^{r_0-r+1},$$

where $\lfloor r\rfloor$ denotes the floor of r. Since $|\boldsymbol{\xi}(r)|_1 \leq \|\boldsymbol{u}\|^2 \leq \|\boldsymbol{u}\|^2\gamma_0^{r_0-r+1}$ for $r \leq r_0+1$ we have that

$$|\boldsymbol{\xi}(r)|_1 \leq \|\boldsymbol{u}\|^2\gamma_0^{r_0-r+1} = \|\boldsymbol{u}\|^2\gamma_0^{r_0+1}\mathrm{e}^{-\ln(\gamma_0)r} \qquad \text{for every } r \geq 0,$$

as claimed. □

Lemma 2.2 *Assume* (V_1) *and let* $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ *be a solution of* (1.1). *Then* $u_i \in W^{2,s}(\mathbb{R}^N) \cap \mathcal{C}^2(\mathbb{R}^N)$ *for every* $s \geq 2$ *and* $i = 1, \ldots, \ell$.

Proof Let $N \geq 3$. The argument for $N = 1, 2$ is similar and easier. For each $i = 1, \ldots, \ell$ set

$$f_i := \sum_{j=1}^{l} \beta_{ij}|u_j|^p|u_i|^{p-2}u_i. \tag{2.4}$$

Since $|u_k| \leq |\boldsymbol{u}| := \sqrt{u_1^2 + \cdots + u_\ell^2}$ for every $k = 1, \ldots \ell$, we have that

$$|f_i| \leq \sum_{i,j=1}^{\ell} |\beta_{ij}||u_j|^p|u_i|^{p-1} \leq \left(\sum_{j=1}^{\ell}|\beta_{ij}|\right)|\boldsymbol{u}|^p|\boldsymbol{u}|^{p-1} \leq \left(\sum_{i,j=1}^{\ell}|\beta_{ij}|\right)|\boldsymbol{u}|^{2p-1}. \tag{2.5}$$

Therefore, $f_i \in L^{s_1}(\mathbb{R}^N)$ for $s_1 := \frac{2^*}{2p-1} > 1$ and, by the standard L^p-elliptic regularity theory, $u_i \in W^{2,s_1}(\mathbb{R}^N)$ for all $i = 1, \ldots, \ell$ (see, e.g., [14, Chapter 9] or [25, Section 3.2]). Using a bootstrapping argument, we conclude the existence of $s > \max\{\frac{N}{2}, 2\}$ such that $u_i \in W^{2,s}(\mathbb{R}^N)$ for all $i = 1, \ldots, \ell$ and thus, by the Sobolev embedding theorem, $u_i \in \mathcal{C}^{1,\alpha}(\mathbb{R}^N)$. Since V_i is Hölder continuous and bounded, applying the Schauder estimates repeatedly, we deduce that u_i is of class $\mathcal{C}^2$ (see [15, Section 1.3]). □

In the rest of the paper, we write $|\cdot|_t$ for the norm in $L^t(\mathbb{R}^N)$, $1 \leq t \leq \infty$. If $\boldsymbol{u} = (u_1, \ldots, u_\ell) \in [L^\infty(\mathbb{R}^N)]^\ell$, then $|\boldsymbol{u}|_\infty := \sum_{i=1}^{\ell} \sup_{\mathbb{R}^N}|u_i|$. Moreover, for a proper open subset Ω of $\mathbb{R}^N$ we denote the usual Sobolev norm in $H^1(\Omega)$ by $\|\cdot\|_{H^1(\Omega)}$, i.e.,

$$\|u\|_{H^1(\Omega)}^2 := \int_\Omega (|\nabla u|^2 + |u|^2).$$

Lemma 2.3 *Assume* (V_1). *Let* $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ *be a solution of* (1.1), $s > \max\{2, \frac{N}{2}\}$ *and* $\Lambda > 0$ *be such that* $|V_i|_\infty \leq \Lambda$ *for* $i = 1, \ldots, \ell$. *Then there is a constant* $C = C(\boldsymbol{\beta}, N, p, \Lambda, s) > 0$ *such that, for any* $x \in \mathbb{R}^N$,

$$\|u_i\|_{W^{2,s}(B_{\frac{1}{2}}(x))} \leq C \left(|u_i|_\infty^{\frac{s-2}{s}} \|u_i\|_{H^1(B_1(x))}^{\frac{2}{s}} + |\boldsymbol{u}|_\infty^{\frac{2ps-(s+2)}{s}} \left(\sum_{j=1}^{\ell} \|u_j\|_{H^1(B_1(x))}^2 \right)^{\frac{p}{s}} \right),$$

where $B_R(x)$ *is the ball of radius* R *centered at* x.

Proof Since $u_i \in W^{2,s}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$, we have that

$$|u_i|^s = |u_i|^{s-2}|u_i|^2 \leq |u_i|_\infty^{s-2}|u_i|^2.$$

Set f_i as in (2.4). By (2.5), there is a constant $C_2 = C_2(\boldsymbol{\beta})$ such that

$$\begin{aligned} |f_i|^s &\leq C_2^s |\boldsymbol{u}|^{(p-1)s} |\boldsymbol{u}|^{ps} = C_2^s |\boldsymbol{u}|^{(p-1)s+p(s-2)} (u_1^2 + \cdots + u_\ell^2)^p \\ &\leq C_2^s |\boldsymbol{u}|_\infty^{2ps-(s+2)} \ell^p (u_1^{2p} + \cdots + u_\ell^{2p}), \end{aligned}$$

where $|\boldsymbol{u}| := \sqrt{u_1^2 + \cdots + u_\ell^2}$ and $(p-1)s + p(s-2) > 0$. Then, by [14, Theorem 9.11], there is a positive constant $C_1 = C_1(s, N, \Lambda)$ such that

$$\|u_i\|_{W^{2,s}(B_{\frac{1}{2}}(x))} \leq C_1 \left(|u_i|_{L^s(B_1(x))} + |f_i|_{L^s(B_1(x))} \right) \quad \text{for any } x \in \mathbb{R}^N.$$

From the previous inequalities we derive

$$\|u_i\|_{W^{2,s}(B_{\frac{1}{2}}(x))} \leq C_1 \left(|u_i|_\infty^{\frac{s-2}{s}} \|u_i\|_{H^1(B_1(x))}^{\frac{2}{s}} + C_2 \ell^{\frac{p}{s}} C_3 |\boldsymbol{u}|_\infty^{\frac{2ps-(s+2)}{s}} \left(\sum_{j=1}^{\ell} \|u_j\|_{H^1(B_1(x))}^2 \right)^{\frac{p}{s}} \right),$$

where $C_3 = C_3(N, p)$ is the constant given by the Sobolev embedding $H^1(B_1) \subset L^{2p}(B_1)$. □

Lemma 2.4 *Assume* $(V_1) - (V_2)$, *let* $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ *be a solution of* (1.1) *and let* f_i *be as in* (2.4). *Then, there are constants* $\eta > 0$, $C_1 > 0$, *and* $C_2 > 0$ *such that*

$$|u_i(x)| \leq C_1 e^{-\eta|x|}, \qquad |f_i(x)| \leq C_2 e^{-(2p-1)\eta|x|}, \qquad \textit{for all } x \in \mathbb{R}^N \textit{ and } i = 1, \ldots, \ell.$$

Proof For $x \in \mathbb{R}^N$ with $|x| \geq 2$, set $r := \frac{1}{2}|x|$. Then, $B_1(x) \subset \mathbb{R}^N \setminus B_r$ and, by Lemma 2.1, there are positive constants $K_1 = K_1(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\beta}, N, \rho, p)$ and $\vartheta = \vartheta(\boldsymbol{\sigma})$, with ρ and σ_i as in (V_2), such that

$$\|u_j\|_{H^1(B_1(x))}^2 \leq \|u_j\|_{H^1(\mathbb{R}^N \setminus B_r)}^2 = \xi_j(r) \leq \sum_{i=1}^{\ell} \xi_i(r) \leq K_1 \mathrm{e}^{-\vartheta r} \quad \text{for every } j = 1, \ldots, \ell.$$

Fix $s > \max\{\frac{N}{2}, 2\}$. By Lemma 2.3 there are positive constants $K_2 = K_2(\boldsymbol{u}, \boldsymbol{\beta}, N, p, \Lambda, s)$ and $K_3 = K_3(\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{\beta}, \rho, N, p, s)$ such that

$$\|u_i\|_{W^{2,s}(B_{\frac{1}{2}}(x))} \leq K_2 \left(\|u_i\|_{H^1(B_1(x))}^{\frac{2}{s}} + \left(\sum_{j=1}^{\ell} \|u_j\|_{H^1(B_1(x))}^2 \right)^{\frac{p}{s}} \right) \leq K_2 K_3 \mathrm{e}^{-\frac{\vartheta}{s} r}.$$

Therefore,

$$|u_i(x)| \leq |u_i|_{L^\infty(B_{\frac{1}{2}}(x))} \leq K_4 \|u_i\|_{W^{2,s}(B_{\frac{1}{2}}(x))} \leq K_2 K_3 K_4 \mathrm{e}^{-\frac{\vartheta}{2s}|x|} \quad \text{for every } x \in \mathbb{R}^N \setminus B_2,$$

where K_4 is the positive constant given by the embedding $W^{2,s}(B_{\frac{1}{2}}) \subset L^\infty(B_{\frac{1}{2}})$. Since u_i is continuous, we may choose $C_1 \geq K_2K_3K_4$ such that $|u_i(x)| \leq C_1 \mathrm{e}^{-\frac{\vartheta}{s}}$ for every $x \in B_2$. So, setting $\eta := \frac{\vartheta}{2s}$, we obtain

$$|u_i(x)| \leq C_1 \mathrm{e}^{-\eta|x|} \quad \text{for every } x \in \mathbb{R}^N.$$

The estimate for f_i follows immediately from (2.5). □

The following result is a particular case of [18, Theorem 2.1]. We include a simplified proof for completeness.

Lemma 2.5 *Assume that* $V : \mathbb{R}^N \to \mathbb{R}$ *satisfies* $\sigma := \inf_{\mathbb{R}^N \setminus B_\rho(0)} V > 0$ *for some* $\rho \geq 0$. *Let* w *be a classical solution of* $-\Delta w + Vw = f$ *in* $\mathbb{R}^N$ *such that*

$$|w(x)| \leq C\mathrm{e}^{-\eta|x|} \quad \text{and} \quad |f(x)| \leq C\mathrm{e}^{-\delta|x|} \qquad \text{for all } x \in \mathbb{R}^N$$

and for some constants $C > 0$, $\eta \in (0, \sqrt{\sigma})$ *and* $\delta \in (\eta, \sqrt{\sigma}]$. *Then, for any* $\mu \in (\eta, \delta)$, *there is* $M = M(\mu, \delta, \rho, \sigma, C) > 0$ *such that*

$$|w(x)| \leq M\mathrm{e}^{-\mu|x|} \qquad \text{for all } x \in \mathbb{R}^N.$$

Proof Let $\rho, \sigma, \eta, \delta, \mu$, and C be as in the statement. Set $v(x) := \mathrm{e}^{-\mu|x|}$ for $x \in \mathbb{R}^N$. Then,

$$\Delta v(x) = v(x)h(|x|) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \qquad \text{where } h(r) := \mu^2 - (N-1)\frac{\mu}{r}.$$

In particular, $V(x) - h(|x|) \geq \sigma - \mu^2 =: \varepsilon > 0$ for $|x| > \rho$. Fix $t \in \mathbb{R}$ satisfying

$$t > \frac{C}{\varepsilon}\mathrm{e}^{(\mu-\delta)\rho} \qquad \text{and} \qquad w(x) < tv(x) \text{ for } |x| = \rho. \tag{2.6}$$

We claim that $w(x) \leq tv(x)$ for all $|x| > \rho$. Indeed, let $z := w - tv$ and assume, by contradiction, that $m := \sup_{|x|\geq\rho} z(x) > 0$. Since $\lim_{|x|\to\infty} z(x) = 0$, there is $R > \rho$ such that $z(x) \leq \frac{m}{2}$ for $|x| \geq R$. Let $\Omega := \{x \in \mathbb{R}^N \ : \ \rho < |x| < R \text{ and } z(x) > 0\}$. Then $z \leq \frac{m}{2}$ on $\partial\Omega$ and, by (2.6),

$$\begin{aligned}
-\Delta z(x) &= -\Delta w(x) + t\Delta v(x) = f(x) - V(x)w(x) + tv(x)h(|x|) \\
&= f(x) - V(x)z(x) + tv(x)(h(|x|) - V(x)) \\
&< C\mathrm{e}^{-\delta|x|} - \varepsilon tv(x) = C\mathrm{e}^{-\delta|x|} - \varepsilon t\mathrm{e}^{-\mu|x|} < 0 \quad \text{for every } x \in \Omega.
\end{aligned}$$

Then, by the maximum principle, $m = \max_\Omega z = \max_{\partial\Omega} z \leq \frac{m}{2}$. This is a contradiction. Therefore $m \leq 0$, namely, $w(x) \leq t\mathrm{e}^{-\mu|x|}$ for all $|x| \geq \rho$. Arguing similarly for $-w$ and using that $w \in L^\infty(\mathbb{R}^N)$ we obtain that $|w(x)| \leq M\mathrm{e}^{-\mu|x|}$ for all $x \in \mathbb{R}^N$, as claimed. □

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Iterating Lemmas 2.4 and 2.5, using that $2p - 1 > 1$, one shows that, for any $\mu_i \in (0, \sqrt{\sigma_i})$, there is $C > 0$ such that $|u_i(x)| \leq Ce^{-\mu_i|x|}$ for all $x \in \mathbb{R}^N$ and for all $i = 1, \ldots, \ell$.

Now, assume that $V_i \equiv 1$ for every $i = 1, \ldots, \ell$ and let $\mu \in (0, 1)$ be such that $(2p-1)\mu > 1$. By Lemma 2.4, we have that $|f_i(x)| \leq C_2\mathrm{e}^{-(2p-1)\mu|x|}$ for all $x \in \mathbb{R}^N$. The claim now follows from [1, Theorem 2.3(*c*)]. □

3 Energy estimates for seminodal solutions

In this section we prove Theorem 1.2. Consider the autonomous system (1.4) where $N \geq 4$, $1 < p < \frac{N}{N-2}$ and β_{ij} satisfy the assumption (B_1) stated in the Introduction. According to the decomposition given by (B_1), a solution $\boldsymbol{u} = (u_1, \ldots, u_\ell)$ to (1.4) may be written in block-form as

$$\boldsymbol{u} = (\overline{u}_1, \ldots, \overline{u}_q) \qquad \text{with } \overline{u}_h = (u_{\ell_{h-1}+1}, \ldots, u_{\ell_h}), \quad h = 1, \ldots, q.$$

$\boldsymbol{u}$ is called *fully nontrivial* if every component u_i is different from zero. We say that $\boldsymbol{u}$ is *block-wise nontrivial* if at least one component in each block $\overline{u}_h$ is nontrivial.

Following [11], we introduce suitable symmetries to produce a change of sign in some components. Let G be a finite subgroup of the group $O(N)$ of linear isometries of $\mathbb{R}^N$ and denote by $Gx := \{gx : g \in G\}$ the G-orbit of $x \in \mathbb{R}^N$. Let $\phi : G \to \mathbb{Z}_2 := \{-1, 1\}$ be a homomorphism of groups. A function $u : \mathbb{R}^N \to \mathbb{R}$ is called *G-invariant* if it is constant on Gx for every $x \in \mathbb{R}^N$ and it is called *ϕ-equivariant* if

$$u(gx) = \phi(g)u(x) \text{ for all } g \in G,\ x \in \mathbb{R}^N. \tag{3.1}$$

Note that, if $\phi \equiv 1$ is the trivial homomorphism and u satisfies (3.1), then u is G-invariant. On the other hand, if ϕ is surjective every nontrivial function satisfying (3.1) is nonradial and changes sign. Define

$$H^1(\mathbb{R}^N)^\phi := \{u \in H^1(\mathbb{R}^N) : u \text{ is } \phi\text{-equivariant}\}.$$

For each $h = 1, \ldots, q$, fix a homomorphism $\phi_h : G \to \mathbb{Z}_2$. Take $\phi_i := \phi_h$ for all $i \in I_h$ and set $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_\ell)$. Denote by

$$\mathcal{H}^{\boldsymbol{\phi}} := H^1(\mathbb{R}^N)^{\phi_1} \times \cdots \times H^1(\mathbb{R}^N)^{\phi_\ell},$$

and let $\mathcal{J}^{\boldsymbol{\phi}} : \mathcal{H}^{\boldsymbol{\phi}} \to \mathbb{R}$ be the functional given by

$$\mathcal{J}^{\boldsymbol{\phi}}(\boldsymbol{u}) := \frac{1}{2}\sum_{i=1}^{\ell} \|u_i\|^2 - \frac{1}{2p}\sum_{i,j=1}^{\ell} \beta_{ij} \int_{\mathbb{R}^N} |u_i|^p |u_j|^p.$$

This functional is of class $\mathcal{C}^1$ and its critical points are the solutions to the system (1.4) satisfying (3.1). The block-wise nontrivial solutions belong to the Nehari set

$$\mathcal{N}^{\boldsymbol{\phi}} := \{\boldsymbol{u} \in \mathcal{H}^{\boldsymbol{\phi}} : \|\overline{u}_h\| \neq 0 \text{ and } \partial_{\overline{u}_h} \mathcal{J}^{\boldsymbol{\phi}}(\boldsymbol{u})\overline{u}_h = 0 \text{ for every } h = 1, \ldots, \ell\}.$$

Note that

$$\partial_{\overline{u}_h} \mathcal{J}^{\boldsymbol{\phi}|K}(\boldsymbol{u})\overline{u}_h = \|\overline{u}_h\|^2 - \sum_{k=1}^{\ell} \sum_{(i,j)\in I_h \times I_k} \beta_{ij} \int_{\mathbb{R}^N} |u_i|^p |u_j|^p,$$

and that $\mathcal{J}^\phi(\boldsymbol{u}) = \frac{p-1}{2p}\|\boldsymbol{u}\|^2$ if $u \in \mathcal{N}^\phi$. Let

$$c^{\boldsymbol{\phi}} := \inf_{\boldsymbol{u} \in \mathcal{N}^{\boldsymbol{\phi}}} \mathcal{J}^{\boldsymbol{\phi}}(\boldsymbol{u}).$$

If $\boldsymbol{s} = (s_1, \ldots, s_q) \in \mathbb{R}^q$ and $\boldsymbol{u} = (\overline{u}_1, \ldots, \overline{u}_q) \in \mathcal{H}^{\boldsymbol{\phi}}$ we write $\boldsymbol{s}\boldsymbol{u} := (s_1\overline{u}_1, \ldots, s_q\overline{u}_q)$. The following facts were proved in [8].

Lemma 3.1 *(i)* $c^{\boldsymbol{\phi}} > 0$.

(ii) If the coordinates of $\boldsymbol{u} \in \mathcal{H}^{\boldsymbol{\phi}}$ *satisfy*

$$\sum_{k=1}^{q} \sum_{(i,j)\in I_h \times I_k} \int_{\mathbb{R}^N} \beta_{ij} |u_i|^p |u_j|^p > 0 \quad \text{for every } h = 1, \dots, q, \tag{3.2}$$

then there exists a unique $\boldsymbol{s_u} \in (0,\infty)^q$ *such that* $\boldsymbol{s_u u} \in \mathcal{N}^{\boldsymbol{\phi}}$*. Furthermore,*

$$\mathcal{J}^{\boldsymbol{\phi}}(\boldsymbol{s_u u}) = \max_{\boldsymbol{s}\in(0,\infty)^q} \mathcal{J}^{\boldsymbol{\phi}}(\boldsymbol{s u}).$$

Proof See [8, Lemma 2.2] or [11, Lemma 2.2]. □

Lemma 3.2 *If* $c^{\boldsymbol{\phi}}$ *is attained, then the system* (1.4) *has a block-wise nontrivial solution* $\boldsymbol{u} = (u_1, \dots, u_\ell) \in \mathcal{H}^{\boldsymbol{\phi}}$*. Furthermore, if* u_i *is nontrivial, then* u_i *is positive if* $\phi_i \equiv 1$ *and* u_i *is nonradial and changes sign if* ϕ_i *is surjective.*

Proof It is shown in [8, Lemma 2.4] that any minimizer of $\mathcal{J}^{\boldsymbol{\phi}}$ on $\mathcal{N}^{\boldsymbol{\phi}}$ is a block-wise nontrivial solution to (1.4). If $u_i \neq 0$ and ϕ_i is surjective, then u_i is nonradial and changes sign. If $\phi_i \equiv 1$ then $|u_i|$ is G-invariant and replacing u_i with $|u_i|$ we obtain a solution with the required properties. □

Set $Q := \{1, \dots, q\}$ and fix a decomposition $Q = Q^+ \cup Q^-$ with $Q^+ \cap Q^- = \emptyset$. From now on, we consider the following symmetries. We write $\mathbb{R}^N \equiv \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$ and a point in $\mathbb{R}^N$ as $(z_1, z_2, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$.

Definitions 3.3 Let i denote the imaginary unit. For each $m \in \mathbb{N}$, let

$$K_m := \{\mathrm{e}^{2\pi \mathrm{i} j/m} : j = 0, \dots, m-1\},$$

G_m be the group generated by $K_m \cup \{\tau\} \cup O(N-4)$, acting on each point $(z_1, z_2, y) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^{N-4}$ as

$$\mathrm{e}^{2\pi \mathrm{i} j/m}(z_1, z_2, y) := (\mathrm{e}^{2\pi \mathrm{i} j/m} z_1, \mathrm{e}^{2\pi \mathrm{i} j/m} z_2, y), \qquad \tau(z_1, z_2, y) := (z_2, z_1, y),$$
$$\alpha(z_1, z_2, y) := (z_1, z_2, \alpha y) \quad \text{if } \alpha \in O(N-4),$$

and $\theta : G_m \to \mathbb{Z}_2$ be the homomorphism satisfying

$$\theta(\mathrm{e}^{2\pi \mathrm{i} j/m}) = 1, \quad \theta(\tau) = -1, \quad \text{and} \quad \theta(\alpha) = 1 \quad \text{for every } \alpha \in \text{O(N-4)}.$$

Define $\phi_h : G_m \to \mathbb{Z}_2$ by

$$\phi_h := \begin{cases} 1 & \text{if } h \in Q^+, \\ \theta & \text{if } h \in Q^-. \end{cases} \tag{3.3}$$

Due to the lack of compactness, $c^{\boldsymbol{\phi}}$ is not always attained; see e.g. [11, Corollary 2.8(i)]. A sufficient condition for this to happen is given by the next lemma. We use the following notation. If $Q' \subset Q := \{1, \dots, q\}$ we consider the subsystem of (1.4) obtained by deleting all components of $\overline{u}_h$ for every $h \notin Q'$, and we denote by $\mathcal{J}^{\boldsymbol{\phi}}_{Q'}$ and $\mathcal{N}^{\boldsymbol{\phi}}_{Q'}$ the functional and the Nehari set associated to this subsystem. We write

$$c^{\boldsymbol{\phi}}_{Q'} := \inf_{\boldsymbol{u}\in\mathcal{N}^{\boldsymbol{\phi}}_{Q'}} \mathcal{J}^{\boldsymbol{\phi}}_{Q'}(\boldsymbol{u}).$$

If $Q' = \{h\}$ we omit the curly brackets and write, for instance, $c^{\boldsymbol{\phi}}_h$ or $\mathcal{J}^{\boldsymbol{\phi}}_h$.

Lemma 3.4 *(**Compactness**) Let $N \neq 5$, $m \geq 5$ and $\phi_h : G_m \to \mathbb{Z}_2$ be as in (3.3). If, for each $h \in Q := \{1, \dots, q\}$, the strict inequality*

$$c^{\phi} < \begin{cases} c^{\phi}_{Q \smallsetminus \{h\}} + m\mu_h \frac{p-1}{2p} \|\omega\|^2, & \text{if } h \in Q^+, \\ c^{\phi}_{Q \smallsetminus \{h\}} + 2m\mu_h \frac{p-1}{2p} \|\omega\|^2, & \text{if } h \in Q^-, \end{cases} \tag{3.4}$$

holds true, then c^{ϕ} is attained, where ω is the positive radial solution to (1.7) and μ_h is given by (1.5).

Proof This statement follows by combining [11, Corollary 2.8(ii)] with [11, Equation (5.1)]. □

To verify condition (3.4) we introduce a suitable test function. Fix $m \geq 5$ and let K_m be as in Definitions 3.3. If $h \in Q^+$, we take $\zeta_h := (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ and, for each $R > 1$, we define

$$\widehat{\sigma}_{hR}(x) := \sum_{g \in K_m} \omega(x - Rg\zeta_h), \qquad x \in \mathbb{R}^N.$$

If $h \in Q^-$ we take $\zeta_h := (1, 0, 0)$ and we define

$$\widehat{\sigma}_{hR}(x) := \sum_{g \in G'_m} \phi_h(g)\, \omega(x - Rg\zeta_h), \qquad x \in \mathbb{R}^N,$$

where ω is the positive radial solution to (1.7) and G'_m is the subgroup of G_m generated by $K_m \cup \{\tau\}$. Note that $\widehat{\sigma}_{hR}(gx) = \phi_h(g)\widehat{\sigma}_{hR}(x)$ for every $g \in G_m$, $x \in \mathbb{R}^N$. Let

$$\sigma_{hR} := t_{hR}\widehat{\sigma}_{hR}, \tag{3.5}$$

where $t_{hR} > 0$ is chosen so that $\|\sigma_{hR}\|^2 = \int_{\mathbb{R}^N} |\sigma_{hR}|^{2p}$.

Lemma 3.5 *If $m \geq 5$, then, for each $h \in \{1, \dots, q\}$, there exist $\bar{t}_h = (t_{\ell_{h-1}+1}, \dots, t_{\ell_h}) \in (0, \infty)^{\ell_h - \ell_{h-1}}$ and $C_0, R_0 > 0$ such that $\bar{t}_h\sigma_{hR} := (t_{\ell_{h-1}+1}\sigma_{hR}, \dots, t_{\ell_h}\sigma_{hR}) \in \mathcal{N}_h^{\phi}$ and*

$$\mathcal{J}_h^{\phi}(\bar{t}_h\sigma_{hR}) \leq |G_m\zeta_h|\, \mu_h \frac{p-1}{2p} \|\omega\|^2 - C_0 \mathrm{e}^{-Rd_m} \qquad \text{for every } R \geq R_0,$$

where $|G_m\zeta_h|$ is the cardinality of the G_m-orbit of ζ_h, i.e., $|G_m\zeta_h| = m$ if $h \in Q^+$ and $|G_m\zeta_h| = 2m$ if $h \in Q^-$, and

$$d_m := |1 - \mathrm{e}^{2\pi \mathrm{i}/m}|. \tag{3.6}$$

Proof Take $\bar{t}_h = (t_{\ell_{h-1}+1}, \dots, t_{\ell_h}) \in (0, \infty)^{\ell_h - \ell_{h-1}}$ such that

$$\sum_{i \in I_h} t_i^2 = \sum_{i,j \in I_h} \beta_{ij} t_j^p t_i^p = \mu_h$$

and apply [11, Proposition 4.1(i) and Lemma 4.4]. □

Proof of Theorem 1.2 Assume (B_1) and let $\phi_h : G_m \to \mathbb{Z}_2$ be given by (3.3). For $q = 1$ and $m \geq 5$ it is proved in [11, Corollary 4.2 and Proposition 4.5] that c^{ϕ} is attained at $\boldsymbol{u} \in \mathcal{N}^{\phi}$ satisfying

$$\|\boldsymbol{u}\|^2 = \mu_1 \|\omega\|^2 \text{ if } Q^+ = \{1\} \qquad \text{and} \qquad \|\boldsymbol{u}\|^2 < 2m\, \mu_1 \|\omega\|^2 \text{ if } Q^- = \{1\}.$$

Taking $m = 5$ gives statement (b).

Fix $m = 6$. We claim that $c^{\boldsymbol{\phi}}$ is attained and that the estimate (c) holds true for every $q \geq 2$. To prove this claim, we proceed by induction. Assume it is true for $q-1$ with $q \geq 2$.

We will show that the compactness condition (3.4) holds true. Using a change of coordinates, it suffices to argue for $h = q$. By induction hypothesis there exists $\boldsymbol{w} = (\overline{w}_1, \dots, \overline{w}_{q-1}) \in \mathcal{N}^{\boldsymbol{\phi}}_{Q\smallsetminus\{q\}}$ such that $\mathcal{J}^{\boldsymbol{\phi}}_{Q\smallsetminus\{q\}}(\boldsymbol{w}) = c^{\boldsymbol{\phi}}_{Q\smallsetminus\{q\}}$. For each $R > 1$ let σ_{qR} be as in (3.5) and take $\bar{t}_q \in (0, \infty)^{\ell-\ell_{q-1}}$ as in Lemma 3.5. Set $\overline{w}_{hR} = \overline{w}_h$ for $h = 1, \dots, q-1$ and $\overline{w}_{qR} = \bar{t}_q\sigma_{qR}$, and define $\boldsymbol{w}_R = (w_{1R}, \dots, w_{\ell R}) := (\overline{w}_{1R}, \dots, \overline{w}_{qR})$. Then, as $\boldsymbol{w} \in \mathcal{N}^{\boldsymbol{\phi}}_{Q\smallsetminus\{q\}}$ and the interaction between the components of $\boldsymbol{w}$ and σ_{qR} tends to 0 as $R \to \infty$, we have that $\boldsymbol{w}_R$ satisfies (3.2) for large enough R and, as a consequence, there exist $R_1 > 0$ and $(s_{1R}, \dots, s_{qR}) \in [1/2, 2]^q$ such that $(s_{1R}\overline{w}_{1R}, \dots, s_{qR}\overline{w}_{qR}) \in \mathcal{N}^{\boldsymbol{\phi}}$ if $R \geq R_1$. Set $\boldsymbol{u}_R = (u_{1R}, \dots, u_{\ell R}) := (s_{1R}\overline{w}_{1R}, \dots, s_{qR}\overline{w}_{qR})$. Using that $\boldsymbol{w} \in \mathcal{N}^{\boldsymbol{\phi}}_{Q\smallsetminus\{q\}}$ and $\bar{t}_q\sigma_{qR} \in \mathcal{N}^{\boldsymbol{\phi}}_q$, from the last statement in Lemma 3.1(ii) and Lemma 3.5 we derive

$$
\begin{aligned}
\mathcal{J}^{\boldsymbol{\phi}}(\boldsymbol{u}_R) &= \frac{1}{2}\sum_{i=1}^{\ell}\|u_{iR}\|^2 - \frac{1}{2p}\sum_{i,j=1}^{\ell}\beta_{ij}\int_{\mathbb{R}^N}|u_{iR}|^p|u_{jR}|^p \\
&\leq \mathcal{J}^{\boldsymbol{\phi}}_{Q\smallsetminus\{q\}}(\boldsymbol{w}) + \mathcal{J}^{\boldsymbol{\phi}}_q(\bar{t}_q\sigma_{qR}) - \frac{1}{p}\sum_{h=1}^{q-1}\sum_{(i,j)\in I_h\times I_q}\beta_{ij}\int_{\mathbb{R}^N}|s_{hR}w_{iR}|^p|s_{qR}w_{jR}|^p \\
&\leq c^{\boldsymbol{\phi}}_{Q\smallsetminus\{q\}} + |G_m\zeta_h|\,\mu_q\,\tfrac{p-1}{2p}\|\omega\|^2 - C_0\mathrm{e}^{-Rd_m} + C_1\sum_{h=1}^{q-1}\sum_{i\in I_h}\int_{\mathbb{R}^N}|w_{iR}|^p|\sigma_{qR}|^p,
\end{aligned}
$$

if $R \geq \max\{R_0, R_1\}$, where C_0 and C_1 are positive constants and d_m is given in (3.6).

It is well known that $|\omega(x)| \leq C\mathrm{e}^{-|x|}$ and, as $\boldsymbol{w}$ solves a subsystem of (1.4), Theorem 1.1 asserts that

$$
|w_{iR}(x)| \leq C\mathrm{e}^{-|x|} \quad \text{for every } i \in I_h \text{ with } h = 1, \dots, q-1.
$$

Therefore, for every $g \in G_m$,

$$
\int_{\mathbb{R}^N}|w_{iR}|^p|\omega(\cdot - Rg\zeta_h)|^p \leq C\int_{\mathbb{R}^N}\mathrm{e}^{-p|x|}\,\mathrm{e}^{-p|x-Rg\zeta_h|}\,\mathrm{d}x \leq C\mathrm{e}^{-Rp}.
$$

So, if $p > d_m$, we conclude that

$$
c^{\boldsymbol{\phi}} < c^{\boldsymbol{\phi}}_{Q\smallsetminus\{q\}} + |G_m\zeta_h|\,\mu_q\,\tfrac{p-1}{2p}\|\omega\|^2
$$

and, by Lemmas 3.4 and 3.2, $c^{\boldsymbol{\phi}}$ is attained at a block-wise nontrivial solution $\boldsymbol{u}$ of (1.4) such that every component of $\bar{u}_h$ is positive if $h \in Q^+$ and every component of $\bar{u}_h$ is nonradial and changes sign if $h \in Q^-$. Furthermore, since we are assuming (B_2) and (B_3) with C_* as in (3.7) below, [11, Theorem 3.3] asserts that $\boldsymbol{u}$ is fully nontrivial.

Finally, note that $p > 1 = d_m$ because $m = 6$. As $|G_m\zeta_h| = 6$ if $h \in Q^+$ and $|G_m\zeta_h| = 12$ if $h \in Q^-$, the estimate in statement (c) follows by induction. □

Remark 3.6 If $m = 5$ and $p > d_m$ we arrive to a similar conclusion, where, in this case, the constant b_h in statement (b) is 5 if $h \in Q^+$ and it is 10 if $h \in Q^-$. Note, however, that numbers p satisfying $d_5 = 2\sin\frac{\pi}{5} < p < \frac{N}{N-2}$ exist only for $N \leq 13$.

Remark 3.7 For ϕ_h as in (3.3), the constant $C_* > 0$ appearing in (B_3) depends on N, p, q, and Q^+. It is explicitly defined in [11, Equation (3.1)] as

$$C_* := \left(\frac{p d_{\boldsymbol{\phi}}}{(p-1) S_{\boldsymbol{\phi}}^{\frac{p}{p-1}}} \right)^p, \tag{3.7}$$

where

$$d_\phi := \frac{p-1}{2p} \inf_{(v_1,\ldots,v_q)\in\mathcal{U}^{\boldsymbol{\phi}}} \sum_{h=1}^{q} \|v_h\|^2$$

with $\mathcal{U}^{\boldsymbol{\phi}} := \{(v_1,\ldots,v_q) \; : \; v_h \in H^1(\mathbb{R}^N)^{\phi_h} \setminus \{0\}, \; \|v_h\|^2 = |v_h|_{2p}^{2p}, \; v_h v_k = 0 \text{ if } h \neq k\}$, and

$$S_{\boldsymbol{\phi}} := \min_{h=1,\ldots,q} \inf_{v\in H^1(\mathbb{R}^N)^{\phi_h}\setminus\{0\}} \frac{\|v\|^2}{|v|_{2p}^2}.$$

Remark 3.8 In the proof of Theorem 1.2 we use [1, Theorem 2.3], which also characterizes the *sharp* decay rate for *positive* components by providing a bound from below. This kind of information can be useful to show uniqueness of positive solutions for some problems, see [4, Section 8.2].

To conclude, we discuss some special cases.

Examples 3.9 Assume (B_1) and let $p \in (1, \frac{2^*}{2})$.

(a) If $q = 1$ the system (1.4) is cooperative and more can be said. Indeed, it is shown in [11, Corollary 4.2 and Proposition 4.5] that, if (B_2) is satisfied, then (1.4) has a synchronized solution $\boldsymbol{u} = (t_1 u, \ldots, t_\ell u)$, where $(t_1,\ldots,t_\ell) \in (0,\infty)^\ell$ is a minimizer for (1.5) and u is a nontrivial ϕ-equivariant least energy solution of the equation

$$-\Delta u + u = |u|^{2p-2} u, \qquad u \in H^1(\mathbb{R}^N)^\phi. \tag{3.8}$$

Here, if $Q^+ = \{1\}$, then $\phi \equiv 1$ (and therefore $u = \omega$) and $\|\boldsymbol{u}\|^2 \leq \mu_1 \|\omega\|^2$. On the other hand, if $Q^- = \{1\}$, then $\phi : G_m \to \mathbb{Z}_2$ is the homomorphism θ given in *Definitions* 3.3 and $\|\boldsymbol{u}\|^2 \leq 10\mu_1\|\omega\|^2$.

(b) If $q = \ell \geq 2$ the system (1.4) is competitive, i.e., $\beta_{ii} > 0$ and $\beta_{ij} < 0$ if $i \neq j$. Assumptions (B_2) and (B_3) are automatically satisfied and, as $\mu_i = \beta_{ii}^{-\frac{1}{p-1}}$, the estimate in *Theorem* 1.2(c) becomes

$$\begin{aligned}\|\boldsymbol{u}\|^2 &< \left(\min_{j\in Q} \left(a_j \beta_{jj}^{-\frac{1}{p-1}} + \sum_{i\in Q\setminus\{i\}} b_i \beta_{ii}^{-\frac{1}{p-1}} \right) \right) \|\omega\|^2 \\ &\leq \begin{cases} \left(6\,|Q^+| + 12\,|Q^-| - 5\right) \beta_0^{-\frac{1}{p-1}} \|\omega\|^2 & \text{if } Q^+ \neq \emptyset, \\ 12\,|Q^-| \beta_0^{-\frac{1}{p-1}} \|\omega\|^2 & \text{if } Q^+ = \emptyset, \end{cases}\end{aligned}$$

where $|Q^\pm|$ denotes the cardinality of $Q^\pm$ and $\beta_0 := \min\{\beta_{11},\ldots,\beta_{\ell\ell}\}$.

(c) Similarly, for any $q \geq 2$, the estimate in *Theorem* 1.2(c) yields

$$\|\boldsymbol{u}\|^2 \leq \begin{cases} \left(6\,|Q^+| + 12\,|Q^-| - 5\right) \mu_* \|\omega\|^2 & \text{if } Q^+ \neq \emptyset, \\ 12\,|Q^-|\, \mu_* \|\omega\|^2 & \text{if } Q^+ = \emptyset. \end{cases}$$

where $\mu_* = \max\{\mu_1, \ldots, \mu_q\}$.

Assumptions (B_2) and (B_3) guarantee that $\boldsymbol{u}$ is fully nontrivial. Note that the left-hand side of the inequality in (B_3) depends only on the entries of the submatrices $(\beta_{ij})_{i,j\in I_h}$, $h = 1, \ldots, q$, whereas the right-hand side only depends on the other entries. So, if the former are large enough with respect to the absolute values of the latter, (B_3) is satisfied. For example, if we take $\ell = 2q$ and the matrix is

$$\begin{pmatrix} \lambda & \lambda & \beta_{13} & \beta_{14} & \beta_{15} & \ldots & \beta_{1\ell} \\ \lambda & \lambda & \beta_{23} & \beta_{24} & \beta_{25} & \ldots & \beta_{2\ell} \\ \beta_{31} & \beta_{32} & \lambda & \lambda & \beta_{35} & \ldots & \beta_{3\ell} \\ \beta_{41} & \beta_{42} & \lambda & \lambda & \beta_{45} & \ldots & \beta_{4\ell} \\ \vdots & \vdots & & & \ddots & & \vdots \\ \beta_{\ell-1\,1} & & & \ldots & \beta_{\ell-1\,\ell-2} & \lambda & \lambda \\ \beta_{\ell 1} & & & \ldots & \beta_{\ell\,\ell-2} & \lambda & \lambda \end{pmatrix}.$$

with $\lambda > 0$ and $\beta_{ji} = \beta_{ij} < 0$, then (B_1) and (B_2) are satisfied. If, additionally,

$$\lambda > 4^{\frac{2p-1}{p-1}}(q-1)C_* \qquad \text{and} \qquad |\beta_{ij}| \leq 1,$$

then, for any $h = 1, \ldots, q$,

$$\Big(\min_{\{i,j\}\in E_h} \beta_{ij}\Big) \left[\frac{\min\limits_{h=1,\ldots,q} \max\limits_{i\in I_h} \beta_{ii}}{\sum\limits_{i,j\in I_h} \beta_{ij}}\right]^{\frac{p}{p-1}} = \lambda \left[\frac{\lambda}{4\lambda}\right]^{\frac{p}{p-1}} > C_* 4(q-1) \geq C_* \sum_{\substack{k=1\\k\neq h}}^{q} \sum_{\substack{i\in I_h\\j\in I_k}} |\beta_{ij}|$$

so (B_3) is satisfied.

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Declarations

Conflicts of interest The authors have no conflicts of interest to disclose. The manuscript has no associated data.

A An auxiliary result

Lemma A.1 *For every $r \geq 1$ there is a linear operator $E_r : H^1(\mathbb{R}^N \smallsetminus B_r) \to H^1(\mathbb{R}^N)$ such that, for every $u \in H^1(\mathbb{R}^N \smallsetminus B_r)$,*

(i) $E_r u = u$ *a.e. in* $\mathbb{R}^N \setminus B_r$,
(ii) $|E_r u|_2^2 \le C_1 |u|^2_{L^2(\mathbb{R}^N \setminus B_r)}$,
(iii) $\|E_r u\|^2 \le C_1 \|u\|^2_{H^1(\mathbb{R}^N \setminus B_r)}$,

for some positive constant C_1 *depending only on* N *and not on* r. *As a consequence, given* $p \in (1, \frac{2^*}{2})$ *there is a positive constant* C *depending only on* N *and* p *such that*

$$|u|_{L^{2p}(\mathbb{R}^N \setminus B_r)} \le C\|u\|_{H^1(\mathbb{R}^N \setminus B_r)} \quad \textit{for every } u \in H^1(\mathbb{R}^N \setminus B_r) \textit{ and every } r \ge 1.$$

Proof Fix a linear (extension) operator $E_1 : H^1(\mathbb{R}^N \setminus B_1) \to H^1(\mathbb{R}^N)$ and a positive constant C_1 satisfying (i), (ii) and (iii) for $r = 1$; see e.g. [16, Theorem 2.3.2]. For $r > 1$, set $\widehat{u}(x) := u(rx)$ and, for $u \in H^1(\mathbb{R}^N \setminus B_r)$, define

$$(E_r u)(y) := (E_1\widehat{u})\Big(\frac{y}{r}\Big).$$

Then, $\widehat{E_r u} = E_1\widehat{u}$. Clearly, E_r satisfies (i). Note that $|\widehat{u}|^2_{L^2(\mathbb{R}^N \setminus B_1)} = r^{-N}|u|^2_{L^2(\mathbb{R}^N \setminus B_r)}$ and that

$$\|\widehat{u}\|^2_{H^1(\mathbb{R}^N \setminus B_1)} = r^{-N}\left(\int_{\mathbb{R}^N \setminus B_r}\left(r^2|\nabla u|^2 + |u|^2\right)\right).$$

Similar identities hold true when we replace $\mathbb{R}^N \setminus B_1$ and $\mathbb{R}^N \setminus B_r$ with $\mathbb{R}^N$. Therefore,

$$r^{-N}|E_r u|_2^2 = |\widehat{E_r u}|_2^2 = |E_1\widehat{u}|_2^2 \le C_1\|\widehat{u}\|^2_{L^2(\mathbb{R}^N \setminus B_1)} = r^{-N}C_1|u|^2_{L^2(\mathbb{R}^N \setminus B_r)},$$

which yields (ii). Furthermore,

$$\begin{aligned} r^{-N}\left(\int_{\mathbb{R}^N}\left(r^2|\nabla(E_r u)|^2 + |E_r u|^2\right)\right) &= \|\widehat{E_r u}\|^2 = \|E_1\widehat{u}\|^2 \\ &\le C_1\|\widehat{u}\|^2_{H^1(\mathbb{R}^N \setminus B_1)} = r^{-N}C_1\left(\int_{\mathbb{R}^N \setminus B_r}\left(r^2|\nabla u|^2 + |u|^2\right)\right). \end{aligned}$$

This inequality, combined with (ii), yields

$$\begin{aligned} r^2\|E_r u\|^2 &= \int_{\mathbb{R}^N}\left(r^2|\nabla(E_r u)|^2 + |E_r u|^2\right) + (r^2 - 1)\int_{\mathbb{R}^N}|E_r u|^2 \\ &\le C_1\int_{\mathbb{R}^N \setminus B_r}\left(r^2|\nabla u|^2 + |u|^2\right) + C_1(r^2 - 1)\int_{\mathbb{R}^N \setminus B_r}|u|^2 = r^2C_1\|u\|^2_{H^1(\mathbb{R}^N \setminus B_r)}, \end{aligned}$$

which gives (iii).

For $p \in (1, \frac{N}{N-2})$ let $C_2 = C_2(N, p)$ be the constant for the Sobolev embedding $H^1(\mathbb{R}^N) \subset L^{2p}(\mathbb{R}^N)$. Then, for any $u \in H^1(\mathbb{R}^N \setminus B_r)$, using statements (i) and (iii) we obtain

$$|u|^2_{L^{2p}(\mathbb{R}^N \setminus B_r)} \le |E_r u|^2_{2p} \le C_2\|E_r u\|^2 \le C_2C_1\|u\|^2_{H^1(\mathbb{R}^N \setminus B_r)},$$

as claimed. □

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