Calc. Var. (2023) 62:151
https://doi.org/10.1007/s00526-023-02458-x

Calculus of Variations

Nonlinear nonlocal Douglas identity

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Received: 10 June 2020 / Accepted: 15 February 2023 / Published online: 15 May 2023

Abstract

We give Hardy–Stein and Douglas identities for nonlinear nonlocal Sobolev–Bregman integral forms with unimodal Lévy measures. We prove that the corresponding Poisson integral defines an extension operator for the Sobolev–Bregman spaces. As an application, we obtain the boundedness of the Dirichlet-to-Neumann operator on weighted L^p spaces. We also show that the Poisson integrals are quasiminimizers of the Sobolev–Bregman forms.

Mathematics Subject Classification 31C05 · 31C45 · 46E35 · 35A15 · 60G51

1 Introduction

In 1931 Douglas [25] established a connection of the energy of the harmonic function u on the unit disc $B(0,1)$ with the "energy" of its boundary trace g, regarded as a function on $[0,2\pi)$:

$$\int_{B(0,1)} |\nabla u(x)|^2 \mathrm{d}x = \frac{1}{8\pi} \iint_{[0,2\pi)\times[0,2\pi)} \frac{(g(\eta)-g(\xi))^2}{\sin^2((\eta-\xi)/2)} \mathrm{d}\eta \mathrm{d}\xi. \tag{1.1}$$

Communicated by Laszlo Szekelyhidi.

Krzysztof Bogdan and Katarzyna Pietruska-Pałuba were supported by the NCN Grant 2018/31/B/ST1/03818. Tomasz Grzywny was supported by the NCN Grant 2017/27/B/ST1/01339. Artur Rutkowski was supported by the NCN Grant 2015/18/E/ST1/00239.

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The formula arose in the study of the so-called Plateau problem—the problem of existence of minimal surfaces posed by J.-L. Lagrange. The identity holds true provided that the left-hand side is finite—for details see, e.g., Chen and Fukushima [15, (2.2.60)]. Thus, under the integrability condition, (1.1) is valid for the solutions of the Dirichlet problem,

$$\begin{cases} \Delta u = 0 & \text{in } B(0,1), \\ u = g & \text{in } \partial B(0,1). \end{cases}$$

In our paper we propose a variant of (1.1), which we call nonlinear nonlocal Douglas identity. The term "nonlocal" means that the Laplace operator Δ above is replaced by a nonlocal operator L. Specifically, we adopt the following setting. Let $d = 1, 2, \ldots$. Suppose that the function $\nu\colon [0,\infty) \to (0,\infty]$ is nonincreasing and, with a slight abuse of notation, let $\nu(z) = \nu(|z|)$ for $z \in \mathbb{R}^d$. In particular, ν is symmetric, i.e., $\nu(z) = \nu(-z)$, $z \in \mathbb{R}^d$. Assume further that

$$\int_{\mathbb{R}^d} \nu(z)\,\mathrm{d}z = \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \left(|z|^2 \wedge 1\right) \nu(z)\,\mathrm{d}z < \infty. \tag{1.2}$$

Thus, ν is a strictly positive density function of an infinite isotropic unimodal Lévy measure on $\mathbb{R}^d$ (in short, ν is unimodal). For $u\colon \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$ we let

$$\begin{aligned} Lu(x) &= \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} (u(y) - u(x))\nu(x,y)\,\mathrm{d}y \\ &= \lim_{\epsilon \to 0^+} \tfrac{1}{2} \int_{|z|>\epsilon} (u(x+z) + u(x-z) - 2u(x))\nu(z)\,\mathrm{d}z. \end{aligned} \tag{1.3}$$

Here, $\nu(x,y) := \nu(y-x)$, and the limit exists, e.g., for u in $C_c^\infty(\mathbb{R}^d)$, the smooth functions with compact support. Operators of the form (1.3) are called nonlocal, because the value of $L\phi(x)$ also depends on the values of ϕ outside of a neighborhood of x. Furthermore, the operators satisfy the maximum principle, meaning that if $\phi(x_0) = \sup\{\phi(x) : x \in \mathbb{R}^d\}$, then $L\phi(x_0) \le 0$. It is well known that such operators may be used to describe transportation of mass, charge, etc. in elliptic and parabolic equations; especially to pose boundary-value problems.

To our nonlocal setting we bring a judicious way of measuring the smoothness of functions for a given set. Let $D \subset \mathbb{R}^d$ be open. For the sake of gradual introduction we first consider the quadratic form

$$\mathcal{E}_D[u] = \frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (u(x) - u(y))^2 \nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{1.4}$$

Such forms appeared in Servadei and Valdinoci [55, 56], where the set $\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c$ was denoted Q, then in Ros-Oton [52, (3.1)] and Dipierro, Ros-Oton and Valdinoci [24, p. 379]. Similar forms were also used in Felsinger, Kassmann and Voigt [29, Definition 2.1 (ii)]. $\mathcal{E}_D$ is the energy functional of the nonlocal Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } D, \\ u = g & \text{on } D^c, \end{cases} \tag{1.5}$$

see [55, 56] and Bogdan, Grzywny, Pietruska-Pałuba and Rutkowski [8]. It should be noted that $\mathcal{E}_D$ is better than the vanilla form $\mathcal{E}_{\mathbb{R}^d}$ for solving (1.5), because it allows for more general external conditions g due to the restriction of integration in (1.4) to $Q = \mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c$, cf. [8, p. 39]. Therefore $\mathcal{E}_D$ constitutes an important step forward in nonlocal variational problems; we refer the reader to [8] for more details and to [55, 56] for applications to nonlinear equations. We note that our results also have consequences for the Dirichlet problem for L on D when $\mathcal{E}_{\mathbb{R}^d}$ is used, see Corollary 4.4 below.

Numerous papers study the nonlocal Dirichlet problem by variational methods for nonlocal operators — in the present setting we should note [29], [52], and Rutkowski [53]. It is known for many Lévy and Lévy-type kernels ν and bounded D [29, 53], [8, Section 5] that a unique weak solution of (1.5) exists provided that $g\colon D^c \to \mathbb{R}$ can be extended to a function $u \in L^2(D)$ from the Sobolev class

$$\mathcal{V}_D := \{u\colon \mathbb{R}^d \to \mathbb{R} \mid \mathcal{E}_D[u] < \infty\}. \tag{1.6}$$

It is therefore important to determine conditions on g that allow for such an extension—in other words—to determine the *trace space*, say, $\mathcal{X}_D$, of $\mathcal{V}_D$. We note in passing that by [8, Lemma 3.4], the functions from $\mathcal{V}_D$ are automatically square integrable on D. For the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ (see Sect. 2.1 for a definition) a solution to this problem was proposed by Dyda and Kassmann [26] by using the Whitney decomposition and the method of reflection. In fact, [26, Theorem 3] concerns general *p-increments*, i.e., $|u(x) - u(y)|^p$ with $p \geq 1$.

In [8] we resolved the extension and trace problem for $p = 2$ for a wide class of unimodal Lévy operators by a different approach based on the (quadratic) nonlocal Douglas identity. Namely, [8, Theorem 2.3] asserts that the trace space $\mathcal{X}_D$ consists of functions $g\colon D^c \to D$ for which the following form on D^c is finite,

$$\mathcal{H}_D[g] := \frac{1}{2} \iint\limits_{D^c \times D^c} (g(z) - g(w))^2 \gamma_D(z, w) \, \mathrm{d}z\mathrm{d}w.$$

Here and afterwards we call

$$\gamma_D(w, z) := \int_D \int_D \nu(w, x) G_D(x, y) \nu(y, z) \, \mathrm{d}x\mathrm{d}y, \quad w, z \in D^c,$$

the *kernel of interaction via* D, or *interaction kernel*, and G_D is the Green function of L for D; see Sect. 2.1 for details. We note that γ_D is the nonlocal normal derivative of the Poisson kernel of L, see (6.1) and (2.8) below, similarly as the kernel in the classical Douglas identity, see Bogdan, Fafuła, and Rutkowski [7, Subsection 2.3]. The nonlocal Douglas identity of [8] can be stated as follows,

$$\mathcal{E}_D[u] = \mathcal{H}_D[g], \tag{1.7}$$

where $g\colon D^c \to \mathbb{R}$, $\mathcal{H}_D[g] < \infty$, and $u = P_D[g]$ is the Poisson integral of g, see Sect. 2.1. Notably, $P_D[g]$ is a harmonic function of L, so the identity (1.7) explains the energy of a harmonic function by the energy of its external values. In the language of Chen and Fukushima [15, Chapter 5], the right-hand side of (1.7) is the *trace form* and $\gamma_D(z, w) \, \mathrm{d}z\mathrm{d}w$ is the *Feller measure* for $(\mathcal{E}_{\mathbb{R}^d}, \mathcal{V}_{\mathbb{R}^d})$ on D^c, but the extension and trace problem for $\mathcal{V}_D$ were not investigated in [15]. We also note that Jacob and Schilling [41] studied Douglas identities for nonlocal censored-type Dirichlet forms.

Our present goal is to extend the nonlocal Douglas formula (1.7) to a more general nonlinear case. The possibility of such a setting occurred to us owing to the recent Hardy–Stein identities of Bogdan, Dyda and Luks [6, Theorem 2]. To this end we will use the following notion, the *French power*:

$$x^{\langle \kappa \rangle} = |x|^\kappa \operatorname{sgn}(x), \quad x \in \mathbb{R}, \quad \kappa \in \mathbb{R}.$$

More precisely, $x^{\langle\kappa\rangle} = x^\kappa$ if $x > 0$, $x^{\langle\kappa\rangle} = -|x|^\kappa$ if $x < 0$, and $0^{\langle\kappa\rangle} = 0$. For example, $x^{\langle 0\rangle} = \operatorname{sgn}(x)$ and $x^{\langle 2\rangle} \neq x^2$ as functions on $\mathbb{R}$. In what follows we fix $1 < p < \infty$, the exponent of the "nonlinearity" alluded to in the title of the paper. Our nonlinear nonlocal Douglas identity is as follows:

$$\begin{aligned}
&\frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (u(x)^{\langle p-1\rangle} - u(y)^{\langle p-1\rangle})(u(x)-u(y))\, \nu(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2} \iint\limits_{D^c\times D^c} (g(w)^{\langle p-1\rangle} - g(z)^{\langle p-1\rangle})(g(w)-g(z))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z, \qquad (1.8)
\end{aligned}$$

where $u = P_D[g]$ and $g\colon D^c \to \mathbb{R}$. For a precise statement see Theorem 4.1 and Remark 4.2 below, since the result hinges on suitable additional assumptions on ν, D and g. No analogue of (1.8) seems to exist in the literature for $p \neq 2$, even for $\Delta^{\alpha/2}$. However, related nonlinear forms $\int u^{\langle p-1\rangle} Lu$ appear often in the literature concerning Markovian semigroups of operators on L^p spaces, see also (2.27) and (7.3) below. This is because for $p \in (1,\infty)$ the dual space of L^p is $L^{p/(p-1)}$ and for $u \in L^p$ we have $u^{\langle p-1\rangle} \in L^{p/(p-1)}$, and $\int |u|^p = \int |u^{\langle p-1\rangle}|^{p/(p-1)} = \int u^{\langle p-1\rangle} u$. Therefore in view of the Lumer–Phillips theorem, $u^{\langle p-1\rangle}$ yields a linear functional on L^p appropriate for testing *dissipativity* of generators, see, e.g., Pazy [48, Section 1.4]. In this connection we note that Davies [18, Chapter 2 and 3] gives some fundamental calculations with forms and powers. For the semigroups generated by local operators we refer to Langer and Maz'ya [45] and Sobol and Vogt [57, Theorem 1.1]. Liskevich and Semenov [46] use the L^p setting to analyze perturbations of Markovian semigroups. For nonlocal operators we refer to Farkas, Jacob and Schilling [28, (2.4)], and to the monograph of Jacob [40, (4.294)].

The following variant of (1.8) is also true, see (2.21), (2.22), and (2.24) below,

$$\begin{aligned}
&\frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} (|u(y)|^p - |u(x)|^p - pu(x)^{\langle p-1\rangle}(u(y)-u(x)))\, \nu(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2} \iint\limits_{D^c\times D^c} (|g(z)|^p - |g(w)|^p - pg(w)^{\langle p-1\rangle}(g(z)-g(w)))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z. \qquad (1.9)
\end{aligned}$$

The integrands in (1.9) come from the second order Taylor remainder of the convex function $x \mapsto |x|^p$, see (2.13), which leads us to the notion of *Bregman divergence*; see Sect. 2.2, see also Bregman [12] for the original contribution or Sprung [58]. Bregman divergence is important for statistical learning, see Nielsen and Nock [47] or Frigyik, Gupta and Srivastava [31] and the references therein. The Bregman divergence based on the power function $|x|^p$ defines the free energy functionals in the studies of Sobolev and Gagliardo–Nirenberg–Sobolev inequalities by Carrillo et al. [14, p. 71] and Bonforte, Dolbeault, Nazaret, and Simonov [11]. It also commonly appears in entropy inequalities, see, e.g., Wang [60].

The present paper indicates further uses of Bregman divergence in PDEs. As we show in Sect. 6, γ_D is the kernel of the Dirichlet-to-Neumann map (6.2) for L. Over the last few years, Dirichlet-to-Neumann map related to nonlocal operators was intensively studied in the context of inverse problems, see, e.g., [3, 17, 33, 34]. The forms in (1.8) are suitable for studying the Dirichlet-to-Neumann map as an operator in L^p. In particular, using our Douglas identity we show that the *normalized* Dirichlet-to-Neumann operator (6.5) is bounded on a certain weighted L^p space. Results in this direction were obtained by Vondraček [59], 63 and Foghem and Kassmann [30] for $p = 2$, but even in this case our approach gives new insights.

As another motivation, we mention that the form on the left-hand side of (1.8) with $D = \mathbb{R}^d$ is appropriate for studying L^p properties of Markovian semigroups. For instance, it was used by Bogdan, Jakubowski, Lenczewska, and Pietruska-Pałuba [9] to characterize the contractivity on $L^p(\mathbb{R}^d)$ of the semigroups generated by the fractional Laplacian with Hardy-type potentials. The interested reader may find insights into the technique in [9, Lemma 7 and Proof of Theorem 3], or even in (2.27) and (7.3) below.

The paper is organized as follows. Section 2 contains definitions and basic facts. Subsection 2.1 introduces notions from the probabilistic potential theory and Sect. 2.2 introduces our nonlinear setting and novel Sobolev–Bregman spaces $\mathcal{V}_D^p$ and $\mathcal{X}_D^p$ defined by the condition of finiteness of the respective sides of (1.8). In (2.19) we collect in one place four (equivalent) approximations for our Bregman divergence, which appear in the literature. In Sect. 3 we generalize the Hardy–Stein identities of [6] and [8] to our present context. This is instrumental for the proof of the Douglas identity in Sect. 4. In Corollary 4.3 we conclude that the Poisson integral P_D and the restriction to D^c are the extension and trace operators between the Sobolev–Bregman spaces. In view toward applications in variational problems, in Sect. 5 we prove the Douglas formula with the remainder for the energy of sufficiently regular nonharmonic functions. We also show that harmonic functions are quasi-minimizers of the considered nonlinear nonlocal forms, but in general not minimizers. In Sect. 6 we apply our results for the analysis of the Dirichlet-to-Neumann operator in L^p for $p \geq 2$. Finally, in Sect. 7 we give, for $p \geq 2$, the following result for Poisson integrals $u = P_D[g]$ and the more usual integral forms based on the p-increments of functions:

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} |u(x)-u(y)|^p \nu(x,y)\,\mathrm{d}x\mathrm{d}y \leq c \iint\limits_{D^c \times D^c} |g(w)-g(z)|^p \gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z. \quad (1.10)$$

It follows that $g \mapsto P_D[g]$ is an extension operator for nonlocal Sobolev-type spaces $\mathcal{W}_D^p$, defined by the finiteness of the left-hand side. In the remainder of Sect. 7 we compare $\mathcal{V}_D^p$ and $\mathcal{W}_D^p$.

2 Preliminaries

All the considered functions, sets and measures are tacitly assumed to be Borel. When we write $f \approx g$ (resp. $f \lesssim g$), we mean that there is a number $c > 0$, i.e. a *constant*, such that $(1/c)f(x) \leq g(x) \leq cf(x)$ (resp. $f(x) \leq cg(x)$) for all arguments x. Important constants will be capitalized: $C_1, C_2, \ldots,$ and their values will not change throughout the paper.

2.1 Processes and potential-theoretic notions

Let L and ν be as in the Introduction. Following [8], we *additionally* assume that:

(A1) ν is twice continuously differentiable on $(0, \infty)$ and there is a constant C_1 such that

$$|\nu'(r)|, |\nu''(r)| \leq C_1 \nu(r), \qquad r > 1.$$

(A2) There exist constants $\beta \in (0, 2)$ and $C_2 > 0$ such that

$$\nu(\lambda r) \leq C_2 \lambda^{-d-\beta} \nu(r), \qquad 0 < \lambda, r \leq 1, \quad (2.1)$$

$$\nu(r) \leq C_2 \nu(r+1), \qquad r \geq 1. \quad (2.2)$$

A prominent representative of *unimodal* Lévy operators L is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. In this case we have $\nu(x, y) = c_{d,\alpha}|y - x|^{-d-\alpha}$, where $\alpha \in (0, 2)$, $x, y \in \mathbb{R}^d$, and

$$c_{d,\alpha} = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2}|\Gamma(-\alpha/2)|}.$$

We refer the reader to Bogdan and Byczkowski [5], Di Nezza, Palatucci and Valdinoci [22], Garofalo [32], and Kwaśnicki [44] for more information on $\Delta^{\alpha/2}$. Clearly, $\nu(r) = c_{d,\alpha} r^{-d-\alpha}$ satisfies both **(A1)** and **(A2)**.

Our results depend in part on martingale properties of harmonic functions, so we introduce the Lévy process $(X_t, t \geq 0)$ on $\mathbb{R}^d$ whose generator is given by (1.3). Let

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos \xi \cdot x)\nu(|x|)\, \mathrm{d}x, \quad \xi \in \mathbb{R}^d,$$

the Lévy–Khinchine exponent of (X_t). Since $\nu(\mathbb{R}^d) = \infty$, by Sato [54, Theorem 27.7] and Kulczycki and Ryznar [43, Lemma 2.5], the densities $p_t(x)$ of (X_t) are continuous on $\mathbb{R}^d \setminus \{0\}$ for $t > 0$, and satisfy

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\xi\cdot x} p_t(x)\, \mathrm{d}x = \mathrm{e}^{-t\psi(\xi)}, \quad t > 0,\ \xi \in \mathbb{R}^d.$$

For $t > 0$ and $x, y \in \mathbb{R}^d$ denote $p_t(x, y) = p_t(y - x)$, the transition density of (X_t) considered as Markov process on $\mathbb{R}^d$. Namely, for starting point $x \in \mathbb{R}^d$, times $0 \leq t_1 < t_2 < \ldots t_n$ and sets $A_1, A_2, \ldots A_n \subset \mathbb{R}^d$ we let, as usual,

$$\begin{aligned}&\mathbb{P}^x(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)\\ &\quad= \int_{A_1}\int_{A_2}\ldots\int_{A_n} p_{t_1}(x, x_1)p_{t_2-t_1}(x_1, x_2)\cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n)\, \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n.\end{aligned}$$

This determines $\mathbb{P}^x$, the distribution of the process (X_t) starting from x, and $\mathbb{E}^x$, the corresponding expectation. In the wording of [54, Section 11], (X_t) is the symmetric Lévy process in $\mathbb{R}^d$ with $(0, \nu, 0)$ as the Lévy triplet. Without losing generality we actually assume that each X_t is the canonical projection $X_t(\omega) = \omega(t)$ on the space of càdlàg functions $\omega\colon [0, \infty) \to \mathbb{R}^d$. We will also use the standard complete right-continuous filtration $(\mathcal{F}_t, t \geq 0)$ to analyze (X_t), see Protter [50, Theorem I.31]. In passing we recall that every Lévy process is a Feller process [50].

Let $\emptyset \neq D \subset \mathbb{R}^d$ be an open set. The time of the first exit of X from D is, as usual,

$$\tau_D = \inf\{t > 0 :\ X_t \notin D\}.$$

The Dirichlet heat kernel $p_t^D(x, y)$ is defined by Hunt's formula, cf. Chung and Zhao [16, Chapter 2.2],

$$p_t^D(x, y) = p_t(x, y) - \mathbb{E}^x(p_{t-\tau_D}(X_{\tau_D}, y);\ \tau_D < t), \quad t > 0,\ x, y \in \mathbb{R}^d.$$

It is the transition density of the process (X_t) killed upon exiting D, i.e.,

$$\mathbb{E}^x[t < \tau_D;\ f(X_t)] = \int_{\mathbb{R}^d} f(y)p_t^D(x, y)\mathrm{d}y, \quad x \in \mathbb{R}^d,\ t > 0,$$

for integrable functions f. The Green function of D is the potential of p_t^D:

$$G_D(x, y) = \int_0^\infty p_t^D(x, y)\, \mathrm{d}t, \quad x, y \in \mathbb{R}^d,$$

and by Fubini–Tonelli we have

$$\mathbb{E}^x \tau_D = \int_{\mathbb{R}^d} G_D(x, y)\, \mathrm{d}y, \quad x \in \mathbb{R}^d. \tag{2.3}$$

The Poisson kernel of D for L is defined by

$$P_D(x, z) = \int_D G_D(x, y)\nu(y, z)\, \mathrm{d}y, \quad x \in D,\ z \in D^c. \tag{2.4}$$

With **(A2)** for bounded set D we easily see that for all $x, y \in D$ and $z \in D^c$ with $\operatorname{dist}(z, D) \geq \rho > 0$,

$$\nu(x, z) \approx \nu(y, z), \tag{2.5}$$

where comparability constants depend on ν, D and ρ. Consequently, (2.4) implies

$$P_D(x, z) \approx \nu(x, z)\mathbb{E}^x \tau_D, \quad x \in D, \quad \operatorname{dist}(z, D) \geq \rho > 0, \tag{2.6}$$

with the same proviso on comparability constants. Note that if D is bounded and $x \in D$ is fixed, then $\mathbb{E}^x \tau_D$ is bounded by a positive constant, see Pruitt [51]. We further note that for $w, z \in D^c$ the interaction kernel satisfies

$$\gamma_D(w, z) = \int_D \int_D \nu(w, x) G_D(x, y)\nu(y, z)\, \mathrm{d}x\mathrm{d}y \tag{2.7}$$

$$= \int_D \nu(w, x) P_D(x, z)\, \mathrm{d}x = \int_D \nu(z, x) P_D(x, w)\, \mathrm{d}x = \gamma_D(z, w). \tag{2.8}$$

Finally, the L-harmonic measure of D for $x \in \mathbb{R}^d$ is, as usual,

$$\omega_D^x(\mathrm{d}z) = \mathbb{P}^x[X_{\tau_D} \in \mathrm{d}z], \tag{2.9}$$

the distribution of the random variable X_{τ_D} with respect to $\mathbb{P}^x$.

From the Ikeda–Watanabe formula (see, e.g., Bogdan, Rosiński, Serafin and Wojciechowski [10, Section 4.2]) it follows that $P_D(x, z)\, \mathrm{d}z$ is the part of $\omega_D^x(\mathrm{d}z)$ which results from the discontinuous exit from D (by a jump). Below, by suitable assumptions on D and ν, we assure that P_D is the density of the whole harmonic measure, that is

$$\int_{D^c} P_D(x, z)\, \mathrm{d}z = 1, \quad x \in D. \tag{2.10}$$

This is true, e.g., if D is bounded, ν satisfies **(A2)**, $|\partial D| = 0$ and D^c has the property (VDC). The latter means that there is $c > 0$ such that for every $r > 0$ and $x \in \partial D$,

$$|D^c \cap B(x, r)| \geq c r^d. \tag{2.11}$$

Here, as usual, $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$. For the proof of (2.10) under the above conditions, see [8, Corollary A.2].

Observe that for $U \subset D$ we have $p^U \leq p^D$ and $G_U \leq G_D$. Therefore, $P_U(x, z) \leq P_D(x, z)$ for $x \in U$, $z \in D^c$, and $\gamma_U(z, w) \leq \gamma_D(z, w)$ for $z, w \in D^c$. These inequalities may be referred to as *domain monotonicity*. For $g\colon D^c \to \mathbb{R}$ we define the Poisson extension of g:

$$P_D[g](x) = \begin{cases} g(x) & \text{for } x \in D^c, \\ \int_{D^c} g(z) P_D(x, z)\, \mathrm{d}z & \text{for } x \in D, \end{cases} \tag{2.12}$$

and we call $\int_{D^c} g(z) P_D(x, z)\, \mathrm{d}z$ the Poisson integral, as long as it is convergent.

2.2 Function F_p and related function spaces

We depend on the two humble real functions:

$$x \mapsto |x|^\kappa \quad \text{and} \quad x \mapsto x^{\langle\kappa\rangle}, \quad x \in \mathbb{R}, \quad \kappa \in \mathbb{R}.$$

Clearly, $|x|^\kappa$ is symmetric, $x^{\langle\kappa\rangle}$ is antisymmetric: $(-x)^{\langle\kappa\rangle} = -x^{\langle\kappa\rangle}$, and their derivatives obey

$$(|x|^\kappa)' = \kappa x^{\langle\kappa-1\rangle} \quad \text{and} \quad (x^{\langle\kappa\rangle})' = \kappa|x|^{\kappa-1}, \quad x \neq 0.$$

Recall that $p > 1$. We let

$$F_p(a,b) = |b|^p - |a|^p - pa^{\langle p-1\rangle}(b-a), \quad a, b \in \mathbb{R}. \tag{2.13}$$

For instance, if $p = 2$, then $F_2(a,b) = (b-a)^2$, and if $p = 4$, then $F_4(a,b) = (b-a)^2(b^2 + 2ab + 3a^2)$. As the second-order Taylor remainder of the convex function $|x|^p$, F_p is nonnegative. In fact,

$$F_p(a,b) \approx (b-a)^2(|b| \vee |a|)^{p-2}, \quad a, b \in \mathbb{R}, \tag{2.14}$$

see [6, Lemma 6]. In particular, for $p \geq 2$ we have

$$F_p(a,b) \approx (b-a)^2(|a|^{p-2} + |b|^{p-2}), \quad a, b \in \mathbb{R}. \tag{2.15}$$

Recall that if X is a random variable with the first moment finite and $a \in \mathbb{R}$, then

$$\mathbb{E}(X-a)^2 = \mathbb{E}(X - \mathbb{E}X)^2 + (\mathbb{E}X - a)^2 = \operatorname{Var} X + (\mathbb{E}X - a)^2. \tag{2.16}$$

Here we do not exclude the case $\mathbb{E}X^2 = \infty$, in which case both sides of (2.16) are infinite, hence equal. This variance formula has the following analogue for F_p.

Lemma 2.1 *Let $p > 1$. Suppose that X is a random variable such that $\mathbb{E}|X| < \infty$. Then,*

(i) $\mathbb{E}F_p(\mathbb{E}X, X) = \mathbb{E}|X|^p - |\mathbb{E}X|^p \geq 0,$
(ii) $\mathbb{E}F_p(a, X) = F_p(a, \mathbb{E}X) + \mathbb{E}F_p(\mathbb{E}X, X) \geq \mathbb{E}F_p(\mathbb{E}X, X), \quad a \in \mathbb{R},$
(iii) $\mathbb{E}F_p(a, X) = \mathbb{E}F_p(b, X) + F_p(a,b) + (pa^{\langle p-1\rangle} - pb^{\langle p-1\rangle})(b - \mathbb{E}X), \quad a, b \in \mathbb{R}.$

Proof The verification is elementary, but we present it to emphasize that the finiteness of the first moment suffices. We have

$$\mathbb{E}F_p(\mathbb{E}X, X) = \mathbb{E}\Big[|X|^p - |\mathbb{E}X|^p - p(\mathbb{E}X)^{\langle p-1\rangle}(X - \mathbb{E}X)\Big] = \mathbb{E}|X|^p - |\mathbb{E}X|^p,$$

where $\mathbb{E}|X|^p = \infty$ is permitted, too. The expression in (i) is nonnegative by Jensen's inequality or because F_p is nonnegative. For all $a \in \mathbb{R}$ we have,

$$\begin{aligned}
\mathbb{E}F_p(a, X) &= \mathbb{E}\Big[|X|^p - |a|^p - pa^{\langle p-1\rangle}(X-a)\Big] \\
&= \mathbb{E}\Big[|X|^p - |\mathbb{E}X|^p - p(\mathbb{E}X)^{\langle p-1\rangle}(X - \mathbb{E}X)\Big] + |\mathbb{E}X|^p - |a|^p - pa^{\langle p-1\rangle}(\mathbb{E}X - a) \\
&= \mathbb{E}F_p(\mathbb{E}X, X) + F_p(a, \mathbb{E}X) \geq \mathbb{E}F_p(\mathbb{E}X, X),
\end{aligned}$$

as claimed in (ii). Finally, for all $a, b \in \mathbb{R}$ the right-hand side of (iii) is

$$\mathbb{E}|X|^p - |b|^p - pb^{\langle p-1\rangle}(\mathbb{E}X - b) + |b|^p - |a|^p - pa^{\langle p-1\rangle}(b-a) + (pa^{\langle p-1\rangle} - pb^{\langle p-1\rangle})(b - \mathbb{E}\mathbb{X}),$$

which simplifies to the left-hand side of (iii). Needless to say, (ii) is a special case of (iii). □

We next propose a simple lemma concerning the p-th moments of random variables, which is another generalization of (2.16).

Lemma 2.2 *For every $p \geq 1$ there exist constants $0 < c_p \leq C_p$ such that for every random variable X with $\mathbb{E}|X| < \infty$ and every number $a \in \mathbb{R}$,*

$$c_p \left(\mathbb{E}|X - \mathbb{E}X|^p + |\mathbb{E}\mathbb{X} - a|^p\right) \leq \mathbb{E}|X - a|^p \leq C_p\left(\mathbb{E}|X - \mathbb{E}X|^p + |\mathbb{E}\mathbb{X} - a|^p\right). \quad (2.17)$$

Proof If $\mathbb{E}|X|^p = \infty$, then all the sides of (2.17) are infinite. Otherwise, by convexity,

$$\mathbb{E}|X - a|^p = \mathbb{E}|(X - \mathbb{E}X) + (\mathbb{E}X - a)|^p \leq 2^{p-1}\left(\mathbb{E}|X - \mathbb{E}X|^p + |\mathbb{E}X - a|^p\right).$$

For the lower bound we make two observations: $|\mathbb{E}X - a|^p \leq \mathbb{E}|X - a|^p$ (Jensen's inequality), and

$$\mathbb{E}|X - \mathbb{E}X|^p = \mathbb{E}|(X - a) - (\mathbb{E}X - a)|^p \leq 2^{p-1}\left(\mathbb{E}|X - a|^p + |\mathbb{E}X - a|^p\right) \leq 2^p\mathbb{E}|X - a|^p.$$

Adding the two, we get that $|\mathbb{E}X - a|^p + \mathbb{E}|X - \mathbb{E}X|^p \leq (1 + 2^p)\mathbb{E}|X - a|^p$. □

The function $F_p(a, b)$ is not symmetric in a, b, but the right-hand side of (2.14) is, so it is natural to consider the symmetrized version of F_p, given by the formula:

$$H_p(a, b) = \tfrac{1}{2}(F_p(a, b) + F_p(b, a)) = \tfrac{p}{2}(b^{\langle p-1\rangle} - a^{\langle p-1\rangle})(b - a), \; a, b \in \mathbb{R}. \quad (2.18)$$

We can relate H_p to a "quadratic" expression as follows.

Lemma 2.3 *For every $p > 1$ we have $F_p(a, b) \approx H_p(a, b) \approx (b^{\langle p/2\rangle} - a^{\langle p/2\rangle})^2$.*

Proof The first comparison follows from (2.14): we have $F_p(a, b) \approx F_p(b, a)$, hence $F_p \approx H_p$. As for the second statement, if either a or b are equal to 0, then the expressions coincide up to constants depending on p. If $a, b \neq 0$, then $a = tb$ with $t \neq 0$. Using this representation we see that the second comparison is equivalent to the following:

$$(t^{\langle p-1\rangle} - 1)(t - 1) \approx (t^{\langle p/2\rangle} - 1)^2, \quad t \in \mathbb{R}.$$

The latter holds because both sides are continuous and positive except at $t = 1$; at infinity both are power functions with the leading term $|t|^p$, and at $t = 1$ their ratio converges to a positive constant. □

Summarizing, by (2.14) and Lemma 2.3 for each $p \in (1, \infty)$ we have

$$F_p(a, b) \approx H_p(a, b) \approx (b - a)^2(|b| \vee |a|)^{p-2} \approx (b^{\langle p/2\rangle} - a^{\langle p/2\rangle})^2, \; a, b \in \mathbb{R}. \quad (2.19)$$

It is hard to trace down the first occurrence of such comparisons in the literature. The one-sided inequality $|b^{p/2} - a^{p/2}|^2 \leq \frac{p^2}{4(p-1)}(b - a)(b^{p-1} - a^{p-1})$ for $a, b \geq 0$ can be found in connection with logarithmic Sobolev inequalities, e.g., in Davies [18, (2.2.9)] for $2 < p < \infty$, and Bakry [2, p. 39] for $p > 1$. The opposite inequality $(b-a)(b^{p-1}-a^{p-1}) \leq (b^{p/2}-a^{p/2})^2$ with $a, b > 0$ and $p > 1$ appears, e.g., in [46, Lemma 2.1].

In fact the following inequalities hold for all $p \in (1, \infty)$ and $a, b \in \mathbb{R}$:

$$\frac{4(p-1)}{p^2}(b^{\langle p/2\rangle} - a^{\langle p/2\rangle})^2 \leq (b - a)(b^{\langle p-1\rangle} - a^{\langle p-1\rangle}) \leq 2(b^{\langle p/2\rangle} - a^{\langle p/2\rangle})^2. \quad (2.20)$$

Indeed, if a and b have opposite signs then it is enough to consider $b = t \geq 1$ and $a = -1$, and to compare $(t+1)(t^{p-1}+1) = t^p + t^{p-1} + t + 1$ with $(t^{p/2}+1)^2 = t^p + 2t^{p/2} + 1$. We have $t^{p/2} = \sqrt{t^{p-1}t} \leq (t^{p-1} + t)/2$, which verifies the left-hand side inequality in (2.20) with constant 1, which is better than $4(p-1)/p^2$. We further get the right-hand side inequality in (2.20), and the constant 2 suffices, because $t^{p-1}+t-(t^p+1)=(1-t)(t^{p-1}-1)\le 0$. Note that the constant 2 is not optimal for individual values of p, e.g., for $p=2$, but the constant 1 does not suffice for $p\in(1,2)\cup(2,\infty)$ because then $1\vee(p-1)>p/2$, and so $t^{p-1}+t>2t^{p/2}$ for large t.

If a and b have the same sign, then we may assume $b=ta$, $a>0$, $t\ge 1$, and consider the quotient

$$H(t)=\frac{(t^{p-1}-1)(t-1)}{(t^{p/2}-1)^2}=1-\frac{t(t^{(p-2)/2}-1)^2}{(t^{p/2}-1)^2}=1-h(s)^2,$$

where $s=\sqrt{t}$, $h(s)=s(s^{p-2}-1)/(s^p-1)$. We see that $h(s)$ is strictly positive for $p>2$, $s>1$ and negative for $p\in(1,2)$. We claim that it decreases in the former case and increases in the latter. The sign of the derivative of h is the same as the sign of the function $l(s)=-s^{2p-2}+(p-1)s^p-(p-1)s^{p-2}+1$. Now, since $l(1)=0$, the sign of l on $(1,\infty)$ is in turn equal to the sign of $l'(s)=(p-1)s^{p-3}(-2s^p+ps^2-(p-2))$, and further equal to the sign of $-2p(s^{p-1}-s)$. Since the last function is negative on $(1,\infty)$ if $p>2$ and positive for $p\in(1,2)$, the claim is proved. Consequently, the function $s\mapsto h(s)^2$ is decreasing on $(1,\infty)$, so we get

$$\lim_{t\to 1^+}H(t)=\frac{4(p-1)}{p^2}<H(t)<1,\quad t>1,$$

and (2.20) follows. The above also shows that the constant $4(p-1)/p^2$ in (2.20) cannot be improved.

We would like to note that for $p\neq 2$, $F_p(a+t,b+t)$ is *not* comparable with $F_p(a,b)$. Indeed, for $a,r>0$ one has $F_p(a,a+r)\approx r^2(a\vee(a+r))^{p-2}=r^2(a+r)^{p-2}$, which is not comparable with $F_p(0,r)=r^2$ for large values of a. Here are one-sided comparisons of $F_p(a,b)$ with the more usual p-increments, see, e.g., Zeidler [61, p. 503].

Lemma 2.4 *If $p\ge 2$ then $F_p(a,b)\gtrsim|b-a|^p$, and if $1<p\le 2$, then $|b-a|^p\gtrsim F_p(a,b)$.*

Proof If $a=b$, then the inequalities are trivial, so assume that $a\neq b$ and consider the quotient

$$\frac{F_p(a,b)}{|b-a|^p}\approx\frac{(|a|\vee|b|)^{p-2}}{|b-a|^{p-2}}.$$

Both parts of the statements now follow from the inequality $|b-a|^r\le 2^r(|a|\vee|b|)^r$, $r>0$. □

In analogy to (1.4) for $u\colon\mathbb{R}^d\to\mathbb{R}$ we define

$$\mathcal{E}_D^{(p)}[u]:=\tfrac{1}{p}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c}F_p(u(x),u(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y. \tag{2.21}$$

By the symmetry of ν and (2.18),

$$\begin{aligned}\mathcal{E}_D^{(p)}[u]&=\tfrac{1}{p}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c}H_p(u(x),u(y))\nu(x,y)\,\mathrm{d}x\mathrm{d}y\\&=\tfrac{1}{2}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c}(u(y)^{\langle p-1\rangle}-u(x)^{\langle p-1\rangle})(u(y)-u(x))\nu(x,y)\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.22}$$

Of course, $\mathcal{E}_D^{(2)} = \mathcal{E}_D$. For $D = \mathbb{R}^d$ we have

$$\mathcal{E}_{\mathbb{R}^d}^{(p)}[u] = \frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} (u(y)^{\langle p-1\rangle} - u(x)^{\langle p-1\rangle})(u(y) - u(x))\nu(x, y)\,\mathrm{d}x\mathrm{d}y. \tag{2.23}$$

Clearly, for $p = 2$ we retrieve the classical Dirichlet form of the operator L.

Let $g\colon D^c \to \mathbb{R}$. To quantify the increments of g, we use the form:

$$\begin{aligned}\mathcal{H}_D^{(p)}[g] &= \frac{1}{p} \iint\limits_{D^c \times D^c} F_p(g(w), g(z))\gamma_D(w, z)\,\mathrm{d}w\mathrm{d}z = \frac{1}{p} \iint\limits_{D^c \times D^c} H_p(g(w), g(z))\gamma_D(w, z)\,\mathrm{d}w\mathrm{d}z \\ &= \frac{1}{2} \iint\limits_{D^c \times D^c} (g(z)^{\langle p-1\rangle} - g(w)^{\langle p-1\rangle})(g(z) - g(w))\gamma_D(w, z)\,\mathrm{d}w\mathrm{d}z. \end{aligned} \tag{2.24}$$

The spaces $\mathcal{V}_D$ and $\mathcal{X}_D$ discussed in the Introduction lend themselves to the following generalizations:

$$\mathcal{V}_D^p := \{u\colon \mathbb{R}^d \to \mathbb{R} \mid \mathcal{E}_D^{(p)}[u] < \infty\}, \tag{2.25}$$

and

$$\mathcal{X}_D^p := \{g\colon D^c \to \mathbb{R} \mid \mathcal{H}_D^{(p)}[g] < \infty\}. \tag{2.26}$$

We call them Sobolev–Bregman spaces, since they involve the Bregman divergence. Our development below indicates that $\mathcal{V}_D^p$ and $\mathcal{X}_D^p$ provide a viable framework for nonlocal nonlinear variational problems. In view of (2.22) for all $u\colon \mathbb{R}^d \to \mathbb{R}$ we have

$$\mathcal{E}_D^{(p)}[u] = \mathcal{E}_D(u^{\langle p-1\rangle}, u), \tag{2.27}$$

where

$$\mathcal{E}_D(v, u) := \frac{1}{2} \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} (v(x) - v(y))(u(x) - u(y))\nu(x, y)\,\mathrm{d}x\mathrm{d}y,$$

if the integral is well defined, which is the case in (2.27) for $v = u^{\langle p-1\rangle}$. For clarity we also note that by (2.20), (2.22) and (2.24), we have the *comparisons*

$$\mathcal{E}_D^{(p)}[u] \approx \mathcal{E}_D[u^{\langle p/2\rangle}], \tag{2.28}$$

and

$$\mathcal{H}_D^{(p)}[g] \approx \mathcal{H}_D[g^{\langle p/2\rangle}], \tag{2.29}$$

for all $u\colon \mathbb{R}^d \to \mathbb{R}$ and $g\colon D^c \to \mathbb{R}$ with the comparability constants depending only on p. Below, however, we focus on genuine *equalities*.

3 Hardy–Stein identity

We first collect properties of harmonic functions that are needed in the proof of the identity (1.8). We mostly follow [8], so our presentation will be brief. We write $U \subset\subset D$ if the closure of U is a compact subset of D.

Definition 3.1 We say that the function $u\colon \mathbb{R}^d \to \mathbb{R}$ is L-harmonic (or harmonic, if L is understood) in D if it has the mean value property inside D, that is: for every open set $U \subset\subset D$,

$$u(x) = \mathbb{E}^x u(X_{\tau_U}), \quad x \in U.$$

If $u(x) = \mathbb{E}^x u(X_{\tau_D})$ for all $x \in D$, then we say that u is regular harmonic.

In the above we assume that the expectations are absolutely convergent.

The strong Markov property of (X_t) implies that if u is regular L-harmonic in D, then it is L-harmonic in D. By [8, Section 4], if u is L-harmonic in D, then $u \in L^1_{loc}(\mathbb{R}^d) \cap C^2(D)$, $Lu(x)$ can be computed pointwise for $x \in D$ as in (1.3), and $Lu(x) = 0$ for $x \in D$. We also note that the Harnack inequality holds for L-harmonic functions (see Grzywny and Kwaśnicki [39, Theorem 1.9]; the assumptions of that theorem follow from **(A2)**).

We will use the following Dynkin-type lemma, proven in our setting in [8, Lemma 4.11].

Lemma 3.2 *Let the set $U \subset\subset D$ be open and Lipschitz. If $\int_{\mathbb{R}^d} |\phi(y)|(1 \wedge \nu(y))\,\mathrm{d}y < \infty$ and $\phi \in C^2(\overline{U})$, then $L\phi$ is bounded on $\overline{U}$ and for every $x \in \mathbb{R}^d$,*

$$\mathbb{E}^x \phi(X_{\tau_U}) - \phi(x) = \int_U G_U(x, y) L\phi(y)\,\mathrm{d}y, \tag{3.1}$$

where the integrals converge absolutely.

The following Hardy–Stein formula extends [6, Lemma 8] and [8, Lemma 4.12], where it was proved, for the fractional Laplacian and $p > 1$, and for unimodal operators L and $p = 2$, respectively.

Proposition 3.3 *If $u\colon \mathbb{R}^d \to \mathbb{R}$ is L-harmonic in D, $p > 1$, and $U \subset\subset D$ is open Lipschitz, then*

$$\mathbb{E}^x |u(X_{\tau_U})|^p = |u(x)|^p + \int_U G_U(x, y) \int_{\mathbb{R}^d} F_p(u(y), u(z))\nu(y, z)\,\mathrm{d}z\mathrm{d}y, \quad x \in U. \tag{3.2}$$

Proof As a guideline, the result follows by taking $\phi = |u|^p$ in the Dynkin formula (3.1). We combine the methods of [6] and [8]. By [8, Lemma 4.9] if u is harmonic in D, then $u \in C^2(D)$. Thus, in particular, $|u|^p$ is bounded in a neighborhood of $\overline{U}$. Let $x \in U$. Consider the complementary cases:

$$\text{(i)} \int_{U^c} |u(z)|^p \nu(x, z)\,\mathrm{d}z = \infty, \quad \text{or} \quad \text{(ii)} \int_{U^c} |u(z)|^p \nu(x, z)\,\mathrm{d}z < \infty.$$

Since $|u|^p$ is bounded in a neighborhood of $\overline{U}$, this dichotomy can be reformulated as

$$\text{(i)} \quad \mathbb{E}^x |u(X_{\tau_U})|^p = \infty, \quad \text{or} \quad \text{(ii)} \quad \mathbb{E}^x |u(X_{\tau_U})|^p < \infty,$$

see the end of the proof of [8, Lemma 4.11] and (2.6).

In case (i), we show that the right-hand side of (3.2) is infinite as well. Assume first that $|u| > 0$ on a subset of U of positive measure. Pick $y \in U$ satisfying $|u(y)| > 0$, and let $A = \{z \in U^c : |u(z)| \geq (2 + \sqrt{2})|u(y)|\}$. Now, since $x, y \in U$ are fixed and ν is positive, continuous, and satisfies (2.2), we have $\nu(x, z) \approx \nu(y, z)$ for $z \in U^c$. Therefore, by (i),

$$\int_{U^c} |u(z)|^p \nu(y, z)\,\mathrm{d}z = \infty$$

as well. Furthermore,

$$\int_{U^c\setminus A} |u(z)|^p \nu(y,z)\,\mathrm{d}z \approx \int_{U^c\setminus A} |u(z)|^p \nu(x,z)\,\mathrm{d}z \le (2+\sqrt{2})^p |u(y)|^p \nu(x,U^c) < \infty,$$

and consequently we must have

$$\int_A |u(z)|^p \nu(y,z)\,\mathrm{d}z = \infty.$$

By the definition of A, for $z \in A$ we have

$$(u(z)-u(y))^2 \ge \frac{1}{2}u(z)^2 \quad \text{and} \quad |u(z)| \ge |u(y)|. \tag{3.3}$$

By (2.14) and (3.3) we therefore obtain

$$\begin{aligned}\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z &\approx \int_{\mathbb{R}^d} (u(z)-u(y))^2(|u(y)| \vee |u(z)|)^{p-2}\nu(y,z)\,\mathrm{d}z\\ &\ge \int_A (u(z)-u(y))^2|u(z)|^{p-2}\nu(y,z)\,\mathrm{d}z\\ &\ge \tfrac{1}{2}\int_A |u(z)|^p\nu(y,z)\,\mathrm{d}z = \infty.\end{aligned}$$

This is true for all points y in a set of positive Lebesgue measure, which proves that the right-hand side of (3.2) is infinite. If, on the other hand, $u \equiv 0$ in U, then $F_p(u(y),u(z)) = c|u(z)|^p$ for all $z \in \mathbb{R}^d$, $y \in U$, and by (i) the right-hand side of (3.2) is infinite again.

We now consider the case (ii). Thus $\mathbb{E}^x|u(X_{\tau_U})|^p < \infty$ and the integrability condition of Lemma 3.2 is satisfied for $\phi = |u|^p$. We will first prove (3.2) for $p \ge 2$. Then ϕ is of class C^2 on D, so we are in a position to use Lemma 3.2 and we get

$$\mathbb{E}^x|u(X_{\tau_U})|^p = |u(x)|^p + \int_U G_U(x,y)L|u|^p(y)\,\mathrm{d}y, \quad x \in U. \tag{3.4}$$

The integral on the right-hand side is absolutely convergent. Furthermore, since u is L-harmonic,

$$\begin{aligned}L|u|^p(y) &= L|u|^p(y) - pu(y)^{\langle p-1\rangle}Lu(y)\\ &= \lim_{\epsilon\to 0+}\int_{|z-y|>\epsilon} (|u(z)|^p - |u(y)|^p - pu(y)^{\langle p-1\rangle}(u(z)-u(y)))\nu(y,z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z \ge 0.\end{aligned}$$

Inserting this to (3.4) gives the statement.

When $p \in (1,2)$, the function $\mathbb{R} \ni r \mapsto |r|^p$ is not twice differentiable, and the above argument needs to be modified. We work under the assumption (ii), and we follow the proof of [6, Lemma 3]. Consider $\varepsilon \in \mathbb{R}$ and the function $\mathbb{R}^d \ni x \mapsto (x^2+\varepsilon^2)^{p/2}$. Let

$$F_p^{(\varepsilon)}(a,b) = (b^2+\varepsilon^2)^{p/2} - (a^2+\varepsilon^2)^{p/2} - pa(a^2+\varepsilon^2)^{(p-2)/2}(b-a), \quad a,b \in \mathbb{R}. \tag{3.5}$$

Since $1 < p < 2$, by [6, Lemma 6],

$$0 \le F_p^{(\varepsilon)}(a,b) \le \frac{1}{p-1}F_p(a,b), \qquad \varepsilon, a, b \in \mathbb{R}, \tag{3.6}$$

Let $\varepsilon > 0$. We note that $(u^2+\varepsilon^2)^{p/2} \in C^2(D)$. Also, the integrability condition in Lemma 3.2 is satisfied for $\phi = (u^2+\varepsilon^2)^{p/2}$ since it is satisfied for $\phi = |u|^p$ by (ii), and

$$(u^2+\varepsilon^2)^{p/2} \le (|u|+\varepsilon)^p \le 2^{p-1}(|u|^p+\varepsilon^p), \tag{3.7}$$

see also (1.2). Furthermore, $\mathbb{E}_x(u(X_{\tau_U})^2+\varepsilon^2)^{p/2} < \infty$. As in the first part of the proof,

$$\begin{aligned} L(u^2+\varepsilon^2)^{p/2}(y) &= L(u^2+\varepsilon^2)^{p/2}(y) - pu(y)(u(y)^2+\varepsilon^2)^{(p-2)/2}Lu(y) \\ &= \int_{\mathbb{R}^d} F_p^{(\varepsilon)}(u(y),u(z))\nu(y,z)\,\mathrm{d}z, \end{aligned} \tag{3.8}$$

therefore by Lemma 3.2,

$$\mathbb{E}^x(u(X_{\tau_U})^2+\varepsilon^2)^{p/2} = (u(x)^2+\varepsilon^2)^{p/2} + \int_U G_U(x,y)\int_{\mathbb{R}^d} F_p^{(\varepsilon)}(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y. \tag{3.9}$$

From the Dominated Convergence Theorem the left-hand side of (3.9) goes to $\mathbb{E}^x|u(X_{\tau_U})|^p < \infty$ as $\varepsilon \to 0^+$. Of course, $F_p^{(\varepsilon)}(a,b) \to F_p(a,b)$ as $\varepsilon \to 0^+$. Furthermore, by Fatou's lemma and (3.9),

$$\begin{aligned} &\int_U G_U(x,y)\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y \\ &\le \liminf_{\varepsilon\to 0^+}\int_U G_U(x,y)\int_{\mathbb{R}^d} F_p^{(\varepsilon)}(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y \\ &= \mathbb{E}^x|u(X_{\tau_U})|^p - |u(x)|^p < \infty. \end{aligned}$$

By (3.6) and the Dominated Convergence Theorem, we obtain (3.2) for $p \in (1,2)$. □

As a consequence, we obtain the the Hardy–Stein identity for D, generalizing and strengthening [6, (16)] and [8, Theorem 2.1].

Proposition 3.4 *Let $p > 1$ be given. If u is L-harmonic in D and $x \in D$, then*

$$\sup_{x\in U\subset\subset D}\mathbb{E}^x|u(X_{\tau_U})|^p = |u(x)|^p + \int_D G_D(x,y)\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y. \tag{3.10}$$

If u is regular L-harmonic in D, then the left-hand side can be replaced with $\mathbb{E}^x|u(X_{\tau_D})|^p$.

Proof As noted in [8, Remark 4.4], $\{u(X_{\tau_U}),\ U \subset D\}$ is a martingale ordered by the inclusion of open subsets of D. By domain monotonicity of the Green function and the nonnegativity of F_p, both sides of (3.2) increase if U increases. Since every open set $U \subset\subset D$ is included in an open Lipschitz set $U \subset\subset D$, the supremum in (3.10) may be taken over open Lipschitz sets $U \subset\subset D$. The first part of the statement follows from the Monotone Convergence Theorem.
If additionally u is regular harmonic, then

$$\mathbb{E}^x|u(X_{\tau_D})|^p = |u(x)|^p + \int_D G_D(x,y)\int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y. \tag{3.11}$$

This is delicate. Indeed, by [8, Remark 4.4], the martingale $\{u(X_{\tau_U}),\ U \subset\subset D\}$ is closed by the integrable random variable $u(X_{\tau_D})$. Therefore Lévy's Martingale Convergence Theorem yields that $u(X_{\tau_U})$ converges almost surely, and in L^1 to a random variable Z, as $U \uparrow D$, and we have $Z = \mathbb{E}^x[u(X_{\tau_D})|\sigma(\bigcup_{U\subset\subset D}\mathcal{F}_{\tau_U})]$, see, e.g., Dellacherie and Meyer [21, Theorem 31 a,b, p. 26]. We claim that the σ-algebra $\sigma(\bigcup_{U\subset\subset D}\mathcal{F}_{\tau_U})$ is equal to $\mathcal{F}_{\tau_D}$. Indeed, by Proposition 25.20 (i),(ii), and Proposition 25.19 (i),(ii) in Kallenberg [42, p. 501], the filtration of (X_t) is quasi-left continuous. Therefore τ_U increases to τ_D as U increases to D, and our claim follows from Dellacherie and Meyer [20, Theorem 83, p. 136]. Consequently, $Z = u(X_{\tau_D})$. Now, if $\sup_{x\in U\subset\subset D}\mathbb{E}^x|u(X_{\tau_U})|^p < \infty$, then [21, Theorem 31 c, p. 26] yields (3.11). Else, if the supremum is infinite, then $\mathbb{E}^x|u(X_{\tau_D})|^p = \infty$ by Jensen's inequality, and (3.11) holds, too. ◻

We note in passing that the case $p = 2$ of (3.11) was stated for less general sets D in the first displayed formula following (5.2) in the proof of Theorem 2.3 in [8]. Accordingly, the proof in [8] was easier.

4 The Douglas identity

We now present our main theorem. It is a counterpart of (1.7) with square increments of the function replaced by "increments" measured in terms F_p or H_p.

Theorem 4.1 *[**Douglas identity**] Let $p > 1$. Assume that the Lévy measure ν satisfies* **(A1)** *and* **(A2)**, *$D \subset \mathbb{R}^d$ is open, D^c satisfies* (VDC), *and $|\partial D| = 0$.*

(i) Let $g\colon D^c \to \mathbb{R}$ be such that $\mathcal{H}_D^{(p)}[g] < \infty$. Then $P_D[g]$ is well-defined and satisfies

$$\mathcal{H}_D^{(p)}[g] = \mathcal{E}_D^{(p)}[P_D[g]]. \tag{4.1}$$

(ii) Furthermore, if $u\colon \mathbb{R}^d \to \mathbb{R}$ satisfies $\mathcal{E}_D^{(p)}[u] < \infty$, then $\mathcal{H}_D^{(p)}[u|_{D^c}] < \infty$.

Here, as usual, $u|_{D^c}$ is the restriction of u to D^c, but in what follows we will abbreviate:

$$\mathcal{H}_D^{(p)}[u] := \mathcal{H}_D^{(p)}[u|_{D^c}]$$

and

$$P_D[u|_{D^c}] = P_D[u].$$

Remark 4.2 The more explicit expression of the Douglas identity (1.8) stated in the Introduction follows from (4.1), (2.22) and (2.24).

To the best of our knowledge the present Douglas identities are completely new, and our approach is original. The proof of Theorem 4.1 is given below in this section.

Recall the space $\mathcal{V}_D^p$, defined in (2.25), which is a natural domain of $\mathcal{E}_D^{(p)}$, and the space $\mathcal{X}_D^p$, defined in (2.26), which is a natural domain of $\mathcal{H}_D^{(p)}$. From Theorem 4.1 we immediately obtain the following trace and extension result in the nonquadratic setting.

Corollary 4.3 *Let* $\operatorname{Ext} g = P_D[g]$, *the Poisson extension, and* $\operatorname{Tr} u = u|_{D^c}$, *the restriction to D^c. Then* $\operatorname{Ext}\colon \mathcal{X}_D^p \to \mathcal{V}_D^p$, $\operatorname{Tr}\colon \mathcal{V}_D^p \to \mathcal{X}_D^p$, *and* $\operatorname{Tr}\operatorname{Ext}$ *is the identity operator on $\mathcal{X}_D^p$.*

It is well justified to call Ext the extension operator and Tr the trace operator for $\mathcal{V}_D^p$.

We next give the Douglas identity for the Poisson extension on D and the form $\mathcal{E}_{\mathbb{R}^d}^{(p)}$ (with the integration over the whole of $\mathbb{R}^d \times \mathbb{R}^d$).

Corollary 4.4 *If $P_D[|g|] < \infty$ on D, in particular if $\mathcal{H}_D^{(p)}[g] < \infty$, then*

$$\mathcal{E}^{(p)}_{\mathbb{R}^d}[P_D[g]] = \frac{1}{p} \iint\limits_{D^c \times D^c} F_p(g(z), g(w))(\gamma_D(z,w) + \nu(z,w))\,\mathrm{d}z\mathrm{d}w.$$

We note that the kernel on the right-hand side of the above identity also appears in [15, Theorem 5.6.3] for $p = 2$, but even the form $\mathcal{E}_D^{(2)}$ and the Douglas identity of Theorem 4.1 with $p = 2$ on full domain $\mathcal{V}_D^2$ do not appear in [15].

The proof of Theorem 4.1 uses the following lemma, which asserts that the condition $\mathcal{H}_D^{(p)}[g] < \infty$ implies the finiteness of $P_D[|g|^p]$ and $P_D[|g|]$ on D.

Lemma 4.5 *Suppose that $g\colon D^c \to \mathbb{R}$ satisfies $\mathcal{H}_D^{(p)}[g] < \infty$. Then for every $x \in D$ we have $\int_{D^c} |g(z)|^p P_D(x,z)\,\mathrm{d}z < \infty$. In particular, the Poisson integral of g is well-defined.*

Proof Denote $I = \int_{D^c} |g(z)|^p P_D(x,z)\,\mathrm{d}z$. If $\mathcal{H}_D^{(p)}[g] < \infty$, then

$$\begin{aligned}
&\iint\limits_{D^c \times D^c} F_p(g(w), g(z))\gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z \\
&= \int_D \int_{D^c} \int_{D^c} F_p(g(w), g(z))\nu(w,x)P_D(x,z)\,\mathrm{d}z\mathrm{d}w\mathrm{d}x < \infty. \qquad (4.2)
\end{aligned}$$

Since $\nu > 0$, for almost all (hence for some) pairs $(w, x) \in D^c \times D$ we get

$$\int_{D^c} F_p(g(w), g(z))P_D(x,z)\,\mathrm{d}z < \infty. \tag{4.3}$$

For the remainder of the proof, we only consider pairs (w, x) satisfying the above condition.

We will use different approaches for $p \geq 2$ and $p \in (1, 2)$. Let $p \geq 2$. From (2.15) we obtain

$$A := \int_{D^c} (g(z) - g(w))^2 |g(z)|^{p-2} P_D(x,z)\,\mathrm{d}z < \infty.$$

For $z \in D^c$, let $g_n(z) = -n \vee g(z) \wedge n$. Clearly $|g_n(z)| \leq |g(z)|$ and $|g_n(z)| \nearrow |g(z)|$ when $n \to \infty$. Since $|g_n(z)| \leq n$, the integral $I_n := \int_{D^c} |g_n(z)|^p P_D(x,z)\,\mathrm{d}z$ is finite. It is also true that the increments of g_n do not exceed those of g, that is $|g_n(z) - g_n(w)| \leq |g(z) - g(w)|$. Consequently,

$$\begin{aligned}
I_n &= \int_{D^c} g_n(z)^2 |g_n(z)|^{p-2} P_D(x,z)\,\mathrm{d}z \\
&\leq 2\int_{D^c} (g_n(z) - g_n(w))^2 |g_n(z)|^{p-2} P_D(x,z) + 2g_n(w)^2 \int_{D^c} |g_n(z)|^{p-2} P_D(x,z)\,\mathrm{d}z \\
&\leq A + 2g(w)^2 \left(\int_{D^c} |g_n(z)|^p P_D(x,z)\,\mathrm{d}z \right)^{\frac{p-2}{p}}.
\end{aligned}$$

The last inequality is obvious for $p = 2$, and follows from Jensen's inequality if $p > 2$. Thus,

$$I_n \leq A + 2g(w)^2 (I_n)^{1-\frac{2}{p}}, \tag{4.4}$$

hence the sequence (I_n) is bounded. By the Monotone Convergence Theorem, $I < \infty$. By Jensen's inequality we also get $\int_{D^c} |g(z)| P_D(x,z)\,\mathrm{d}z < \infty$. By the Harnack inequality, the finiteness of the Poisson integral of $|g|$ or $|g|^p$ at any point $x \in D$ guarantees its finiteness at every point of D, see, e.g., [8, Lemma 4.6], therefore the proof is finished for $p \geq 2$.

Now let $p \in (1,2)$. If $g \equiv 0$ a.e. on D^c, then the statement is trivial. Otherwise, pick $w \in D^c$ such that $0 < |g(w)| < \infty$. Let $B = \{z \in D^c : |g(z)| > |g(w)|\}$. We have

$$\int_{D^c \setminus B} |g(z)|^p P_D(x,z)\, \mathrm{d}z \leq |g(w)|^p < \infty.$$

Using (2.14) and (4.3) we get

$$\begin{aligned}
&\int_B |g(z)|^p P_D(x,z)\, \mathrm{d}z = \int_B g(z)^2 |g(z)|^{p-2} P_D(x,z)\, \mathrm{d}z \\
&\leq 2\int_B (g(z)-g(w))^2 |g(z)|^{p-2} P_D(x,z)\, \mathrm{d}z + 2g(w)^2 \int_B |g(z)|^{p-2} P_D(x,z)\, \mathrm{d}z \\
&\approx \int_B F_p(g(w),g(z)) P_D(x,z)\, \mathrm{d}z + 2|g(w)|^p < \infty.
\end{aligned}$$

Thus, $P_D[|g|^p](x) < \infty$. The rest of the proof is the same as in the case $p \geq 2$. □

Proof of Theorem 4.1 To prove (i) we let $\mathcal{H}_D^{(p)}[g] < \infty$ and we have (4.2). Let $u = P_D[g]$. By (2.12), $u = g$ on D^c. By Lemma 4.5, u is well-defined, and it is regular L-harmonic in D, that is $\mathbb{E}^x[u(X_{\tau_D})] = u(x)$ for $x \in D$, cf. Definition 3.1 and (2.10). In particular, we have $\mathbb{E}^x|u(X_{\tau_D})| < \infty$.

For $x \in D$ consider the integral $\int_{D^c} F_p(u(w),u(z)) P_D(x,z)\, \mathrm{d}z$. By (2.10), $P_D(x,z)$ is the density of the distribution of X_{τ_D} under $\mathbb{P}^x$, hence

$$\int_{D^c} F_p(u(w),u(z)) P_D(x,z)\, \mathrm{d}z = \mathbb{E}^x[F_p(u(w),u(X_{\tau_D}))].$$

By Lemma 2.1 (ii) applied to $a = u(w)$, $X = u(X_{\tau_D})$ and $\mathbb{E} = \mathbb{E}^x$, the above expression is equal to

$$F_p(u(w),\mathbb{E}^x u(X_{\tau_D})) + \mathbb{E}^x F_p(u(x),u(X_{\tau_D})) = F_p(u(w),u(x)) + \mathbb{E}^x F_p(u(x),u(X_{\tau_D})). \tag{4.5}$$

By integrating the first term on the right-hand side of (4.5) against $\nu(x,w)\, \mathrm{d}x\mathrm{d}w$ we obtain

$$\iint_{D^c \times D} F_p(u(w),u(x)) \nu(x,w)\, \mathrm{d}x\mathrm{d}w. \tag{4.6}$$

For the second term in (4.5) we use Lemma 2.1 (i) and Proposition 3.4:

$$\begin{aligned}
\mathbb{E}^x F_p(u(x),u(X_{\tau_D})) &= \mathbb{E}^x |u(X_{\tau_D})|^p - |u(x)|^p \\
&= \int_D G_D(x,y) \int_{\mathbb{R}^d} F_p(u(y),u(z)) \nu(y,z)\, \mathrm{d}z\mathrm{d}y.
\end{aligned}$$

We integrate the latter expression against $\nu(x,w)\, \mathrm{d}x\mathrm{d}w$. By Fubini–Tonelli, (2.4) and (2.10),

$$\begin{aligned}
&\int_{D^c}\int_D\int_D\int_{\mathbb{R}^d} G_D(x,y) F_p(u(y),u(z)) \nu(y,z)\nu(x,w)\, \mathrm{d}z\mathrm{d}y\mathrm{d}x\mathrm{d}w \\
&= \int_D\int_{\mathbb{R}^d} F_p(u(y),u(z)) \left(\int_{D^c} \left(\int_D G_D(x,y)\nu(x,w)\, \mathrm{d}x\right)\mathrm{d}w\right) \nu(y,z)\, \mathrm{d}z\mathrm{d}y \\
&= \int_D\int_{\mathbb{R}^d} F_p(u(y),u(z)) \left(\int_{D^c} P_D(y,w)\, \mathrm{d}w\right) \nu(y,z)\, \mathrm{d}z\mathrm{d}y
\end{aligned}$$

$$= \int_D \int_{\mathbb{R}^d} F_p(u(y), u(z))\nu(y, z)\, dz dy. \tag{4.7}$$

Since the sum of (4.6) and (4.7) equals $p\mathcal{E}_D^{(p)}[u]$, we obtain the Douglas identity.

We now prove (ii). It is not obvious how to directly conclude that $\mathcal{E}_D^{(p)}[u] < \infty$ implies $P_D[|u|] < \infty$ on D, thus we cannot apply Lemma 2.1. Instead we use another approach: by Lemma 2.3, $\mathcal{E}_D^{(p)}[u] < \infty$ is equivalent to $\mathcal{E}_D[u^{\langle p/2\rangle}] < \infty$. By the trace theorem for $p = 2$, see [8, Theorem 2.3], $\mathcal{H}_D[u^{\langle p/2\rangle}] < \infty$. By Lemma 2.3 we get (ii). □

5 Douglas and Hardy–Stein identities with remainders

Throughout this section we assume that D is bounded. In the (quadratic) case $p = 2$, under a mild additional assumption on D, the Poisson integral $P_D[g]$ was shown to be the minimizer of the form $\mathcal{E}_D$ among all Borel functions with a fixed exterior condition $g \in \mathcal{X}_D$ (see [8, Proposition 5.4 and Theorem 5.5]). This needs not be the case when $p \neq 2$, and in this section we give an example of D and $g \in \mathcal{X}_D^p$ for which $P_D[g]$ does not minimize $\mathcal{E}_D^{(p)}$ among functions in $\mathcal{V}_D^p$ equal to g on D^c. However, $P_D[g]$ is always a *quasiminimizer*, if we adopt the following definition:

Definition 5.1 Let $K \geq 1$. Function $u \in \mathcal{V}_D^p$ is a K-quasiminimizer of $\mathcal{E}_D^{(p)}$, if $\mathcal{E}_U^{(p)}[u] \leq K\mathcal{E}_U^{(p)}[v]$ for every nonempty open set $U \subset D$ satisfying (VDC) and $|\partial U| = 0$, and every $v \in \mathcal{V}_U^p$ equal to u on U^c. We say that u is a quasiminimizer if it is a K-quasiminimizer for some $K \in [1, \infty)$.

The definition is inspired by the classical one given by Giaquinta and Giusti [35, (5.26)]. To avoid technical complications and to make this digression short we require regular test sets U above. However, to be prudent we note that the choice of admissible sets U may affect the definition of quasiminimizers and should be carefully considered, cf. Giusti [36, Example 6.5]. In the classical PDEs, quasiminimizers display many regularity properties similar to minimizers, see, e.g., Adamowicz and Toivanen [1], DiBenedetto and Trudinger [23], and Ziemer [62]. The main motivation for studying quasiminimizers is the fact that the solution of a complicated variational problem may be a quasiminimizer of a better understood functional see, e.g., [35, Theorem 2.1].

Proposition 5.2 *Suppose that the assumptions of Theorem 4.1 are satisfied, D is bounded, and let $g \in \mathcal{X}_D^p$. Then $P_D[g]$ is a K-quasiminimizer of $\mathcal{E}_D^{(p)}$ with K independent of g.*

Proof Fix a subset $U \subset D$ satisfying (VDC) and $|\partial U| = 0$, and let $v \in \mathcal{V}_U^p$ be equal to $u := P_D[g]$ on U^c. According to (2.28) we have $v^{\langle p/2\rangle} \in \mathcal{V}_U$ and

$$\mathcal{E}_U^{(p)}[v] \approx \mathcal{E}_U[v^{\langle p/2\rangle}],$$

with constants independent of U and v. Note that $v^{\langle p/2\rangle}$ agrees with $u^{\langle p/2\rangle}$ on U^c. Since U^c satisfies (VDC), by [8, Proposition 5.4 and Theorem 5.5],

$$\mathcal{E}_U[v^{\langle p/2\rangle}] \geq \mathcal{E}_U[P_U[u^{\langle p/2\rangle}]]. \tag{5.1}$$

By applying the Douglas identity for the set U, first with exponent 2, and then with exponent p, and by (2.29) we get that the right-hand side of (5.1) is equal to

$$\mathcal{H}_U[u^{\langle p/2\rangle}] \approx \mathcal{H}_U^{(p)}[u] = \mathcal{E}_U^{(p)}[P_U[u]] = \mathcal{E}_U^{(p)}[u].$$

In the last equality we use the identity $P_U[u] = u$, see (2.10). The proof is complete. $\square$

To prove that Poisson integrals need not be minimizers, we first extend the Hardy–Stein and Douglas identities to functions that are not harmonic. The results are new even for $p = 2$ and $\Delta^{\alpha/2}$.

Recall that D is bounded, hence $\mathbb{E}^x \tau_D$ is bounded. In what follows by $\lim\limits_{U \uparrow D}$ we denote the limit over an arbitrary ascending sequence of Lipschitz open sets $U_n \subset\subset D$ such that $\bigcup_n U_n = D$. Here is an extended version of the Hardy–Stein formula.

Proposition 5.3 *Let $p > 1$ and assume that ν satisfies* **(A1)** *and* **(A2)**. *Let $u\colon \mathbb{R}^d \to \mathbb{R}$. If $u \in C^2(D)$ and u and Lu are bounded in D, then for every $x \in D$,*

$$\lim_{U \uparrow D} \mathbb{E}^x |u(X_{\tau_U})|^p = |u(x)|^p + \int_D G_D(x, y) \int_{\mathbb{R}^d} F_p(u(y), u(z))\nu(y, z)\, dz dy \tag{5.2}$$

$$+ p \int_D G_D(x, y) u(y)^{\langle p-1 \rangle} Lu(y)\, dy. \tag{5.3}$$

If in addition D^c satisfies (VDC) *and $|\partial D| = 0$, then $\lim\limits_{U \uparrow D} \mathbb{E}^x |u(X_{\tau_U})|^p = \mathbb{E}^x |u(X_{\tau_D})|^p$.*

Proof Let $x \in D$. Since u, Lu, and $\mathbb{E}^x \tau_D$ are bounded on D, by (2.3) we get that the integral in (5.3) is finite. Therefore, using the arguments from the proof of Proposition 3.4, in what follows we may and do assume that $\int_{\mathbb{R}^d} |u(x)|^p (1 \wedge \nu(x))\, dx < \infty$, because otherwise both sides of (5.2) are infinite. With this in mind we first consider open Lipschitz $U \subset\subset D$ so large that $x \in U$.

Let $p \geq 2$. Since $u \in C^2(D)$, we get that $L|u|^p(x)$ and $\mathbb{E}^x |u(X_{\tau_U})|^p$ are finite for $x \in U$, and (3.4) holds. Furthermore, since Lu is finite in D, the following manipulations are justified for $y \in D$:

$$\begin{aligned} L|u|^p(y) &= L|u|^p(y) - pu(y)^{\langle p-1 \rangle} Lu(y) + pu(y)^{\langle p-1 \rangle} Lu(y) \\ &= \lim_{\epsilon \to 0^+} \int_{|z-y|>\epsilon} \big(|u(z)|^p - |u(y)|^p - pu(y)^{\langle p-1 \rangle}(u(z) - u(y))\big)\nu(z, y)\, dz \\ &\quad + pu(y)^{\langle p-1 \rangle} Lu(y) \\ &= \int_{\mathbb{R}^d} F_p(u(y), u(z))\nu(y, z)\, dz + pu(y)^{\langle p-1 \rangle} Lu(y). \end{aligned} \tag{5.4}$$

Consequently, (3.4) takes on the form

$$\mathbb{E}^x |u(X_{\tau_U})|^p = |u(x)|^p + \int_U G_U(x, y) \int_{\mathbb{R}^d} F_p(u(y), u(z))\nu(y, z)\, dz dy \tag{5.5}$$

$$+ \int_U G_U(x, y) u(y)^{\langle p-1 \rangle} Lu(y)\, dy. \tag{5.6}$$

For clarity we note that the left-hand side of (5.5) is finite and the integral in (5.6) is absolutely convergent, so the integral in (5.5) is finite as well.

For $p \in (1, 2)$ we proceed as in the proof of Proposition 3.3, that is, instead of $|u(x)|^p$ we consider $\varepsilon > 0$ and the function $x \mapsto (u(x)^2 + \varepsilon^2)^{p/2}$. We obtain (cf. (3.8) and (5.4)),

$$\mathbb{E}^x (u(X_{\tau_U})^2 + \varepsilon^2)^{p/2} = (u(x)^2 + \varepsilon^2)^{p/2} \quad + \int_U G_U(x, y) \int_{\mathbb{R}^d} F_p^{(\varepsilon)}(u(y), u(z))\nu(y, z)\, dz dy \tag{5.7}$$

$$+\,p\int_U G_U(x,y)u(y)(u(y)^2+\varepsilon^2)^{(p-2)/2}Lu(y)\,\mathrm{d}y. \tag{5.8}$$

As in the proof of Proposition 3.4, the left-hand side tends to $\mathbb{E}^x|u(X_{\tau_U})|^p$ as $\varepsilon \to 0^+$. Furthermore, since Lu and u are bounded in D, the integral in (5.8) converges to that in (5.6). Then we apply Fatou's lemma and the Dominated Convergence Theorem to the integral on the right-hand side of (5.7) and we obtain (5.5) for $p \in (1,2)$, too.

We let $U \uparrow D$ in (5.5). By the boundedness of u and Lu in D, the integral in (5.6) tends to the one in (5.3), which is absolutely convergent. The integral on the right-hand side of (5.5) converges to the one on the right-hand side of (5.2) by the domain monotonicity and the Monotone Convergence Theorem. Since the limit on the right-hand side of (5.2) exists, the limit on the left-hand side must exist as well. This proves (5.2).

If D^c satisfies (VDC) and $|\partial D| = 0$, then (2.10) holds true. Furthermore, we have

$$\mathbb{E}^x|u(X_{\tau_U})|^p = \mathbb{E}^x(|u(X_{\tau_U})|^p;\ \tau_U \neq \tau_D) + \mathbb{E}^x(|u(X_{\tau_D})|^p;\ \tau_U = \tau_D).$$

The first term on the right converges to 0 by the boundedness of u on D and the fact that $\mathbb{P}^x(\tau_U \neq \tau_D)$ decreases to 0 as $U \uparrow D$ (see the remark preceding (2.10); see also the proof of Lemma 17 in Bogdan [4] and the proof of Lemma A.1 in [8]). The second term converges to $\mathbb{E}^x|u(X_{\tau_D})|^p$ by the Monotone Convergence Theorem. Thus the left-hand side of (5.5) tends to $\mathbb{E}^x|u(X_{\tau_D})|^p$. □

We next provide a Douglas-type identity for a class of nonharmonic functions:

Theorem 5.4 *Suppose that the assumptions of Theorem 4.1 hold with the addition that D is bounded. Let $u\colon \mathbb{R}^d \to \mathbb{R}$ be bounded, $u \in C^2(D)$, and Lu be bounded in D. Then*

$$\mathcal{E}_D^{(p)}[P_D[u]] = \mathcal{E}_D^{(p)}[u] + A_D(u), \tag{5.9}$$

where

$$A_D(u) = \int_D u(x)^{\langle p-1\rangle}Lu(x)\,\mathrm{d}x + \int_D\int_{D^c} u(w)^{\langle p-1\rangle}(u(x)-P_D[u](x))\,\nu(w,x)\,\mathrm{d}w\mathrm{d}x.$$

Proof Since u is bounded on $\mathbb{R}^d$, we have $\int_{\mathbb{R}^d}|u(x)|(1\wedge\nu(x))\,\mathrm{d}x < \infty$.

Assume first that $\mathcal{H}_D^{(p)}[u] < \infty$. From Theorem 4.1 we have

$$\mathcal{E}_D^{(p)}[P_D[u]] = \mathcal{H}_D^{(p)}[u].$$

By (2.8) and Fubini–Tonelli,

$$p\mathcal{H}_D^{(p)}[u] = \int_D\int_{D^c}\int_{D^c} F_p(u(w),u(z))P_D(x,z)\nu(x,w)\,\mathrm{d}z\mathrm{d}w\mathrm{d}x.$$

We apply Lemma 2.1 (iii) to $a = u(w)$, $b = u(x)$, with $w \in D^c$ and $x \in D$, $X = u(X_{\tau_D})$, and $\mathbb{E} = \mathbb{E}^x$. Note that $\mathbb{E}X = P_D[u](x)$. This yields:

$$\begin{aligned}&\int_{D^c} F_p(u(w),u(z))P_D(x,z)\,\mathrm{d}z\\ &= \int_{D^c} F_p(u(x),u(z))P_D(x,z)\,\mathrm{d}z + F_p(u(w),u(x))\\ &\quad + (pu(w)^{\langle p-1\rangle} - pu(x)^{\langle p-1\rangle})(u(x)-P_D[u](x)).\end{aligned}$$

After integration, we obtain

$$
\begin{aligned}
p\mathcal{H}_D^{(p)}[u] &= \int_D \int_{D^c} \int_{D^c} F_p(u(x), u(z)) P_D(x,z)\nu(x,w)\,\mathrm{d}z\mathrm{d}w\mathrm{d}x \\
&\quad + \int_D \int_{D^c} F_p(u(w), u(x))\nu(x,w)\,\mathrm{d}w\mathrm{d}x \\
&\quad + \int_D \int_{D^c} (pu(w)^{\langle p-1\rangle} - pu(x)^{\langle p-1\rangle})(u(x) - P_D[u](x))\,\nu(x,w)\,\mathrm{d}w\mathrm{d}x \\
&=: A_1(u) + A_2(u) + A_3(u).
\end{aligned}
$$

Note that every term above is finite. Indeed, by the boundedness of u,

$$
|A_3(u)| \lesssim \int_D \int_{D^c} |u(x) - P_D[u](x)|\nu(x,w)\,\mathrm{d}w\mathrm{d}x.
$$

To prove that this is finite, let $v = u - P_D[u]$. We have $Lv = Lu = f \in L^\infty(D)$ and $v = 0$ on D^c. Note that $v \in C^2(D)$ and $\int_{\mathbb{R}^d} |v(x)|(1 \wedge \nu(x))\,\mathrm{d}x < \infty$, cf. [8, Lemma 3.6]. Let $U \subset\subset D$. By Lemma 3.2,

$$
\mathbb{E}^x v(X_{\tau_U}) - v(x) = \int_U G_U(x,y) f(y)\,\mathrm{d}y, \quad x \in U.
$$

Since u is bounded on $\mathbb{R}^d$, we have $\mathbb{E}^x u(X_{\tau_U}) \to \mathbb{E}^x u(X_{\tau_D}) = P_D[u](x)$ as $U \uparrow D$, cf. the last part of the proof of Proposition 5.3. Hence, the boundedness of f, the domain monotonicity, and the Dominated Convergence Theorem yield

$$
v(x) = -\int_D G_D(x,y) f(y)\,\mathrm{d}y, \quad x \in D.
$$

This allows us to further estimate A_3:

$$
|A_3(u)| \lesssim \int_D \int_{D^c} \int_D G_D(x,y)\nu(w,x)\,\mathrm{d}y\mathrm{d}w\mathrm{d}x = \int_D \int_{D^c} P_D(y,w)\,\mathrm{d}w\mathrm{d}y = |D| < \infty.
$$

Since $A_1(u)$ and $A_2(u)$ are nonnegative, they must be finite as well, because $\mathcal{H}_D^{(p)}[u] < \infty$. We then have

$$
\begin{aligned}
\int_{D^c} F_p(u(x), u(z)) P_D(x,z)\,\mathrm{d}z &= \mathbb{E}^x F_p(u(x), u(X_{\tau_D})) \\
&= \mathbb{E}^x |u(X_{\tau_D})|^p - |u(x)|^p - pu(x)^{\langle p-1\rangle}(P_D[u](x) - u(x)).
\end{aligned}
$$

Thus, by Proposition 5.3 we obtain

$$
\begin{aligned}
A_1(u) = A_4(u) &+ p \int_D \int_{D^c} \int_D G_D(x,y) u(y)^{\langle p-1\rangle} Lu(y)\nu(x,w)\,\mathrm{d}y\mathrm{d}w\mathrm{d}x \\
&- p \int_D \int_{D^c} u(x)^{\langle p-1\rangle}(P_D[u](x) - u(x))\nu(x,w)\,\mathrm{d}w\mathrm{d}x,
\end{aligned} \tag{5.10}
$$

where $A_4(u)$ is the integral in (4.7). Note that $A_2(u) + A_4(u) = p\mathcal{E}_D^{(p)}[u]$. Also, all the expressions in (5.10) are finite, see the discussion of $A_3(u)$. To finish the proof of (5.9) in the case $\mathcal{H}_D^{(p)}[u] < \infty$, we simply note that $pA_D(u) = A_1(u) - A_4(u) + A_3(u)$.

The situation $\mathcal{H}_D^{(p)}[u] = \infty$ remains to be considered. Since $P_D[u]$ is bounded in D, by arguments similar to those in the estimates of $A_3(u)$ above, we prove that $A_D(u)$ is finite. Therefore by Theorem 4.1 the identity (5.9) holds with both sides infinite. □

Knowing the form of the remainder $A_D(u)$ in the Douglas identity (5.9), we may provide an example which shows that the Poisson integral need not be a minimizer of $\mathcal{E}_D^{(p)}$ for $p \neq 2$; it is only a quasiminimizer by Proposition 5.2.

Example 5.5 [The Poisson extension need not be a minimizer for $p \neq 2$] Let $p > 2$ and consider $0 < R < R_1$ such that $D \subset\subset B_R$. Define

$$g_n(z) = \left(\left(|z| - R\right)^{-1/(p-1)} \wedge n\right)\mathbf{1}_{B_{R_1} \setminus B_R}(z).$$

Since each g_n is bounded with support separated from D, we have $g_n \in \mathcal{X}_D^p \cap \mathcal{X}_D$; see the discussion following Example 2.4 in [8]. By (2.6) there exists $c > 0$ such that

$$P_D(x, z) \le c, \quad x \in D,\ z \in B_{R_1} \setminus B_R. \tag{5.11}$$

Furthermore, for every $U \subset\subset D$ there is $\epsilon > 0$ such that

$$P_D(x, z) \ge \epsilon, \quad x \in U,\ z \in B_{R_1} \setminus B_R. \tag{5.12}$$

For $x \in D$ we let

$$u_n(x) = G_D[1](x) + P_D[g_n](x).$$

Obviously u_n are bounded on $\mathbb{R}^d$. We will verify that $G_D[1] \in C^2(D)$. For this purpose we let f be a smooth, compactly supported, nonnegative function equal to 1 on D. By the Hunt's formula and Fubini–Tonelli we get

$$G_D[f](x) = G_D[1](x) = \int_{\mathbb{R}^d} G(x - y) f(y)\,\mathrm{d}y - \mathbb{E}^x \int_{\mathbb{R}^d} G(X_{\tau_D}, y) f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d. \tag{5.13}$$

Here G is either the potential kernel or the compensated potential kernel of (X_t); see Grzywny, Kassmann and Leżaj [38, Appendix A] for details. In particular, by [38, Corollary A.3] and [54, Theorem 35.4] G is locally integrable, thus the first term in (5.13) is finite and smooth in D. Since the latter term in (5.13) is a harmonic function, we get that $G_D[1] \in C^2(D)$. In particular, by [8, Lemma 4.10] and Dynkin [27, Lemma 5.7] we have $Lu_n = -1$ in D. We are now in a position to apply Theorem 5.4. Fix open $U \subset\subset D$. We get

$$\begin{aligned} A_D(u_n) &= -\int_D u_n(x)^{p-1}\,\mathrm{d}x + \int_D \int_{D^c} u_n(w)^{p-1} G_D[1](x)\nu(x, w)\,\mathrm{d}w\mathrm{d}x \\ &= \int_D \left(\mathbb{E}^x u_n(X_{\tau_D})^{p-1} - \left(\mathbb{E}^x u_n(X_{\tau_D}) + G_D[1](x)\right)^{p-1}\right)\mathrm{d}x = \int_U + \int_{D\setminus U}. \end{aligned} \tag{5.14}$$

We claim that $A_D(u_n) > 0$ for large n. Indeed, recall that $G_D[1](x) = \mathbb{E}^x \tau_D$ is bounded. Since the integrals $\int_{D^c} g_n(x)\,\mathrm{d}x$ are bounded, by (5.11) there is $M > 0$ such that $\mathbb{E}^x u_n(X_{\tau_D}) < M$ for every $x \in D$ and $n \in \mathbb{N}$. Therefore the integral $\int_{D\setminus U}$ in (5.14) is bounded from below, independently of n. Note that $\int_{D^c} g_n(x)^{p-1}\mathrm{d}x \to \infty$ as $n \to \infty$. Thus, by (5.12) we obtain that $\int_U \to \infty$ in (5.14) as $n \to \infty$. Hence, for sufficiently large n we get that $A_D(u_n) > 0$, which proves that $\mathcal{E}_D^{(p)}[P_D[u_n]] > \mathcal{E}_D^{(p)}[u_n]$ for some n, as needed. The case $p \in (1, 2)$ may be handled similarly, by using $g_n(z) = ((|z| - R)^{-1} \wedge n)\mathbf{1}_{B_{R_1} \setminus B_R}(z)$ and $u_n = P_D[g_n] - G_D[1]$.

6 Applications to Dirichlet-to-Neumann map

In this section we adopt the assumptions of Theorem 4.1. In addition, we assume that the set D is bounded and $p \in [2, \infty)$. We define the nonlocal normal derivative as an analogue of the fractional version of Dipierro, Ros-Oton, and Valdinoci [24, (1.2)], see also Vondraček [59]:

$$\mathcal{N}f(z) = \int_D (f(z) - f(x))\nu(x, z)\,\mathrm{d}x. \tag{6.1}$$

Note that the increments of f are integrated on D, but the integral is evaluated for $z \in D^c$, if convergent. For instance, if $f \in L^1(\mathbb{R}^d, 1 \wedge \nu)$, then $\mathcal{N}f \in L^1_{loc}(D^c)$.

Assume that $g \in \mathcal{X}^p_D$. Then $u = P_D[g]$ solves the Dirichlet problem (1.5). By definition, the Dirichlet-to-Neumann operator DN maps the exterior condition g to the nonlocal normal derivative $h := \mathcal{N}u$. So, u solves the Neumann problem

$$\begin{cases} Lu = 0 & \text{in } D, \\ \mathcal{N}u = h & \text{on } D^c, \end{cases}$$

and $DN := \mathcal{N} \circ P_D$ on $\mathcal{X}^p_D$. In fact, for almost every $z \in D^c$,

$$\begin{aligned} DNg(z) = \mathcal{N}u(z) &= \int_D (u(z) - u(x))\nu(x, z)\,\mathrm{d}x \qquad (6.2)\\ &= \int_D \int_{D^c} (u(z) - u(w))P_D(x, w)\nu(x, z)\,\mathrm{d}w\mathrm{d}x \\ &= \int_{D^c} (u(z) - u(w))\gamma_D(z, w)\,\mathrm{d}w \\ &= \int_{D^c} (g(z) - g(w))\gamma_D(z, w)\,\mathrm{d}w, \qquad (6.3) \end{aligned}$$

where we have used the definition of γ_D, the fact that $u = P_D[g]$, and the Fubini–Tonelli theorem (justified by the estimates in the proof of Proposition 6.1). For $z \in \mathrm{Int}(D^c) = \mathbb{R}^d \setminus \overline{D}$ we let

$$m(z) := \int_{D^c} \gamma_D(w, z)\,\mathrm{d}w = \int_D \nu(x, z)\,\mathrm{d}x < \infty. \tag{6.4}$$

For example, for $L = \Delta^{\alpha/2}$ and (bounded) D of class $C^{1,1}$, with $\delta_D(z) := d(z, D)$ we have

$$m(z) = c_{d,\alpha} \int_D |z - x|^{-d-\alpha}\,\mathrm{d}x \approx \begin{cases} \delta_D(z)^{-\alpha}, & \delta_D(z) \le 1, \\ \delta_D(z)^{-d-\alpha}, & \delta_D(z) > 1. \end{cases}$$

Back to general L, we note that sharp estimates of γ_D are known for bounded $C^{1,1}$ domains and the half-space, see [8, Theorem 2.6 and 6.1]. We next define the *normalized* Dirichlet-to-Neumann operator, for $g \in \mathcal{X}^p_D$ and $a.e.\ z \in D^c$,

$$\widetilde{DN}g(z) = \frac{DNg(z)}{m(z)} = \int_{D^c} (g(z) - g(w))\frac{\gamma_D(z, w)}{m(z)}\,\mathrm{d}z. \tag{6.5}$$

In what follows we give several results for the Dirichlet-to-Neumann operator on L^p. In particular, we show that DN is well-defined: $\mathcal{X}^p_D \to L^p(D^c, m^{1-p})$ and $\widetilde{DN}$ is bounded on $L^p(D^c, m)$. We also relate the form $\mathcal{H}^{(p)}_D$ to the operator DN in (6.9).

Proposition 6.1 *Assume that $g \in \mathcal{X}_D^p$. Then $DNg \in L^p(D^c, m^{1-p})$ and $\widetilde{DN}g \in L^p(D^c, m)$. Furthermore, there exists a constant C, independent of g, such that*

$$\|DNg\|^p_{L^p(D^c,m^{1-p})} = \|\widetilde{DN}g\|^p_{L^p(D^c,m)} \leq C\mathcal{H}_D^{(p)}[g].$$

Proof Using (6.3) and Jensen's inequality we get

$$\begin{aligned}\int_{D^c} |DNg(z)|^p m(z)^{1-p}\,\mathrm{d}z &= \int_{D^c} \left(\frac{|DNg(z)|}{m(z)}\right)^p m(z)\,\mathrm{d}z \\ &\leq \int_{D^c}\int_{D^c} |g(w) - g(z)|^p \gamma_D(z, w)\,\mathrm{d}z\mathrm{d}w.\end{aligned}$$

Since $p \geq 2$, we have $|a - b|^p \leq (a - b)^2(|a|^{p-2} + |b|^{p-2})$. So, by (2.19),

$$\int_{D^c}\int_{D^c} |g(w) - g(z)|^p \gamma_D(z, w)\,\mathrm{d}z\mathrm{d}w \lesssim \mathcal{H}_D^{(p)}[g] < \infty, \tag{6.6}$$

which ends the proof. □

Proposition 6.2 *If $g \in L^p(D^c, m)$, then $g \in \mathcal{X}_D^p$ and there is $C > 0$, independent of g, such that*

$$\mathcal{H}_D^{(p)}[g] \leq C\|g\|^p_{L^p(D^c,m)}.$$

Proof Following [30, Remark 2.37], we let $\widetilde{g}$ be the function g extended to D by 0. Then,

$$\begin{aligned}\mathcal{E}_D^{(p)}[\widetilde{g}] &= \frac{1}{p}\iint_{(D^c\times D^c)^c} F_p(\widetilde{g}(z), \widetilde{g}(w))\nu(z, w)\,\mathrm{d}z\mathrm{d}w = \int_D\int_{D^c} |g(z)|^p \nu(z, w)\,\mathrm{d}w\mathrm{d}z \\ &= \int_{D^c} |g(z)|^p m(z)\,\mathrm{d}z < \infty. \end{aligned}\tag{6.7}$$

In particular, $\widetilde{g} \in \mathcal{V}_D^p$. By Proposition 5.2 we get that there exists a constant C, independent of g, such that $\mathcal{E}_D^{(p)}[P_D[g]] \leq C\mathcal{E}_D^{(p)}[\widetilde{g}]$. Using this, Theorem 4.1, and (6.7), we find that

$$\mathcal{H}_D^{(p)}[g] = \mathcal{E}_D^{(p)}[P_D[g]] \leq C\mathcal{E}_D^{(p)}[\widetilde{g}] = C\int_{D^c} |g(z)|^p m(z)\,\mathrm{d}z,$$

which proves the result. □

Corollary 6.3 *The normalized Dirichlet-to-Neumann map $\widetilde{DN}$ is bounded on $L^p(D^c, m)$.*

The following is an analogue of the formula (7.3) below.

Proposition 6.4 *Let $f \in L^p(D^c, m)$ and $g \in \mathcal{X}_D^p$. Then $\int_{D^c} |DNg(z)||f(z)|^{p-1}\,\mathrm{d}z < \infty$ and*

$$\int_{D^c} DNg(z) f(z)^{\langle p-1\rangle} dz = \frac{1}{2}\int_{D^c}\int_{D^c} (g(z) - g(w))(f(z)^{\langle p-1\rangle} - f(w)^{\langle p-1\rangle})\gamma_D(z, w) dz dw. \tag{6.8}$$

Furthermore, if $g \in L^p(D^c, m)$, then

$$\int_{D^c} DNg(z)g(z)^{\langle p-1\rangle}\,\mathrm{d}z = \int_{D^c} \widetilde{DN}g(z)g(z)^{\langle p-1\rangle}m(z)\,\mathrm{d}z = \mathcal{H}_D^{(p)}[g]. \tag{6.9}$$

Proof By Hölder's inequality with exponents p and $p' = \frac{p}{p-1}$, and by Proposition 6.1,

$$\int_{D^c} |DNg(z)||f(z)|^{p-1}\,\mathrm{d}z = \int_{D^c} |DNg(z)|m(z)^{\frac{1-p}{p}} m(z)^{\frac{p-1}{p}} |f(z)|^{p-1}\,\mathrm{d}z$$
$$\le \left(\int_{D^c} |DNg(z)|^p m(z)^{1-p}\,\mathrm{d}z \right)^{\frac{1}{p}} \left(\int_{D^c} |f(z)|^p m(z)\,\mathrm{d}z \right)^{\frac{p-1}{p}} < \infty.$$

It suffices to prove (6.8). By the symmetry of γ_D,

$$\begin{aligned}\int_{D^c} DNg(z) f(z)^{\langle p-1\rangle}\,\mathrm{d}z &= \int_{D^c}\int_{D^c} (g(z)-g(w)) f(z)^{\langle p-1\rangle}\gamma_D(z,w)\,\mathrm{d}w\mathrm{d}z \\ &= \int_{D^c}\int_{D^c} (g(z)-g(w)) f(z)^{\langle p-1\rangle}\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w \\ &= \int_{D^c}\int_{D^c} (g(w)-g(z)) f(w)^{\langle p-1\rangle}\gamma_D(z,w)\,\mathrm{d}w\mathrm{d}z.\end{aligned}$$

The above application of the Fubini–Tonelli theorem is justified by using Hölder's inequality with exponents p and $p/(p-1)$, and (6.6); see also (6.4). The first and the last lines above yield (6.8). □

Let us discuss related results for $p = 2$. In [59, Proposition 3.2], Vondraček shows that the normalized Dirichlet-to-Neumann operator map is bounded on $L^2(D^c, m)$; our Corollary 6.3 extends this result to L^p. As observed by Foghem and Kassmann [30, Remark 2.37], the space $L^2(D^c, m)$ can be smaller than the trace space $\mathcal{X}_D$. In [30, Section 4.4], the authors investigate the Dirichlet-to-Neumann operator for the equation $Lu = \lambda u + f$, where $\lambda \in \mathbb{R}$ is not a Dirichlet eigenvalue of L in D. They prove the boundedness of the Dirichlet-to-Neumann operator from the trace space into its dual. If we let DN_{FK} be the Dirichlet-to-Neumann operator defined in [30] for $\lambda = 0$ and $f = 0$, then using our Douglas identity and $\widetilde{v} = u_g = P_D[g]$ in [30, Definition 4.18], for $g \in \mathcal{X}_D$ we get

$$\langle DN_{FK}g, g\rangle = \mathcal{E}_D[P_D[g]] = \mathcal{H}_D[g] = \frac{1}{2}\int_{D^c}\int_{D^c} (g(z)-g(w))^2\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w.$$

Here $\langle\cdot,\cdot\rangle$ is the pairing between $\mathcal{X}_D$ and its dual, see [30, Section 2.6]. Then, by polarization,

$$\langle DN_{FK}g_1, g_2\rangle = \mathcal{H}_D(g_1, g_2) = \frac{1}{2}\int_{D^c}\int_{D^c} (g_1(z)-g_1(w))(g_2(z)-g_2(w))\gamma_D(z,w)\,\mathrm{d}z\mathrm{d}w, \tag{6.10}$$

for $g_1, g_2 \in \mathcal{X}_D$. Both (6.3) and (6.10) give explicit integral representations for the Dirichlet-to-Neumann operator, which are more direct than (6.2). They were not stated in [30, 59], although similar formulas appear in [59, Section 3 and (4.2)] and the author of [59] was probably aware of the explicit versions.

On an informal level, (6.3) and (6.10) mean that the Dirichlet-to-Neumann map is the negative of the Lévy-type operator on D^c with jump kernel γ_D, and $\mathcal{H}_D$ is the corresponding Dirichlet form. Despite being smaller than $\mathcal{X}_D^2$, the space $L^2(D^c, m)$, used by Vondraček, is suitable for studying the (negative of the) normalized Dirichlet-to-Neumann operator $\widetilde{DN}$ as a generator of a Markov process on D^c. In fact, $-\widetilde{DN}$ is the generator of the so-called trace process, see [59, (4.2)]. In this connection, the reader may compare (6.9), DN and $\mathcal{H}_D^{(p)}$ with (7.3), $-L$ and $\mathcal{E}_D^{(p)}$; see also [9, Lemma 7] for a detailed discussion of $\mathcal{E}_{\mathbb{R}^d}^{(p)}$ for $L = \Delta^{\alpha/2}$.

7 Further discussion

As usual, D is a nonempty open set in $\mathbb{R}^d$. We define

$$\mathcal{W}_D^p = \left\{ u\colon \mathbb{R}^d \to \mathbb{R} \,\middle|\, \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} |u(x)-u(y)|^p \nu(x,y)\,\mathrm{d}x\mathrm{d}y < \infty \right\}, \tag{7.1}$$

and

$$\mathcal{Y}_D^p = \left\{ g\colon D^c \to \mathbb{R} \,\middle|\, \iint\limits_{D^c\times D^c} |g(w)-g(z)|^p \gamma_D(w,z)\,\mathrm{d}w\mathrm{d}z < \infty \right\}.$$

Proposition 7.1 *If $p \geq 2$ then (1.10) holds true under the assumptions on D and ν from Theorem 4.1, and the Poisson extension acts from $\mathcal{Y}_D^p$ to $\mathcal{W}_D^p$.*

Proof Assume that $g \in \mathcal{Y}_D^p$, i.e., the right-hand side of (1.10) is finite. By a simple modification of the proof of [8, Lemma 4.6] we get that $g \in L^p(D^c, P_D(x,z)\,\mathrm{d}z)$ for every $x \in D$, in particular the Poisson integral $P_D[g](x)$ converges absolutely. By (2.8), the right-hand side of (1.10) equals

$$\int_{D^c}\int_{D^c}\int_D |g(w)-g(z)|^p \nu(w,x) P_D(x,z)\,\mathrm{d}x\mathrm{d}w\mathrm{d}z.$$

We use Fubini–Tonelli and consider the integral

$$\int_{D^c} |g(w)-g(z)|^p P_D(x,z)\,\mathrm{d}z = \mathbb{E}^x \left|u(X_{\tau_D}) - g(w)\right|^p.$$

By Lemma 2.2 we get that for $x \in D$ and $w \in D^c$,

$$\mathbb{E}^x \left|u(X_{\tau_D}) - g(w)\right|^p \approx \mathbb{E}^x |u(X_{\tau_D}) - u(x)|^p + |u(x)-g(w)|^p \geq \mathbb{E}^x |u(X_{\tau_D}) - u(x)|^p.$$

We apply Proposition 3.4, to $\widetilde{u}(z) := u(z) - u(x)$. It is L-harmonic on D and $\widetilde{u}(x) = 0$, therefore

$$\mathbb{E}^x \left|u(X_{\tau_D}) - u(x)\right|^p = \int_D G_D(x,y) \int_{\mathbb{R}^d} F_p(\widetilde{u}(y), \widetilde{u}(z)) \nu(z,y)\,\mathrm{d}z\mathrm{d}y.$$

For $p \neq 2$ it is not true that $F_p(a+t, b+t)$ is comparable with $F_p(a,b)$, but since $p \geq 2$, by Lemma 2.4 we have $F_p(a+t,b+t) \geq c|a+t-b-t|^p = c|a-b|^p$. It follows that

$$F_p(\widetilde{u}(y), \widetilde{u}(z)) \gtrsim |u(y)-u(z)|^p,$$

and thus

$$\mathbb{E}^x \left|u(X_{\tau_D}) - g(w)\right|^p \gtrsim \int_D G_D(x,y) \int_{\mathbb{R}^d} |u(y)-u(z)|^p \nu(z,y)\,\mathrm{d}z\mathrm{d}y.$$

We integrate the inequality on $D^c \times D$ against $\nu(w,x)\,\mathrm{d}w\mathrm{d}x$ as in (4.7), and the right-hand side is

$$\int_D \int_{\mathbb{R}^d} |u(x)-u(y)|^p \nu(x,y)\,\mathrm{d}x\mathrm{d}y \geq \frac{1}{2} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus(D^c\times D^c)} |u(x)-u(y)|^p \nu(x,y)\,\mathrm{d}x\mathrm{d}y.$$

The result follows. □

We remark that in general (1.10) fails for $p \in (1, 2)$; see Lemma 7.4 and Example 7.5.

In the remainder of this section we compare $\mathcal{W}^p_D$ and $\mathcal{V}^p_D$, see (2.25), by using $C_c^\infty(\mathbb{R}^d)$.

Lemma 7.2 *For every $p > 1$ we have $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{V}^p_{\mathbb{R}^d} \subseteq \mathcal{V}^p_D$.*

Proof The inclusion $\mathcal{V}^p_{\mathbb{R}^d} \subseteq \mathcal{V}^p_D$ follows from the definition. To prove that $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{V}^p_{\mathbb{R}^d}$, we let $\phi \in C_c^\infty(\mathbb{R}^d)$. We have

$$|\phi(x+z) + \phi(x-z) - 2\phi(x)| \le M(1 \wedge |z|^2), \quad x, z \in \mathbb{R}^d.$$

It follows that $L\phi$ is bounded on $\mathbb{R}^d$, cf. (1.3) and (1.2). Thus,

$$\int_{\mathbb{R}^d} |\phi(x)|^{p-1} |L\phi(x)| \, \mathrm{d}x < \infty. \tag{7.2}$$

Furthermore, by the Dominated Convergence Theorem and the symmetry of ν,

$$\begin{aligned}\int_{\mathbb{R}^d} \phi(x)^{\langle p-1\rangle} L\phi(x)\,\mathrm{d}x &= \tfrac{1}{2}\int_{\mathbb{R}^d} \phi(x)^{\langle p-1\rangle} \lim_{\epsilon\to 0^+} \int_{|z|>\epsilon} (\phi(x+z)+\phi(x-z)-2\phi(x))\nu(z)\,\mathrm{d}z\mathrm{d}x \\ &= \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^d}\int_{|z|>\epsilon} \phi(x)^{\langle p-1\rangle}(\phi(x+z)-\phi(x))\nu(z)\,\mathrm{d}z\mathrm{d}x.\end{aligned}$$

By Fubini's theorem, the substitutions $z \to -z$ and $x \to x+z$, and the symmetry of ν,

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{|z|>\epsilon} \phi(x)^{\langle p-1\rangle}(\phi(x+z)-\phi(x))\nu(z)\,\mathrm{d}z\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\int_{|z|>\epsilon} \phi(x+z)^{\langle p-1\rangle}(\phi(x)-\phi(x+z))\nu(z)\,\mathrm{d}z\mathrm{d}x \\ &= -\tfrac{1}{2}\int_{|z|>\epsilon}\int_{\mathbb{R}^d} (\phi(x+z)^{\langle p-1\rangle}-\phi(x)^{\langle p-1\rangle})(\phi(x+z)-\phi(x))\,\mathrm{d}x\,\nu(z)\,\mathrm{d}z\end{aligned}$$

for every $\epsilon > 0$. By (2.23), the Monotone Convergence Theorem and the above,

$$\begin{aligned}\mathcal{E}^{(p)}_{\mathbb{R}^d}[\phi] &= \tfrac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (\phi(x+z)^{\langle p-1\rangle}-\phi(x)^{\langle p-1\rangle})(\phi(x+z)-\phi(x))\nu(z)\,\mathrm{d}x\mathrm{d}z \\ &= -\int_{\mathbb{R}^d} \phi(x)^{\langle p-1\rangle} L\phi(x)\,\mathrm{d}x.\end{aligned} \tag{7.3}$$

The result follows from (7.2) and (2.25). □

The inclusion $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{V}^p_D$ indicates that the Sobolev–Bregman spaces will be useful in variational problems posed in L^p.

The situation with the spaces $\mathcal{W}^p_D$ is more complicated. While for $p \ge 2$ we have a result similar to that of Lemma 7.2, for $p \in (1, 2)$ it is not so. More precisely, we have the following two lemmas:

Lemma 7.3 *For $p \ge 2$ we have $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{W}^p_{\mathbb{R}^d} \subseteq \mathcal{W}^p_D$.*

Proof For $\phi \in C_c^\infty(\mathbb{R}^d)$ let $K = \operatorname{supp}\phi$. Then we have $|\phi(x) - \phi(y)| = 0$ on $K^c \times K^c$ and

$$|\phi(x) - \phi(y)|^p \lesssim 1 \wedge |x-y|^p \le 1 \wedge |x-y|^2, \quad x, y \in \mathbb{R}^d \times \mathbb{R}^d \setminus K^c \times K^c.$$

It follows that $\phi \in \mathcal{W}^p_{\mathbb{R}^d}$. The inclusion $\mathcal{W}^p_{\mathbb{R}^d} \subseteq \mathcal{W}^p_D$ is clear from the definition of the spaces. □

Lemma 7.4 *Let $p \in (1,2)$ and assume that for some $r > 0$ we have $\nu(y) \gtrsim |y|^{-d-p}$ for $|y| < r$. If $u \in \mathcal{W}^p_D$ has compact support in $\mathbb{R}^d$ and vanishes on D^c, then $u \equiv 0$.*

Results of this type are well-known for the spaces with integration over $D \times D$, where D is connected. Brezis [13, Proposition 2] shows that any measurable function must be constant in this case; a simpler proof of this fact was given by De Marco, Mariconda and Solimini [19, Theorem 4.1]. Lemma 7.4 follows by taking $\Omega = \mathbb{R}^d$ in the aforementioned results, but we present a different proof. Such facts also hold true in the context of metric spaces, see, e.g., Pietruska-Pałuba [49]. We will see in the proof of Lemma 7.4 that the result reduces to that with $D = \mathbb{R}^d$.

Proof of Lemma 7.4 We may assume that u is bounded, because the p-increments of $(0\vee u)\wedge 1$ do not exceed those of u. Thus, since u is compactly supported, we get that $u \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let

$$\widehat{u}(\xi) = \int_{\mathbb{R}^d} u(x) e^{-2\pi i \xi x}\, \mathrm{d}x, \quad \xi \in \mathbb{R}^d.$$

The Hausdorff–Young inequality asserts that for $u \in L^p(\mathbb{R}^d)$ we have

$$\|u\|_p \geq \|\widehat{u}\|_{p'}, \tag{7.4}$$

where $p' = \frac{p}{p-1}$, see, e.g., Grafakos [37, Proposition 2.2.16]. We estimate the left-hand side of (1.10) by using (7.4):

$$\begin{aligned}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d\setminus D^c\times D^c} |u(x)-u(y)|^p \nu(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |u(x)-u(x+y)|^p \nu(y)\,\mathrm{d}x\mathrm{d}y \\ &\geq \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |(u(\cdot)-u(\cdot+y))^\wedge(\xi)|^{p'}\mathrm{d}\xi\right)^{\frac{p}{p'}} \nu(y)\mathrm{d}y \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |1-e^{-2\pi i\xi y}|^{p'}|\widehat{u}(\xi)|^{p'}\mathrm{d}\xi\right)^{\frac{p}{p'}} \nu(y)\mathrm{d}y.\end{aligned}$$

By (7.4), $|\widehat{u}(\xi)|^{p'}\,\mathrm{d}\xi$ is a finite measure on $\mathbb{R}^d$. As we have $p/p' < 1$, by Jensen and Fubini–Tonelli,

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |1-e^{-2\pi i\xi y}|^{p'}|\widehat{u}(\xi)|^{p'}\mathrm{d}\xi\right)^{\frac{p}{p'}} \nu(y)\mathrm{d}y &\gtrsim \int_{\mathbb{R}^d}\nu(y)\int_{\mathbb{R}^d} |1-e^{-2\pi i\xi y}|^{p}|\widehat{u}(\xi)|^{p'}\mathrm{d}\xi\mathrm{d}y \\ &= \int_{\mathbb{R}^d}|\widehat{u}(\xi)|^{p'}\int_{\mathbb{R}^d} |1-e^{2\pi i\xi y}|^{p}\nu(y)\mathrm{d}y\mathrm{d}\xi.\end{aligned}$$

Since $|1-e^{2\pi i\xi y}| \geq |\sin 2\pi\xi y|$ and $\nu(y) \gtrsim |y|^{-d-p}$ for small $|y|$, the integral is infinite, unless $u = 0$ a.e. in $\mathbb{R}^d$. □

As a comment to Lemmas 7.2 and 7.4 we recall that $\mathcal{V}^p_D$ is defined in terms of F_p. When a is close to b then, regardless of $p > 1$, the Bregman divergence $F_p(a,b)$ is of order $(b-a)^2$ rather than $|b-a|^p$. Thus $\mathcal{V}^p_D$ agrees with the Lévy measure condition (1.2) better than $\mathcal{W}^p_D$ does.

The following example indicates that the scale of linear spaces $\mathcal{W}^p_D$ may not be suitable for analysis of harmonic functions when $p \leq 2$:

Example 7.5 Let ν and p be as in Lemma 7.4. Let $B = B(0,1)$ and assume that D is bounded and $\operatorname{dist}(D, B) > 0$. Then there is $g \in \mathcal{Y}_D^p$ such that $u := P_D[g] \notin \mathcal{W}_D^p$, i.e.,

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} |u(x) - u(y)|^p \nu(x, y)\, \mathrm{d}x\mathrm{d}y = \infty. \tag{7.5}$$

Let $g(z) = \mathbf{1}_B(z)$ for $z \in D^c$. Then $g \in \mathcal{Y}_D^p$, cf. the arguments following [8, Example 2.4]. Clearly, u is bounded in D. By the positivity of P_D [39, Lemma 2.2], $u(x) > 0$ for every $x \in D$. Of course, B, $D^c \setminus B = B^c \setminus D$ and D form a partition of $\mathbb{R}^d$. Therefore their Cartesian products partition $\mathbb{R}^d \times \mathbb{R}^d$; in fact also $B^c \times B^c$ and $\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c$ (see below). Since u vanishes on $D^c \setminus B$, $u(x) - u(y)$ vanishes on $(D^c \setminus B) \times (D^c \setminus B)$. It follows that

$$\int_{B^c} \int_{B^c} |u(x) - u(y)|^p \nu(x, y)\, \mathrm{d}x\mathrm{d}y \le \iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} |u(x) - u(y)|^p \nu(x, y)\, \mathrm{d}x\mathrm{d}y. \tag{7.6}$$

Define $\widetilde{u} = u$ on B^c and $\widetilde{u} = 0$ on B. Then, $\widetilde{u} = u$ on D and $\widetilde{u} = 0$ on D^c, and

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^c \times D^c} |\widetilde{u}(x) - \widetilde{u}(y)|^p \nu(x, y)\, \mathrm{d}x\mathrm{d}y = \int_D \int_D + \int_D \int_{D^c \setminus B} + \int_{D^c \setminus B} \int_D + \int_B \int_D + \int_D \int_B$$

$$= \int_{B^c} \int_{B^c} |u(x) - u(y)|^p \nu(x, y)\, \mathrm{d}x\mathrm{d}y + 2 \int_D |u(y)|^p \int_B \nu(x, y)\, \mathrm{d}x\mathrm{d}y.$$

By the boundedness of u, the boundedness of D and the separation of D and B, the last integral is finite. Furthermore, since $\widetilde{u}$ is not constant and vanishes on D^c, the left-hand side is infinite by Lemma 7.4. Therefore the left-hand side of (7.6) is infinite, which yields (7.5).

Acknowledgements We thank Tomasz Adamowicz, Włodzimierz Bąk, Artur Bogdan, Bartłomiej Dyda, Agnieszka Kałamajska, Moritz Kassmann, Mateusz Kwaśnicki, René Schilling and Enrico Valdinoci for discussions, comments or suggestions.

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