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Calculus of Variations

ℓ-convex Legendre curves and geometric inequalities

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Abstract

In this paper we consider ℓ-convex Legendre curves, which are natural generalizations of strictly convex curves. We generalize various optimal geometric inequalities, isoperimetric inequality, Bonnesen's inequality and Green–Osher inequality, for strictly convex curves to ones for ℓ-convex Legendre curves. Moreover we generalize the inverse curvature curve flow for this class of Legendre curves and prove that it always converges to a compact soliton after rescaling. Unlike in the class of regular curves, there are infinitely many compact solitons, which include circles and astroids.

1 Introduction

In this paper we are interested in Legendre curves, which are in general not regular, but have a smooth unit normal vector field. Such a curve is also called frontal, especially in the theory of singularities. Precisely, a curve (or one dimensional map) $\gamma : I \to \mathbb{R}^2$ is a Legendre curve, if there exists a unit vector field $\nu : I \to \mathbb{S}^1$ such that

$$\langle \gamma'(\theta), \nu(\theta) \rangle = 0, \quad \text{for any } \theta \in I, \tag{1.1}$$

where $\gamma'(\theta) = \frac{d\gamma}{d\theta}$. It is easy to see that this condition is equivalent to the one that $(\gamma, \nu) : I \to \mathbb{R}^2 \times \mathbb{S}^1$ is a Legendre curve w.r.t. the canonical contact structure of $\mathbb{R}^2 \times \mathbb{S}^1$. Let J be the standard complex structure and define $\mu := J\nu$. Then one can define a *curvature pair* for γ:

$$\ell := \langle \nu', \mu \rangle, \qquad \beta := \langle \gamma', \mu \rangle. \tag{1.2}$$

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See for example [4, 17]. It is clear that a Legendre curve is regular if and only if β never vanishes. In this case γ has the usual curvature κ and it is easy to see that $\kappa = \ell/|\beta|$. In general β has zeros, and hence κ could not be defined or is infinite.

In our paper we focus on a special class of Legendre curves, which have positive ℓ. We call such a curve an *ℓ-convex curve.* An ℓ-convex curve is in general not convex, see [5]. Our first observation is that any closed ℓ-convex Legendre curve can be reparametrized as a map

$$\gamma(\theta) = p(\theta)(\cos\theta, \sin\theta) + p'(\theta)(-\sin\theta, \cos\theta), \tag{1.3}$$

for a smooth function $p : \mathbb{S}^1 \to \mathbb{R}$, which will be called a support function for the ℓ-positive Legendre curve. It is well-known that a closed, strictly convex curve can be represented by (1.3) with $p > 0$ and $p + p'' > 0$ and p is the Minkowski support function. Hence the class of ℓ-convex Legendre curves is much larger than the one of strictly convex curves. The study of this class goes back at least to the work of Geppert [9], where the author called such a curve *stützbar* (supportable). Such a curve is also called *hedgehog*. For the recent study on such curves see [14]. A map with form (1.3) is clearly a Legendre curve with

$$\ell = 1, \qquad \beta = p + p''.$$

For a strictly convex curve it is clear that β is the principal radius, i.e., $\beta = 1/\kappa$. Hence with the terminology of the curvature pair (ℓ, β) for Legendre curves we have a very natural generalization of the concept of principal radius, which in general now could be zero or even negative.

The paper consists of three parts. The first part of this paper is to generalize isoperimetric inequalities (isoperimetric type inequalities) for convex curves to ones for ℓ-convex curves. One of inequalities we obtain is

Theorem 1.1 *For any ℓ-convex Legendre curve, there holds*

$$0 \le \left(\int_{\mathbb{S}^1} \beta^2 d\theta - 2A\right) - 8\left(\frac{L^2}{4\pi} - A\right) \le \frac{1}{12}\int_{\mathbb{S}^1} \beta'^2 d\theta - 2\left(\frac{L^2}{4\pi} - A\right). \tag{1.4}$$

Moreover, equality in the first inequality holds if and only if γ is generated by

$$p = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta \tag{1.5}$$

and equality in the second inequality holds if and only if

$$p = a_0 + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + a_3\cos 3\theta + b_3\sin 3\theta. \tag{1.6}$$

Here L and A are the algebraic length and the algebraic area of an ℓ-convex Legendre curve, which are natural generalizations of the length and the enclosed area of a regular planar curve. See Definition 3.4 below. We remark that L and A could be zero or negative. Indeed the class of Legendre curves with $L = 0$ (which implies $A < 0$ by Theorem 1.1) is an important one which we are interested in. When γ is a regular convex curve, the first inequality was proved in [13], while the second in a very recent work [11]. There are a series of optimal inequalities in the latter paper by using a higher order Wirtinger inequality for regular convex curves. When $L = 0$, an ℓ-convex Legendre curve is nerve a regular curve. As a direct consequence we have a characterization of astroid in the class of ℓ-convex Legendre curves with $L = 0$.

Corollary 1.2 *If γ is an ℓ-convex Legendre curve with $L = 0$, then*

$$\int_{\mathbb{S}^1} \beta^2 d\theta \ge -6A = 6|A|$$

with equality if and only if γ is an astroid, i.e.,

$$p = a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta.$$

For a figure of an astroid centered at the origin, see Fig. 1 below. For further discussion see below.

Another consequence (1.4) is a

$$\int_{\mathbb{S}^1} \beta^2 d\theta \geq 2A. \tag{1.7}$$

Moreover equality holds iff γ is a circle. When γ is a regular, (1.7) is equivalent to

$$\int_\gamma \frac{1}{\kappa} ds \geq 2A,$$

where s is the arc-length element and κ the curvature of γ. In this case, it is a known inequality. Its higher dimensional analogue is the Heintze-Karcher-Ros inequality.

In the second part we prove Bonnesen's inequality and Green–Osher's inequality [8] for ℓ-convex curves.

Theorem 1.3 *(Bonnesen's inequality) If γ is an ℓ-convex and not a circle, then*

$$\pi t^2 - tL + A < 0, \quad t \in [r_{in}, r_{out}], \tag{1.8}$$

$$L^2 - 4\pi A > \pi^2 (r_{out} - r_{in})^2. \tag{1.9}$$

Here r_{in} and r_{out} are the generalization of the ordinary inradius r_{in} and outradius r_{out} for regular curves. See Definition 5.1 below. We emphasize that r_{in} and r_{out} are not necessarily positive. Example 5.1 shows that our Bonnesen's inequality is sharper than ordinary Bonnesen's inequality for ℓ-convex curves.

Theorem 1.4 *[Green–Osher inequality] Let F be a positive convex function on $\mathbb{R}$, i.e. $F''(x) \geq 0$, for any $x \in \mathbb{R}$. Then*

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} F(\beta) d\theta \geq \frac{1}{2}(F(-t_1) + F(-t_2)), \tag{1.10}$$

where

$$t_1 = -\frac{L}{2\pi} + \frac{\sqrt{L^2 - 4\pi A}}{2\pi}, \quad t_2 = -\frac{L}{2\pi} - \frac{\sqrt{L^2 - 4\pi A}}{2\pi}.$$

It is easy to see that (1.7) follows also from (1.10) by choosing $F(x) = x^2$. We prove this result following closely the method given in [8], while generalizing certain geometric concepts to ℓ-convex curves. In fact, we use the support function p to replace the curve itself and use analytic properties of p to replace the geometric properties of the curve. It seems that this way is more flexible.

In the third part of this paper we observe that the inverse curvature curve flow is well-defined for ℓ-convex curves and prove the global convergence.

Theorem 1.5 *The inverse curvature flow has a global convergence for any initial ℓ-convex curve with algebraic length L_0. Moreover we have*

(1) If $L_0 > 0$, the flow expands exponentially and the evolving curve becomes convex after a certain time. After a suitable rescaling, the flow converges to a circle.

(2) *If $L_0 = 0$, then the flow shrinks and after a suitable rescaling it converges to a Legendre curve with a form*

$$a_1 \cos\theta + b_1 \sin\theta + a_k \cos k\theta + b_k \sin k\theta \tag{1.11}$$

for some integer $k \geq 2$.
The curves given by (1.11), together with circles, are all closed solitons of the inverse curvature flow.

Here L_0 is the algebraic length of the initial curve, which we may always assume $L_0 \geq 0$. See Definition 3.4 and the following remark.

Though the proof is elementary, the phenomenon that Theorem 1.5 shows is rather interesting. For the convex regular curves, it is well-known that the inverse curvature curve flow expands and admits only circles as its compact solitons [1]. Now for ℓ-convex curves we have two cases: when $L_0 > 0$, the inverse curvature flow first evolves to a regular convex curve and then expands as in the case of regular curves; when $L_0 = 0$ then the flow shrinks to a point and converses to a curve with form (1.11) after rescaling. The curves given by (1.11) are compact solitons. Therefore the inverse curvature flow admits infinitely many compact solitons. For non-closed solitons see [2]. In (1.11) in Theorem 1.5 (2), a_k and b_k could be both zero. In this case the flow starts from a constant curve (i.e. a point) and keeps fixed. We remark that a point $(a_1, b_1) \in \mathbb{R}^2$ is also a Legendre curve with support function $p = a_1 \cos\theta + b_1 \sin\theta$. See Sect. 3.

It would be interesting to ask if one can generalize these results to higher dimensional non-regular hypersurfaces. We will consider this problem later.

The curves with support function in form (1.11) play an important role in this paper. They are called astroids for $k = 2$ and deltoids for $k = 3$. See Figs. 1 and 2. Actually one of main aims of this paper is to show these curves play a very similar role for Legendre curves as the circles for regular planar curves.

The paper is organized: In Sect. 2 we recall notions of frontals, fronts and Legendre curves and their fundamental theorems. In Sect. 3 we introduce ℓ-convex curves. The isoperimetric inequalities for ℓ-convex curves are presented in Sect. 4, while Bonnesen's inequality and Green–Osher's inequality are proved in Sect. 5. We study the inverse curvature curve flow in Sect. 6.

2 Fronts and Legendre curves

As defined in the Introduction, a Legendre curve is a one dimensional map $\gamma : I \to \mathbb{R}^2$, together with a unit vector field $\nu : I \to \mathbb{S}^1$, satisfying

$$\langle \gamma'(\theta), \nu(\theta) \rangle = 0, \quad \text{for any } \theta \in I. \tag{2.1}$$

It is also called a frontal. If $|\gamma'(\theta)|^2 + |\nu'(\theta)|^2 \neq 0, \forall \theta \in I$, then it is called a front.

For frontals and fronts, there is a complete equivalent definition by using the language of Legendre curves on the contact manifold $T_1\mathbb{R}^2$, the unit circle bundle of $\mathbb{R}^2$. To describe the structure of interest here, we first review the geometry of tangent bundle $\pi_0 : T\mathbb{R}^2 \to \mathbb{R}^2$ of $\mathbb{R}^2$, that is $T\mathbb{R}^2$ is the tangential bundle of $\mathbb{R}^2$, which is equivalent to $\mathbb{R}^2 \times \mathbb{R}^2$. If (x_1, x_2) are local coordinates on $\mathbb{R}^2$, then (x_1, x_2) together with the fibre coordinates (y_1, y_2) form local coordinates on $T\mathbb{R}^2$, which is Kählerian with the standard inner product g_0, the integrable complex structure J_0 and the Liouville form η_0:

$$g_0=\sum_{i=1}^{2}(dx_i\otimes dx_i+dy_i\otimes dy_i),\quad J_0\frac{\partial}{\partial x_i}=\frac{\partial}{\partial y_i},\quad J_0\frac{\partial}{\partial y_i}=-\frac{\partial}{\partial x_i},\quad \eta_0=\sum_{i=1}^{2}y_idx_i.$$

The unit circle bundle, $\pi : T_1\mathbb{R}^2=\mathbb{R}^2\times\mathbb{S}^1\to\mathbb{R}^2$, is the hypersurface of $T\mathbb{R}^2$ defined by

$$\sum_{i=1}^{2}y_i^2=1.$$

The vector field $\sum_{i=1}^{2}y_i\frac{\partial}{\partial y_i}$ is a unit normal as well as the position vector for points in $T_1\mathbb{R}^2$. $T_1\mathbb{R}^2$ is a contact manifold with a contact metric structure (g,ϕ,η,ξ) that is given by

$$\eta=\eta_0|_{T_1\mathbb{R}^2},\quad \xi=y_i\frac{\partial}{\partial x_i}\Big|_{T_1\mathbb{R}^2},\quad g=g_0|_{T_1\mathbb{R}^2},\quad \phi X=J_0X-\eta_0(X)(y_i\frac{\partial}{\partial y_i}).$$

We remark that $T_1\mathbb{R}^2$ is not Sasakian, since one can check that ξ is not a Killing vector field. See also Remark 2.2.

Definition 2.1 Let I be an interval or $\mathbb{R}$. A curve $\Gamma : I\to T_1\mathbb{R}^2$ is called a *Legendre curve*, if

$$\Gamma^*(\eta)=0,\qquad \text{on } I.$$

That is, the pull-back of the canonical contact form η vanishes everywhere on I. If it is immersion, then it is called Legendre immersion.

For a Legendre curve $\Gamma : I\to T_1\mathbb{R}^2=\mathbb{R}^2\times\mathbb{S}^1$, one can decompose it into

$$\Gamma=(\gamma,\nu).$$

It is easy to see that

$$\Gamma^*(\eta)=\langle\gamma',\nu\rangle.$$

Hence, Γ is a Legendre curve if and only if γ, together with ν, is a frontal. Moreover, if Γ is an immersion, then γ is a front.

As a curve in a 3-dimensional manifold, we have the ordinary definition of its second fundamental form. We present them in terms of ℓ, β and their derivatives. Along a Legendre immersion $\Gamma=(\gamma,\nu) : I\to T_1\mathbb{R}^2$, we choose a unit orthogonal adapted frames

$$\begin{cases} e_1=\frac{1}{\sqrt{\ell^2+\beta^2}}(\gamma',\nu')=\frac{1}{\sqrt{\ell^2+\beta^2}}(\beta\mu,\ell\mu),\\ e_2=\phi e_1=\frac{1}{\sqrt{\ell^2+\beta^2}}(-\nu',\gamma')=\frac{1}{\sqrt{\ell^2+\beta^2}}(-\ell\mu,\beta\mu),\\ e_3=(\nu,0)=\xi. \end{cases}\tag{2.2}$$

Hence

$$\Gamma'=\sqrt{\ell^2+\beta^2}e_1,\quad \Gamma''\Big|_{T_1\mathbb{R}^2}=\frac{\ell\ell'+\beta\beta'}{\sqrt{\ell^2+\beta^2}}e_1+(\ell^2+\beta^2)(h_{11}^2e_2+h_{11}^3e_3),$$

where

$$h_{11}^2=\frac{\ell'\beta-\ell\beta'}{(\sqrt{\ell^2+\beta^2})^3},\quad h_{11}^3=-\frac{\ell\beta}{\ell^2+\beta^2}$$

are coefficients of the second fundamental form of the Legendre curve Γ in $T_1\mathbb{R}^2$.

Remark 2.2 A Legendrian surface (or curve) in a Sasakain manifolds has a property that its second fundamental form with respect to the Reeb field ξ vanishes, since ξ is Killing. The above computation shows that $T_1\mathbb{R}^2$ is not Sasakian.

We now give an example of Legendre curves. For more examples see Sect. 3 below.

Example 2.1 Let m, n be two natural numbers with $k = n - m \geq 1$. Then $\Gamma = (\gamma, \nu) : \mathbb{R} \to T_1\mathbb{R}^2$ defined by

$$\gamma(\theta) = \left(\frac{1}{m}\theta^m, \frac{1}{n}\theta^n\right), \quad \nu(\theta) = \frac{1}{\sqrt{\theta^{2k}+1}}(-\theta^k, 1)$$

is a Legendre curve. It is a Legendre immersion if and only if $m = 1$ or $k = 1$. Such a curve is called a curve of type (m, n), which is important in the theory of singularities. It is easy to check that

$$\ell(\theta) = \frac{k\theta^{k-1}}{\theta^{2k}+1}, \quad \beta(\theta) = -\theta^{m-1}\sqrt{\theta^{2k}+1}.$$

It is clear that when $k = 1$ then $\ell(\theta) = \frac{1}{\theta^2+1} > 0$, while β changes sign near 0, if m is even. Hence the curve is not convex, by [5]. One can easily show that the curve is convex for odd m.

3 ℓ-convex Legendre curves

First we generalize convex curves to ℓ-convex curves.

Definition 3.1 *A Legendre curve is called* ℓ-convex *if* $\ell > 0$.

It was proved in [5] that a Legendre curve is convex if and only if both ℓ and β do not change sign. By changing the orientation, it is equivalent to $\ell \geq 0$ and either $\beta \geq 0$ or $\beta \leq 0$. In this paper we consider the set of ℓ-convex Legendre curves, which is much more larger than the set of strictly convex regular curves. This can be seen clearly in the following Lemmas.

Lemma 3.2 *If a front $\gamma : I = [a, b] \to \mathbb{R}^2$ is ℓ-convex, then there exists a parametrization $\varphi : J \to I$ such that $\tilde{\gamma} = \gamma \circ \varphi$ satisfies $\tilde{\ell} = 1$.*

Proof Let $\psi(t) = \int_a^t 1/\ell(s)ds$ and $\varphi : J \to I$ its inverse. Here $J = [0, \int_a^b \frac{1}{\ell(s)}ds]$. Set

$$\tilde{\gamma}(t) := \gamma \circ \varphi, \quad \tilde{\nu} = \nu \circ \varphi.$$

It is clear that $(\tilde{\gamma}, \tilde{\nu})$ is a Legendre curve with $\tilde{\mu} = \mu \circ \varphi$. Moreover

$$\tilde{\ell} = \langle \tilde{\nu}', \tilde{\mu} \rangle = \varphi' \ell \circ \varphi = 1.$$

□

We call the parametrization founded in Lemma 3.2 *canonical parametrization* for ℓ-convex curves. For ℓ-convex curves we always use the canonical parametrization and hence consider only Legendre curves with $\ell = 1$.

Lemma 3.3 *If a closed Legendre curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ has $\ell(\theta) = 1$ for any $\theta \in \mathbb{S}^1$, then there exists a function $p : \mathbb{S}^1 \to \mathbb{R}$ and $\theta_0 \in \mathbb{S}^1$ such that*

$$\gamma(\theta + \theta_0) = p(\theta)(\cos\theta, \sin\theta) + p'(\theta)(-\sin\theta, \cos\theta).$$

Proof Let $f(\theta)$ be the angle function of ν, i.e., $\nu(\theta) = (\cos f(\theta), \sin f(\theta))$. It is clear that $\mu(\theta) = (-\sin f(\theta), \cos f(\theta))$. We can represent γ by

$$\gamma(\theta) = p(\theta)\nu(\theta) + q(\theta)\mu(\theta).$$

Since $\langle \gamma', \nu \rangle = p' - q$, the Legendrian condition (2.1) is equivalent to $q = p'$. The condition $\ell = 1$ is equivalent to

$$1 = \ell(\theta) = \langle \nu', \mu \rangle = f',$$

i.e., $f(\theta) = \theta - \theta_0$, for some $\theta_0 \in \mathbb{S}^1$. Therefore we have

$$\gamma(\theta) = p(\theta - \theta_0)(\cos(\theta - \theta_0), \sin(\theta - \theta_0)) + p'(\theta - \theta_0)(-\sin(\theta - \theta_0), \cos(\theta - \theta_0)).$$

□

Moreover, it is clear that a map with the following form

$$\gamma(\theta) = p(\theta)(\cos\theta, \sin\theta) + p'(\theta)(-\sin\theta, \cos\theta) \tag{3.1}$$

is a Legendre curve with $\nu(\theta) = (\cos\theta, \sin\theta)$ and $\ell = 1$. Hence to consider ℓ-convex Legendre curves we only need to consider curves with form (3.1). For a convex regular curve the associated function p is the so-called Minkowski support function. Therefore, we call the function p in (3.1) *support function* of γ. From now on all curves we consider are closed Legendre curves with form (3.1).

Definition 3.4 Let γ be a closed Legendre curve with support function p. We define its (algebraic) length L and (algebraic) enclosed area A by

$$L = \int_{\mathbb{S}^1} (p + p'')d\theta = \int_{\mathbb{S}^1} p d\theta, \qquad A = \frac{1}{2}\int_{\mathbb{S}^1} p(p + p'')d\theta.$$

We emphasize that the length and the area of a Legendre curve could be zero or even negative. By replacing p to $-p$ we can change the sign of L and keep the sign of A. Hence we only need to consider ℓ-convex curves with $L \geq 0$ in this paper. If a curve γ is strictly convex, then it is well-known that p can be chosen such that

$$p(\theta) > 0, \quad p(\theta) + p''(\theta) > 0, \quad \theta \in \mathbb{S}^1.$$

Remark 3.5 An ℓ-convex curve with support p has

$$\beta = p + p''. \tag{3.2}$$

One can easily check that β satisfies

$$\int_{\mathbb{S}^1} \beta(\theta)\cos\theta d\theta = \int_{\mathbb{S}^1} \beta(\theta)\sin\theta d\theta = 0. \tag{3.3}$$

It follows from $\gamma'(\theta) = \beta(\theta)\mu(\theta)$. Vice versa, if a function $\beta : \mathbb{S}^1 \to \mathbb{R}$ satisfies (3.3), then there exists an ℓ-convex curve with support function p such that its β-curvature satisfies (3.2).

Now we give examples of ℓ-convex Legendre curves γ with support function p.

Example 3.1 (Points) The first example is the most trivial one, points. A point $(a_1, b_1) \in \mathbb{R}^2$ can be represented by an ℓ-convex curve with support function $p = a_1\cos\theta + b_1\sin\theta$.

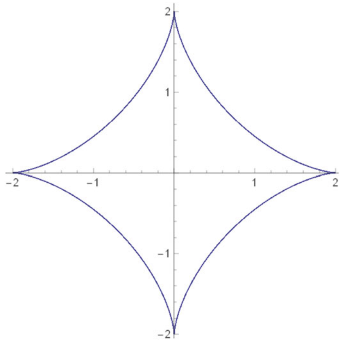

Fig. 1 Astroid with $p = \sin 2\theta$

Example 3.2 (Circles) Any constant $a_0 \in \mathbb{R}$ defines an ℓ-convex Legendre curve with support $p = a_0$. Its corresponding curve γ is $\gamma(\theta) = a_0(\cos\theta, \sin\theta)$. If $a_0 > 0$, then it is the circle of radius a_0 centered at the origin. If $a_0 < 0$, we view it as a circle of negative radius a_0. If $a_0 = 0$, we also see it as a circle of zero radius. Certainly in this case, i.e., $p = 0$, it can also be seen as a point, the origin.

For any two ℓ-convex curves with support functions p_1 and p_2, we define their Minkowski sum, the curve with support function $p = p_1 + p_2$. For strictly convex regular it is well-known that the Minkowski sum of two curves has this property. For the definition of Minkowski sum and its properties, we refer to the classical book [18]. With the Minkowski sum a circle of radius a_0 centered at (a_1, b_1) can also be seen as a Minkowski sum of a circle and a point, with the support function $a_0 + a_1 \cos\theta + b_1 \sin\theta$.

For a Legendre curve (γ, ν) one can define parallel curves by

$$\gamma_\lambda := \gamma + \lambda\nu,$$

for $\lambda \in \mathbb{R}$. When this curve is an ℓ-convex curve with support function p, then it is easy to check that the support functions of its parallel curves are

$$p_\lambda = p + \lambda.$$

Example 3.3 (Astroid) Except circles, i.e., the curves with $p = a_0 + a_1 \cos\theta + b_1 \sin\theta$, we are also interested in astroids, which are ℓ-convex curves with support function

$$a_2 \cos 2\theta + b_2 \sin 2\theta.$$

See Fig. 1. It is a superellipse, i.e., it satisfies

$$x^{2/3} + y^{2/3} = a^{2/3},$$

for certain constant a.

Astroids have β-curvature

$$-3(a_2 \cos 2\theta + b_2 \sin 2\theta).$$

Since the β-curvature changes sign, it is not convex. Moreover, it has 4 singular points. An ℓ-convex curve with support function

$$a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta$$

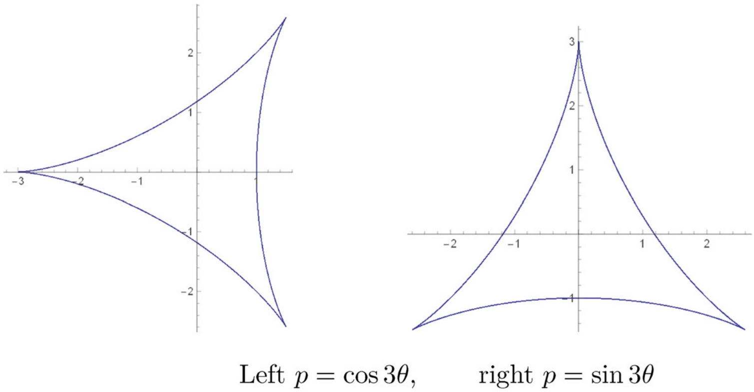

Fig. 2 Left $p = \cos 3\theta$, right $p = \sin 3\theta$

is a parallel curve of an astroid centered at (a_1, b_1). It has β-curvature

$$a_0 - 3(a_2 \cos 2\theta + b_2 \sin 2\theta).$$

Hence, if $a_0 > 0$ is large enough, it is a strictly convex regular curve.

In general, for any $k \in \mathbb{N}$, a curve with support function

$$p = a_k \cos k\theta + b_k \sin k\theta$$

is an interesting figure. See [12]. For $k = 3$ the corresponding curve is called deltoid. See Fig. 2.

We remark that the deltoid has in fact 6 singularities. It runs twice as θ goes from 0 to 2π.

In Sect. 4 we will be interested in isoperimetric type inequalities and geometric problems, in which astroids or deltoid achieve optimality.

4 Isoperimetric inequalities for ℓ-convex Legendre curves

We start this section with the isoperimetric inequality.

Theorem 4.1 *[The isoperimetric inequality] Suppose that γ is ℓ-convex. Then*

$$\Delta_1 := \frac{L^2}{4\pi} - A \geq 0,$$

with equality if and only if γ is a circle.

Its proof is standard by using the Fourier series, which was first used by Hurwitz [10]. It is well known that this classical inequality is true even under a very weaker condition. We remark that $|L|$ is less than or equals to the length of γ, $\int_{\mathbb{S}^1} |\gamma'| d\theta$. There has been a lot of work on the isoperimetric inequality. Here we just refer to one nice survey by Osserman [16] and a recent work [15].

Moreover, L could be zero. When $L = 0$, then A must be negative. Indeed, Theorem 4.1 implies that in this case $A \leq 0$. If $A = 0$, i.e., $\Delta_1 = 0$, then again from Theorem 4.1 γ is a circle, which is a contradiction. Here, Δ_1 is called *deficit* of the isoperimetric inequality.

With the concept of curvature pair (ℓ, β) we have another simple inequality.

Proposition 4.2 *A closed, ℓ-convex front γ in the plane satisfies*

$$\Delta_2 := \int_{\mathbb{S}^1} \beta^2 d\theta - 2A \geq 0, \tag{4.1}$$

equality holds if and only if γ is a circle.

Proof A direct computation gives

$$\begin{aligned}\int_{\mathbb{S}^1} \beta^2 d\theta &= \int_{\mathbb{S}^1} (p+p'')^2 d\theta = 2A + \int_{\mathbb{S}^1} p''(p+p'')d\theta \\ &= 2A + \int_{\mathbb{S}^1} \left(p''^2 - p'^2\right) d\theta.\end{aligned}$$

Since $\int_{\mathbb{S}^1} p' d\theta = 0$, the Proposition follows from the Wirtinger inequality. □

If γ is a convex regular curve with curvature κ, then the above inequality has the following equivalent form

$$\int_{\mathbb{S}^1} \frac{1}{\kappa} ds \geq 2A, \tag{4.2}$$

where s is the arc-length element and $ds = (p+p'')d\theta$. In this case, it is a known inequality and follows from the isoperimetric inequality

$$\int_{\mathbb{S}^1} \frac{1}{\kappa} ds = \frac{1}{2\pi} \int_{\mathbb{S}^1} \frac{1}{\kappa} ds \cdot \int_{\mathbb{S}^1} \kappa ds \geq \frac{1}{2\pi} L^2 \geq 2A.$$

Its higher dimensional analogue is the Heintze-Karcher-Ros inequality. When the algebraic area is non-positive, (4.1) is trivial. We can improve this proposition in the following stronger form.

Theorem 4.3 *For any ℓ-convex Legendre curve, there holds*

$$0 \leq \Delta_3 := \left(\int_{\mathbb{S}^1} \beta^2 d\theta - 2A\right) - 8\left(\frac{L^2}{4\pi} - A\right) \leq \frac{1}{12} \int_{\mathbb{S}^1} \beta'^2 d\theta - 2\left(\frac{L^2}{4\pi} - A\right) =: \Delta_4. \tag{4.3}$$

Moreover equality in the first inequality holds, i.e., $\Delta_3 = 0$, iff γ is generated by

$$p = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta, \tag{4.4}$$

and equality in the second inequality holds, i.e., $\Delta_4 = \Delta_3$, iff

$$p = a_0 + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta + a_3 \cos 3\theta + b_3 \sin 3\theta. \tag{4.5}$$

Proof Let $p = a_0 + \sum_{k=1}^{\infty}(a_k \cos k\theta + b_k \sin k\theta)$. An elementary computation yields

$$\begin{aligned}
L &= 2\pi a_0, \\
A &= \pi a_0^2 + \frac{1}{2}\pi \sum_{k\geq 2}(1-k^2)(a_k^2+b_k^2), \\
\int_{\mathbb{S}^1} \beta^2 d\theta &= 2\pi a_0^2 + \pi \sum_{k\geq 2}(k^2-1)^2(a_k^2+b_k^2), \\
\Delta_1 &= \frac{L^2}{4\pi} - A = \frac{1}{2}\pi \sum_{k\geq 2}(k^2-1)(a_k^2+b_k^2), \\
\Delta_2 &= \int_{\mathbb{S}^1} \beta^2 d\theta - 2A = \pi \sum_{k\geq 2}(k^2-1)k^2(a_k^2+b_k^2), \\
\Delta_3 &= \left(\int_{\mathbb{S}^1} \beta^2 d\theta - 2A\right) - 8\left(\frac{L^2}{4\pi} - A\right) \\
&= \pi \sum_{k\geq 3}(k^2-1)(k^2-4)(a_k^2+b_k^2) \geq 0.
\end{aligned}$$

In the last line equality holds if and only if p has form (4.4). Recall that $\beta = p + p''$, thus

$$\beta = a_0 + \sum_{k\geq 2}(1-k^2)(a_k \cos k\theta + b_k \sin k\theta) \tag{4.6}$$

and

$$\int_{\mathbb{S}^1} \beta'^2 d\theta = \pi \sum_{k\geq 2}(k^2-1)^2k^2(a_k^2+b_k^2). \tag{4.7}$$

It follows

$$\Delta_4 = \frac{1}{12}\int_{\mathbb{S}^1} \beta'^2 d\theta - 2\left(\frac{L^2}{4\pi} - A\right) = \frac{\pi}{12}\sum_{k\geq 3}(k^2-1)(k^4-k^2-12)(a_k^2+b_k^2) \geq 0,$$

and

$$\begin{aligned}
&\frac{1}{12}\int_{\mathbb{S}^1} \beta'^2 d\theta - 2\left(\frac{L^2}{4\pi} - A\right) - \left(\int_{\mathbb{S}^1} \beta^2 d\theta - 2A\right) + 8\left(\frac{L^2}{4\pi} - A\right) \\
&= \frac{\pi}{12}\sum_{k\geq 4}(k^2-1)(k^2-4)(k^2-9)(a_k^2+b_k^2) \geq 0,
\end{aligned}$$

and equality holds if and only if its support function has form (4.5). □

From (4.6) and (4.7), we get

$$\beta'' = \sum_{k\geq 2} k^2(k^2-1)(a_k \cos k\theta + b_k \sin k\theta)$$

and thus

$$\int_{\mathbb{S}^1} \beta''^2 d\theta = \pi \sum_{k\geq 2} k^4(k^2-1)^2(a_k^2+b_k^2).$$

Now we can estimate further

$$\int_{\mathbb{S}^1} \beta''^2 d\theta \geq 4 \int_{\mathbb{S}^1} \beta'^2 d\theta + 72\Delta_4, \tag{4.8}$$

where

$$\begin{aligned}&\int_{\mathbb{S}^1} \beta''^2 d\theta - 4 \int_{\mathbb{S}^1} \beta'^2 d\theta - 72\Delta_4 - 216(\Delta_4 - \Delta_3)\\ &= \pi \sum_{k\geq 5} (k^2-1)(k^2-4)(k^2-9)(k^2-16)(a_k^2+b_k^2).\end{aligned}$$

When γ is a regular convex curve, the first inequality in Theorem 4.3 was proved in [13], while the second in a very recent work [11]. There is a series of sharp geometric inequalities for closed curves in the latter paper by using a higher order Wirtinger inequality.

Remark 4.4 All terms in (4.3) have geometric meaning. We explain the geometric meaning of $\int_{\mathbb{S}^1} \beta'^2 d\theta$. It was observed in [6] that the concept of evolute of a convex curve can be naturally generalized to ℓ-convex curves:

$$\gamma_E := \gamma - \frac{\beta}{\ell}\nu = \gamma - \beta\nu.$$

Due to $\beta = p + p''$ there holds

$$\begin{aligned}\gamma_E(\theta) &= p(\theta)(\cos\theta, \sin\theta) + p'(\theta)(-\sin\theta, \cos\theta) - (p(\theta) + p''(\theta))(\cos\theta, \sin\theta)\\ &= -p''(\theta)(\cos\theta, \sin\theta) + p'(\theta)(-\sin\theta, \cos\theta)\\ &= p'(\theta)(\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})) + p''(\theta)(-\sin(\theta + \frac{\pi}{2}), \cos(\theta + \frac{\pi}{2})).\end{aligned}$$

Hence $\gamma_E(\theta - \frac{\pi}{2})$ is an ℓ-convex curve with support function $p'(\theta - \frac{\pi}{2})$, and the associated β-curvature satisfies $\beta_{\gamma_E} = \beta'(\theta - \frac{\pi}{2})$. Thus

$$\int_{\mathbb{S}^1} \beta'^2 d\theta = \int_{\mathbb{S}^1} \beta_{\gamma_E}^2 d\theta.$$

Moreover one can easily check that the algebraic area A_E of γ_E has the following relation to $\int_{\mathbb{S}^1} \beta^2 d\theta$

$$-2A_E = -\int_{\mathbb{S}^1} (p'^2 - p''^2) d\theta = \frac{1}{2} \int_{\mathbb{S}^1} \beta^2 d\theta - 2A.$$

Together with Bonnesen's inequality proved in the next Section we have

Corollary 4.5 *For any ℓ-convex Legendre curve, there holds*

$$\int_{\mathbb{S}^1} \beta^2 d\theta - 2A \geq 8\left(\frac{L^2}{4\pi} - A\right) \geq 2\pi(r_{out} - r_{in})^2. \tag{4.9}$$

For the definition of r_{out} and r_{in} see the next Section. Hence if the deficit of inequality (4.1), $\int_{\mathbb{S}^1} \beta^2 d\theta - 2A$, is small, then the curve is close to a circle. This is a stability result for inequality (4.1).

Problem 1 It would be interesting to ask the stability of (4.3): when $\frac{1}{12}\int_{\mathbb{S}^1} \beta'^2 d\theta - 2(\frac{L^2}{4\pi} - A)$ is close to zero, is γ close to a parallel curve of an astroid?

Corollary 4.6 *If γ is an ℓ-convex Legendre curve with $L=0$, then*

$$\int_{\mathbb{S}^1} \beta^2 d\theta \geq -6A = 6|A|$$

with equality holding iff γ is an astroid.

It would also be interesting to ask if Theorem 4.3 is true for higher dimensions.

Conjecture 1 (Improved Heintze-Karcher-Ros inequality). Let $\Sigma \subset \mathbb{R}^{n+1}$ be a closed hypersurface with positive mean curvature $H>0$ and Ω its enclosed domain. There exists a positive constant C_n depending only on n such that

$$\int_\Sigma \frac{n}{H} \geq (n+1)|\Omega| + C_n \left(\frac{1}{(n+1)\omega_n^{\frac{1}{n}}} |\Sigma|^{\frac{n+1}{n}} - |\Omega| \right), \tag{4.10}$$

where ω_n is the area of the unit n-sphere, $|\Sigma|$ the area of Σ and $|\Omega|$ the volume of Ω. Here the mean curvature H is defined by the sum of all principal curvatures.

The classical isoperimetric inequality is

$$\frac{1}{(n+1)\omega_n^{\frac{1}{n}}} |\Sigma|^{\frac{n+1}{n}} - |\Omega| \geq 0.$$

Hence (4.10) is an improved inequality. If it is true, (4.10), together with a work of Figalli-Maggi-Pratelli [3], implys a stability result of the Heintze-Karcher-Ros inequality.

5 Bonnesen's inequality and Green–Osher's inequality

In this section we generalize Bonnesen's inequality and Green–Osher's inequality [8] for convex curves to ℓ-convex Legendre curves. The proof is very close to the one given in [8]. The main modification is that we use the analytic properties of the support function to replace the geometric ones of the corresponding curve, which could be different. See Example 5.1 below.

Let $A(t)$ denote the area of closed fronts $\gamma(t,\theta)$ under the following constant speed flow

$$\gamma(t,\theta) = \gamma(\theta) + t\nu(\theta). \tag{5.1}$$

If the initial curve γ_0 has $\ell_0 = 1$, then for any t,

$$\gamma'(t,\theta) = \gamma'(\theta) + t\nu'(\theta) = (\beta(\theta)+t)\mu(\theta),$$

which implies (5.1) preserves the Legendrian condition (2.1) and

$$\nu(t,\theta) = \nu(\theta), \quad \ell(t,\theta) = 1.$$

Therefore under flow (5.1)

$$p(t,\theta) = \langle \gamma(t,\theta), \nu(t,\theta) \rangle = p(\theta) + t.$$

Following Definition 3.4

$$\begin{aligned}A(t) &= \frac{1}{2}\int_{\mathbb{S}^1} p(t,\theta)(p(t,\theta)+p''(t,\theta))d\theta = \frac{1}{2}\int_{\mathbb{S}^1}(p(\theta)+t)(p(\theta)+t+p''(\theta))d\theta \\ &= \frac{1}{2}\int_{\mathbb{S}^1} p(\theta)(p(\theta)+p''(\theta))d\theta + \frac{t}{2}\int_{\mathbb{S}^1}(2p(\theta)+p''(\theta))d\theta + \frac{t^2}{2}\int_{\mathbb{S}^1} d\theta \\ &= A + tL + \pi t^2,\end{aligned}$$

which can be rewritten as

$$A(t) = \pi(t+\frac{L}{2\pi})^2 + A - \frac{L^2}{4\pi}.$$

Let $t_1 \geq t_2$ be the roots of $A(t)$, that is

$$t_1 = -\frac{L}{2\pi} + \frac{\sqrt{L^2-4\pi A}}{2\pi}, \quad t_2 = -\frac{L}{2\pi} - \frac{\sqrt{L^2-4\pi A}}{2\pi}.$$

For a convex curve one can define inradius r_{in} and outradius r_{out}. In this case r_{in} and r_{out} are defined by the radii of the largest inscribed and smallest circumscribed circles of the domain of γ respectively. See for example [8]. The same definition can be used for any Jordon curves. For our ℓ-convex Legendre curves we need to generalize these concepts for ℓ-convex Legendre curves by using the support function. Let $r \in \mathbb{R}$ and $z = (a_1, b_1) \in \mathbb{R}^2 = \mathbb{C}$ and define

$$c_{(z,r)}(\theta) = r + a_1 \cos\theta + b_1 \sin\theta.$$

If there is no confusion, we denote the Legendre curve with support function $c_{(z,r)}$ also by $c_{(z,r)}$. If $r > 0$, $c_{(z,r)}$ is the circle of radius r centered at z. For $r \leq 0$ we view $c_{(z,r)}$ as a circle of radius $r \leq 0$ centered at z, as mentioned above. For any ℓ-convex Legendre curve γ with potential p we call $c_{(z,r)}$ an *inscribed circle* of γ if

$$c_{(z,r)}(\theta) \leq p(\theta), \quad \forall\theta \in \mathbb{S}^1.$$

A circumscribed circle of γ can be defined similarly. Now we define the inradius r_{in} and outradius r_{out} for ℓ-convex curves.

Definition 5.1 Let γ be an ℓ-convex curve with support function p. The inradius r_{in} and outradius r_{out} of γ are defined by

$$r_{in} = \max\{r \in \mathbb{R} \,|\, c_{(z,r)}(\theta) \leq p(\theta), \forall\theta \in \mathbb{S}^1\}$$

and

$$r_{out} = \min\{r \in \mathbb{R} \,|\, c_{(z,r)}(\theta) \geq p(\theta), \forall\theta \in \mathbb{S}^1\}$$

respectively.

It is easy to prove that there exists a maximal inscribed circle $c_{z,r}$ such that $r = r_{in}$. An inscribed circle $c_{(z,r)}$ of γ is called tangent to γ at $\gamma(\theta_0)$ if $c_{(z,r)}(\theta_0) = p(\theta_0)$. One can consider circumscribed circles analogously. We remark that r_{in} could be negative. It is easy to see that if γ is a regular convex curve then the definitions of r_{in} and r_{out} are the same as the ordinary ones. However in general it could be quite different. See Example 5.1 below.

With these notations we can generalize Bonnesen's inequality for ℓ-convex Legendre curves.

Theorem 5.2 *[Bonnesen's inequality] If γ is an ℓ-convex and not a circle, then*

$$\pi t^2 - tL + A < 0, \quad t \in [r_{in}, r_{out}], \tag{5.2}$$
$$L^2 - 4\pi A > \pi^2 (r_{out} - r_{in})^2. \tag{5.3}$$

As in [8] we need the following Lemma

Lemma 5.3 *Let $c_{(z,r)}$ be either a maximal inscribed or minimal circumscribed circle for an ℓ-convex Legendre curve. Then for any θ_0 there exists $\tilde{\theta} \in [\theta_0, \theta_0 + \pi]$ such that $c_{(z,r)}$ is tangent to γ at $\gamma(\tilde{\theta})$.*

Proof We follow the proof given in [8], with a different setting, and consider also only the case of the inscribed circle. The case of circumscribed circle is the same. By a translation we may assume $z = 0$. Assume, by contradiction, that there exists θ_0 such that $c_{(z,r)}$ is not tangent to γ at $\gamma(\theta)$ for any $\theta \in [\theta_0, \theta_0 + \pi]$. Wlog, we may assume that $\theta_0 = 0$. Hence we have $r < p(\theta)$ for any $\theta \in [0, \pi]$. Since $[0, \pi]$ is compact, there exist $\epsilon > 0$ such that $r + 2\epsilon < p(\theta)$ for $\theta \in [0, \pi]$. Consider a new circle with the radius r defined by the support function $r + \epsilon \sin\theta$. Since $r + \epsilon \sin\theta < r \leq p(\theta)$ for any $\theta \in (\pi, 2\pi)$, we have

$$r + \epsilon \sin\theta < p(\theta), \qquad \forall\theta \in [0, 2\pi].$$

This contradicts the maximality of r. □

Now we prove Theorem 5.2.

Proof of Theorem 5.2 Choosing points of tangency $\theta_1, \theta_2, ..., \theta_N$ on $[0, 2\pi]$ so we get

$$\theta_{i+1} - \theta_i \leq \pi, \quad p(\theta_i) = r_{in}, \ i = 1, 2, ..., N.$$

Then integration by parts implies

$$\begin{aligned} A(-r_{in}) &= \frac{1}{2}\int_{\mathbb{S}^1} (p - r_{in})(p + p'' - r_{in})d\theta \\ &= \frac{1}{2}\int_{\mathbb{S}^1} [(p - r_{in})^2 - p'^2]d\theta \\ &= \frac{1}{2}\sum_{i=1}^{N}\int_{\theta_i}^{\theta_{i+1}} [(p - r_{in})^2 - p'^2]d\theta \leq 0. \end{aligned}$$

In the last inequality we have used the Wirtinger inequality. The same argument works for r_{out}. Hence we have finised the proof of (5.2). (5.2) and the isoperimetric inequality means that

$$t_2 < -r_{out} < -\frac{L}{2\pi} < -r_{in} < t_1.$$

It implies

$$r_{out} - r_{in} < t_1 - t_2 = \frac{1}{\pi}\sqrt{L^2 - 4\pi A},$$

and hence (5.3), which completes the proof. □

The Bonnesen inequality holds for a general regular simple curve. Though our inequalities (5.2) and (5.3) have the same form as the classical Bonnesen inequality, the definitions of r_{in} and r_{out} are not the same for non-convex ℓ-convex Legendre curves. Our inequalities are stronger for non-convex curves. This is showed in the following example.

Example 5.1 Let γ with $p = \sin 2\theta$, namely γ is an astroid. Then it is easy to see that its algebraic length and area are $L = 0$, $A = -\frac{3}{2}\pi$. Moreover, it is easy to see that $r_{in} = -1$ and $r_{out} = 1$. Hence we have

$$L^2 - 4\pi A = 6\pi^2, \quad \pi^2(r_{out} - r_{in})^2 = 4\pi^2. \tag{5.4}$$

The curve is given by

$$(x, y) := (2\sin^3\theta, 2\cos^3\theta).$$

We have

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2^{\frac{2}{3}}.$$

Its absolute length is $\bar{L} = 12$ and the absolute (or enclosed) area $\bar{A} = \frac{3}{2}\pi$. This curve is a Jordan curve and one can define the inradius $\bar{r}_{in}$ and outradius $\bar{r}_{out}$ by a usual way. Now one can easily check that $\bar{r}_{in} = 1$ and $\bar{r}_{out} = 2$. Hence

$$\bar{L}^2 - 4\pi\bar{A} = 144 - 6\pi^2 = 6\pi^2 + 12(12 - \pi^2), \quad \pi^2(\bar{r}_{out} - \bar{r}_{in})^2 = \pi^2.$$

It is now clear to see that the usual Bonnesen inequality $\bar{L}^2 - 4\pi\bar{A} \geq \pi^2(\bar{r}_{out} - \bar{r}_{in})^2$ is less precise than our inequality $L^2 - 4\pi A \geq \pi^2(r_{out} - r_{in})^2$, for

$$\bar{L}^2 - 4\pi\bar{A} > L^2 - 4\pi A > \pi^2(r_{out} - r_{in})^2 > \pi^2(\bar{r}_{out} - \bar{r}_{in})^2.$$

Problem 2 From Example 5.1 it would be interesting to ask: is there an improved Bonnesen's inequality for ℓ-convex curves with $L = 0$ such that the above example is an optimal one, namely an improved Bonnesen's inequality with equality iff it is an astroid. For example, is

$$L^2 - 4\pi A \geq \frac{3}{2}\pi^2(r_{out} - r_{in})^2$$

true? (5.4) shows that the astroid achieves equality.

Now we prove the Green–Osher inequality for ℓ-convex curves.

Theorem 5.4 *Let $F(\theta)$ be a convex function on $\mathbb{R}$, i.e. $F''(\theta) \geq 0$. Then*

$$\frac{1}{2\pi}\int_{\mathbb{S}^1} F(\beta)d\theta \geq \frac{1}{2}[F(-t_1) + F(-t_2)].$$

To prove this theorem, we make the following definition and notation: Consider

$$\sup\left\{\int_I \beta(\theta)d\theta \mid I \subset \mathbb{S}^1, \int_I d\theta = \pi\right\}.$$

Let I_1 denote a subset of $\mathbb{S}^1$ of measure π realizing this bound, and let I_2 be its complement. There exists a real number a such that

$$I_1 \subseteq \{\theta, \beta(\theta) \geq a\}, \quad I_2 \subseteq \{\theta, \beta(\theta) \leq a\}.$$

Set

$$\beta_1 = \frac{1}{\pi}\int_{I_1} \beta(\theta)d\theta, \quad \beta_2 = \frac{1}{\pi}\int_{I_2} \beta(\theta)d\theta.$$

Note that

$$\beta_1 + \beta_2 = \frac{L}{\pi}, \quad \beta_1 \geq \beta_2,$$

and we may write

$$\beta_1 = \frac{L}{2\pi} + b, \quad \beta_2 = \frac{L}{2\pi} - b$$

for some $b \geq 0$.

γ is called *symmetric* if $p(\theta) = p(\theta + \pi)$ for all $\theta \in \mathbb{S}^1$.

Proposition 5.5 *Let γ be symmetric, then*

$$\beta_1 \geq -t_2.$$

Proof Since γ is symmetric, then for any given θ_0, we get

$$\begin{aligned}
A(-p(\theta_0)) &= \frac{1}{2}\int_{\mathbb{S}^1} (p - p(\theta_0))(p + p'' - p(\theta_0))d\theta \\
&= \frac{1}{2}\int_{\mathbb{S}^1} [(p - p(\theta_0))^2 - p'^2]d\theta \\
&= \frac{1}{2}\int_{\theta_0}^{\theta_0+\pi} [(p - p(\theta_0))^2 - p'^2]d\theta + \frac{1}{2}\int_{\theta_0+\pi}^{\theta_0+2\pi} [(p - p(\theta_0))^2 - p'^2]d\theta \\
&\leq 0,
\end{aligned}$$

which means for all θ,

$$-t_1 \leq p(\theta) \leq -t_2.$$

In other words

$$-u \leq p(\theta) - \frac{L}{2\pi} \leq u,$$

where $u = \frac{\sqrt{L^2 - 4\pi A}}{2\pi}$. The inequality we are trying to prove is equivalent to

$$b \geq u.$$

On I_1, it follows from $\beta(\theta) - a \geq 0$ that

$$-\left(p(\theta) - \frac{L}{2\pi}\right)(\beta(\theta) - a) \leq u(\beta(\theta) - a).$$

Integrating, we have

$$-\frac{1}{\pi}\int_{I_1} \left(p(\theta) - \frac{L}{2\pi}\right)(\beta(\theta) - a)d\theta \leq u(\beta_1 - a). \tag{5.5}$$

On I_2, since $\beta(\theta) - a \leq 0$,

$$-\left(p(\theta) - \frac{L}{2\pi}\right)(\beta(\theta) - a) \leq -u(\beta(\theta) - a).$$

Integrating, we have

$$-\frac{1}{\pi}\int_{I_2}\left(p(\theta)-\frac{L}{2\pi}\right)(\beta(\theta)-a)d\theta \le -u(\beta_2-a). \tag{5.6}$$

Adding (5.5) and (5.6) gives

$$-\frac{1}{\pi}\int_{\mathbb{S}^1}\left(p(\theta)-\frac{L}{2\pi}\right)(\beta(\theta)-a)d\theta \le u(\beta_1-\beta_2).$$

The left-hand side simplifies to

$$\frac{1}{2\pi^2}(L^2-4\pi A)=2u^2.$$

The right-hand side is $2ub$, and thus $u \le b$ as desired. □

For the symmetrization we can follow their idea again, but in our setting. Given θ_0, let us want to do for the half curve $\gamma:[\theta_0,\theta_0+\pi]$ as in their proof by rotating this half curve by the line connecting $\gamma(\theta_0)$ and $\gamma(\theta_0+\pi)$ with respect to their middle point. In term of the support function, one can show that the translated curve $\tilde{\gamma}$ has a support function

$$\tilde{p}(\theta)=p(\theta)-\frac{p(\theta_0)-p(\theta_0+\pi)}{2}\cos(\theta-\theta_0)-\frac{p'(\theta_0)-p'(\theta_0+\pi)}{2}\sin(\theta-\theta_0).$$

It is clear that $\tilde{\gamma}$ admits the same β, length L and area A with γ on interval $[0,2\pi]$, and

$$\tilde{p}(\theta_0)=\tilde{p}(\theta_0+\pi)=\frac{1}{2}(p(\theta_0)+p(\theta_0+\pi)),\quad \tilde{p}'(\theta_0)=\frac{1}{2}(p'(\theta_0)+p'(\theta_0+\pi)).$$

Proposition 5.6 *If γ is an ℓ-convex curve with $\ell=1$, then*

$$\beta_1 \ge -t_2.$$

Proof We proceed by a symmetrization argument as in [8]. For any θ, γ can be divided into two curves by joining the points on γ corresponding to $\theta,\theta+\pi$ by a straight line. Let L_1 and L_2 be the lengths of the two pieces of γ, and denote

$$A_1=\frac{1}{2}\int_{\theta}^{\theta+\pi}p(p+p'')d\theta,\ A_2=\frac{1}{2}\int_{\theta+\pi}^{\theta+2\pi}p(p+p'')d\theta,$$

then

$$L=L_1+L_2,\ A=A_1+A_2.$$

Choose θ_0 such that

$$L_1^2-2\pi A_1=L_2^2-2\pi A_2.$$

We assume γ satisfies

$$p(\theta_0+\pi)=p(\theta_0).$$

Otherwise we can translate it via $\tilde{p}$, which owns the same β, length L and area A with γ.

Let γ_1 and γ_2 be two symmetric curves that have support functions

$$p_1(\theta)=\begin{cases}p(\theta), & \theta\in[\theta_0,\theta_0+\pi],\\ p(\theta-\pi), & \theta\in[\theta_0+\pi,\theta_0+2\pi]\end{cases}$$

and

$$p_2(\theta) = \begin{cases} p(\theta + \pi), & \theta \in [\theta_0, \theta_0 + \pi], \\ p(\theta), & \theta \in [\theta_0 + \pi, \theta_0 + 2\pi] \end{cases}$$

respectively. It is clear that γ_i has perimeter $2L_i$, area $2A_i$, and $p_i(\theta + \pi) = p_i(\theta)$ for any $\theta, i = 1, 2$.

Of importance we notice, for symmetric curves γ_1 and γ_1, their subsets $I_1(\gamma_1)$ and $I_1(\gamma_2)$ are both symmetric. Since $\int_I \beta(\theta)d\theta$ is maximized by I_1 among all subsets of measure π, it follows that

$$\beta_1 \geq \frac{1}{2}\left(\int_{I_1(\gamma_1)} \beta(\theta)d\theta + \int_{I_1(\gamma_2)} \beta(\theta)d\theta\right) = \frac{1}{2}(\beta_1(\gamma_1) + \beta_1(\gamma_2)).$$

By Proposition 5.5,

$$\beta_1(\gamma_1) \geq \frac{2L_1}{2\pi} + \frac{\sqrt{(2L_1)^2 - 8\pi A_1}}{2\pi}, \quad \beta_1(\gamma_2) \geq \frac{2L_2}{2\pi} + \frac{\sqrt{(2L_2)^2 - 8\pi A_2}}{2\pi}.$$

Taking the average

$$\beta_1 \geq \frac{L_1 + L_2}{2\pi} + \frac{\sqrt{(2L_1)^2 - 8\pi A_1} + \sqrt{(2L_2)^2 - 8\pi A_2}}{4\pi} = \frac{L}{2\pi} + \frac{\sqrt{(2L_1)^2 - 8\pi A_1}}{2\pi}.$$

The inequality we want is

$$\beta_1 \geq \frac{L}{2\pi} + \frac{\sqrt{L^2 - 4\pi A}}{2\pi} = \frac{L}{2\pi} + \frac{\sqrt{(L_1 + L_2)^2 - 4\pi(A_1 + A_2)}}{2\pi},$$

which holds since

$$\begin{aligned} (2L_1)^2 - 8\pi A_1 &= 2L_1^2 - 4\pi A_1 + 2L_2^2 - 4\pi A_2 \\ &= (L_1 + L_2)^2 + (L_1 - L_2)^2 - 4\pi(A_1 + A_2). \end{aligned}$$

This proves the proposition. □

To prove Theorem 5.4, we need the elementary lemma:

Lemma 5.7 *If $F(\theta)$ is convex, then if $b \geq a \geq 0$ and c arbitrary, then*

$$F(c - a) + F(c + a) \leq F(c - b) + F(c + b).$$

Proof of Theorem 5.4. By Jensen's inequality, applied to I_i, we have

$$\frac{1}{\pi}\int_{I_i} F(\beta(\theta))d\theta \geq F\left(\frac{1}{\pi}\int_{I_i} \beta(\theta)d\theta\right) = F(\beta_i),\ i = 1, 2.$$

Taking the average of these two inequalities for $i = 1, 2$ gives

$$\frac{1}{2\pi}\int_{\mathbb{S}^1} F(\beta(\theta))d\theta \geq \frac{1}{2}(F(\beta_1) + F(\beta_2)).$$

By Lemma 5.7,

$$F(\beta_1) + F(\beta_2) \geq F(-t_1) + F(-t_2).$$

The Theorem then holds. □

When F is strictly convex, one can show that equality in Green–Osher's inequality implies that the curve is a circle. This follows from [7]. It would be interesting to ask if one can find an optimal Green–Osher type inequalities in the class of ℓ-convex curves with $L = 0$, which include Corollary 4.6 as a special case. Moreover equality in such an inequality holds iff the curve is an astroid.

6 An inverse curve flow for ℓ-convex curves

In this section we consider an inverse curve flow for ℓ-convex curves, which preserves the Legendrian condition, namely (2.1). In order to construct such a flow, instead of γ it would be better to consider the pair (γ, ν) as a curve into the contact manifold $\mathbb{R}^2 \times \mathbb{S}^1$. In our recent paper we propose a general approach to consider curvature flows for Legendre curves. It was proved that a suitable flow with a normal velocity, which means keeping the Legendrian condition (2.1), must have the following form

$$\begin{cases} \dfrac{\partial \gamma(t,\theta)}{\partial t} = \dfrac{\ell f'}{\ell^2+\beta^2}\mu + f\nu, \\ \dfrac{\partial \nu(t,\theta)}{\partial t} = -\dfrac{\beta f'}{\ell^2+\beta^2}\mu, \end{cases} \tag{6.1}$$

where f is some smooth function defined on the evolving Legendre curve, for example the curvature. Since adding a tangential vector field does not change the flow, we can normalize (6.1) by adding $(\frac{\beta h}{\sqrt{\ell^2+\beta^2}}\mu, \frac{\ell h}{\sqrt{\ell^2+\beta^2}}\mu)$, which is a tangential vector field in view of (2.2), to (6.1) for any h

$$\begin{cases} \partial_t \gamma = \dfrac{\beta h}{\sqrt{\ell^2+\beta^2}}\mu + \dfrac{\ell f'}{\ell^2+\beta^2}\mu + f\nu, \\ \partial_t \nu = \dfrac{\ell h}{\sqrt{\ell^2+\beta^2}}\mu - \dfrac{\beta f'}{\ell^2+\beta^2}\mu. \end{cases} \tag{6.2}$$

When $\ell > 0$, we can choose h satisfying

$$h = \frac{\beta}{\ell}\frac{f'}{\sqrt{\ell^2+\beta^2}},$$

and obtain a simpler flow

$$\begin{cases} \partial_t \gamma = \dfrac{f'}{\ell}\mu + f\nu, \\ \partial_t \nu = 0. \end{cases} \tag{6.3}$$

It follows that

$$\partial_t \mu = 0, \quad \partial_t \ell = 0, \quad \partial_t \beta = \left(\frac{f'}{\ell}\right)' + fl.$$

Lemma 6.1 *If the initial curve is ℓ-convex, then flow (6.3) preserves the normal field ν, and hence μ. Moreover if the initial curve γ_0 has $\ell_0 = 1$, then for any t, $\ell(t,\theta) = 1$ and $\gamma(t,\theta)$ has form (3.1) with support function $p(t,\theta)$.*

Now we consider an inverse curvature type Legendrian flow for ℓ-convex curves by choosing

$$f = \frac{\beta}{\ell}. \tag{6.4}$$

Remember that for regular curves it is the reciprocal of the curvature κ. Therefore, the corresponding flow is a natural generalization of the inverse curvature flow. By Lemma 6.1 we can consider $f = \beta$ for the initial curves with $\ell_0 = 1$. By Lemma 6.1 again we can write the flow by using support functions.

Lemma 6.2 *If the initial curve γ_0 has form* (3.1) *with support function p_0, then flow* (6.3) *with velocity* (6.4) *is equivalent to*

$$\partial_t p = p + p'' \tag{6.5}$$

with $p(0) = p_0$.

Proof The proof is now obvious with

$$\partial_t p = \partial_t \langle \gamma, \nu \rangle = \langle \partial_t \gamma, \nu \rangle + \langle \gamma, \partial_t \nu \rangle = \beta = p + p''.$$

□

This is a very simple parabolic equation. Now we compute the evolution equation for various geometric quantities under this flow.

$$\frac{dL}{dt} = \int_{\mathbb{S}^1} \frac{\partial p}{\partial t} d\theta = \int_{\mathbb{S}^1} \beta d\theta = \int_{\mathbb{S}^1} p d\theta = L. \tag{6.6}$$

Since the area function is

$$A = \frac{1}{2} \int_{\mathbb{S}^1} p(p + p'') d\theta = \frac{1}{2} \int_{\mathbb{S}^1} (p^2 - p'^2) d\theta,$$

then

$$\begin{aligned} \frac{dA}{dt} &= \int_{\mathbb{S}^1} \left(p \partial_t p - p' \frac{\partial^2 p}{\partial \theta \partial t} \right) d\theta \\ &= \int_{\mathbb{S}^1} p(p + p'') d\theta - \int_{\mathbb{S}^1} p' d(p + p'') \\ &= \int_{\mathbb{S}^1} \beta^2 d\theta. \end{aligned}$$

The evolution equation for β is

$$\partial_t \beta = \beta + \beta''. \tag{6.7}$$

Hence,

$$\frac{d}{dt} \int_{\mathbb{S}^1} \beta^2 d\theta = 2 \int_{\mathbb{S}^1} (\beta^2 - \beta'^2) d\theta.$$

If $L_0 = 0$, i.e, $\int_{\mathbb{S}^1} \beta_0 d\theta = 0$, we have $\int_{\mathbb{S}^1} \beta d\theta = L(t) = 0$. Moreover we have

$$\int_{\mathbb{S}^1} \beta \cos\theta d\theta = \int_{\mathbb{S}^1} \beta \sin\theta d\theta = 0.$$

Hence for such a β we know that

$$4\int_{\mathbb{S}^1}\beta^2 d\theta \leq \int_{\mathbb{S}^1}\beta'^2 d\theta.$$

It follows that

$$\frac{d}{dt}\left(\int_{\mathbb{S}^1}\beta^2 d\theta + 6A\right) = 2\int_{\mathbb{S}^1}\left(4\beta^2 - \beta'^2\right) d\theta \leq 0.$$

Theorem 6.3 *The inverse curvature flow has a global convergence for any initial ℓ-convex curve. Moreover we have*

(1) If $L_0 > 0$, the flow expands exponentially and after a certain time the evolving curve becomes convex. After a suitable rescaling, the flow converges to a circle.

(2) If $L_0 = 0$, then the flow shrinks and after a suitable rescaling it converges to a Legendre curve with a form

$$a_1\cos\theta + b_1\sin\theta + a_k\cos k\theta + b_k\sin k\theta \tag{6.8}$$

for a certain $k \geq 2$, which is determined by γ_0.

The curves given by (6.8), together with circles, are all closed solitons of the inverse curvature flow.

Proof If one expand $p(t)$ into the Fourier series

$$p(t) = a_0(t) + \sum_k a_k(t)\cos k\theta + b_k(t)\sin k\theta,$$

then it is easy to see that along the flow

$$\begin{aligned}
\frac{d}{dt}a_0(t) &= a_0(t)\\
\frac{d}{dt}a_1(t) &= 0\\
\frac{d}{dt}a_k(t) &= (1-k^2)a_k, \quad \forall k \geq 2,
\end{aligned}$$

and the same equations for b_k. Hence

$$p(t) = a_0e^t + a_1(0)\cos\theta + b_1(0)\sin\theta + \sum_{k=2}^{\infty} e^{-(k^2-1)t}\left(a_k(0)\cos k\theta + b_k(0)\sin k\theta\right)$$

is the solution of the inverse curvature flow. Consider

$$\tilde{p}(t) := p(t) - a_1(0)\cos\theta - b_1(0)\sin\theta.$$

Then it is easy to see that after a suitable rescaling one can prove (1) and (2).

One can prove that a homothetic soliton of the flow is a solution of

$$p + p'' = cp,$$

for some constant c. It is easy to see that $c = 0$ or $c = k$ for some positive integer $k \geq 2$. $c = 0$ corresponds to a circle, while $c = k$ a solution in form (6.8). □

Under the inverse curvature flow for ℓ-convex Legendre curves we have a monotone property for isoperimetric quantities.

Theorem 6.4 *The isoperimetric ratio* $\frac{L^2}{A}$, Δ_1, Δ_2, Δ_3 *and* Δ_4*, which are defined in Section 4, are all exponentially decreasing under the inverse curvature flow. Moreover* Δ_1, Δ_2, Δ_3 *and* Δ_4 *converges to zero as the time t goes to infinity.*

Proof With the help of the evolution equations for L and A, we compute

$$\frac{d}{dt}\left(\frac{L^2}{A}\right) = \frac{2L}{A}\frac{dL}{dt} - \frac{L^2}{A^2}\frac{dA}{dt} = \frac{L^2}{A^2}\left(2A - \frac{dA}{dt}\right) = \frac{L^2}{A^2}\left(2A - \int_{\mathbb{S}^1}\beta^2 d\theta\right) \le 0.$$

According to (4.3), there holds

$$\begin{aligned}\frac{d\Delta_1}{dt} &= \frac{L^2}{2\pi} - \int_{\mathbb{S}^1}\beta^2 d\theta = \frac{L^2}{2\pi} - 2A - \left(\int_{\mathbb{S}^1}\beta^2 d\theta - 2A\right)\\ &= 2\Delta_1 - \Delta_2 = -6\Delta_1 - \Delta_3 \le -6\Delta_1.\end{aligned}$$

Now we use (6.7) to deduce

$$\frac{d\Delta_2}{dt} = 2\int_{\mathbb{S}^1}\beta\frac{\partial\beta}{\partial t}d\theta - 2\frac{dA}{dt} = 2\int_{\mathbb{S}^1}\beta(\beta+\beta'')d\theta - 2\int_{\mathbb{S}^1}\beta^2 d\theta = -2\int_{\mathbb{S}^1}\beta'^2 d\theta.$$

By (4.3), it suffices

$$\frac{d\Delta_2}{dt} = -24\Delta_4 - 48\Delta_1 = -6\Delta_2 - 18\Delta_3 + 24(\Delta_3 - \Delta_4) \le -6\Delta_2.$$

When comes to Δ_3, we have

$$\frac{d\Delta_3}{dt} = \frac{d\Delta_2}{dt} - 8\frac{d\Delta_1}{dt} = -24\Delta_4 - 48\Delta_1 - 8(-6\Delta_1 - \Delta_3) = -24\Delta_4 + 8\Delta_3 \le -16\Delta_3.$$

Since

$$\frac{d}{dt}\int_{\mathbb{S}^1}\beta'^2 d\theta = 2\int_{\mathbb{S}^1}\beta'\frac{\partial\beta}{\partial\theta\partial t}d\theta = 2\int_{\mathbb{S}^1}\beta'(\beta+\beta'')'d\theta = 2\int_{\mathbb{S}^1}(\beta'^2 - \beta''^2)d\theta,$$

due to (4.8) there holds

$$\frac{d}{dt}\int_{\mathbb{S}^1}\beta'^2 d\theta \le -6\int_{\mathbb{S}^1}\beta'^2 d\theta - 144\Delta_4 = -72(2\Delta_1 + 3\Delta_4),$$

and hence

$$\frac{d\Delta_4}{dt} \le -6(2\Delta_1 + 3\Delta_4) - 2(-6\Delta_1 - \Delta_3) = -18\Delta_4 + 2\Delta_3 \le -16\Delta_4.$$

Integrating $\frac{d\Delta_i}{dt} \le -n_i\Delta_i$ with $n_1 = n_2 = 6, n_3 = n_4 = 16$ yields

$$\Delta_i(t) \le \Delta_i(0)e^{-n_i t},$$

which means that $\Delta_i, i = 1, 2, 3, 4$ of the evolving curve converges to zero as t goes to infinity. □

Theorem 6.4 indicates that all inequalities given in Sect. 4 are stable, at least under the inverse curvature flow.

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