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Calculus of Variations

Two short closed geodesics on a sphere of odd dimension

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Abstract
We show that for an open and dense set of *non-reversible* Finsler metrics on a sphere S^n of odd dimension $n = 2m - 1 \geq 3$ there is a second closed geodesic with Morse index $\leq 4(m+2)(m-1)+2$.

Mathematics Subject Classification 53C22 · 58E10

1 Introduction

In this paper we consider the sphere S^n of dimension $n \geq 2$ carrying a non-reversible Finsler metric f. Hence the length of a curve in general depends on the orientation. The *reversibility* $\lambda = \max\{f(-X)\,;\; f(X) = 1\}$ was introduced in [17]. Then $\lambda \geq 1$ and $\lambda = 1$ if and only if the Finsler metric is *reversible,* i.e. $f(-X) = f(X)$ for all tangent vectors X. For a tangent vector $X \in TS^n$ we denote by $f_0(X) = \sqrt{g_0(X, X)}$ the length of a vector with respect to the standard Riemannian metric g_0 of constant sectional curvature 1 on S^n. Let $D = D(f)$ be the smallest positive number such that

$$D^{-1} f_0(X) \leq f(X) \leq D f_0(X) \tag{1}$$

holds for all tangent vectors X. We call this invariant the *distortion* of the Finsler metric f. Obviously $D^2 \geq \lambda$. Let $L = L(f)$ be the critical value of a generator of the non-trivial homology class $H_{n-1}\left(\Lambda S^n / S^1; \mathbb{Q}\right) \cong \mathbb{Q}$ in dimension $(n-1)$ in the free loop space ΛS^n. Lyusternik and Fet [12] used an idea by Birkhoff to show the existence of a closed geodesic c_1 whose length $l(c_1)$ equals L and whose Morse index satisfies $\operatorname{ind}(c_1) \leq n-1$. Inequality (1) implies that $2\pi/D \leq L = l(c_1) \leq 2\pi D$. It follows from a result by Fet [6] that there exists a second closed geodesic for a *reversible* Finsler metric which is bumpy, i.e. all its closed geodesics are non-degenerate.

In this paper we consider the existence of a second closed geodesic for a *non-reversible* Finsler metric. On a 2-sphere with a bumpy metric there always exists a second closed

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geodesic c_2 geometrically distinct from c_1 as shown in [16, (4.1)]. Bangert and Long were able to show in [2] that this statement holds for *any* non-reversible Finsler metric. There is a family $f_\mu, \mu \in [0,1), \mu \notin \mathbb{Q}$ of Katok metrics on S^2 which are bumpy, have constant flag curvature 1 and carry exactly two geometrically distinct closed geodesics c_1, c_2 with $\text{ind}(c_1) = 1, \lim_{\mu\to 1} \text{ind}(c_2) = \infty, L(c_1) < 2\pi$, and $\lim_{\mu\to 1} L(c_2) = \infty$. Hence there exists in general only one *short* closed geodesic on S^2.

In higher dimensions there are many results on the existence of a second closed geodesic, cf. for example [4, 19, 20], [5, Cor. 1.2], and [1, Cor. 1.14]. Compare also the recent survey [11]. For existence results for closed geodesics in Riemannian and Finsler geometry we also refer to the surveys [13, 22]. Under curvature assumptions one can give bounds for the index of the second closed geodesic, cf. for example [18]. But we are not aware of estimates for the index of the second closed geodesic holding on an open and dense subset of metrics on an n-dimensional sphere with $n \geq 3$.

We state our main result which shows in particular that for an odd-dimensional sphere of dimension $n = 2m-1 \geq 3$ endowed with a bumpy metric there are two geometrically distinct short closed geodesics, with index $\leq 4(m+2)(m-1)+2$. More precisely we show:

Theorem 1.1 *Let f be a non-reversible Finsler metric on the odd-dimensional sphere S^n of dimension $n = 2m-1 \geq 3$ with distortion $D = D(f)$. Let p_m be the smallest prime number which is neither a divisor of $(m-1)$ nor of m, cf. Lemma 3.3, in particular $3 \leq p_m \leq m+2$ for all $m \geq 2$. Assume that all closed geodesics with length $\leq L_3 := 2\pi\, p_m D^3$ are non-degenerate. Then there are two geometrically distinct closed geodesics with index $\leq 4p_m(m-1)+2$ and of length $\leq L_3$.*

For $n = 3, m = 2, p_2 = 3$ we obtain for the second closed geodesic c_2 on S^3 : $\text{ind} c_2 \leq 14$. For $n = 6k+3 = 2m-1$ resp. $m \equiv 2 \pmod 3$ we have $p_m = 3$, hence for the second closed geodesic c_2 on S^{2m-1} : $\text{ind}(c_2) \leq 12m - 10$. The proof of Theorem 1.1 is given in Sect. 4. We use the computation of the homomorphism in homology induced by the projection of the free loop space ΛS^n onto the quotient space $\Lambda S^n / S^1$ as given in Lemma 2.1 for $n = 2m-1$. An analogous result is not available for even dimension n. Recall that f_0 is the Finsler metric defined by the standard Riemannian metric of constant sectional curvature 1. There is a one-parameter family $f_\mu, \mu \in [0,1)$ of Finsler metrics on S^n starting at the standard metric f_0 with the following properties: For every irrational μ the metric is non-reversible and bumpy and carries exactly $2m$ geometrically distinct closed geodesics. For $n = 2m-1$ of these closed geodesics the index is at most $6(m-1)$ but the index of one of these closed geodesics can be arbitrarily large. This example is explained in detail in Sect. 5, these metrics were first studied by Katok, cf. [23].

The set of metrics satisfying the assumptions of Theorem 1.1 contains an open and dense subset. This follows from the following

Theorem 1.2 *Let M be a compact manifold endowed with a Finsler metric f_0. For an arbitrary non-reversible Finsler metric f the distortion $D = D(f)$ is the smallest positive number satisfying Eq. (1) for all tangent vectors. For a positive number L let $\mathcal{F}_1(L)$ be the set of Finsler metrics f on M all of whose closed geodesics of length $\leq D^3(f)L$ are non-degenerate. Then $\mathcal{F}_1(L)$ is an open and dense subset of the space $\mathcal{F}(M)$ of all Finsler metrics on M with respect to the (strong) C^r-topology for $r \geq 4$.*

We give the proof in Sect. 6. The essential ingredient is the *bumpy metrics theorem* for Finsler metrics, cf. [21, Thm. 4].

Using Theorem 1.2 we obtain from Theorem 1.1 the following

Corollary 1.3 *Let p_m be the smallest prime number which is neither a divisor of $(m-1)$ nor of m. Then there is an open and dense subset of non-reversible Finsler metrics on the sphere S^n of odd dimension $n = 2m-1 \geq 3$ carrying two geometrically distinct closed geodesics with index $\leq 4p_m(m-1)+2$.*

2 Homology of the free loop space

Closed geodesics on S^n with a Finsler metric f are the critical points of the functional

$$F : \Lambda S^n \longrightarrow \mathbb{R}\,;\; F(\sigma) := \left(\int_0^1 f^2(\sigma'(t))\,dt\right)^{1/2},$$

cf. [10, Sect. 1] and [16, ch. 1]. We denote by $\Lambda = \Lambda S^n$ the free loop space, i.e. the space of H^1-maps $\sigma : S^1 = \mathbb{R}/\mathbb{Z} \to S^n$. The function F is up to a factor $1/2$ the square root of the energy functional $E(\sigma) = 1/2 \int_0^1 f^2(\sigma'(t))\,dt$. The functional F agrees with the length functional $l(\gamma) = \int_0^1 f\left(\gamma'(t)\right)\,dt$ on loops parametrized proportional to arc length. The *Morse index* $\mathrm{ind}(c)$ is the maximal dimension of a subspace of the tangent space $T_c\Lambda S^n$ on which the hessian d^2F_c is negative definite, cf. for example [16, ch. 1]. For a closed geodesic c the iterates $c^k, k \geq 1$ with $c^k(t) = c(kt)$ are closed geodesics, too. These closed geodesics are *geometrically equivalent.* Note that in general the curve c^{-1} with opposite orientation, i.e. $c^{-1}(t) = c(-t)$, is not a closed geodesic since the metric is assumed to be non-reversible.

For $f = f_0$ we use the following notation:

$$F_0(\sigma) =: \left(\int_0^1 f_0^2(\sigma'(t))\,dt\right)^{1/2}\;;\; l_0(\sigma) = \int_0^1 f_0\left(\sigma'(t)\right)\,dt\,.$$

For the sublevel sets of the functional F we use the following notation: $\Lambda^R = \{\sigma \in \Lambda\,;\; F(\sigma) \leq R\}$. The free loop space Λ carries a canonical S^1-action by linear reparametrization of the curves, i.e. shift of the initial point. We use the following notation for quotient spaces with respect to the S^1-action and its sublevel spaces: $\overline{\Lambda} = \Lambda/S^1$ and $\overline{\Lambda}^R = \{\sigma \in \overline{\Lambda}\,;\; F(\sigma) \leq R\}$. For the sublevel sets with respect to the functional F_0 we use the following notation: $\Lambda_0^R = \{\sigma \in \Lambda\,;\; F_0(\sigma) \leq R\}$, and $\overline{\Lambda}_0^R = \{\sigma \in \overline{\Lambda}\,;\; F_0(\sigma) \leq R\}$. The set of prime closed geodesics of positive length of the standard metric f_0 equals the subset $BS^n \subset \Lambda S^n$ of great circles which can be identified with the unit tangent bundle T^1S^n. Then the set of closed geodesics equals the union $\bigcup_{j\geq 1} B^j$. Here $B^j := \{c_0^j, c_0 \in BS^n\}$ is the set of j-fold covered great circles, i.e. great circles c_0 parametrized proportional to arc length with $l_0(c_0^j) = jl_0(c_0) = 2\pi j$. The functional $F_0 : \Lambda S^n \longrightarrow \mathbb{R}$ is a Morse-Bott function, i.e. the subsets B^j are non-degenerate critical submanifolds. This follows since the dimension of the kernel of the hessian of a great circle equals the dimension $2n-1$ of the manifold $BS^n = T^1S^n$. For $n = 2m-1 \geq 3$ we have

$$H_j\left(T^1S^{2m-1};\mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}\,;\; j = 0, 2m-2, 2m-1, 4m-3 \\ 0\;;\; \text{otherwise} \end{cases}.$$

If $\nu_k : N_k \longrightarrow B^k$ is the negative normal bundle of the critical submanifold B^k of dimension $\mathrm{ind}(c^k) = (4k-2)(m-1)$ with the associated disc bundle $\nu_k : DN_k \longrightarrow B^k$, resp. sphere bundle $SN_k \longrightarrow B^k$, then the generalized Morse lemma implies

$$H_j\left(\Lambda_0^{2\pi k}, \Lambda_0^{2\pi(k-1)};\mathbb{Z}\right) \cong H_j(DN_k, SN_k;\mathbb{Z})\,,$$

cf. [15, Sect. 4]. The negative normal bundle ν_k is oriented for all k, since $\operatorname{ind}(c_0^2) - \operatorname{ind}(c_0) = 4(m-1)$ is even resp. $\gamma_{c_0} = 1$ for a great circle c_0, cf. [16, Prop. 2.2]. Hence the Thom-isomorphism implies

$$H_j\left(\Lambda_0^{2\pi k}, \Lambda_0^{2\pi(k-1)}; \mathbb{Z}\right) \cong H_{j-(4k-2)(m-1)}\left(T^1 S^{2m-1}; \mathbb{Z}\right) . \tag{2}$$

The functional F_0 is *perfect,* i.e.

$$H_j(\Lambda, \Lambda^0; \mathbb{Z}) \cong \bigoplus_{k\geq 1} H_j\left(\Lambda_0^{2\pi k}, \Lambda_0^{2\pi(k-1)}; \mathbb{Z}\right) \tag{3}$$

which follows for $m \geq 2$ from the long exact homology sequence. Hence

$$H_j\left(\Lambda, \Lambda^0; \mathbb{Z}\right) \cong \begin{cases} \mathbb{Z} \ ; \ j = 2r(m-1); r \geq 1 \\ \mathbb{Z} \ ; \ j = 2r(m-1)+1, r \geq 2 \\ 0 \ ; \ \text{otherwise} \end{cases} , \tag{4}$$

and the homomorphism

$$H_j\left(\Lambda_0^{2\pi k}, \Lambda^0; \mathbb{Q}\right) \longrightarrow H_j\left(\Lambda, \Lambda^0, \mathbb{Q}\right) \tag{5}$$

induced by the inclusion is an isomorphism for all $k \geq 1$ and $j < i(k+1) = (4k+2)(m-1)$. This follows since $i(k+1) = \operatorname{ind}(c_0^{k+1}) = i(k) + 4(m-1)$.

The quotient space $T^1 S^n / S^1$ of unparametrized oriented great circles can be identified with the Grassmannian $\widetilde{G}(2, 2m-2)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^{2m}$.

The equivariant Morse Lemma implies

$$H_j\left(\overline{\Lambda}_0^{2\pi k}, \overline{\Lambda}_0^{2\pi(k-1)}; \mathbb{Z}\right) \cong H_j\left(\overline{DN}_k, \overline{SN}_k; \mathbb{Z}\right) ,$$

cf. [15, Sect. 4]. Here the quotient bundle $\nu_k : \overline{DN}_k \longrightarrow \overline{B}_k$ resp. $\nu_k : \overline{SN}_k \longrightarrow \overline{B}_k$ is a bundle with fibre $D^{i(k)}/\mathbb{Z}_k$ resp. $S^{i(k)-1}/\mathbb{Z}_k$. Here $i(k) = \operatorname{ind}(c_0^k) = (4k-2)(m-1)$ is the Morse index of a k-fold covered great circle c_0^k as a closed geodesic of the standard metric f_0. Then

$$H_*\left(D^{i(k)}/\mathbb{Z}_k, S^{i(k)-1}/\mathbb{Z}_k; \mathbb{Q}\right) \cong H_*\left(D^{i(k)}, S^{i(k)-1}; \mathbb{Q}\right)$$

and the Thom isomorphism implies

$$H_*\left(\overline{\Lambda}_0^{2\pi k}, \overline{\Lambda}_0^{2\pi(k-1)}; \mathbb{Q}\right) \cong H_{*-i(k)}\left(\widetilde{G}\left(2, 2m-2\right), \mathbb{Q}\right) .$$

Non-trivial homology only occurs in even dimensions since

$$H_j(\widetilde{G}(2, 2m-2); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & ; \ j = 0, 4m-4 \\ \mathbb{Z} \oplus \mathbb{Z} & ; \ j = 2m-2 \\ 0 & ; \ \text{otherwise} \end{cases} . \tag{6}$$

This follows from the Gysin sequence of the S^1-bundle $T^1 S^{2m-1} \longrightarrow \widetilde{G}(2, 2m)$, cf. [14, Beweis Satz 4.9]. Hence we obtain:

$$H_*\left(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q}\right) \cong \bigoplus_{k\geq 1} H_*\left(\overline{\Lambda}_0^{2\pi k}, \overline{\Lambda}_0^{2\pi(k-1)}; \mathbb{Q}\right) ,$$

which implies

$$H_j\left(\overline{\Lambda},\overline{\Lambda}^0;\mathbb{Q}\right)\cong\begin{cases}\mathbb{Q} & ;\ j\geq 2(m-1),\ j \text{ even },\ j\neq 2k(m-1),\ k\geq 2\\ \mathbb{Q}\oplus\mathbb{Q} & ;\ j=2k(m-1),\ k\geq 2\\ 0 & ;\ \text{otherwise}\end{cases}, \tag{7}$$

[16, Rem. 2.5(a)]. Therefore the functional $F_0:\overline{\Lambda}\longrightarrow\mathbb{R}$ can be seen as a perfect Morse Bott function for rational coefficients, too. In particular the homomorphism

$$i_*:H_j\left(\overline{\Lambda}_0^{2\pi k},\overline{\Lambda}^0;\mathbb{Q}\right)\longrightarrow H_j\left(\overline{\Lambda},\overline{\Lambda}^0;\mathbb{Q}\right)$$

induced by the inclusion is an isomorphism for all $k\geq 1$ and $j<i(k+1)=(4k+2)(m-1)$. This follows since $i(k+1)=\operatorname{ind}(c_0^{k+1})=i(k)+4(m-1)$.

Lemma 2.1 $n=2m-1, m\geq 2$. *Let* $a_k\in H_{4k(m-1)}(\Lambda,\Lambda^0;\mathbb{Z})\cong\mathbb{Z}, k\geq 1$ *be a generator. Then the canonical projection* $\rho:\left(\Lambda,\Lambda^0\right)\longrightarrow(\overline{\Lambda},\overline{\Lambda}^0)$ *induces an injective homomorphism*

$$\rho_*:H_{4k(m-1)}\left(\Lambda,\Lambda^0;\mathbb{Z}\right)\cong\mathbb{Z}\longrightarrow H_{4k(m-1)}\left(\overline{\Lambda},\overline{\Lambda}^0;\mathbb{Z}\right)$$

with $\rho_*(a_k)=k\tilde{a}_k\neq 0$ *and* $\tilde{a}_k$ *is not a torsion element.*

Proof The projection $\rho:\Lambda\longrightarrow\overline{\Lambda}$ induces the homomorphism

$$\rho_*:H_{4k(m-1)}\left(\Lambda,\Lambda^0;\mathbb{Z}\right)\cong\mathbb{Z}\longrightarrow H_{4k(m-1)}\left(\overline{\Lambda},\overline{\Lambda}^0;\mathbb{Z}\right).$$

The homomorphism

$$\rho_*:H_{4k(m-1)}\left(\Lambda_0^{2\pi k},\Lambda_0^{2\pi(k-1)};\mathbb{Z}\right)\longrightarrow H_{4k(m-1)}\left(\overline{\Lambda}_0^{2\pi k},\overline{\Lambda}_0^{2\pi(k-1)};\mathbb{Z}\right)$$

can be expressed by the homomorphism

$$\rho_*:H_{4k(m-1)}\left(DN_k,SN_k;\mathbb{Z}\right)\cong\mathbb{Z}\longrightarrow H_{4k(m-1)}\left(\overline{DN}_k,\overline{SN}_k;\mathbb{Z}\right)$$

which is a multiplication with the number k, i.e. for a generator a_k' with $0\neq a_k'\in H_{4k(m-1)}(DN_k,SN_k;\mathbb{Z})\cong\mathbb{Z}$ we have $\rho_*(a_k')=ks\overline{a}_k$ for a generator $\overline{a}_k\in H_{4k(m-1)}\left(\overline{DN}_k,\overline{SN}_k;\mathbb{Z}\right)$ and an integer $s>0$. This follows since the homomorphism

$$H_{4k(m-1)}(D^{4k(m-1)},S^{4k(m-1)-1};\mathbb{Z})\cong\mathbb{Z}$$
$$\longrightarrow H_{4k(m-1)}\left(D^{4k(m-1)}/\mathbb{Z}_k,S^{4k(m-1)-1}/\mathbb{Z}_k;\mathbb{Z}\right)\cong\mathbb{Z}$$

induced by the canonical projection is a multiplication by k. This follows since the isometric $\mathbb{Z}_k$-action on the disc $D^{4k(m-1)}$ is free on an open and dense subset, which we see as follows: For any divisor $d|k, d<k$ we have the following inequality for the indices of coverings c_0^k of a great circle $c_0:\operatorname{ind}(c_0^d)<\operatorname{ind}(c_0^k)$. Actually one can show $s=1$, i.e. $\rho_*(a_k)=k\tilde{a}_k$. This follows from the Gysin sequence of the S^1-bundle $T^1S^{2m-1}\longrightarrow\widetilde{G}(2,2m-2)$ and Eq. (6). □

Remark 2.2 The S^1-action on Λ induces the homomorphism

$$\Delta:H_{4k(m-1)}\left(\Lambda,\Lambda^0;\mathbb{Z}\right)\cong\mathbb{Z}\cdot a_k\longrightarrow H_{4k(m-1)+1}\left(\Lambda,\Lambda^0;\mathbb{Z}\right),$$

cf. [8, (17.1)]. The homomorphism is used to define the *Batalin Vilkovisky algebra,* cf. [3, Thm. 5.4].

It can be expressed as composition $\Delta = \tau \circ \rho_*$ of the homomorphism

$$\rho_* : H_{4k(m-1)}\left(\Lambda, \Lambda^0; \mathbb{Z}\right) \cong \mathbb{Z} \cdot a_k \longrightarrow H_{4k(m-1)}\left(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Z}\right)$$

induced by the canonical projection and the *transfer map*

$$\tau : H_{4k(m-1)}\left(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Z}\right) \longrightarrow H_{4k(m-1)+1}\left(\Lambda, \Lambda^0; \mathbb{Z}\right) \cong \mathbb{Z} \cdot \tilde{a}_k \,.$$

Hence we obtain $\Delta(a_k) = \tau(\rho_*(a_k)) = k\tilde{s}\tilde{a}_k$ for a positive integer $\tilde{s}$ and a generator $\tilde{a}_k \in H_{4k(m-1)+1}(\Lambda, \Lambda^0; \mathbb{Z})$. The homomorphism can be computed, cf. [14, Satz 4.13] resp. [10, Lem. 6.2], it follows that $\tilde{s} = 2$.

Remark 2.3 Since c is prime and since for all divisors q of r with $q < r$ the inequality $\operatorname{ind}(c^q) < \operatorname{ind}(c^r)$ holds, we can conclude that for $r \geq 1$ the following holds: There are generators

$$s_r, t_r \in H_*\left(\Lambda^{rL}, \Lambda^{(r-1)L}; \mathbb{Z}\right) \; ; \; S_r \in H_*\left(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Z}\right) \tag{8}$$

with $\deg(s_r) = \deg(S_r) = \deg(t_r) - 1 = \operatorname{ind}(c^r) = j$ such that the induced projection

$$\rho_* : H_j\left(\Lambda^{rL}, \Lambda^{(r-1)L}; \mathbb{Z}\right) \cong \mathbb{Z} \cdot s_r \longrightarrow H_j\left(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Z}\right) \cong \mathbb{Z} \cdot S_r \tag{9}$$

satisfies

$$\rho_*(s_r) = r \cdot S_r \,, \tag{10}$$

cf. [15, Sect. 3]. This will be crucial in the Proof of Theorem 1.1 given in Sect. 4. For the transfer homomorphism

$$\Delta : H_j\left(\Lambda^{rL}, \Lambda^{(r-1)L}; \mathbb{Z}\right) \cong \mathbb{Z} \cdot s_r \longrightarrow H_{j+1}\left(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Z}\right) \cong \mathbb{Z} \cdot t_r$$

one obtains $\Delta(s_r) = r \cdot t_r$.

3 Morse theory for a metric with only one closed geodesic

In this section we study non-reversible Finsler metrics on S^{2m-1} for which all closed geodesics with length $\leq 2\pi p_m D^3(f)$ are geometrically equivalent to the closed geodesic c of length $L = l(c)$. We will show that this assumption determines the sequence $\operatorname{ind}(c^r), rL \leq 2\pi p_m D^3$ completely.

Lemma 3.1 *Let f be a non-reversible Finsler metric on the sphere $S^n, n \geq 2$ with distortion $D = D(f)$. We assume that all closed geodesics with length $\leq 2\pi D$ are non-degenerate. Then there exists a prime closed geodesic c whose length satisfies $L := l(c) \leq 2\pi D$ and with* $\operatorname{ind}(c) \leq n - 1$.

Proof Equation (1) implies that $\Lambda_0^{2\pi} \subset \Lambda^{2\pi D}$. Since

$$H_{n-1}\left(\Lambda_0^{2\pi}, \Lambda^0; \mathbb{Q}\right) \longrightarrow H_{n-1}\left(\Lambda, \Lambda^0; \mathbb{Q}\right) \cong \mathbb{Q}$$

is an isomorphism, cf. Eq. (5), we conclude that the homomorphism

$$H_{n-1}\left(\Lambda^{2\pi D}, \Lambda^0; \mathbb{Q}\right) \longrightarrow H_{n-1}\left(\Lambda, \Lambda^0; \mathbb{Q}\right) \cong \mathbb{Q}$$

is surjective, i.e. $\dim H_{n-1}\left(\Lambda^{2\pi D}, \Lambda^0; \mathbb{Q}\right) \geq 1$. It follows from the Morse inequalities for the space $\Lambda^{2\pi D}$ that there is a closed geodesic c with length $l(c) \leq 2\pi D$ and index $\text{ind}(c) \leq n-1$, cf. [16, Sect. 2]. □

We later use the following

Assumption 3.2 For $m \geq 2$ let p_m be the smallest prime number which does not divide $(m-1)$ nor m, cf. Lemma 3.3. Given a non-reversible Finsler metric f on a sphere of dimension $n = 2m-1 \geq 3$ with distortion $D = D(f)$ we assume that all closed geodesics γ with $L(\gamma) \leq L_3 := 2\pi p_m D^3$ are non-degenerate and that all closed geodesics with length $\leq L_3 = 2\pi p_m D^3$ and index $\leq 4p_m(m-1)+2$ are geometrically equivalent.

Hence we conclude from Lemma 3.1 that there is a prime closed geodesic c such that every closed geodesic γ with $l(\gamma) \leq L_3 = 2\pi p_m D^3$ and $\text{ind}(\gamma) \leq 4p_m(m-1)+2$ is up to the choice of the initial point a covering of the closed geodesic c, i.e. there is a positive integer $r \geq 1$ and an element $z \in S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\{0,1\}$ such that $\gamma = z.c^r$. Here $z.c(t) = c(t+z)$ defines the canonical S^1-action on the free loop space $\Lambda = \Lambda S^n$ leaving the functional F invariant.

Lemma 3.3 *For $m \geq 2$ denote by p_m the smallest prime number, which is neither a divisor of $(m-1)$ nor of m. Then $p_2 = 3$, $p_3 = 5$, and for $m \geq 4 : 3 \leq p_m \leq m+1$.*

Proof For $m \leq 5$ we have: $p_2 = 3$, $p_3 = p_4 = 5$. Assume $m \geq 5$. If $m \equiv 2 \pmod 3$ then $p_m = 3$. If $m-2 \neq 2^s$ for some s choose a prime factor $q \geq 3$ of $m-2 \geq 4$. If $m-2 = 2^s$, choose a prime factor $q \geq 3$ of $m+1$. Then $p_m \leq q \leq m+1$, and hence $3 \leq p_m \leq m+2$ for all $m \geq 2$. □

The invariant $\gamma_c \in \{\pm 1/2, \pm 1\}$ of a prime closed geodesic is defined as follows: $\gamma_c = \pm 1$ if and only if $\text{ind}(c^2) - \text{ind}(c)$ is even and $\gamma_c > 0$ if and only if $\text{ind}(c)$ is even, cf. [16, Def. 1.6].

Lemma 3.4 *Let Assumption 3.2 be satisfied, i.e. there exists a prime closed geodesic c with $L = l(c) \leq 2\pi D$ such that all closed geodesics γ with length $l(\gamma) \leq 2\pi p_m D^3$ and index $\text{ind}(\gamma) \leq 4p_m(m-1)+2$ are geometrically equivalent to c, cf. Lemma 3.1.*

We use the following notation for Betti numbers of the quotients $\overline{\Lambda}^{2\pi D p_m}$ and $\overline{\Lambda}^{2\pi D^3 p_m}$ of the sublevel sets by the canonical S^1-action:

$$\overline{\beta}_j := \dim H_j\left(\overline{\Lambda}^{2\pi D p_m}, \overline{\Lambda}^0; \mathbb{Q}\right), \ \overline{\beta}_j^* := \dim H_j\left(\overline{\Lambda}^{2\pi D^3 p_m}, \overline{\Lambda}^0; \mathbb{Q}\right).$$

Let $L_1 = 2\pi p_m D$, $L_3 = 2\pi p_m D^3 = L_1 D^2$ and let

$$v_j := \#\{1 \leq r \leq L_1/L;\ \text{ind}(c^r) = j, r \equiv 1 \pmod 2 \text{ or } \gamma_c = \pm 1\}$$
$$v_j^* := \#\{1 \leq r \leq L_3/L;\ \text{ind}(c^r) = j, r \equiv 1 \pmod 2 \text{ or } \gamma_c = \pm 1\}.$$

Then for all even $j \leq 4p_m(m-1)+2$:

$$\overline{\beta}_j = v_j\,;\ \overline{\beta}_j^* = v_j^*$$

and $\overline{\beta}_j = \overline{\beta}_j^* = 0$ for all odd $j \leq 4p_m(m-1)+2$.

Proof We conclude from [16, Sect. 2] and [16, Def. 1.6] or [19, Sec.2]:
Let $v_j(c^r) = b_j(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Q})$. If $l(c^r) = rl(c) \le L_3$ we conclude from Assumption 3.2 for all $j \le 4p_m(m-1)+2 : v_j(c^r) \in \{0,1\}$ with $v_j(c^r) = 1$ if and only if $j = \text{ind}(c^r)$ and r is odd or $\text{ind}(c^2) \equiv \text{ind}(c) \pmod 2$. It follows that for $1 \le r \le L_3/L$:

$$v_j(c^r) = 1 \Rightarrow j = \text{ind}(c^r) \equiv \text{ind}(c) \pmod 2\,. \tag{11}$$

The *Morse inequalities* for the functional F on the space $\overline{\Lambda}^{L_1} = \overline{\Lambda}^{L_1} S^n$ resp. $\overline{\Lambda}^{L_3} = \overline{\Lambda}^{L_3} S^n$ give a relation between the number of (homologically visible) critical points v_j, resp. v_j^* with index j and length $l \le L_1$, resp. $\le L_3$ with the Betti numbers $\overline{\beta}_j$, resp. $\overline{\beta}_j^*$. We obtain:

$$v_j = \overline{\beta}_j + q_j + q_{j-1} \text{ resp. } v_j^* = \overline{\beta}_j^* + q_j^* + q_{j-1}^*$$

for a non-negative sequence $q_j,\, j \ge 0$, resp. $q_j^*,\, j \ge 0$, cf. [16, Sec. 2]. Equation (11) implies the following for all $j \le 4p_m(m-1)+2,\, j \equiv 1 \pmod 2$

$$v_j = v_j^* = 0 \tag{12}$$

and $q_j = q_j^* = 0$ for all j. Here we have used that under the assumptions of the Lemma there is up to geometric equivalence only one closed geodesic of length $\le 2\pi p_m D^3$, and that an iterate c^r can have non-trivial local homology in degree j only for even j, cf. Eq. (11). Hence

$$v_j = \overline{\beta}_j;\, v_j^* = \overline{\beta}_j^* \tag{13}$$

for all $j \le 4p_m(m-1)+2$. □

For a topological pair (X, A) with singular homology $H_j(X, A; \mathbb{Z})$ with integer coefficients let $\text{Tor}_j \subset H_j(X, A)$ be the torsion submodule. We denote by $FH_j(X, A; \mathbb{Z}) = H_j(X, A; \mathbb{Z})/\text{Tor}_j$ the associated free module. Then $H_j(X, A; \mathbb{Q}) \cong H_j(X, A; \mathbb{Z}) \otimes \mathbb{Q} \cong FH_j(X, A; \mathbb{Z}) \otimes \mathbb{Q}$.

Lemma 3.5 *If the Finsler metric f on S^{2m-1} satisfies Assumption 3.2 and $p = p_m$ then the homomorphism*

$$H_j\left(\overline{\Lambda}^{2\pi pD}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q}\right)$$

induced by the inclusion is an isomorphism for all $j \le 4p(m-1)+2$. Using the notation from Lemma 3.4 we obtain for the Betti numbers $\overline{b}_j := \dim H_j(\overline{\Lambda}, \overline{\Lambda}^0, \mathbb{Q})$ for all $j \le 4p(m-1)+2$:

$$\overline{\beta}_j = \overline{b}_j\,. \tag{14}$$

Proof From the definition of the distortion given in Eq. (1) we obtain the following inclusions:

$$\Lambda_0^{2\pi p} \subset \Lambda^{2\pi pD} \subset \Lambda_0^{2\pi pD^2} \subset \Lambda^{2\pi pD^3} \tag{15}$$

and

$$\overline{\Lambda}_0^{2\pi p} \subset \overline{\Lambda}^{2\pi pD} \subset \overline{\Lambda}_0^{2\pi pD^2} \subset \overline{\Lambda}^{2\pi pD^3}\,.$$

It follows that the composition

$$H_j\left(\overline{\Lambda}_0^{2\pi p}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}^{2\pi pD}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}_0^{2\pi pD^2}, \overline{\Lambda}^0; \mathbb{Q}\right) \cong H_j\left(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q}\right) \tag{16}$$

is an isomorphism for $j \le 4p(m-1)+2$, cf. Eq. (7) and the arguments below. Therefore we conclude that the homomorphism

$$i_{1*} : H_j\left(\overline{\Lambda}^{2\pi pD}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q}\right) \tag{17}$$

induced by the inclusion is surjective for $j \le 4p(m-1)+2$ since $4p(m-1)+2 < i(p+1) = (4p+2)(m-1)$. From Assumption 3.2 and Lemma 3.4 we conclude

$$H_j\left(\overline{\Lambda}^{2\pi pD}, \overline{\Lambda}^0; \mathbb{Q}\right) = H_j\left(\overline{\Lambda}^{2\pi pD^3}, \overline{\Lambda}^0; \mathbb{Q}\right) = 0 \tag{18}$$

for all odd $j \le 4p(m-1)+2$.

If the homomorphism given in Eq. (17) is not injective for some $j \le 4p(m-1)+2$ then there is a non-trivial class

$$Z \in H_j\left(\overline{\Lambda}^{2\pi pD} S^n, \overline{\Lambda}^0 S^n; \mathbb{Q}\right)$$

with $\deg(Z) = j \le 4p(m-1)+2$ such that $i_{1*}(Z) = 0$.

We consider the homomorphisms induced by the respective inclusions

$$\begin{aligned} i_{2*} &: H_j\left(\overline{\Lambda}^{2\pi pD}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}_0^{2\pi pD^2}, \overline{\Lambda}^0; \mathbb{Q}\right)\\ i_{3*} &: H_j\left(\overline{\Lambda}_0^{2\pi pD^2}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}^{2\pi pD^3}, \overline{\Lambda}^0; \mathbb{Q}\right)\\ i_{4*} &: H_j\left(\overline{\Lambda}^{2\pi pD^3}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q}\right). \end{aligned}$$

Then $i_{1*} = i_{4*} \circ i_{3*} \circ i_{2*}$. Since the homomorphism

$$i_{4*} \circ i_{3*} : H_j\left(\overline{\Lambda}_0^{2\pi D^2}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Q}\right) \tag{19}$$

is an isomorphism for all $j \le 4p(m-1)+2$, cf. Eq. (16), we conclude that Z lies in the kernel of the homomorphism

$$i_{3*} \circ i_{2*} : H_j\left(\overline{\Lambda}^{2\pi pD}, \overline{\Lambda}^0; \mathbb{Q}\right) \longrightarrow H_j\left(\overline{\Lambda}^{2\pi pD^3}, \overline{\Lambda}^0; \mathbb{Q}\right), \tag{20}$$

i.e. $(i_3 \circ i_2)_*(Z) = 0$. The exactness of the long homology sequence of the triple $(\overline{\Lambda}^{2\pi pD^3} S^n, \overline{\Lambda}^{2\pi pD} S^n, \overline{\Lambda}^0 S^n)$ implies that there exists a non-trivial class

$$Y \in H_{j+1}\left(\overline{\Lambda}^{2\pi pD^3} S^n, \overline{\Lambda}^{2\pi pD} S^n; \mathbb{Q}\right)$$

with $\partial_* Y = Z$. Here ∂_* is the boundary operator of the long homology sequence of the triple. But since j is even this leads to a contradiction to Eq. (18). □

Let $L = F(c) = l(c)$ be the length of the prime closed geodesic c. Then we obtain for the Betti numbers $b_j(c^r) = \text{rk} H_j(\Lambda^{rL}, \Lambda^{(r-1)L}; \mathbb{Z})$ of the critical group of c^r, $r \le L_3/L$:

$$b_k\left(c^r\right) = \begin{cases} 1\ ;\ k = \text{ind}(c^r), r \text{ odd, or } \gamma_c = 1\\ 1\ ;\ k = \text{ind}(c^r)+1, r \text{ odd, or } \gamma_c = 1\\ 0\ ;\ \text{otherwise} \end{cases}.$$

The Betti numbers $\overline{b}_k(c^r) = \mathrm{rk} H_k(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Z}) = \dim H_k(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}; \mathbb{Q})$ of the S^1-critical group of c^r for $r \leq L_3/L$:

$$\overline{b}_k(c^r) = \begin{cases} 1 \ ; \ k = \mathrm{ind}(c^r) \ , \ \gamma_c = 1 \text{ or } r \text{ odd} \\ 0 \ ; \ \text{ otherwise} \end{cases} .$$

The Betti numbers $b_k = \mathrm{rk} H_k(\Lambda S^n, \Lambda^0 S^n; \mathbb{Z}) = \dim H_k(\Lambda S^n, \Lambda^0 S^n; \mathbb{Q})$ are given by

$$b_k = \begin{cases} 1 \ ; \ k = 2s(m-1), s \geq 1 \\ 1 \ ; \ k = 2s(m-1)+1, s \geq 2 \\ 0 \ ; \ k \text{ otherwise} \end{cases} ,$$

cf. Eq. (4). The Betti numbers $\overline{b}_k = \mathrm{rk} H_k(\overline{\Lambda} S^{2m-1}, \overline{\Lambda}^0 S^{2m-1}; \mathbb{Z})$
$= \dim H_k(\overline{\Lambda} S^{2m-1}, \overline{\Lambda}^0 S^{2m-1}; \mathbb{Q})$ of the S^1-quotient space are as follows:

$$\overline{b}_k = \begin{cases} 2 \ ; \ k = 2s(m-1), s \geq 2 \\ 1 \ ; \ k \geq 2m-2, k \text{ even}, k \neq 2s(m-1), s \geq 2 \\ 0 \ ; \ k \text{ otherwise} \end{cases} , \tag{21}$$

cf. Eq. (7).

Bott's formula for the sequence $(\mathrm{ind}(c^r))_{r \geq 1}$ of indices of the iterates c^r implies (cf. for example [19]):

$$\mathrm{ind}(c^r) \geq \mathrm{ind}(c), r \geq 1 . \tag{22}$$

Lemma 3.4, Eqs. (22) and (21) imply that $\mathrm{ind}(c) = n - 1$ and that the sequence $\mathrm{ind}(c^r)$ is monotone increasing, i.e. for all $r \geq 1$:

$$\mathrm{ind}\left(c^{r+1}\right) \geq \mathrm{ind}\left(c^r\right) , \tag{23}$$

cf. [20] or [19]. Bott's formula implies that $v_j > 0$ resp. $v_j^* > 0$ for $j \leq 4p(m-1)+2$ holds only for even j. Since

$$v_j = v_j^* = \overline{\beta}_j = \overline{\beta}_j^* = 0 \tag{24}$$

for all odd $j \leq 4p(m-1)+2$ the Morse inequalities take the simple form for all $j \leq 4p(m-1)+2$, cf. Eq. (13):

$$v_j = \overline{\beta}_j = \overline{b}_j \ ; \ v_j^* = \overline{\beta}_j^* . \tag{25}$$

If $\gamma_c = 1/2$, i.e. if $\mathrm{ind} c^2 = 2m - 1 = \mathrm{ind} c + 1$, we obtain from Bott's formula for $\mathrm{ind}(c^r)$ that the sequence $\mathrm{ind}(c^r), r \geq 1$ is strictly monotone increasing. But $\overline{b}_{4m-4} = 2$, cf. Eq. (21). This contradicts Eq. (25). Hence $\gamma_c = 1$, resp. $\mathrm{ind}(c^2) = 2m$. The sequence $\mathrm{ind}(c^r)_{1 \leq r \leq L_1/L}$ is uniquely determined by Eq. (23) and Eq. (25):

$$\begin{aligned} \left(\mathrm{ind}(c^r)\right)_{r \geq 1} = & \\ & (2m-2, 2m, 2m+2, \ldots, 4m-6, 4m-4, 4m-4, 4m-2, \ldots \\ & \ldots, 6m-8, 6m-6, 6m-6, 6m-4, \ldots) \end{aligned}$$

From Lemma 3.5 and Eq. (21) we conclude

$$\overline{\beta}_{4p(m-1)} = \dim H_{4p(m-1)}\left(\overline{\Lambda}^{2\pi pD}, \overline{\Lambda}^0; \mathbb{Q}\right) = \overline{b}_{4p(m-1)} = 2 .$$

Therefore we obtain the following, cf. [20, Eq. (13)]:

Lemma 3.6 *If Assumption 3.2 holds then for $p = p_m$ we have*

$$\operatorname{ind}\left(c^{(2p-1)m}\right) = \operatorname{ind}\left(c^{(2p-1)m+1}\right) = 4p(m-1)\,. \tag{26}$$

and $l(c^{(2p-1)m+1}) = ((2p-1)m+1)L \le 2\pi pD$.

Lemma 3.7 *If Assumption 3.2 holds, $p = p_m$ then for all $j \le 4p(m-1)+1$:*

$$H_j\left(\Lambda^{2\pi pD^3}, \Lambda^{2\pi pD}; \mathbb{Q}\right) = 0\,;\; H_j\left(\overline{\Lambda}^{2\pi pD^3}, \overline{\Lambda}^{2\pi pD}; \mathbb{Q}\right) = 0\,, \tag{27}$$

and the homomorphism

$$H_j\left(\Lambda^{2\pi pD}, \Lambda^0; \mathbb{Q}\right) \longrightarrow H_j(\Lambda, \Lambda^0, \mathbb{Q}) \tag{28}$$

induced by the inclusion is an isomorphism for all $j \le 4p(m-1)$.

Proof Since $\operatorname{ind}(c^r) \ge 4p(m-1)+2$ for all $r \ge (2p-1)m+2$ it also follows that for $j \le 4p(m-1)+1 : v_j = v_j^*$, hence Eq. (27) follows, cf. Lemma 3.6. The inclusion Eq. (15) together with the isomorphism (3) imply that the homomorphism (28) is an isomorphism for $j \le 4p(m-1)$. □

4 Proof of Theorem 1.1

In this proof we use as coefficient ring for homology the ring $\mathbb{Z}$ of integers if not otherwise stated. We assume that Assumption 3.2 holds and derive a contradiction. Let $p = p_m$. Because of the Morse inequalities (25) and Lemma 3.5 we obtain for $j \le 4p(m-1)+2$:

$$FH_j(\overline{\Lambda}, \overline{\Lambda}^0; \mathbb{Z}) \cong FH_j\left(\overline{\Lambda}^{2\pi Dp}, \overline{\Lambda}^0\right) \cong \bigoplus_{rL \le 2\pi p_m D} FH_j\left(\overline{\Lambda}^{rL}, \overline{\Lambda}^{(r-1)L}\right).$$

We have shown that $\operatorname{ind}(c) = 2m-2$, $\operatorname{ind}(c^2) = 2m$ and $\operatorname{ind}(c^{(2p-1)m}) = \operatorname{ind}(c^{(2p-1)m+1}) = 4p(m-1)$, cf. Eq. (26). It also follows that $((2p-1)m+1)L \le 2\pi pD$. Let $s_{(2p-1)m}, s_{(2p-1)m+1}$ denote generators of the local critical groups, cf. Eq. (8). It follows that

$$\begin{aligned}
&H_{4p(m-1)}\left(\Lambda^{((2p-1)m+1)L}, \Lambda^{((2p-1)m-1)L}\right)\\
&\cong H_{4p(m-1)}\left(\Lambda^{((2p-1)m+1)L}, \Lambda^{(2p-1)mL}\right) \oplus H_{4p(m-1)}\left(\Lambda^{(2p-1)mL}, \Lambda^{((2p-1)m-1)L}\right)\\
&\cong \mathbb{Z}\cdot s_{(2p-1)m+1} \oplus \mathbb{Z}\cdot s_{(2p-1)m}\,.
\end{aligned}$$

We consider the following commutative diagram, the vertical homomorphisms are induced by inclusions, the horizontal ones by the canonical projection with respect to the S^1-action:

$$\begin{array}{ccc}
H_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^{((2p-1)m-1)L}) & \xrightarrow{\rho_{1*}} & FH_{4p(m-1)}\left(\overline{\Lambda}^{((2p-1)m+1)L}, \overline{\Lambda}^{((2p-1)m-1)L}\right)\\
\uparrow h_{1*} & & j_{1*}\uparrow \cong\\
H_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^0) & \xrightarrow{\rho_{2*}} & FH_{4p(m-1)}\left(\overline{\Lambda}^{((2p-1)m+1)L}, \overline{\Lambda}^0\right)\\
\cong \downarrow h_{2*} & & \cong \downarrow j_{2*}\\
H_{4p(m-1)}(\Lambda, \Lambda^0) & \xrightarrow{\rho_*} & FH_{4p(m-1)}(\overline{\Lambda}, \overline{\Lambda}^0)
\end{array}$$

j_{1*}, j_{2*} are isomorphisms, which follows from the following arguments: For $r \le (2p-1)m-1$: $\operatorname{ind}(c^r) \le 4p(m-1)-2$, hence $H_j(\overline{\Lambda}^{((2p-1)m-1)L}, \overline{\Lambda}^0) = 0$ for $j = 4p(m-1), 4p(m-1)-1$. Therefore j_{1*} is an isomorphism. The homomorphism j_{2*} is an isomorphism since $FH_j(\overline{\Lambda}, \overline{\Lambda}^{((2p-1)m+1)L}) = 0$ for $j = 4p(m-1), 4p(m-1)+1$, cf. Eq. (25) and Eq. (24).

Lemma 3.7 implies that h_{2*} is an isomorphism.

Since $H_{4p(m-1)}(\Lambda^{((2p-1)m-1)L}, \Lambda^0) = 0$ the map h_{1*} is injective and the image of a generator $a'_p \in H_{4p(m-1)}(\Lambda^{((2p-1)m+1)L}, \Lambda^0) \cong \mathbb{Z}$ is a prime element. Let $a_p \in H_{4p(m-1)}\left(\Lambda, \Lambda^0\right)$ be a generator. Then we conclude from Lemma 2.1:

$$\rho_*(a_p) = p\tilde{a}_p$$

for a generator $\tilde{a}_p \in H_{4p(m-1)}(\overline{\Lambda}, \overline{\Lambda}^0)$.

Define $a'_p = (h_{2*})^{-1}a_p$ and for $\eta \in \{0, 1\}$: $s'_{(2p-1)m+\eta} = (j_{2*})^{-1}s_{(2p-1)m+\eta}$, $s''_{(2p-1)m+\eta} = (j_{1*})^{-1}s'_{(2p-1)m+\eta}$. Then there are coprime integers α, β

$$h_{1*}(a'_p) = \alpha s_{(2p-1)m+1} + \beta s_{(2p-1)m}.$$

Here we also allow $\alpha = 0$, which implies $\beta = \pm 1$ resp. $\beta = 0$, which implies $\alpha = \pm 1$. By Eq. (10) we have

$$\rho_{1*}(s_{(2p-1)m+1}) = ((2p-1)m+1)\overline{s}_{(2p-1)m+1}\,;\ \rho_{1*}\left(s_{(2p-1)m}\right) = (2p-1)m\overline{s}_{(2p-1)m}.$$

Since $\overline{s}''_{(2p-1)m+1}, \overline{s}''_{(2p-1)m}$ form a basis for $FH_{4p(m-1)}(\overline{\Lambda}, \overline{\Lambda}^0)$ there are integers $w, z \in \mathbb{Z}$ with

$$\tilde{a}_p = w\overline{s}''_{(2p-1)m+1} + z\overline{s}''_{(2p-1)m}\,.$$

We obtain the following explicit description of the last commutative diagram with respect to the given basis elements

$$\begin{array}{ccc}
\mathbb{Z}s_{(2p-1)m+1} \oplus \mathbb{Z}s_{(2p-1)m} & \xrightarrow[\rho_{1*}]{\begin{pmatrix} (2p-1)m+1 & 0 \\ 0 & (2p-1)m \end{pmatrix}} & \mathbb{Z}\overline{s}_{(2p-1)m+1} \oplus \mathbb{Z}\overline{s}_{(2p-1)m} \\
(\alpha,\ \beta)\ \Big\uparrow\ h_{1*} & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ \Big\uparrow\ j_{1*} \\
\mathbb{Z}a'_p & \xrightarrow{\rho_{2*}} & \mathbb{Z}\overline{s}'_{(2p-1)m+1} \oplus \mathbb{Z}\overline{s}'_{(2p-1)m} \\
\cong\ \Big\downarrow\ h_{2*} & & j_{2*}\ \Big\downarrow\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\mathbb{Z}a_p & \xrightarrow{\rho_*} & \mathbb{Z}\overline{s}''_{(2p-1)m+1} \oplus \mathbb{Z}\overline{s}''_{(2p-1)m}
\end{array}$$

We conclude from this diagram

$$\rho_*(a_p) = p\tilde{a}_p = p\left(w\overline{s}''_{(2p-1)m+1} + z\overline{s}''_{(2p-1)m}\right)$$

$$= j_{2*}\rho_{2*}h_{2*}^{-1}(a_p) = j_{2*}\rho_{2*}(a'_p) = j_{2*}j_{1*}^{-1}\rho_{1*}h_{1*}(a'_p)$$
$$= j_{2*}j_{1*}^{-1}\rho_{1*}\left(\alpha\cdot s_{(2p-1)m+1} + \beta\cdot s_{(2p-1)m}\right)$$
$$= \alpha((2p-1)m+1)\cdot \overline{s}''_{(2p-1)m+1} + \beta(2p-1)m\cdot \overline{s}''_{(2p-1)m}\,.$$

Since $\overline{s}''_{(2p-1)m+1}, \overline{s}''_{(2p-1)m}$ form a basis we obtain:

$$pw = ((2p-1)m+1)\alpha\ ;\ \ pz = (2p-1)m\beta$$

which is equivalent to

$$p(2m\alpha - w) = (m-1)\alpha\ ;\ \ p(2m\beta - z) = m\beta. \tag{29}$$

Equation (29) implies that p is a common divisor of the numbers α and β since $p = p_m$ neither divides m nor $m-1$, cf. Lemma 3.3.

But the numbers α, β are by assumption coprime, hence we arrive at a contradiction. Note that this argument is also valid for the cases $\alpha = 0, \beta = \pm 1$, resp. $\alpha = \pm 1, \beta = 0$.

5 Katok metrics

Choose numbers $p_1 < \ldots < p_m$ which are relatively prime and let $p = p_1 \cdots p_m$.

Let

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

be the rotation in $\mathbb{R}^2$ with angle ϕ. Let $\mathbb{R}^{2m} = V_1 \oplus \ldots \oplus V_m$ be an orthogonal decomposition into 2-dimensional subspaces and let $A(t) \in S\mathbb{O}(2m)$ be the 1-parameter family of isometries of S^{2m-1} with $A(t)|V_j = R(pt/p_j)$, $j = 1, \ldots, m$. This one-parameter group of isometries generates a Killing field V on S^{2m-1} with norm

$$\|V\| = a(p_1, \ldots, p_m) = p\left(\sum_{j=1}^{m} p_j^{-2}\right)^{1/2}.$$

For $\mu < a(p_1, \ldots, p_m)^{-1}$ we define the Killing field $V_\mu = \mu V$ with $\|V_\mu\| < 1$. Then the sphere bundle determined by $\{\xi \in T_x S^n\ ;\ \|\xi - V_\mu(x)\| = 1\}$ determines the unit sphere bundle of a non-reversible Finsler metric f_μ. These metrics are called *Katok metrics,* cf. [23, p.139] or *Zermelo deformation* of the standard metric [7]. For the flow ψ_t of the Killing field V_μ the geodesics of the Finsler metric f_μ are of the form $t \longmapsto \psi_t(c(t))$ for a great circle $c(t)$ on S^n. For irrational μ the Katok metric f_μ determined by the Killing field V_μ has exactly $2m$ closed geodesics $c_j^{\pm}$, $j = 1, \ldots, m$ with $c_j^-(t) = c_j^+(-t)$ for all t, cf. [23, p.139] and [7]. These are the great circles invariant under the flow ψ_t. Since μ is irrational these metrics are bumpy. As remarked in [7, 17] these metrics have constant flag curvature 1. The lengths $l(c_j^{\pm})$, $j = 1, \ldots, m$ of the closed geodesics $c_j^{\pm}$ are given by

$$l\left(c_j^{\pm}\right) = \frac{2\pi}{1 \pm \frac{p}{p_j}\mu}\,,\ j = 1, \ldots, m\,.$$

The distortion is given by:

$$D = \lambda = \frac{1}{1 - \max\{\|V_\mu(x)\|, x \in S^n\}} = \frac{1}{1 - \mu a(p_1, \ldots, p_m)}\,.$$

If we choose $p_1 = 1$ then for an arbitrary $N \in \mathbb{N}$ one can choose $N < p_2 < \ldots < p_m$ and an irrational μ satisfying

$$\frac{1}{p}\frac{N-1}{N} < \mu < \frac{1}{a(1, p_2, \ldots, p_m)}.$$

This implies that $l(c_1^-) \geq 2\pi N$, $\text{ind}(c_1^{-1}) \geq 2N(m-1)$ and the distortion satisfies $D \geq N$. One can also show that $l(c_j^+) < 2\pi$, $\text{ind}(c_j^+) \leq 4(m-1)$, $1 \leq j \leq m$ and for $2 \leq j \leq m$ we obtain $l(c_j^{-1}) < 2\pi N/(N-1)$, $\text{ind}(c_j^{-1}) \leq 6(m-1)$.

For $n = 3, m = 2$ we obtain for any N a bumpy Katok metric f_μ with exactly four closed geodesics $c_1^\pm, c_2^\pm$ with the following (in)equalities for the indices resp. lengths of these closed geodesics: $l(c_1), l(c_2) < 2\pi$, $\text{ind}(c_1) = 2$, $\text{ind}(c_2) = 4$, $l(c_2^-) \leq 2\pi N/(N-1)$; $\text{ind}(c_2^{-1}) \in \{4, 6\}$, and $l(c_1^{-1}) \geq 2\pi N$, $\text{ind}(c_1^{-1}) \geq 2N(m-1)$.

In a certain sense one can say that these examples show that the minimal number of *short* closed geodesics on a sphere of dimension $n = 2m - 1$ resp. $n = 2m$ is $2m - 1$. Here *short* closed geodesics posess an a priori bound for the index.

6 Genericity statement

The set $\mathcal{F}(T)$ of Finsler metrics on a compact manifold M for which all closed geodesics of length $\leq T$ are non-degenerate is an open and dense subset of the space $\mathcal{F} = \mathcal{F}(M)$ of Finsler metrics on M with the strong C^r-topology for $r \geq 4$, cf.[21, Thm. 4].

Proof of Theorem 1.2 Let $f_1 \in \mathcal{F}_1(L)$, hence by definition all closed geodesics of the Finsler metric f_1 of length $\leq D^3(f_1)L$ are non-degenerate. Let $\phi_{f_1}^t : TM \longrightarrow TM$ be the *geodesic flow* of the Finsler metric f_1. If $\tau : TM \longrightarrow M$ is the tangent bundle projection then $t \longmapsto \tau(\phi_{f_1}^t(v))$ is the geodesic c_v determined by the initial condition $c_v'(0) = v$. Let $HM = (TM - M)/\mathbb{R}^+$ be the bundle of oriented directions in the tangent bundle TM. We consider instead of the geodesic flow $\phi_{f_1}^t : TM \longrightarrow TM$ the map $\Phi_{f_1}^t : HM \longrightarrow HM$ with $\Phi_{f_1}^t(v) = \phi_{f_1}^t(v/f_1(v))$, hence the geodesic $t \in \mathbb{R} \longmapsto \tau(\Phi_{f_1}^t(v)) \in M$ is parametrized by arc length. If $\Phi_{f_1}^t(v)$ is a periodic flow line of period a, then $t \in [0, a] \longmapsto \tau(\Phi_{f_1}^t(v))$ is a closed geodesic of length a. The *minimal period* is then the length of the underlying prime closed geodesic.

Let $\Phi_{f_1}^t(v_1), \ldots, \Phi_{f_1}^t(v_N)$ be the periodic flow lines of the geodesic flow $\Phi_{f_1}^t : HM \longrightarrow HM$ corresponding to the closed geodesics of period (resp. length) $a_1, \ldots, a_N$ which satisfy $a_i \leq D(f_1)^3 L$.

Then there is an open neighborhood $\mathcal{U} \subset \mathcal{F}$ of f_1 such that the following holds: There are continuous maps $v_i : f \in \mathcal{U} \mapsto v_i(f) \in HS^n$, $a_i : f \in \mathcal{U} \mapsto a_i(f) \in (0, \infty)$, $i = 1, 2, \ldots, N$ with $v_i = v_i(f_1)$, $a_i = a_i(f_1)$, $i = 1, \ldots, N$ such that for all $f \in \mathcal{U}$ the sets $\Phi_f^t(v_i(f))$, $t \in [0, a_i(f)]$, $i = 1, 2, \ldots, N$ are periodic and non-degenerate flow lines of the geodesic flow of f of period $a_1(f), \ldots, a_N(f)$ and there are no further periodic flow lines of f of length $\leq D^3(f)L$. This holds since the distortion

$$f \in \mathcal{F} \longmapsto D(f) \in (0, \infty) \tag{30}$$

is a continuous function. Hence the set $\mathcal{F}_1(L)$ is an open subset of $\mathcal{F}$.

Choose $T = 2D^3(f)L$. Since $\mathcal{F}(T)$ is a dense subset of $\mathcal{F}$ we find a sequence $(f_k)_{k\geq 2} \subset \mathcal{F}(T)$ converging to f_1. Since the function given in Eq. (30) is continuous it follows that also $\mathcal{F}_1(L)$ is dense in $\mathcal{F}$. □

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