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Calculus of Variations

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Upper and lower bounds for the Dunkl heat kernel

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Abstract
On $\mathbb{R}^N$ equipped with a normalized root system R, a multiplicity function $k(\alpha) > 0$, and the associated measure

$$dw(\mathbf{x}) = \prod_{\alpha\in R} |\langle \mathbf{x}, \alpha\rangle|^{k(\alpha)}\, d\mathbf{x},$$

let $h_t(\mathbf{x}, \mathbf{y})$ denote the heat kernel of the semigroup generated by the Dunkl Laplace operator Δ_k. Let $d(\mathbf{x}, \mathbf{y}) = \min_{g\in G} \|\mathbf{x} - g(\mathbf{y})\|$, where G is the reflection group associated with R. We derive the following upper and lower bounds for $h_t(\mathbf{x}, \mathbf{y})$: for all $c_l > 1/4$ and $0 < c_u < 1/4$ there are constants $C_l, C_u > 0$ such that

$$C_l w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_l \frac{d(\mathbf{x},\mathbf{y})^2}{t}} \Lambda(\mathbf{x}, \mathbf{y}, t) \le h_t(\mathbf{x}, \mathbf{y}) \le C_u w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_u \frac{d(\mathbf{x},\mathbf{y})^2}{t}} \Lambda(\mathbf{x}, \mathbf{y}, t),$$

where $\Lambda(\mathbf{x}, \mathbf{y}, t)$ can be expressed by means of some rational functions of $\|\mathbf{x} - g(\mathbf{y})\|/\sqrt{t}$. An exact formula for $\Lambda(\mathbf{x}, \mathbf{y}, t)$ is provided.

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1 Introduction and statement of the results

On the Euclidean space $\mathbb{R}^N$ equipped with a normalized root system R and a multiplicity function $k(\alpha) > 0$, let Δ_k denote the Dunkl Laplace operator (see Sect. 2). Let $dw(\mathbf{x}) = w(\mathbf{x})\, d\mathbf{x}$ be the associated measure, where

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$$w(\mathbf{x}) = \prod_{\alpha \in R} |\langle \mathbf{x}, \alpha \rangle|^{k(\alpha)} = \prod_{\alpha \in R_+} |\langle \mathbf{x}, \alpha \rangle|^{2k(\alpha)}, \tag{1.1}$$

where R_+ is a fixed positive subsystem of R. It is well-known that Δ_k generates a semigroup $\{e^{t\Delta_k}\}_{t \geq 0}$ of linear operators on $L^2(dw)$ which has the form

$$e^{t\Delta_k} f(\mathbf{x}) = \int_{\mathbb{R}^N} h_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, dw(\mathbf{y}),$$

where $0 < h_t(\mathbf{x}, \mathbf{y})$ is a smooth function called the Dunkl heat kernel.

The main goal of this paper is to prove upper and lower bounds for $h_t(\mathbf{x}, \mathbf{y})$. In order to state the result we need to introduce some notation.

For $\alpha \in R$, let

$$\sigma_\alpha(\mathbf{x}) = \mathbf{x} - 2\frac{\langle \mathbf{x}, \alpha \rangle}{\|\alpha\|^2}\alpha \tag{1.2}$$

stand for the reflection with respect to the subspace perpendicular to α. Let G denote the Coxeter (reflection) group generated by the reflections σ_α, $\alpha \in R_+$. We define the distance of the orbit of $\mathbf{x}$ to the orbit of $\mathbf{y}$ by

$$d(\mathbf{x}, \mathbf{y}) = \min\{\|\mathbf{x} - g(\mathbf{y})\| : g \in G\}.$$

Obviously,

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, g(\mathbf{y})) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^N \text{ and } g \in G.$$

It is well known that $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - g(\mathbf{y})\|$ if and only if $\mathbf{x}$ and $g(\mathbf{y})$ belong to the same (closed) Weyl chamber (see [6, Chapter VII, proof of Theorem 2.12]). Let

$$B(\mathbf{x}, r) = \{\mathbf{x}' \in \mathbb{R}^N : \|\mathbf{x} - \mathbf{x}'\| \leq r\}$$

stand for the (closed) Euclidean ball centered at $\mathbf{x}$ and radius r. We denote by $w(B(\mathbf{x}, r))$ the dw-volume of the ball $B(\mathbf{x}, r)$.

For a finite sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ of elements of R_+, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$, let

$$\ell(\boldsymbol{\alpha}) := m \tag{1.3}$$

be the length of $\boldsymbol{\alpha}$,

$$\sigma_{\boldsymbol{\alpha}} := \sigma_{\alpha_m} \circ \sigma_{\alpha_{m-1}} \circ \dots \circ \sigma_{\alpha_1}, \tag{1.4}$$

and

$$\begin{aligned} &\rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}, t) \\ &:= \left(1 + \frac{\|\mathbf{x} - \mathbf{y}\|}{\sqrt{t}}\right)^{-2} \left(1 + \frac{\|\mathbf{x} - \sigma_{\alpha_1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} \left(1 + \frac{\|\mathbf{x} - \sigma_{\alpha_2} \circ \sigma_{\alpha_1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} \cdot \ldots \cdot \\ &\quad \cdot \left(1 + \frac{\|\mathbf{x} - \sigma_{\alpha_{m-1}} \circ \dots \circ \sigma_{\alpha_1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2}. \end{aligned} \tag{1.5}$$

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, let $n(\mathbf{x}, \mathbf{y}) = 0$ if $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ and

$$n(\mathbf{x}, \mathbf{y}) = \min\{m \in \mathbb{Z} : d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \sigma_{\alpha_m} \circ \dots \circ \sigma_{\alpha_2} \circ \sigma_{\alpha_1}(\mathbf{y})\|, \quad \alpha_j \in R_+\} \tag{1.6}$$

otherwise. In other words, $n(\mathbf{x}, \mathbf{y})$ is the smallest number of reflections σ_α which are needed to move $\mathbf{y}$ to a (closed) Weyl chamber which contains $\mathbf{x}$ (see Sect. 2.3). We also allow $\boldsymbol{\alpha}$ to be the empty sequence, denoted by $\boldsymbol{\alpha} = \emptyset$. Then for $\boldsymbol{\alpha} = \emptyset$, we set: $\sigma_{\boldsymbol{\alpha}} = I$ (the identity operator), $\ell(\boldsymbol{\alpha}) = 0$, and $\rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}, t) = 1$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$.

We say that a finite sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of positive roots is *admissible for the pair* $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^N \times \mathbb{R}^N$ if $n(\mathbf{x}, \sigma_{\boldsymbol{\alpha}}(\mathbf{y})) = 0$. In other words, the composition

$$\sigma_{\boldsymbol{\alpha}} = \sigma_{\alpha_m} \circ \sigma_{\alpha_{m-1}} \circ \ldots \circ \sigma_{\alpha_1}$$

of the reflections σ_{α_j} maps $\mathbf{y}$ to a Weyl chamber containing also $\mathbf{x}$.

The set of the all admissible sequences $\boldsymbol{\alpha}$ for the pair $(\mathbf{x}, \mathbf{y})$ will be denoted by $\mathcal{A}(\mathbf{x}, \mathbf{y})$. Note that if $n(\mathbf{x}, \mathbf{y}) = 0$, then $\boldsymbol{\alpha} = \emptyset \in \mathcal{A}(\mathbf{x}, \mathbf{y})$.

Let us define

$$\Lambda(\mathbf{x}, \mathbf{y}, t) := \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \mathbf{y}),\ \ell(\boldsymbol{\alpha}) \leq 2|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}, t). \tag{1.7}$$

Note that for any $c > 1$ and for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we have

$$c^{-2|G|}\Lambda(\mathbf{x}, \mathbf{y}, ct) \leq \Lambda(\mathbf{x}, \mathbf{y}, t) \leq \Lambda(\mathbf{x}, \mathbf{y}, ct). \tag{1.8}$$

We are now in a position to state our main result about upper and lower bounds for the Dunkl heat kernel which are given by means of w-volumes of Euclidean balls, the function $\Lambda(\mathbf{x}, \mathbf{y}, t)$, and $d(\mathbf{x}, \mathbf{y})$. Recall that $k(\alpha) > 0$ in the whole paper.

Theorem 1.1 *Assume that* $0 < c_u < 1/4$ *and* $c_l > 1/4$. *Then there are constants* $C_u, C_l > 0$ *such that for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ *and* $t > 0$ *we have*

$$C_l w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_l \frac{d(\mathbf{x}, \mathbf{y})^2}{t}} \Lambda(\mathbf{x}, \mathbf{y}, t) \leq h_t(\mathbf{x}, \mathbf{y}), \tag{1.9}$$

$$h_t(\mathbf{x}, \mathbf{y}) \leq C_u w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_u \frac{d(\mathbf{x}, \mathbf{y})^2}{t}} \Lambda(\mathbf{x}, \mathbf{y}, t). \tag{1.10}$$

Let us remark that this way of expressing estimates of the heat kernel is convenient in handling real harmonic analysis problems, because it allows us to apply methods from analysis on spaces of homogeneous type in the sense of Coifman and Weiss.

The proof of the theorem is based on an iteration procedure. In order to illustrate the method we start by proving upper and lower bounds for $h_t(\mathbf{x}, \mathbf{y})$ in the case where the root system is associated with symmetries of a regular m-sided polygon in $\mathbb{R}^2$, e.g. when G is the dihedral group. In this case the formulation of the estimates and they proofs are much simpler.

Theorem 1.2 *Assume that G is the group of symmetries of a regular m-sided polygon in* $\mathbb{R}^2$ *centered at the origin and let R be the associated root system. Fix a positive subsystem R_+ of R and set*

$$\Lambda_D(\mathbf{x}, \mathbf{y}, t) := \begin{cases} 1 & \text{if } n(\mathbf{x}, \mathbf{y}) = 0, \\ \left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} & \text{if } n(\mathbf{x}, \mathbf{y}) = 1, \\ \left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} \sum_{\alpha \in R_+} \left(1 + \frac{\|\mathbf{x}-\sigma_\alpha(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} & \text{if } n(\mathbf{x}, \mathbf{y}) = 2. \end{cases}$$

Let $0 < c_u < 1/4$ *and* $c_l > 1/4$. *There are constants* $C_u, C_l > 0$ *such that for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ *and* $t > 0$ *we have*

$$C_l w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_l \frac{d(\mathbf{x}, \mathbf{y})^2}{t}} \Lambda_D(\mathbf{x}, \mathbf{y}, t) \leq h_t(\mathbf{x}, \mathbf{y}), \tag{1.11}$$

$$h_t(\mathbf{x},\mathbf{y}) \leq C_u w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c_u \frac{d(\mathbf{x},\mathbf{y})^2}{t}} \Lambda_D(\mathbf{x},\mathbf{y},t). \tag{1.12}$$

Theorems 1.1 and 1.2 can be consider as improvements of the following estimates

$$C^{-1} w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c^{-1}\|\mathbf{x}-\mathbf{y}\|^2/t} \leq h_t(\mathbf{x},\mathbf{y}) \leq C w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-cd(\mathbf{x},\mathbf{y})^2/t} \tag{1.13}$$

obtained in [2, Theorems 3.1 and 4.4] (see Sect. 3 for more details).

A natural problem one can post is to define a positive function $H(\mathbf{x},\mathbf{y},t)$ by means of volumes of balls, reflections, and distances, such that $C^{-1} \leq h_t(\mathbf{x},\mathbf{y})/H(\mathbf{x},\mathbf{y},t) \leq C$.

2 Preliminaries and notation

2.1 Basic definitions of Dunkl theory

In this section we present basic facts concerning the theory of the Dunkl operators. For more details we refer the reader to [3, 8, 10, 11].

We consider the Euclidean space $\mathbb{R}^N$ with the scalar product $\langle \mathbf{x},\mathbf{y}\rangle = \sum_{j=1}^N x_j y_j$, where $\mathbf{x} = (x_1,\dots,x_N)$, $\mathbf{y} = (y_1,\dots,y_N)$, and the norm $\|\mathbf{x}\|^2 = \langle \mathbf{x},\mathbf{x}\rangle$.

A *normalized root system* in $\mathbb{R}^N$ is a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $R \cap \alpha\mathbb{R} = \{\pm\alpha\}$, $\sigma_\alpha(R) = R$, and $\|\alpha\| = \sqrt{2}$ for all $\alpha \in R$, where σ_α is defined by (1.2). Each root system can be written as a disjoint union $R = R_+ \cup -R_+$, where R_+, $-R_+$ are separated by a hyperplane through the origin. Such a set R_+ is called a *positive subsystem*. Its choice is not unique. In this paper, we will work with a fixed positive subsystem R_+.

The finite group G generated by the reflections σ_α, $\alpha \in R_+$ is called the *Coxeter group* (*reflection group*) of the root system. Clearly, $|G| > |R_+|$.

A *multiplicity function* is a G-invariant function $k : R \to \mathbb{C}$ which will be fixed and > 0 throughout this paper.

Let $\mathbf{N} = N + \sum_{\alpha\in R_+} 2k(\alpha)$. Then,

$$w(B(t\mathbf{x},tr)) = t^{\mathbf{N}} w(B(\mathbf{x},r)) \quad \text{for all } \mathbf{x}\in\mathbb{R}^N,\ t,r>0,$$

where w is the associated measure defined in (1.1). Observe that there is a constant $C > 0$ such that for all $\mathbf{x} \in \mathbb{R}^N$ and $r > 0$ we have

$$C^{-1} w(B(\mathbf{x},r)) \leq r^N \prod_{\alpha\in R_+} (|\langle \mathbf{x},\alpha\rangle| + r)^{2k(\alpha)} \leq C w(B(\mathbf{x},r)), \tag{2.1}$$

so $dw(\mathbf{x})$ is doubling, that is, there is a constant $C > 0$ such that

$$w(B(\mathbf{x},2r)) \leq C w(B(\mathbf{x},r)) \quad \text{for all } \mathbf{x}\in\mathbb{R}^N,\ r>0. \tag{2.2}$$

For $\xi \in \mathbb{R}^N$, the *Dunkl operators* T_ξ are the following k-deformations of the directional derivatives ∂_ξ by difference operators:

$$T_\xi f(\mathbf{x}) = \partial_\xi f(\mathbf{x}) + \sum_{\alpha\in R_+} k(\alpha)\langle\alpha,\xi\rangle \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle\alpha,\mathbf{x}\rangle}.$$

The Dunkl operators T_ξ, which were introduced in [3], commute and are skew-symmetric with respect to the G-invariant measure dw.

Let us denote $T_j = T_{e_j}$, where $\{e_j\}_{1\leq j\leq N}$ is a canonical orthonormal basis of $\mathbb{R}^N$.

For fixed $\mathbf{y} \in \mathbb{R}^N$ the *Dunkl kernel* $E(\mathbf{x}, \mathbf{y})$ is a unique analytic solution to the system

$$T_\xi f = \langle \xi, \mathbf{y} \rangle f, \quad f(0) = 1.$$

The function $E(\mathbf{x}, \mathbf{y})$, which generalizes the exponential function $e^{\langle \mathbf{x}, \mathbf{y} \rangle}$, has a unique extension to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$. Moreover, $E(\mathbf{z}, \mathbf{w}) = E(\mathbf{w}, \mathbf{z})$ for all $\mathbf{z}, \mathbf{w} \in \mathbb{C}^N$.

2.2 Dunkl Laplacian and Dunkl heat semigroup

The *Dunkl Laplacian* associated with R and k is the differential-difference operator $\Delta_k = \sum_{j=1}^N T_j^2$, which acts on $C^2(\mathbb{R}^N)$-functions by

$$\Delta_k f(\mathbf{x}) = \Delta_{\mathrm{eucl}} f(\mathbf{x}) + \sum_{\alpha \in R_+} 2k(\alpha) \delta_\alpha f(\mathbf{x}),$$

$$\delta_\alpha f(\mathbf{x}) = \frac{\partial_\alpha f(\mathbf{x})}{\langle \alpha, \mathbf{x} \rangle} - \frac{\|\alpha\|^2}{2} \frac{f(\mathbf{x}) - f(\sigma_\alpha(\mathbf{x}))}{\langle \alpha, \mathbf{x} \rangle^2}.$$

The operator Δ_k is essentially self-adjoint on $L^2(dw)$ (see for instance [1, Theorem 3.1]) and generates a semigroup H_t of linear self-adjoint contractions on $L^2(dw)$. The semigroup has the form

$$H_t f(\mathbf{x}) = \int_{\mathbb{R}^N} h_t(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) \, dw(\mathbf{y}),$$

where the heat kernel

$$h_t(\mathbf{x}, \mathbf{y}) = \boldsymbol{c}_k^{-1} (2t)^{-\mathbf{N}/2} E\left(\frac{\mathbf{x}}{\sqrt{2t}}, \frac{\mathbf{y}}{\sqrt{2t}} \right) e^{-(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)/(4t)} \tag{2.3}$$

is a C^∞-function of all the variables $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$, $t > 0$, and satisfies

$$0 < h_t(\mathbf{x}, \mathbf{y}) = h_t(\mathbf{y}, \mathbf{x}).$$

Here and subsequently,

$$\boldsymbol{c}_k = \int_{\mathbb{R}^N} e^{-\|\mathbf{x}\|^2/2} \, dw(\mathbf{x}).$$

The following specific formula for the Dunkl heat kernel was obtained by Rösler [9]:

$$h_t(\mathbf{x}, \mathbf{y}) = \boldsymbol{c}_k^{-1} 2^{-\mathbf{N}/2} t^{-\mathbf{N}/2} \int_{\mathbb{R}^N} \exp(-A(\mathbf{x}, \mathbf{y}, \eta)^2/4t) \, d\mu_{\mathbf{x}}(\eta) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^N, t > 0. \tag{2.4}$$

Here

$$A(\mathbf{x}, \mathbf{y}, \eta) = \sqrt{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{y}, \eta \rangle} = \sqrt{\|\mathbf{x}\|^2 - \|\eta\|^2 + \|\mathbf{y} - \eta\|^2} \tag{2.5}$$

and $\mu_{\mathbf{x}}$ is a probability measure, which is supported in the convex hull conv $\mathcal{O}(\mathbf{x})$ of the orbit $\mathcal{O}(\mathbf{x}) = \{g(\mathbf{x}) : g \in G\}$.

One can easily check that

$$d(\mathbf{x}, \mathbf{y}) \le A(\mathbf{x}, \mathbf{y}, \eta) \quad \text{for all } \eta \in \operatorname{conv} \mathcal{O}(\mathbf{x}). \tag{2.6}$$

2.3 Weyl chambers and their properties

The closures of connected components of

$$\{\mathbf{x} \in \mathbb{R}^N \ : \ \langle \mathbf{x}, \alpha \rangle \neq 0 \text{ for all } \alpha \in R_+\}$$

are called (closed) *Weyl chambers*. Below we present some properties of the reflections and the Weyl chambers, which will be used in next sections.

Lemma 2.1 *Fix* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ *and* $g \in G$. *Then* $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - g(\mathbf{y})\|$ *if and only if* $g(\mathbf{y})$ *and* $\mathbf{x}$ *belong to the same Weyl chamber.*

Proof See [6, Chapter VII, proof of Theorem 2.12]. □

Lemma 2.2 *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ *and assume that* $n(\mathbf{x}, \mathbf{y}) \geq 1$. *Then there is* $\alpha \in R_+$ *such that*

$$\|\mathbf{x} - \mathbf{y}\| > \|\mathbf{x} - \sigma_\alpha(\mathbf{y})\|. \tag{2.7}$$

Proof If for a fixed $\alpha \in R_+$, $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \sigma_\alpha(\mathbf{y})\|$, then $\mathbf{x}$ and $\mathbf{y}$ are situated in the same half space with the boundary $\alpha^\perp$ [6, Chapter VII, proof of Theorem 2.12]. Now, suppose towards a contradiction that $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - \sigma_\alpha(\mathbf{y})\|$ for all $\alpha \in R_+$. Then $\mathbf{x}$ and $\mathbf{y}$ belong to the same Weyl chamber, hence $n(\mathbf{x}, \mathbf{y}) = 0$. This contradicts our assumption. □

Corollary 2.3 *For any* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ *such that* $n(\mathbf{x}, \mathbf{y}) > 0$ *there are:* $1 \leq m \leq |G|$ *and* $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m)$ *such that*

$$\|\mathbf{x}-\mathbf{y}\| > \|\mathbf{x}-\sigma_{\alpha_1}(\mathbf{y})\| > \|\mathbf{x}-\sigma_{\alpha_2} \circ \sigma_{\alpha_1}(\mathbf{y})\| > \dots > \|\mathbf{x}-\sigma_{\alpha_m} \circ \sigma_{\alpha_{m-1}} \circ \dots \circ \sigma_{\alpha_1}(\mathbf{y})\| = d(\mathbf{x}, \mathbf{y}) \tag{2.8}$$

and

$$n(\mathbf{x}, \sigma_{\boldsymbol{\alpha}}(\mathbf{y})) = 0. \tag{2.9}$$

3 Auxiliary estimates for the heat kernel

In the present section we establish auxiliary estimates for the heat kernel which will be used for proving Theorems 1.1 and 1.2. Our starting point is the following proposition which is an improvement of the estimates (1.13).

Proposition 3.1 *For any constants* $\widetilde{c}_\ell > 1/4$ *and* $0 < \widetilde{c}_u < 1/4$ *there are positive constants* $\widetilde{C}_\ell, \widetilde{C}_u$, *such that for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ *and* $t > 0$ *we have*

$$\widetilde{C}_\ell w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-\widetilde{c}_\ell \|\mathbf{x}-\mathbf{y}\|^2/t} \leq h_t(\mathbf{x}, \mathbf{y}) \leq \widetilde{C}_u w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-\widetilde{c}_u d(\mathbf{x}, \mathbf{y})^2/t}. \tag{3.1}$$

Proof To obtain the upper bound with any constant $0 < \widetilde{c}_u < 1/4$ arbitrarily close to $1/4$, we apply (2.4) together with (2.6) and get

$$\begin{aligned} h_t(\mathbf{x}, \mathbf{y}) &\leq \boldsymbol{c}_k^{-1} (2t)^{-\mathbf{N}/2} \int_{\mathbb{R}^N} \exp\Big(- \frac{(1-4\widetilde{c}_u) A(\mathbf{x}, \mathbf{y}, \eta)^2}{4t} \Big) \exp\big(- \widetilde{c}_u d(\mathbf{x}, \mathbf{y})^2/t \big) \, d\mu_{\mathbf{x}}(\eta) \\ &= (1-4\widetilde{c}_u)^{-\mathbf{N}/2} h_{t/(1-4\widetilde{c}_u)}(\mathbf{x}, \mathbf{y}) \exp\big(- \widetilde{c}_u d(\mathbf{x}, \mathbf{y})^2/t \big) \\ &\leq \widetilde{C}_u w(B(\mathbf{x}, \sqrt{t}))^{-1} \exp\big(- \widetilde{c}_u d(\mathbf{x}, \mathbf{y})^2/t \big), \end{aligned}$$

where in the last inequality we have used the second inequality in (1.13) and the doubling property (2.2).

The lower bound in (3.1) with any constant $\widetilde{c}_\ell > 1/4$ is Corollary 2.3 of Jiu and Li [7]. □

We now turn to deriving estimates for the heat kernel which will be used for an iteration procedure.

Proposition 3.2 *Let $\widetilde{c}_\ell, \widetilde{c}_u$ be the constants from Proposition 3.1 and let $c_1 < \widetilde{c}_u$. There is $C_1 \geq 1$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we have*

$$C_1^{-1}\left(w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-\widetilde{c}_\ell \frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}} + \left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} \sum_{\alpha \in R_+} h_t(\mathbf{x}, \sigma_\alpha(\mathbf{y}))\right) \leq h_t(\mathbf{x}, \mathbf{y}), \tag{3.2}$$

$$h_t(\mathbf{x}, \mathbf{y}) \leq C_1\left(w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_1 \frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}} + \left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} \sum_{\alpha \in R_+} h_t(\mathbf{x}, \sigma_\alpha(\mathbf{y}))\right). \tag{3.3}$$

Proof The following formula was proved in [4, formula (3.5)]: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$, we have

$$\partial_t h_t(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{x}-\mathbf{y}\|^2}{(2t)^2} h_t(\mathbf{x}, \mathbf{y}) - \frac{N}{2t} h_t(\mathbf{x}, \mathbf{y}) - \frac{1}{t} \sum_{\alpha \in R_+} k(\alpha) h_t(\mathbf{x}, \sigma_\alpha(\mathbf{y})). \tag{3.4}$$

On the other hand, by (2.4),

$$\begin{aligned} \partial_t h_t(\mathbf{x}, \mathbf{y}) &= -\frac{\mathbf{N}}{2} t^{-1} h_t(\mathbf{x}, \mathbf{y}) + \boldsymbol{c}_k^{-1} 2^{-\mathbf{N}/2} t^{-1} t^{-\mathbf{N}/2} \int_{\mathbb{R}^N} \frac{A(\mathbf{x}, \mathbf{y}, \eta)^2}{4t} e^{-A(\mathbf{x}, \mathbf{y}, \eta)^2/4t} \, d\mu_{\mathbf{x}}(\eta) \\ &=: I_1(t, \mathbf{x}, \mathbf{y}) + I_2(t, \mathbf{x}, \mathbf{y}). \end{aligned} \tag{3.5}$$

Combining (3.4) with (3.5) we get

$$\left(2\mathbf{N} - 2N + \frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}\right) h_t(\mathbf{x}, \mathbf{y}) = 4t I_2(t, \mathbf{x}, \mathbf{y}) + 4 \sum_{\alpha \in R_+} k(\alpha) h_t(\mathbf{x}, \sigma_\alpha(\mathbf{y})). \tag{3.6}$$

Note that $I_2(t, \mathbf{x}, \mathbf{y}) \geq 0$, and, thanks to our assumption on $k(\alpha) > 0$, we have $\mathbf{N} > N$. So, by (3.6),

$$h_t(\mathbf{x}, \mathbf{y}) \geq 4(2\mathbf{N} - 2N + 1)^{-1} \left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}\right)^{-1} \sum_{\alpha \in R_+} k(\alpha) h_t(\mathbf{x}, \sigma_\alpha(\mathbf{y})). \tag{3.7}$$

Now, taking the arithmetic mean of the lower bound in (3.1) with (3.7) we obtain (3.2), since there is a constant $C > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we have

$$C^{-1}\left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}\right)^{-1} \leq \left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} \leq C\left(1 + \frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}\right)^{-1}.$$

In order to prove (3.3), set $\varepsilon = (\widetilde{c}_u - c_1)/(2\widetilde{c}_u)$. Clearly, by the assumption $c_1 < \widetilde{c}_u$, we have $0 < \varepsilon < \frac{1}{2}$. To obtain (3.3), we split the integral for $t I_2(t, \mathbf{x}, \mathbf{y})$ as follows:

$$\begin{aligned} t I_2(t, \mathbf{x}, \mathbf{y}) &= \boldsymbol{c}_k^{-1} 2^{-\mathbf{N}/2} t^{-\mathbf{N}/2} \int_{A(\mathbf{x}, \mathbf{y}, \eta)^2 \leq (1-\varepsilon)\|\mathbf{x}-\mathbf{y}\|^2} \cdots \\ &\quad + \boldsymbol{c}_k^{-1} 2^{-\mathbf{N}/2} t^{-\mathbf{N}/2} \int_{A(\mathbf{x}, \mathbf{y}, \eta)^2 > (1-\varepsilon)\|\mathbf{x}-\mathbf{y}\|^2} \cdots \\ &=: t I_{2,1}(t, \mathbf{x}, \mathbf{y}) + t I_{2,2}(t, \mathbf{x}, \mathbf{y}). \end{aligned} \tag{3.8}$$

Clearly,

$$
\begin{aligned}
tI_{2,1}(t,\mathbf{x},\mathbf{y}) &\le \boldsymbol{c}_k^{-1}2^{-\mathbf{N}/2}(1-\varepsilon)t^{-\mathbf{N}/2}\int_{\mathbb{R}^N}\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}e^{-A(\mathbf{x},\mathbf{y},\eta)^2/4t}\,d\mu_{\mathbf{x}}(\eta)\\
&=(1-\varepsilon)\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}h_t(\mathbf{x},\mathbf{y}),
\end{aligned}
\tag{3.9}
$$

where for the equality we have applied (2.4). In order to estimate $I_{2,2}(t,\mathbf{x},\mathbf{y})$, note that there are $C_\varepsilon, C'_\varepsilon, C''_\varepsilon > 0$ such that

$$
\begin{aligned}
tI_{2,2}(t,\mathbf{x},\mathbf{y}) &\le C_\varepsilon t^{-\mathbf{N}/2}\int_{A(\mathbf{x},\mathbf{y},\eta)^2>(1-\varepsilon)\|\mathbf{x}-\mathbf{y}\|^2} e^{-(1-2\varepsilon)A(\mathbf{x},\mathbf{y},\eta)^2/(4t(1-\varepsilon))}e^{-\varepsilon A(\mathbf{x},\mathbf{y},\eta)^2/(8t(1-\varepsilon))}\,d\mu_{\mathbf{x}}(\eta)\\
&\le C_\varepsilon e^{-(1-2\varepsilon)\|\mathbf{x}-\mathbf{y}\|^2/4t}t^{-\mathbf{N}/2}\int_{\mathbb{R}^N} e^{-\varepsilon A(\mathbf{x},\mathbf{y},\eta)^2/(8t(1-\varepsilon))}\,d\mu_{\mathbf{x}}(\eta)\\
&= C'_\varepsilon e^{-(1-2\varepsilon)\|\mathbf{x}-\mathbf{y}\|^2/4t}h_{2(1-\varepsilon)t/\varepsilon}(\mathbf{x},\mathbf{y}) \le C''_\varepsilon w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-(1-2\varepsilon)\|\mathbf{x}-\mathbf{y}\|^2/4t}.
\end{aligned}
\tag{3.10}
$$

In the last inequality we have used Proposition 3.1. Combining (3.6), (3.9), and (3.10) we obtain

$$
\begin{aligned}
\left(2\mathbf{N}-2N+\frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}\right)h_t(\mathbf{x},\mathbf{y}) \le{}& (1-\varepsilon)\frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}h_t(\mathbf{x},\mathbf{y})\\
&+4C''_\varepsilon w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-(1-2\varepsilon)\|\mathbf{x}-\mathbf{y}\|^2/4t}\\
&+4\sum_{\alpha\in R_+}k(\alpha)h_t(\mathbf{x},\sigma_\alpha(\mathbf{y})),
\end{aligned}
\tag{3.11}
$$

which finally leads to (3.3), because, by our assumption, $\mathbf{N} > N$. □

Observe that our basic upper and lower bounds [see (3.2) and (3.3)] are of the same type and they differ by the constants in the exponent of the first component.

From now on the constants C_1, c_1 from Proposition 3.2 are fixed.

Remark 3.3 The estimate (3.3) together with (1.13) imply the known bounds

$$
h_t(\mathbf{x},\mathbf{y}) \le Cw(B(\mathbf{x},\sqrt{t}))^{-1}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}e^{-cd(\mathbf{x},\mathbf{y})^2/t}
\tag{3.12}
$$

see [4, Theorem 3.1]. An alternative proof of (3.12) which uses a Poincaré inequality was announced by W. Hebisch.

4 The case of the dihedral group: proof of Theorem 1.2

Let D_m be a regular m-polygon in $\mathbb{R}^2$, $m \ge 3$, such that the related root system R consists of $2m$ vectors

$$
\alpha_j = \sqrt{2}\left(\sin\left(\frac{\pi j}{m}\right), \cos\left(\frac{\pi j}{m}\right)\right), \quad j \in \{0, 1, \dots, 2m-1\},
$$

and the reflection group G acts either by the symmetries σ_{α_j}, or by the rotations $\sigma_{\alpha_j} \circ \sigma_{\alpha_i}$, $0 \le i, j \le 2m-1$. Consequently, $\max_{\mathbf{x},\mathbf{y}\in\mathbb{R}^2} n(\mathbf{x},\mathbf{y}) = 2$.

Proof of Theorem 1.2 Fix $0 < c_u < c_1$, where c_1 is a constant from Proposition 3.2. Let us consider three cases depending on the value of $n(\mathbf{x},\mathbf{y})$.

Case $n(\mathbf{x},\mathbf{y})=0$. By the definition of $n(\mathbf{x},\mathbf{y})$ [see (1.6)], in this case $\|\mathbf{x}-\mathbf{y}\|=d(\mathbf{x},\mathbf{y})$. Hence Proposition 3.1 reads

$$\widetilde{C}_\ell w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-\widetilde{c}_\ell d(\mathbf{x},\mathbf{y})^2/t}\leq h_t(\mathbf{x},\mathbf{y})\leq \widetilde{C}_u w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-\widetilde{c}_u d(\mathbf{x},\mathbf{y})^2/t}, \tag{4.1}$$

which are the desired estimates, since $\Lambda_D(\mathbf{x},\mathbf{y},t)=1$ in this case.

Case $n(\mathbf{x},\mathbf{y})=1$. Then, by the definition of $n(\mathbf{x},\mathbf{y})$ [see (1.6)], there is $\alpha_0\in R_+$ such that $n(\mathbf{x},\sigma_{\alpha_0}(\mathbf{y}))=0$, that is, $\|\mathbf{x}-\sigma_{\alpha_0}(\mathbf{y})\|=d(\mathbf{x},\mathbf{y})$. Using (3.2), we get

$$\begin{aligned} h_t(\mathbf{x},\mathbf{y}) &\geq C_1^{-1}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}\sum_{\alpha\in R_+}h_t(\mathbf{x},\sigma_\alpha(\mathbf{y}))\\ &\geq C_1^{-1}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}h_t(\mathbf{x},\sigma_{\alpha_0}(\mathbf{y}))\\ &\geq C_1^{-1}\widetilde{C}_\ell w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-\widetilde{c}_\ell d(\mathbf{x},\mathbf{y})^2/t}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}, \end{aligned} \tag{4.2}$$

where in the last inequality we have used (4.1).

In order to prove the upper bound, we use (3.3), Proposition 3.1 together with the inequality $d(\mathbf{x},\mathbf{y})\leq\|\mathbf{x}-\mathbf{y}\|$ and obtain

$$h_t(\mathbf{x},\mathbf{y})\leq C_u w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-c_u d(\mathbf{x},\mathbf{y})^2/t}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}. \tag{4.3}$$

Case of $n(\mathbf{x},\mathbf{y})=2$. In the proof of the upper and lower bounds we use the fact that, in this case, $n(\mathbf{x},\sigma_\alpha(\mathbf{y}))=1$ for all $\alpha\in R_+$.

We start by proving the lower bound. Using (3.2) we have

$$\begin{aligned} h_t(\mathbf{x},\mathbf{y}) &\geq C_1^{-1}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}\sum_{\alpha\in R_+}h_t(\mathbf{x},\sigma_\alpha(\mathbf{y})))\\ &\geq C_1^{-2}\widetilde{C}_\ell w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-\widetilde{c}_\ell\frac{d(\mathbf{x},\mathbf{y})^2}{t}}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}\sum_{\alpha\in R_+}\left(1+\frac{\|\mathbf{x}-\sigma_\alpha(\mathbf{y})\|}{\sqrt{t}}\right)^{-2}, \end{aligned} \tag{4.4}$$

where in the last inequality we have used (4.2), since $n(\mathbf{x},\boldsymbol{\sigma_\alpha}(\mathbf{y}))=1$ for all $\alpha\in R_+$.

In order to obtain the upper bound, we apply (3.3) and then (4.3), and get

$$\begin{aligned} h_t(\mathbf{x},\mathbf{y}) &\leq C_1\left(w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-c_1\frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}}+\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}\sum_{\alpha\in R_+}h_t(\mathbf{x},\sigma_\alpha(\mathbf{y}))\right)\\ &\leq C_1 w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-c_1\frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}}\\ &\quad+C_1C_u w(B(\mathbf{x},\sqrt{t}))^{-1}e^{-c_1 d(\mathbf{x},\mathbf{y})^2/t}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2}\sum_{\alpha\in R_+}\left(1+\frac{\|\mathbf{x}-\sigma_\alpha(\mathbf{y})\|}{\sqrt{t}}\right)^{-2}. \end{aligned} \tag{4.5}$$

Let now $\alpha_0\in R_+$ be such that $\|\mathbf{x}-\sigma_{\alpha_0}(\mathbf{y})\|=\min_{\alpha\in R_+}\|\mathbf{x}-\sigma_\alpha(\mathbf{y})\|$. Then

$$d(\mathbf{x},\mathbf{y})\leq\|\mathbf{x}-\sigma_{\alpha_0}(\mathbf{y})\|\leq\|\mathbf{x}-\mathbf{y}\|$$

(see Lemma 2.2). Thus, from (4.5) we conclude that

$$h_t(\mathbf{x},\mathbf{y}) \leq C'_u w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c_u d(\mathbf{x},\mathbf{y})^2/t} \left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} \left(1+\frac{\|\mathbf{x}-\sigma_{\alpha_0}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2}, \tag{4.6}$$

which implies the desired estimate (1.12).

□

5 Proof of Theorem 1.1

5.1 Proof of the lower bound (1.9)

The proposition below combined with Corollary 2.3 imply (1.9).

Proposition 5.1 *Assume that $\widetilde{C}_\ell, \widetilde{c}_\ell$ are the constants from Proposition 3.1 and C_1 is the constant from Proposition 3.2. For all $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$, $t>0$, and $\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x},\mathbf{y})$ we have*

$$\begin{aligned} h_t(\mathbf{x},\mathbf{y}) &\geq C_1^{-\ell(\boldsymbol{\alpha})} \rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t) h_t(\mathbf{x},\sigma_{\boldsymbol{\alpha}}(\mathbf{y})) \\ &\geq C_1^{-\ell(\boldsymbol{\alpha})} \widetilde{C}_\ell w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-\widetilde{c}_\ell \frac{d(\mathbf{x},\mathbf{y})^2}{t}} \rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t) \end{aligned} \tag{5.1}$$

Proof The proof is by induction with respect to $m=\ell(\boldsymbol{\alpha})$. For $m=0$ and $m=1$ the claim is a consequence of Proposition 3.1 and (3.2) [see also (4.2)]. Assume that (5.1) holds for all $\mathbf{x}_1,\mathbf{y}_1 \in \mathbb{R}^N$, $t_1>0$, and $\widetilde{\boldsymbol{\alpha}} \in \mathcal{A}(\mathbf{x}_1,\mathbf{y}_1)$ such that $\ell(\widetilde{\boldsymbol{\alpha}})=m$. Let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_{m+1}) \in \mathcal{A}(\mathbf{x},\mathbf{y})$ be such that $\ell(\boldsymbol{\alpha})=m+1$. By (3.2) we have

$$h_t(\mathbf{x},\mathbf{y}) \geq C_1^{-1}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} h_t(\mathbf{x},\sigma_{\alpha_1}(\mathbf{y})).$$

Note that $\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x},\mathbf{y})$ implies that the sequence $\widetilde{\boldsymbol{\alpha}}=(\alpha_2,\dots,\alpha_{m+1})$ belongs to $\mathcal{A}(\mathbf{x},\sigma_{\alpha_1}(\mathbf{y}))$ and, obviously, $\ell(\widetilde{\boldsymbol{\alpha}})=m$. Therefore, the claim is a consequence of the induction hypothesis applied to $\mathbf{x}$, $\sigma_{\alpha_1}(\mathbf{y})$, and $\widetilde{\boldsymbol{\alpha}}$, and the fact that, by the definition of $\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t)$ and $\rho_{\widetilde{\boldsymbol{\alpha}}}(\mathbf{x},\sigma_{\alpha_1}(\mathbf{y}),t)$ (see (1.5)), we have

$$\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t)=\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} \rho_{\widetilde{\boldsymbol{\alpha}}}(\mathbf{x},\sigma_{\alpha_1}(\mathbf{y}),t).$$

□

5.2 Proof of the upper bound (1.10)

Let us begin with a corollary which follows by Proposition 5.1.

Corollary 5.2 *Assume that $\widetilde{c}_\ell$ is the constant from Proposition 3.1. Then there is a constant $C_2>0$ such that for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$ and $t>0$ we have*

$$C_2^{-1} w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-\widetilde{c}_\ell \frac{d(\mathbf{x},\mathbf{y})^2}{t}} \left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2|G|} \leq h_t(\mathbf{x},\mathbf{y}). \tag{5.2}$$

Proof If $n(\mathbf{x},\mathbf{y})=0$, then (5.2) holds by Proposition 3.1, because $d(\mathbf{x},\mathbf{y})=\|\mathbf{x}-\mathbf{y}\|$ in this case. For fixed $\mathbf{x},\mathbf{y} \in \mathbb{R}^N$ such that $n(\mathbf{x},\mathbf{y}) \geq 1$, let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$, $m \leq |G|$, be as in Corollary 2.3. Then, thanks to (2.8), we have

$$\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t) \geq \left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2|G|},$$

so the claim follows by Proposition 5.1. □

From now on the constant C_2 from Corollary 5.2 is fixed.

Proposition 5.3 *Let $C_1 \geq 1$ and $0 < c_1 < \widetilde{c}_u$ be the constants from Proposition 3.2. There is a constant $c_2 > 4C_1|G|$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$ satisfying*

$$\|\mathbf{x}-\mathbf{y}\| > c_2 d(\mathbf{x},\mathbf{y}) \text{ and } \|\mathbf{x}-\mathbf{y}\| > c_2\sqrt{t} \tag{5.3}$$

we have

$$h_t(\mathbf{x},\mathbf{y}) \leq 2C_1 \left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2} \sum_{\alpha\in R_+} h_t(\mathbf{x},\sigma_\alpha(\mathbf{y})). \tag{5.4}$$

Remark 5.4 The condition"$c_2 > 4C_1|G|$"occurs in the formulation of the proposition for some technical reasons and it will be used later on in the proof of (1.10).

Proof Thanks to (3.3) and the fact that $0 < h_t(\mathbf{x},\mathbf{y}) < \infty$ it is enough to show that

$$w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c_1\frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}} \leq \frac{1}{2C_1} h_t(\mathbf{x},\mathbf{y}). \tag{5.5}$$

To this end, by Corollary 5.2, we get

$$h_t(\mathbf{x},\mathbf{y}) \geq C_2^{-1} w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-\widetilde{c}_\ell\frac{d(\mathbf{x},\mathbf{y})^2}{t}} \left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2|G|},$$

so (5.5) is a consequence of the fact that taking $c_2 > 0$ large enough in (5.3), we have

$$C_2^{-1}\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2|G|} e^{-\widetilde{c}_\ell\frac{d(\mathbf{x},\mathbf{y})^2}{t}} \geq 2C_1 e^{-c_1\frac{\|\mathbf{x}-\mathbf{y}\|^2}{t}}.$$

□

From now on the constant c_2 from Proposition 5.3 is fixed.

Proposition 5.5 *Assume that $\widetilde{c}_u$ is the constant from Proposition 3.1 and c_2 is the same as in Proposition 5.3. Let $0 < c_3 < \widetilde{c}_u$. Then there is a constant $C_3 > 0$ such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $t > 0$ such that*

$$\|\mathbf{x}-\mathbf{y}\| \leq c_2\sqrt{t} \text{ or } \|\mathbf{x}-\mathbf{y}\| \leq c_2 d(\mathbf{x},\mathbf{y})$$

there is $\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x},\mathbf{y})$, $\ell(\boldsymbol{\alpha}) \leq |G|$, such that

$$h_t(\mathbf{x},\mathbf{y}) \leq C_3 w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c_3\frac{d(\mathbf{x},\mathbf{y})^2}{t}} \rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t). \tag{5.6}$$

Proof If $n(\mathbf{x},\mathbf{y}) = 0$, then one can take $\boldsymbol{\alpha} = \emptyset$ and the claim is a consequence of Proposition 3.1. Assume that $n(\mathbf{x},\mathbf{y}) > 0$. For fixed $\mathbf{x}, \mathbf{y}$, let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $m \leq |G|$, be as in Corollary 2.3. If $\|\mathbf{x}-\mathbf{y}\| \leq c_2\sqrt{t}$, then the claim is satisfied by Proposition 3.1 and (2.8), and we may take even $c_3 = \widetilde{c}_u$ in the inequality (5.6). If $\|\mathbf{x}-\mathbf{y}\| \leq c_2 d(\mathbf{x},\mathbf{y})$, then by Proposition 3.1 we get

$$h_t(\mathbf{x},\mathbf{y}) \leq \widetilde{C}_u w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-\widetilde{c}_u\frac{d(\mathbf{x},\mathbf{y})^2}{t}} = \widetilde{C}_u w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c_3\frac{d(\mathbf{x},\mathbf{y})^2}{t}} e^{-(\widetilde{c}_u-c_3)\frac{d(\mathbf{x},\mathbf{y})^2}{t}}.$$

Moreover, the assumption $\|\mathbf{x}-\mathbf{y}\| \leq c_2 d(\mathbf{x},\mathbf{y})$ implies

$$e^{-(\widetilde{c}_u-c_3)\frac{d(\mathbf{x},\mathbf{y})^2}{t}} \leq e^{-(\widetilde{c}_u-c_3)\frac{\|\mathbf{x}-\mathbf{y}\|^2}{c_2^2 t}},$$

so the claim follows by the fact that there is $C>0$ such that

$$e^{-(\widetilde{c}_u-c_3)\frac{\|\mathbf{x}-\mathbf{y}\|^2}{c_2^2 t}} \leq C\left(1+\frac{\|\mathbf{x}-\mathbf{y}\|}{\sqrt{t}}\right)^{-2|G|} \leq C\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t),$$

where the second inequality is a consequence of (2.8). □

From now on the constants C_3, c_3 from Proposition 5.5 are fixed.

Proof of (1.10) Let c_2 be the constant from Proposition 5.3. Fix $\mathbf{x},\mathbf{y}\in\mathbb{R}^N$ and $t>0$ and consider

$$G_0 := \{g \in G \ : \ \text{the assumption (5.3) is not satisfied for } \mathbf{x}, g(\mathbf{y}), t\}.$$

Note that $G_0\neq\emptyset$, because there is $g_0\in G$ such that $\|\mathbf{x}-g_0(\mathbf{y})\|=d(\mathbf{x},\mathbf{y})=d(\mathbf{x},g_0(\mathbf{y}))$, so the assumption (5.3) is not satisfied for $\mathbf{x}$, $g_0(\mathbf{y})$, t. We will prove (1.10) for $h_t(\mathbf{x},g(\mathbf{y}))$ for all $g\in G$. Note that, by the definition of G_0, if $g\in G_0$, then by Proposition 5.5 we have

$$h_t(\mathbf{x},g(\mathbf{y})) \leq C_3 w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c_3\frac{d(\mathbf{x},\mathbf{y})^2}{t}} \sum_{\boldsymbol{\alpha}\in\mathcal{A}(\mathbf{x},g(\mathbf{y}))\,:\,\ell(\boldsymbol{\alpha})\leq|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x},g(\mathbf{y}),t). \tag{5.7}$$

If $G=G_0$, the proof is complete. Assume that $G_0\neq G$. Consider all the values $h_t(\mathbf{x},g(\mathbf{y}))$ for all $g\notin G_0$ and list them in a decreasing sequence, that is, $G\setminus G_0=\{g_1,g_2,\dots,g_m\}$ and

$$h_t(\mathbf{x},g_1(\mathbf{y}))\geq h_t(\mathbf{x},g_2(\mathbf{y}))\geq\cdots\geq h_t(\mathbf{x},g_m(\mathbf{y})). \tag{5.8}$$

For $1\leq j\leq m$ let us denote

$$G_j := G_0\cup\{g_1,\dots,g_j\}. \tag{5.9}$$

We will prove by induction on j that for all $1\leq j\leq m$ we have

$$h_t(\mathbf{x},g_j(\mathbf{y})) \leq C_3(2C_1|G|)^j w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c_3\frac{d(\mathbf{x},\mathbf{y})^2}{t}} \sum_{\boldsymbol{\alpha}\in\mathcal{A}(\mathbf{x},g_j(\mathbf{y}))\,:\,\ell(\boldsymbol{\alpha})\leq|G|+j} \rho_{\boldsymbol{\alpha}}(\mathbf{x},g_j(\mathbf{y}),t), \tag{5.10}$$

where C_3, c_3 are the constants from Proposition 5.5 and C_1 is the constant from Proposition 3.2. We have already remarked that (5.10) is satisfied for $g\in G_0$ with $j=0$, [see (5.7)].

Fix $0\leq j\leq m-1$ and suppose that the estimate (5.10) holds for all $g\in G_j$. We will prove (5.10) for g_{j+1}. Since $g_{j+1}\notin G_0$,

$$\|\mathbf{x}-g_{j+1}(\mathbf{y})\|>c_2 d(\mathbf{x},g_{j+1}(\mathbf{y})) \quad\text{and}\quad \|\mathbf{x}-g_{j+1}(\mathbf{y})\|>c_2\sqrt{t} \tag{5.11}$$

[cf. (5.3)]. Hence, by Proposition 5.3, we get

$$h_t(\mathbf{x},g_{j+1}(\mathbf{y})) \leq 2C_1\left(1+\frac{\|\mathbf{x}-g_{j+1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} \sum_{\alpha\in R_+} h_t(\mathbf{x},\sigma_\alpha\circ g_{j+1}(\mathbf{y})). \tag{5.12}$$

Further, from (5.11) and the fact that $c_2>4C_1|G|$, we conclude that

$$2C_1\left(1+\frac{\|\mathbf{x}-g_{j+1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} < \frac{1}{8C_1|G|^2} < \frac{1}{2|G|}. \tag{5.13}$$

Let $\alpha_0\in R_+$ be such that

$$h_t(\mathbf{x},\sigma_{\alpha_0}\circ g_{j+1}(\mathbf{y})) = \max_{\alpha\in R_+} h_t(\mathbf{x},\sigma_\alpha\circ g_{j+1}(\mathbf{y})). \tag{5.14}$$

It follows from (5.12) and (5.14) that

$$h_t(\mathbf{x}, g_{j+1}(\mathbf{y})) \leq 2C_1 \left(1 + \frac{\|\mathbf{x} - g_{j+1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} |R_+| h_t(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y})). \tag{5.15}$$

We claim that $\sigma_{\alpha_0} \circ g_{j+1} \in G_j$. To prove the claim, aiming for a contradiction, suppose that $\sigma_{\alpha_0} \circ g_{j+1} \in G \setminus G_j$. Then $\sigma_{\alpha_0} \circ g_{j+1} \in \{g_{j+2}, \ldots, g_m\}$, because $\sigma_{\alpha_0} \circ g_{j+1} \neq g_{j+1}$. Consequently, by (5.8), $h_t(\mathbf{x}, g_{j+1}(\mathbf{y})) \geq h_t(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y}))$. Hence, using (5.15), we obtain

$$h_t(\mathbf{x}, g_{j+1}(\mathbf{y})) \leq 2C_1 \left(1 + \frac{\|\mathbf{x} - g_{j+1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} |R_+| h_t(\mathbf{x}, g_{j+1}(\mathbf{y})). \tag{5.16}$$

Since $h_t > 0$, applying (5.16) together with (5.13), we get

$$1 \leq 2C_1 \left(1 + \frac{\|\mathbf{x} - g_{j+1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} |R_+| \leq \frac{|R_+|}{2|G|} \leq 1/2$$

and we arrive at a contradiction. Thus the claim is established.

Thanks to the claim and by the induction hypothesis, the estimate (5.10) already holds for $\sigma_{\alpha_0} \circ g_{j+1}$, in particular, since $2C_1|G| > 1$, we have,

$$\begin{aligned} &h_t(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y})) \\ &\leq C_3 (2C_1|G|)^j w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_3 \frac{d(\mathbf{x},\mathbf{y})^2}{t}} \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y}))\,:\,\ell(\boldsymbol{\alpha}) \leq |G|+j} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y}), t) \end{aligned} \tag{5.17}$$

Hence, utilizing (5.15) combined with (5.17), we obtain

$$\begin{aligned} h_t(\mathbf{x}, g_{j+1}(\mathbf{y})) \leq{}& 2C_1 \left(1 + \frac{\|\mathbf{x} - g_{j+1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} |R_+| C_3 (2C_1|G|)^j w(B(\mathbf{x}, \sqrt{t}))^{-1} \\ &\cdot e^{-c_3 \frac{d(\mathbf{x},\mathbf{y})^2}{t}} \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y}))\,:\,\ell(\boldsymbol{\alpha}) \leq |G|+j} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y}), t). \end{aligned} \tag{5.18}$$

For any sequence

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

from $\mathcal{A}(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y}))$ with $\ell(\boldsymbol{\alpha}) \leq |G| + j$, we define the new sequence

$$\widetilde{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$$

from $\mathcal{A}(\mathbf{x}, g_{j+1}(\mathbf{y}))$ satisfying $\ell(\widetilde{\boldsymbol{\alpha}}) \leq |G| + j + 1$. Moreover,

$$\left(1 + \frac{\|\mathbf{x} - g_{j+1}(\mathbf{y})\|}{\sqrt{t}}\right)^{-2} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \sigma_{\alpha_0} \circ g_{j+1}(\mathbf{y}), t) = \rho_{\widetilde{\boldsymbol{\alpha}}}(\mathbf{x}, g_{j+1}(\mathbf{y}), t). \tag{5.19}$$

So (5.10) for g_{j+1} follows from (5.18) and (5.19). The proof is complete. □

6 Applications of Theorem 1.1

6.1 Regularity of the heat kernel

The following theorem can be consider as an improvement of the estimates [2, Theorem 4.1 (b)].

Theorem 6.1 *Let m be a non-negative integer. There are constants $C_4, c_4 > 0$ such that for all $\mathbf{x}, \mathbf{y}, \mathbf{y}' \in \mathbb{R}^N$ and $t > 0$ satisfying $\|\mathbf{y}-\mathbf{y}'\| < \frac{\sqrt{t}}{2}$ we have*

$$|\partial_t^m h_t(\mathbf{x},\mathbf{y}) - \partial_t^m h_t(\mathbf{x},\mathbf{y}')| \leq C_4 t^{-m} \frac{\|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}} h_{c_4 t}(\mathbf{x},\mathbf{y}). \tag{6.1}$$

The constant c_4 does not depend on m.

In the proof of Theorem 6.1 we will need the following lemma (which is a Harnack type inequality).

Lemma 6.2 *There are constants $C_5, c_5 > 0$ such that for all $\mathbf{x}, \mathbf{y}, \mathbf{y}' \in \mathbb{R}^N$ and $t > 0$ satisfying $\|\mathbf{y}-\mathbf{y}'\| < \frac{\sqrt{t}}{2}$ we have*

$$h_t(\mathbf{x},\mathbf{y}) \leq C_5 h_{c_5 t}(\mathbf{x},\mathbf{y}'). \tag{6.2}$$

Proof Fix $\mathbf{x}, \mathbf{y}, \mathbf{y}' \in \mathbb{R}^N$ and $t > 0$ such that $\|\mathbf{y}-\mathbf{y}'\| < \frac{\sqrt{t}}{2}$. By Theorem 1.1 we have

$$h_t(\mathbf{x},\mathbf{y}) \leq C_u w(B(\mathbf{x},\sqrt{t}))^{-1} e^{-c_u \frac{d(\mathbf{x},\mathbf{y})^2}{t}} \Lambda(\mathbf{x},\mathbf{y},t), \tag{6.3}$$

where

$$\Lambda(\mathbf{x},\mathbf{y},t) = \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x},\mathbf{y}),\ \ell(\boldsymbol{\alpha}) \leq 2|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t). \tag{6.4}$$

Let us consider a sequence $\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x},\mathbf{y})$ such that $\ell(\boldsymbol{\alpha}) \leq 2|G|$. We shall prove that

$$\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t) \leq 2^{2\ell(\boldsymbol{\alpha})} \rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y}',t). \tag{6.5}$$

If $\ell(\boldsymbol{\alpha}) = 0$, then (6.5) is trivial. If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathcal{A}(\mathbf{x},\mathbf{y})$ and $\|\mathbf{y}-\mathbf{y}'\| < \frac{\sqrt{t}}{2}$, then for any $1 \leq j \leq \ell(\boldsymbol{\alpha})$, we have

$$\begin{aligned}
&\frac{\|\mathbf{x} - \sigma_{\alpha_j} \circ \sigma_{\alpha_{j-1}} \circ \ldots \circ \sigma_{\alpha_1}(\mathbf{y}')\|}{\sqrt{t}} + 1 \\
&\leq \frac{\|\mathbf{x} - \sigma_{\alpha_j} \circ \sigma_{\alpha_{j-1}} \circ \ldots \circ \sigma_{\alpha_1}(\mathbf{y})\| + \|\sigma_{\alpha_j} \circ \sigma_{\alpha_{j-1}} \circ \ldots \circ \sigma_{\alpha_1}(\mathbf{y}) - \sigma_{\alpha_j} \circ \sigma_{\alpha_{j-1}} \circ \ldots \circ \sigma_{\alpha_1}(\mathbf{y}')\|}{\sqrt{t}} + 1 \\
&= \frac{\|\mathbf{x} - \sigma_{\alpha_j} \circ \sigma_{\alpha_{j-1}} \circ \ldots \circ \sigma_{\alpha_1}(\mathbf{y})\| + \|\mathbf{y}-\mathbf{y}'\|}{\sqrt{t}} + 1 \leq 2\left(\frac{\|\mathbf{x} - \sigma_{\alpha_j} \circ \sigma_{\alpha_{j-1}} \circ \ldots \circ \sigma_{\alpha_1}(\mathbf{y})\|}{\sqrt{t}} + 1\right).
\end{aligned}$$

Hence, by the definition of $\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y},t)$ [see (1.5)], we obtain (6.5).

Now, for $\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x},\mathbf{y})$, $\ell(\boldsymbol{\alpha}) \leq 2|G|$, we are going to define a new sequence $\widetilde{\boldsymbol{\alpha}}$ of elements of R_+ such that $\widetilde{\boldsymbol{\alpha}} \in \mathcal{A}(\mathbf{x},\mathbf{y}')$, $\ell(\widetilde{\boldsymbol{\alpha}}) \leq 4|G|$, and

$$\rho_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{y}',t) \leq 2^{2|G|} \left(1 + \frac{d(\mathbf{x},\mathbf{y})}{\sqrt{t}}\right)^{2|G|} \rho_{\widetilde{\boldsymbol{\alpha}}}(\mathbf{x},\mathbf{y}',t). \tag{6.6}$$

To this end, let us consider two cases.

Case 1. $\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x},\mathbf{y}')$. Then we set $\widetilde{\boldsymbol{\alpha}} := \boldsymbol{\alpha}$. Clearly, in this case (6.6) is satisfied.

Case 2. $\boldsymbol{\alpha} \notin \mathcal{A}(\mathbf{x}, \mathbf{y}')$. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{m_1})$ be a sequence from Corollary 2.3 chosen for the points $\mathbf{x}$ and $\sigma_{\boldsymbol{\alpha}}(\mathbf{y}')$. In particular, by (2.8), then by the fact that $\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \mathbf{y})$ and $\|\mathbf{y} - \mathbf{y}'\| \leq \frac{\sqrt{t}}{2}$, for any $1 \leq j \leq \ell(\boldsymbol{\beta})$, we have

$$\begin{aligned}
\|\mathbf{x} - \sigma_{\beta_j} \circ \sigma_{\beta_{j-1}} \circ \ldots \circ \sigma_{\beta_1} \circ \sigma_{\boldsymbol{\alpha}}(\mathbf{y}')\| &\leq \|\mathbf{x} - \sigma_{\boldsymbol{\alpha}}(\mathbf{y}')\| \\
&\leq \|\mathbf{x} - \sigma_{\boldsymbol{\alpha}}(\mathbf{y})\| + \|\sigma_{\boldsymbol{\alpha}}(\mathbf{y}) - \sigma_{\boldsymbol{\alpha}}(\mathbf{y}')\| \\
&= \|\mathbf{x} - \sigma_{\boldsymbol{\alpha}}(\mathbf{y})\| + \|\mathbf{y} - \mathbf{y}'\| \\
&\leq d(\mathbf{x}, \mathbf{y}) + \frac{\sqrt{t}}{2}.
\end{aligned}$$

Consequently,

$$1 \geq \rho_{\boldsymbol{\beta}}(\mathbf{x}, \sigma_{\boldsymbol{\alpha}}(\mathbf{y}'), t) \geq 2^{-2|G|}\left(1 + \frac{d(\mathbf{x}, \mathbf{y})}{\sqrt{t}}\right)^{-2|G|}. \tag{6.7}$$

We set

$$\begin{aligned}
\widetilde{\boldsymbol{\alpha}} &:= (\beta_1, \beta_2, \ldots, \beta_{m_1}) \quad \text{if } \ell(\boldsymbol{\alpha}) = 0, \\
\widetilde{\boldsymbol{\alpha}} &:= (\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_{m_1}) \quad \text{if } \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m).
\end{aligned}$$

Then, by the choice of $\boldsymbol{\beta}$, we have $\widetilde{\boldsymbol{\alpha}} \in \mathcal{A}(\mathbf{x}, \mathbf{y}')$, $\ell(\widetilde{\boldsymbol{\alpha}}) \leq 4|G|$. Moreover, by the definition of $\rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}', t)$ and $\rho_{\widetilde{\boldsymbol{\alpha}}}(\mathbf{x}, \mathbf{y}', t)$, and (6.7) we have

$$\rho_{\widetilde{\boldsymbol{\alpha}}}(\mathbf{x}, \mathbf{y}', t) = \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}', t)\rho_{\boldsymbol{\beta}}(\mathbf{x}, \sigma_{\boldsymbol{\alpha}}(\mathbf{y}'), t) \geq 2^{-2|G|}\left(1 + \frac{d(\mathbf{x}, \mathbf{y})}{\sqrt{t}}\right)^{-2|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}', t),$$

which implies (6.6).

Applying (6.5) and (6.6), we have

$$\begin{aligned}
\Lambda(\mathbf{x}, \mathbf{y}, t) &= \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \mathbf{y}),\ \ell(\boldsymbol{\alpha}) \leq 2|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}, t) \\
&\leq 2^{6|G|}\left(1 + \frac{d(\mathbf{x}, \mathbf{y})}{\sqrt{t}}\right)^{2|G|} \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \mathbf{y}'),\ \ell(\boldsymbol{\alpha}) \leq 4|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}', t),
\end{aligned}$$

which, together with (6.3), gives

$$\begin{aligned}
h_t(\mathbf{x}, \mathbf{y}) &\leq 2^{6|G|} C_u \left(1 + \frac{d(\mathbf{x}, \mathbf{y})}{\sqrt{t}}\right)^{2|G|} w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_u \frac{d(\mathbf{x}, \mathbf{y})^2}{t}} \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \mathbf{y}'),\ \ell(\boldsymbol{\alpha}) \leq 4|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}', t) \\
&\leq C_u' w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_u' \frac{d(\mathbf{x}, \mathbf{y})^2}{t}} \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \mathbf{y}'),\ \ell(\boldsymbol{\alpha}) \leq 4|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}', t).
\end{aligned} \tag{6.8}$$

Note that $\|\mathbf{y} - \mathbf{y}'\| < \frac{\sqrt{t}}{2}$ implies

$$2d(\mathbf{x}, \mathbf{y})^2 \geq d(\mathbf{x}, \mathbf{y}')^2 - 2\|\mathbf{y} - \mathbf{y}'\|^2 \geq d(\mathbf{x}, \mathbf{y}')^2 - \frac{t}{2}.$$

Therefore, by (6.8) we get

$$h_t(\mathbf{x}, \mathbf{y}) \leq C_u'' w(B(\mathbf{x}, \sqrt{t}))^{-1} e^{-c_u' \frac{d(\mathbf{x}, \mathbf{y}')^2}{2t}} \sum_{\boldsymbol{\alpha} \in \mathcal{A}(\mathbf{x}, \mathbf{y}'),\ \ell(\boldsymbol{\alpha}) \leq 4|G|} \rho_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}', t). \tag{6.9}$$

Finally, (6.2) is a consequence of Proposition 5.1, (6.9), and (1.8). □

Proof of Theorem 6.1 For $x \in \mathbb{R}$ and $t > 0$ let us denote

$$\widetilde{h_t}(x) := \boldsymbol{c}_k^{-1} 2^{-\mathbf{N}/2} t^{-\mathbf{N}/2} \exp\left(-\frac{x^2}{4t}\right).$$

Then the formula (2.4) reads

$$h_t(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{R}^N} \widetilde{h_t}(A(\mathbf{x}, \mathbf{y}, \eta))\, d\mu_{\mathbf{x}}(\eta). \tag{6.10}$$

Observe that $\tilde{\mathfrak{h}}_t(x) := \partial_x \partial_t^m \widetilde{h_t}(x)$ is equal to $\frac{x}{t^{m+1}} \tilde{h}_t(x)$ times a polynomial in $\frac{x^2}{t}$. Hence, for any non-negative integer m there is a constant $C_m > 0$ such that for all $x \in \mathbb{R}$ and $t > 0$ we have

$$\left|\tilde{\mathfrak{h}}_t(x)\right| \leq C_m t^{-m-1/2} \widetilde{h_{2t}}(x)\,. \tag{6.11}$$

Further, by (6.10) and (6.11) we get

$$\begin{aligned} |\partial_t^m h_t(\mathbf{x}, \mathbf{y}) - \partial_t^m h_t(\mathbf{x}, \mathbf{y}')| &= \left|\int_{\mathbb{R}^N} \left\{\partial_t^m \widetilde{h_t}(A(\mathbf{x}, \mathbf{y}, \eta)) - \partial_t^m \widetilde{h_t}(A(\mathbf{x}, \mathbf{y}', \eta))\right\} d\mu_{\mathbf{x}}(\eta)\right| \\ &= \left|\int_{\mathbb{R}^N} \int_0^1 \frac{\partial}{\partial s} \partial_t^m \widetilde{h_t}(A(\mathbf{x}, \underbrace{\mathbf{y}' + s(\mathbf{y} - \mathbf{y}')}_{\mathbf{y}_s}, \eta))\, ds\, d\mu_{\mathbf{x}}(\eta)\right| \\ &\leq \|\mathbf{y} - \mathbf{y}'\| \int_0^1 \int_{\mathbb{R}^N} \left|\tilde{\mathfrak{h}}_t(A(\mathbf{x}, \mathbf{y}_s, \eta))\right| d\mu_{\mathbf{x}}(\eta)\, ds \\ &\leq C_m t^{-m} \frac{\|\mathbf{y} - \mathbf{y}'\|}{\sqrt{t}} \int_0^1 h_{2t}(\mathbf{x}, \mathbf{y}_s)\, ds. \end{aligned} \tag{6.12}$$

Finally, note that for any $s \in [0, 1]$, we have

$$\|\mathbf{y} - \mathbf{y}_s\| \leq \|\mathbf{y} - \mathbf{y}'\| < \frac{\sqrt{t}}{2},$$

so, by Lemma 6.2, we get

$$h_{2t}(\mathbf{x}, \mathbf{y}_s) \leq C_5 h_{2c_5 t}(\mathbf{x}, \mathbf{y}). \tag{6.13}$$

Now (6.12) together with (6.13) imply the desired estimate (6.1). □

Remark 6.3 In the proof of Theorem 6.1, we partially repeat the argument from [2, Theorem 4.1 (b)]. The novelty of the approach is using Lemma 6.2 instead of Proposition 3.1 in estimating the last integral of (6.12).

6.2 Remark on a theorem of Gallardo and Rejeb

In [5], the authors proved that the points $g(\mathbf{x})$, $g \in G$, belong to the support of the measure $\mu_{\mathbf{x}}$ (see [5, Theorem A 3)]). Below, as an application of the estimates (1.9) and (1.10), we provide another proof of this theorem. The proof, at the same time, gives a more precise behavior of the measure $\mu_{\mathbf{x}}$ around these points.

For $\mathbf{y} \in \mathbb{R}^N$ and $t > 0$ we set

$$U(\mathbf{y}, t) := \{\eta \in \operatorname{conv} \mathcal{O}(\mathbf{y}) : \|\mathbf{y}\|^2 - \langle \mathbf{y}, \eta \rangle \leq t\}, \tag{6.14}$$

$$V(\mathbf{y}, t) := (\operatorname{conv} \mathcal{O}(\mathbf{y})) \setminus U(\mathbf{y}, t) = \{\eta \in \operatorname{conv} \mathcal{O}(\mathbf{y}) : \|\mathbf{y}\|^2 - \langle \mathbf{y}, \eta \rangle > t\}. \tag{6.15}$$

Theorem 6.4 *There is a constant $C_6 > 0$ such that for all $\mathbf{x} \in \mathbb{R}^N$, $t > 0$, and $g \in G$ we have*

$$C_6^{-1}\frac{t^{\mathbf{N}/2}\Lambda(\mathbf{x}, g(\mathbf{x}), t)}{w(B(\mathbf{x}, \sqrt{t}))} \leq \mu_{\mathbf{x}}(U(g(\mathbf{x}), t)) \leq C_6\frac{t^{\mathbf{N}/2}\Lambda(\mathbf{x}, g(\mathbf{x}), t)}{w(B(\mathbf{x}, \sqrt{t}))}. \tag{6.16}$$

Proof Let $\mathbf{y} = g(\mathbf{x})$. Then $d(\mathbf{x}, \mathbf{y}) = 0$. Moreover, by the definition of $A(\mathbf{x}, \mathbf{y}, \eta)$ [see (2.5)] and the fact that $\|\mathbf{y}\| = \|\mathbf{x}\|$ we have

$$A(\mathbf{x}, \mathbf{y}, \eta)^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{y}, \eta\rangle = 2\|\mathbf{y}\|^2 - 2\langle \mathbf{y}, \eta\rangle.$$

We first prove the upper bound in (6.16). Observe that $(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{y}, \eta\rangle)/4t \leq 1/2$ for all $\eta \in U(\mathbf{y}, t)$. Thus, applying (2.4), we get

$$\begin{aligned}\mu_{\mathbf{x}}(U(\mathbf{y}, t)) = \int_{U(\mathbf{y},t)} d\mu_{\mathbf{x}}(\eta) &\leq e^{1/2}\int_{U(\mathbf{y},t)} e^{-(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\langle \mathbf{y},\eta\rangle)/4t}\, d\mu_{\mathbf{x}}(\eta)\\ &\leq e^{1/2}\int_{\mathbb{R}^N} e^{-(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\langle \mathbf{y},\eta\rangle)/4t}\, d\mu_{\mathbf{x}}(\eta)\\ &\leq Ct^{\mathbf{N}/2}h_t(\mathbf{x}, \mathbf{y}) \leq C'\frac{t^{\mathbf{N}/2}\Lambda(\mathbf{x}, \mathbf{y}, t)}{w(B(\mathbf{x}, \sqrt{t}))},\end{aligned} \tag{6.17}$$

where in the last inequality we have used (1.10).

We now turn to prove the lower bound in (6.16). From Theorem 1.1, the fact that $d(\mathbf{x}, \mathbf{y}) = 0$, (1.8), and the doubling property (2.2), we deduce that there is a constant $C > 0$ being independent of $\mathbf{x}$, $g \in G$, and $t > 0$ such that

$$\begin{aligned}h_{2t}(\mathbf{x}, g(\mathbf{x})) &\leq C_u w(B(\mathbf{x}, \sqrt{2t}))^{-1}\Lambda(\mathbf{x}, g(\mathbf{x}), 2t)\\ &\leq C_u' w(B(\mathbf{x}, \sqrt{t}))^{-1}\Lambda(\mathbf{x}, g(\mathbf{x}), t) \leq Ch_t(\mathbf{x}, g(\mathbf{x})).\end{aligned}$$

Hence, using (2.4) applied to $h_{2t}(\mathbf{x}, g(\mathbf{x}))$ and $h_t(\mathbf{x}, g(\mathbf{x}))$ together with (1.8) and the doubling property (2.2), we conclude that there is a constant $\widetilde{c} \geq 0$ independent of $\mathbf{x}$, $g \in G$, and $t > 0$ such that

$$\int_{\mathbb{R}^N} e^{-(2\|\mathbf{y}\|^2-2\langle \mathbf{y},\eta\rangle)/8t}\, d\mu_{\mathbf{x}}(\eta) \leq e^{\widetilde{c}}\int_{\mathbb{R}^N} e^{-(2\|\mathbf{y}\|^2-2\langle \mathbf{y},\eta\rangle)/4t}\, d\mu_{\mathbf{x}}(\eta). \tag{6.18}$$

Let $M = 4(\widetilde{c} + 1)$. We rewrite (6.18) by splitting the areas of the integration:

$$\begin{aligned}I_U + I_V :&= \int_{U(\mathbf{y},Mt)} e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/4t}\, d\mu_{\mathbf{x}}(\eta) + \int_{V(\mathbf{y},Mt)} e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/4t}\, d\mu_{\mathbf{x}}(\eta)\\ &\leq e^{\widetilde{c}}\int_{U(\mathbf{y},Mt)} e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/2t}\, d\mu_{\mathbf{x}}(\eta) + e^{\widetilde{c}}\int_{V(\mathbf{y},Mt)} e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/2t}\, d\mu_{\mathbf{x}}(\eta)\\ &=: J_U + J_V.\end{aligned} \tag{6.19}$$

Observe that, by the definition of $V(\mathbf{y}, Mt)$ [see (6.15)], and the fact that $M = 4(\widetilde{c} + 1)$, for all $\eta \in V(\mathbf{y}, Mt)$ we have

$$\begin{aligned}\frac{1}{2}e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/4t} &= \frac{1}{2}e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/2t}e^{(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/4t} \geq \frac{1}{2}e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/2t}e^{M/4}\\ &= \frac{1}{2}e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/2t}e^{\widetilde{c}+1} \geq e^{\widetilde{c}}e^{-(\|\mathbf{y}\|^2-\langle \mathbf{y},\eta\rangle)/2t}.\end{aligned} \tag{6.20}$$

Consequently, (6.20) implies $J_V \leq \frac{1}{2} I_V$. Therefore, by (6.19), we get

$$I_U + \frac{1}{2} I_V \leq J_U.$$

Applying Theorem 1.1, we deduce that

$$\begin{aligned} \frac{C_l}{2} \frac{\Lambda(\mathbf{x}, \mathbf{y}, 2t)}{w(B(\mathbf{x}, \sqrt{2t}))} &\leq \frac{1}{2} h_{2t}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \boldsymbol{c}_k^{-1} (4t)^{-\mathbf{N}/2} (I_U + I_V) \leq \boldsymbol{c}_k^{-1} (4t)^{-\mathbf{N}/2} J_U \\ &\leq \boldsymbol{c}_k^{-1} (4t)^{-\mathbf{N}/2} e^{\widetilde{c}} \mu_{\mathbf{x}}(U(\mathbf{y}, Mt)). \end{aligned} \tag{6.21}$$

Now the claim follows from (1.8) and the doubling property (2.2). □

We want to remark that Theorem 6.4 extends the result of Jiu and Li [7, Theorem 2.1], where the behavior of $\mu_{\mathbf{x}}$ around $\mathbf{x}$ is studied.

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