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Calculus of Variations

On the spectrum of a mixed-type operator with applications to rotating wave solutions

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Abstract

We study rotating wave solutions of the nonlinear wave equation

$$\begin{cases} \partial_t^2 v - \Delta v + mv = |v|^{p-2}v & \text{in } \mathbb{R} \times \mathbf{B} \\ v = 0 & \text{on } \mathbb{R} \times \partial\mathbf{B} \end{cases}$$

where $2 < p < \infty$, $m \in \mathbb{R}$ and $\mathbf{B} \subset \mathbb{R}^2$ denotes the unit disk. If the angular velocity α of the rotation is larger than 1, this leads to a semilinear boundary value problem on $\mathbf{B}$ involving a mixed-type operator, whose spectrum is related to the zeros of Bessel functions and could generally be badly behaved. Based on new estimates for these zeros, we find values of α such that the spectrum only consists of eigenvalues with finite multiplicity and has no accumulation point. Combined with suitable spectral estimates, this allows us to formulate an appropriate indefinite variational setting and find ground state solutions of the reduced equation for $p \in (2, 4)$. Using a minimax characterization of the ground state energy, we ultimately show that these ground states are nonradial and thus yield nontrivial rotating waves, provided m is sufficiently large.

Mathematics Subject Classification Primary 35M12; Secondary 35B06 · 35P15 · 47J30 · 35P20.

1 Introduction

We consider time-periodic solutions of the nonlinear wave equation

$$\begin{cases} \partial_t^2 v - \Delta v + mv = |v|^{p-2}v & \text{in } \mathbb{R} \times \mathbf{B} \\ \qquad v = 0 & \text{on } \mathbb{R} \times \partial\mathbf{B} \end{cases} \tag{1.1}$$

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where $2 < p < \infty$, $m \in \mathbb{R}$ and $\mathbf{B} \subset \mathbb{R}^2$ denotes the unit disk. In the case $m > 0$, this is also commonly referred to as a nonlinear Klein-Gordon equation. A well-known class of such solutions is given by standing wave solutions, which reduce (1.1) either to a stationary nonlinear Schrödinger or a nonlinear Helmholtz equation and have been studied extensively on the whole space $\mathbb{R}^N$, see [17, 34]. Note that this yields complex-valued solutions whose amplitude remains stationary, however, while other types of time-periodic solutions are significantly less well understood. In particular, much less is known about the dynamics of nonlinear wave equations in general bounded domains.

In the one-dimensional setting, which typically describes the forced vibrations of a nonhomogeneous string, the existence of time-periodic solutions satisfying either Dirichlet or periodic boundary conditions has been treated in the seminal works of Rabinowitz [33] and Brézis, Coron and Nirenberg [7] by variational methods, but the results in higher dimensions are more sparse. On balls centered at the origin, the existence of radially symmetric time-periodic solutions was first studied by Ben-Naoum and Mawhin [3] for sublinear nonlinearities and subsequently received further attention, see e.g. the recent works of Chen and Zhang [9–11] and the references therein.

In this paper, we study *rotating wave solutions* as introduced in [19], which are time-periodic real-valued solutions of (1.1) given by the ansatz

$$v(t, x) = u(R_{\alpha t}(x)), \tag{1.2}$$

where $R_\theta \in O(2)$ describes a rotation in $\mathbb{R}^2$ with angle $\theta > 0$, i.e.,

$$R_\theta(x) = (x_1 \cos\theta + x_2 \sin\theta, -x_1 \sin\theta + x_2 \cos\theta) \qquad \text{for } x \in \mathbb{R}^2. \tag{1.3}$$

In particular, the constant $\alpha > 0$ in (1.2) is the angular velocity of the rotation. Consequently, such solutions can be interpreted as rotating waves in a nonlinear medium. We note that a related ansatz for generalized traveling waves on manifolds has also been considered in [27, 28, 37], while a class of spiral shaped solutions for a nonlinear Schrödinger equation on $\mathbb{R}^3$ has been treated in [1].

In the following, we let θ denote the angular variable in two-dimensional polar coordinates and note that the ansatz (1.2) reduces (1.1) to

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + mu = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B} \end{cases} \tag{1.4}$$

where $\partial_\theta = x_1\partial_{x_2} - x_2\partial_{x_1}$ then corresponds to the angular derivative. Note that this equation has solutions which are independent of θ, but these correspond to stationary and therefore non-rotating solutions of (1.1). In the following, our goal is to prove the existence of nonradial, i.e., θ-dependent, solutions of (1.4).

In the case $\alpha \leq 1$, this question has been studied in great detail in [19], where a connection to degenerate Sobolev inequalities is explored. In particular, it has been observed that the ground states, i.e., minimizers of the associated Rayleigh quotient, are nonradial in certain parameter regimes for p and α.

The main purpose of the present paper is the study of nonradial solutions of (1.4) for $\alpha > 1$. However, the direct variational methods employed in [19] cannot be extended to this case since the operator

$$L_\alpha := -\Delta + \alpha^2 \partial_\theta^2$$

is neither elliptic nor degenerate elliptic, and the associated Rayleigh quotient becomes unbounded from below, see [19, Remark 4.3]. Indeed, note that in polar coordinates $(r, \theta) \in (0, 1) \times (-\pi, \pi)$ we have

$$L_\alpha u = -\partial_r^2 u - \frac{1}{r}\partial_r u - \left(\frac{1}{r^2} - \alpha^2\right)\partial_\theta^2 u$$

and hence the operator is in fact of mixed-type for $\alpha > 1$: It is elliptic in the smaller ball $B_{1/\alpha}(0)$ of radius $1/\alpha$, parabolic on the sphere of radius $1/\alpha$ and hyperbolic in the annulus $\mathbf{B} \setminus \overline{B_{1/\alpha}(0)}$. In general, such operators are difficult to deal with via variational methods, and instead results often rely on separate treatments of the different regions of specific type and then gluing the solutions together, see e.g. [26, 29] for more details.

From a functional analytic viewpoint, the quadratic form associated to L_α is strongly indefinite, i.e., it is negative on an infinite-dimensional subspace. Classically, related problems have been treated for operators of the form $-\Delta - E$ on $\mathbb{R}^N$ where $E \in \mathbb{R}$ lies in a spectral gap of the Laplacian. In this direction, we mention the use of a dual variational framework in order to prove the existence of nonzero solutions of a nonlinear stationary Schrödinger equation in [2], as well as abstract operator theoretic methods used in [8] for a related problem. However, both of these exemplary approaches require specific assumptions regarding spectral properties of the associated operator. Moreover, the sole existence of nonzero solutions to (1.4) is insufficient in our case since we are interested in nontrivial *rotating* wave solutions.

In the present case of problem (1.4), a main obstruction, in addition to the unboundedness of the spectrum of the linear operator L_α from above and below, is the possible existence of finite accumulation points of this spectrum. As a first step, we therefore analyze the spectrum of L_α in detail, which is closely related to the spectrum of the Laplacian and thus the zeros of Bessel functions. In fact, the Dirichlet eigenvalues of L_α are given by

$$j_{\ell,k}^2 - \alpha^2\ell^2,$$

where $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ and $j_{\ell,k}$ denotes the k-th zero of the Bessel function of the first kind J_ℓ. Here and in the following, $\mathbb{N}_0$ denotes the natural numbers extended by zero. In particular, the structure of the spectrum therefore heavily depends on the asymptotic behavior of the zeros of these Bessel functions. Despite this explicit characterization, it is not clear whether the spectrum of L_α only consists of isolated points. Indeed, known results on the asymptotics of the zeros of Bessel functions turn out to be insufficient to exclude accumulation points or even density in $\mathbb{R}$. In fact, similar spectral issues arise in the study of radially symmetric time-periodic solutions of (1.1) on balls $B_a(0)$, where the spectral properties of the radial wave operator are intimately connected to the arithmetic properties of the ratio between the radius $a > 0$ and the period length, see e.g. [4, 23] and the references therein for more details.

This turns out to be a serious obstruction for the use of variational methods and thus necessitates a detailed analysis of the asymptotic behavior of different sequences of zeros. Our first main result then characterizes the spectrum of L_α as follows.

Theorem 1.1 *For any $\alpha > 1$ the spectrum of L_α is unbounded from above and below. Moreover, there exists an unbounded sequence $(\alpha_n)_n \subset (1, \infty)$ such that the following properties hold for $n \in \mathbb{N}$:*

(i) *The spectrum of L_{α_n} consists of eigenvalues with finite multiplicity.*
(ii) *There exists $c_n > 0$ such that for each $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ we either have $j_{\ell,k}^2 - \alpha_n^2\ell^2 = 0$ or*

$$|j_{\ell,k}^2 - \alpha_n^2\ell^2| \geq c_n j_{\ell,k}. \tag{1.5}$$

(iii) *The spectrum of L_{α_n} has no finite accumulation points.*

The proof of this result is based on the observation that the formula

$$j_{\ell,k}^2 - \alpha^2\ell^2 = (j_{\ell,k} + \alpha\ell)\ell\left(\frac{j_{\ell,k}}{\ell} - \alpha\right)$$

implies that for any unbounded sequences $(\ell_i)_i$, $(k_i)_i$, the corresponding sequence of eigenvalues $j_{\ell_i,k_i}^2 - \alpha^2\ell_i^2$ can only remain bounded if

$$\frac{j_{\ell_i,k_i}}{\ell_i} - \alpha \to 0 \tag{1.6}$$

as $i \to \infty$. It turns out that (1.6) can only hold if $\ell_i/k_i \to \sigma$, where $\sigma = \sigma(\alpha) > 0$ is uniquely determined and can be characterized via a transcendental equation. This motivates a more detailed investigation of $j_{\sigma k,k}$, $k \in \mathbb{N}$ which gives rise to a new estimate for $j_{\ell,k}$, $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$, see Lemma 3.3 and Remark 3.4 below. In order to estimate arbitrary sequences in (1.6), we are then forced to restrict the problem to velocities $\alpha = \alpha_n$ such that the associated values $\sigma_n = \sigma(\alpha_n)$ are suitable rational numbers. The fact that such a restriction is necessary is not surprising when compared to similar properties observed for the radial wave operator as mentioned above.

Theorem 1.1 then plays a central role in the formulation of a variational framework for (1.4) and allows us to recover sufficient regularity properties for L_{α_n}. More specifically, for $\alpha = \alpha_n$ we may then define a suitable Hilbert space $E_{\alpha,m}$ whose norm is related to the quadratic form

$$u \mapsto \int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\, dx,$$

see Sect. 5 below for details. The space $E_{\alpha,m}$ admits a decomposition of the form

$$E_{\alpha,m} = E_{\alpha,m}^+ \oplus F_{\alpha,m},$$

where the spaces $E_{\alpha,m}^+$ and $F_{\alpha,m}$ essentially correspond to the eigenspaces of positive and nonpositive eigenvalues of $-\Delta + \alpha^2\partial_\theta^2 + m$, respectively. Crucially, the estimate (1.5) and fractional Sobolev embeddings allow us to deduce that $E_{\alpha,m}$ compactly embeds into $L^p(\mathbf{B})$ for $p \in (2, 4)$.

We may then find solutions of (1.4) as critical points of the associated energy functional $\Phi_{\alpha,m} : E_{\alpha,m} \to \mathbb{R}$ given by

$$\Phi_{\alpha,m}(u) := \frac{1}{2}\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\, dx - \frac{1}{p}\int_{\mathbf{B}} |u|^p\, dx.$$

Due to the strongly indefinite nature of (1.4), $\Phi_{\alpha,m}$ is unbounded from above and below and does not possess a mountain pass structure so, in particular, the classical mountain pass theorem and its variants are not applicable. Instead, we consider the generalized Nehari manifold introduced by Pankov [31]

$$\mathcal{N}_{\alpha,m} := \left\{u \in E_{\alpha,m} \setminus F_{\alpha,m} :\ \Phi_{\alpha,m}'(u)u = 0 \text{ and } \Phi_{\alpha,m}'(u)v = 0 \text{ for all } v \in F_{\alpha,m}\right\}.$$

Using further abstract results due to Szulkin and Weth [36], we can then show that

$$c_{\alpha,m} = \inf_{u \in \mathcal{N}_{\alpha,m}} \Phi_{\alpha,m}(u)$$

is positive and attained by a critical point of $\Phi_{\alpha,m}$ for $\alpha = \alpha_n$ as in Theorem 1.1 and $m \in \mathbb{R}$. In particular, such a minimizer then necessarily has minimal energy among all critical points of $\Phi_{\alpha,m}$, and is therefore referred to as a *ground state solution* or *ground state* of (1.4).

In general, it is not clear whether such a ground state is nonradial. Our second main result further states that (1.4) has nonradial ground state solutions for certain choices of parameters.

Theorem 1.2 *Let $p \in (2,4)$ and let the sequence $(\alpha_n)_n \subset (1,\infty)$ be given by* Theorem 1.1. *Then the following properties hold:*

(i) *For any $n \in \mathbb{N}$ and $m \in \mathbb{R}$ there exists a ground state solution of* (1.4) *for $\alpha = \alpha_n$.*
(ii) *For any $n \in \mathbb{N}$ there exists $m_n > 0$ such that the ground state solutions of* (1.4) *are nonradial for $\alpha = \alpha_n$ and $m > m_n$.*

In fact, we can prove a slightly more general result in the sense that the statement of Theorem 1.2 holds whenever the kernel of L_α is finite-dimensional and an inequality of the form (1.5) holds. The proof is essentially based on an energy comparison, noting that the minimal energy of the unique positive radial solution can be estimated from below in terms of m. Using a minimax characterization of $c_{\alpha,m}$, we can then show that this ground state energy grows slower than the radial energy as $m \to \infty$.

Throughout the paper, we only consider real-valued solutions and consequently let all function spaces be real. Nonradial complex-valued solutions of (1.4), on the other hand, can be found much more easily using constrained minimization over suitable eigenspaces. This technique has been applied to a related problem in [37]. We point out, however, that the modulus of such solutions is necessarily radial, while Theorem 1.2 yields solutions with nonradial modulus. With our methods, by combining (1.2) with a standing wave ansatz, we can also prove the existence of genuinely complex-valued ground states with nonradial modulus, see the appendix of this paper.

The paper is organized as follows. In Sect. 2, we introduce Sobolev spaces via their spectral characterization and collect several known results on the properties of the zeros of Bessel functions. In Sect. 3 we then prove a crucial technical estimate for certain sequences of such zeros. This result is subsequently used in Sect. 4 to investigate the asymptotics of the zeros of Bessel functions in detail and, in particular, prove Theorem 1.1. Section 5 is then devoted to the rigorous formulation of the variational framework outlined earlier and the proof of Theorem 1.2. In Appendix A, we discuss the results for complex-valued solutions mentioned above.

2 Preliminaries

We first collect some general facts on eigenvalues and eigenfunctions of the Laplacian on $\mathbf{B}$, we refer to [18] for a more comprehensive overview. Recall that the eigenvalues of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B} \end{cases}$$

are given by $j_{\ell,k}^2$, where $j_{\ell,k}$ denotes the k-th zero of the Bessel function of the first kind J_ℓ with $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$. To each eigenvalue $j_{\ell,k}^2$ correspond two linearly independent eigenfunctions

$$\begin{aligned}\varphi_{\ell,k}(r,\theta) &:= A_{\ell,k}\cos(\ell\theta)J_\ell(j_{\ell,k}r)\\ \psi_{\ell,k}(r,\theta) &:= B_{\ell,k}\sin(\ell\theta)J_\ell(j_{\ell,k}r),\end{aligned} \tag{2.1}$$

where the constants $A_{\ell,k}, B_{\ell,k} > 0$ are chosen such that $\|\varphi_{\ell,k}\|_2 = \|\psi_{\ell,k}\|_2 = 1$. These functions constitute an orthonormal basis of $L^2(\mathbf{B})$ and we can then characterize Sobolev spaces as follows:

$$H_0^1(\mathbf{B}) := \left\{ u \in L^2(\mathbf{B}) : \|u\|_{H^1}^2 := \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} j_{\ell,k}^2 \left(|\langle u, \varphi_{\ell,k}\rangle|^2 + |\langle u, \psi_{\ell,k}\rangle|^2 \right) < \infty \right\}.$$

It can be shown that this is consistent with the usual definition of $H^1(\mathbf{B})$. By classical Sobolev embeddings, $H_0^1(\mathbf{B})$ compactly maps into $L^p(\mathbf{B})$ for any $1 \le p < \infty$.

Similarly, we consider the fractional Sobolev spaces

$$H_0^s(\mathbf{B}) := \left\{ u \in L^2(\mathbf{B}) : \|u\|_{H^s}^2 := \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} j_{\ell,k}^{2s} \left(|\langle u, \varphi_{\ell,k}\rangle|^2 + |\langle u, \psi_{\ell,k}\rangle|^2 \right) < \infty \right\}$$

for $s \in (0,1)$. Using interpolation, it can be shown that this is equivalent to the classical definition and $H_0^s(\mathbf{B})$ compactly maps into $L^p(\mathbf{B})$ for $p < \frac{2}{1-s}$, i.e., there exists $C_s > 0$ such that

$$\|u\|_p \le C_s \|u\|_{H_0^s(\mathbf{B})}$$

holds for $u \in H_0^1(\mathbf{B})$.

Next, we collect several results on the properties of zeros Bessel functions, see e.g. [13] for a more extensive overview. In the following, we let $j_{\nu,k}$ denote the k-th zero of the Bessel function J_ν, where $\nu \ge 0$, $k \in \mathbb{N}$. By definition, $j_{\nu,k} < j_{\nu,k+1}$. In the following, we let $\mathrm{Ai} : \mathbb{R} \to \mathbb{R}$ denote the Airy function given by

$$\mathrm{Ai}(x) = \frac{1}{\pi} \lim_{b\to\infty} \int_0^b \cos\left(\frac{t^3}{3} + xt\right) dt.$$

Importantly, Ai is oscillating on $(-\infty, 0)$ and we let a_k denote its k-th negative zero.

Proposition 2.1 *For each fixed $k \in \mathbb{N}$, $j_{\nu,k}$ is increasing with respect to ν. Moreover, the following properties hold:*

(i) [32] *We have*

$$\nu + \frac{|a_k|}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} < j_{\nu,k} < \nu + \frac{|a_k|}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} + \frac{3}{20}|a_k|^2 \frac{2^{\frac{1}{3}}}{\nu^{\frac{1}{3}}}$$

where a_k denotes the k-th negative zero of the Airy function $\mathrm{Ai}(x)$.

(ii) [24] *For each fixed $k \in \mathbb{N}$ the map*

$$\nu \mapsto \frac{j_{\nu,k}}{\nu}$$

is strictly decreasing on $(0,\infty)$.

(iii) [16] *For $k \in \mathbb{N}$ it holds that*

$$\pi k - \frac{\pi}{4} < j_{0,k} \le \pi k - \frac{\pi}{4} + \frac{1}{8\pi(k - \frac{1}{4})}.$$

(iv) [14] *For each fixed* $k \in \mathbb{N}$ *the map* $\nu \mapsto j_{\nu,k}$ *is differentiable on* $(0, \infty)$ *and*

$$\frac{dj_{\nu,k}}{d\nu} \in \left(1, \frac{\pi}{2}\right)$$

for $\nu \geq 0$.

The zeros of the Airy function can in turn be estimated (see [6]) by

$$\left(\frac{3\pi}{8}(4k - 1.4)\right)^{\frac{2}{3}} < |a_k| < \left(\frac{3\pi}{8}(4k - 0.965)\right)^{\frac{2}{3}}$$

for $k \in \mathbb{N}$, which yields the following result:

Corollary 2.2 *Let* $j_{\nu,k} \in \mathbb{R}$ *be defined as above. Then*

$$\nu + \frac{\left(\frac{3\pi}{8}(4k-2)\right)^{\frac{2}{3}}}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} < j_{\nu,k} < \nu + \frac{\left(\frac{3\pi}{2}k\right)^{\frac{2}{3}}}{2^{\frac{1}{3}}}\nu^{\frac{1}{3}} + \frac{3}{20}\left(\frac{3\pi}{2}k\right)^{\frac{4}{3}}\frac{2^{\frac{1}{3}}}{\nu^{\frac{1}{3}}}.$$

3 Asymptotics of the zeros of Bessel functions

In order to study L_α in Sect. 4, we will be particularly interested in the asymptotics of the zeros $j_{\nu,k}$ when the ratio ν/k remains fixed. For this case, we note the following result by Elbert and Laforgia:

Theorem 3.1 [15] *Let* $x > -1$ *be fixed. Then*

$$\lim_{k\to\infty} \frac{j_{xk,k}}{k} =: \iota(x)$$

exists. Moreover, $\iota(x)$ *is given by*

$$\iota(x) = \begin{cases} \pi, & x = 0, \\ \frac{x}{\sin\varphi} & x \neq 0 \end{cases}$$

where $\varphi = \varphi(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *denotes the unique solution of*

$$\frac{\sin\varphi}{\cos\varphi - (\frac{\pi}{2} - \varphi)\sin\varphi} = \frac{x}{\pi}. \tag{3.1}$$

Moreover, we note the following properties of a function associated to ι.

Lemma 3.2 *The map*

$$f : (0, \infty) \to \mathbb{R}, \qquad f(x) = \frac{\iota(x)}{x}$$

is strictly decreasing and satisfies

$$\lim_{x\to 0} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = 1.$$

Moreover, its inverse is explicitly given by

$$f^{-1} : (1, \infty) \to \mathbb{R}, \qquad f^{-1}(y) = \frac{\pi}{\sqrt{y^2 - 1} - \left(\frac{\pi}{2} - \arcsin\frac{1}{y}\right)}.$$

Proof Note that the left hand side of (3.1) is strictly increasing with respect to φ, and the right hand side is strictly increasing with respect to x, so that φ is necessarily an increasing function of x. In particular, we then have $f(x) = \frac{\iota(x)}{x} = \frac{1}{\sin\varphi}$ which clearly implies the monotonicity of f.

Next, we note that $y = f(x) = \frac{1}{\sin\varphi}$ implies $\varphi = \arcsin\frac{1}{y}$ and hence

$$\frac{x}{\pi} = \frac{\frac{1}{y}}{\cos\left(\arcsin\frac{1}{y}\right) - (\frac{\pi}{2} - \arcsin\frac{1}{y})\frac{1}{y}}.$$

The identity $\cos(\arcsin(t)) = \sqrt{1-t^2}$ then gives

$$\frac{x}{\pi} = \frac{\frac{1}{y}}{\sqrt{1-\frac{1}{y^2}} - (\frac{\pi}{2} - \arcsin\frac{1}{y})\frac{1}{y}} = \frac{1}{\sqrt{y^2-1} - \left(\frac{\pi}{2} - \arcsin\frac{1}{y}\right)}$$

and thus the claim follows. □

In order to characterize the eigenvalues of L_α later on, we need more information on the order of convergence in Theorem 3.1. To this end, we first recall some ingredients of the proof of this result. By the Watson integral formula [38, p. 508], for fixed $k \in \mathbb{N}$ the function $\nu \mapsto j_{\nu,k}$ satisfies

$$\frac{d}{d\nu} j_{\nu,k} = 2 j_{\nu,k} \int_0^\infty K_0(2 j_{\nu,k} \sinh(t)) e^{-2\nu t}\, dt,$$

where K_0 denotes the modified Bessel function of the second kind of order zero. It then follows that the function

$$\iota_k(x) := \frac{j_{kx,k}}{k}$$

satisfies

$$\frac{d}{dx}\iota_k(x) = 2\iota_k \int_0^\infty K_0\left(t 2\iota_k \frac{\sinh\left(\frac{t}{k}\right)}{\left(\frac{t}{k}\right)}\right) e^{-2xt}\, dt =: F_k(\iota_k, x) \tag{3.2}$$

for $k \in \mathbb{N}$ and $x \in (-1, \infty)$. In [15] it is then shown that ι_k converges pointwise to the solution of

$$\begin{cases} \frac{d}{dx}\iota(x) = 2\iota \int_0^\infty K_0\,(t 2\iota)\, e^{-2xt}\, dt =: G(\iota, x) \\ \iota(0) = \pi, \end{cases} \tag{3.3}$$

which is precisely given by the function ι discussed in Theorem 3.1. Moreover, it is shown that

$$\iota_k(x) < \iota(x) \tag{3.4}$$

holds for all $k \in \mathbb{N}$.

We now give a more precise characterization of this convergence in the case $x > 0$.

Lemma 3.3 *For any $x > 0$ and $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that*

$$-\exp\left(\left(\frac{1}{3}+\varepsilon\right)x\right)\frac{\pi}{4k} \le \frac{j_{xk,k}}{k} - \iota(x) \le -(1-\varepsilon)\frac{\pi}{4k}$$

holds for $k \ge k_0$.

Proof Recall that we set $\iota_k(x) = \frac{j_{xk,k}}{k}$ and the functions satisfy

$$\frac{d}{dx}\iota_k = F_k(\iota_k, x)$$
$$\frac{d}{dx}\iota = G(\iota, x)$$

in $(-1, \infty)$ with F_k and G defined in (3.2) and (3.3), respectively. Now consider $u_k(x) := \iota_k(x) - \iota(x)$ so that

$$\frac{d}{dx}u_k = \frac{F_k(\iota_k, x) - G(\iota, x)}{\iota_k(x) - \iota(x)} u_k(x) = \beta_k(x) u_k(x)$$

where we set

$$\beta_k(x) := \frac{F_k(\iota_k, x) - G(\iota, x)}{\iota_k(x) - \iota(x)}.$$

Note that β_k is well-defined by (3.4). In particular, we find that

$$u_k(x) = u_k(0) \exp\left(\int_0^x \beta_k(t)\, dt\right).$$

Next, we note that the monotonicity of K_0 and the fact that $\sinh(t) > t$ holds for $t > 0$ imply

$$F_k(\iota_k, x) = 2\iota_k \int_0^\infty K_0\left(t2\iota_k \frac{\sinh\left(\frac{t}{k}\right)}{\left(\frac{t}{k}\right)}\right) e^{-2xt}\, dt$$
$$< 2\iota \int_0^\infty K_0\,(t2\iota)\, e^{-2xt}\, dt = G(\iota_k, x)$$

where [38, p. 388] implies

$$G(y, x) = 2y \int_0^\infty K_0\,(t2y)\, e^{-2xt}\, dt = \frac{\arccos \frac{x}{y}}{\sqrt{1 - \left(\frac{x}{y}\right)^2}} \qquad \text{if } \left|\frac{x}{y}\right| < 1. \tag{3.5}$$

Importantly, the function

$$g : (1, \infty) \mapsto \mathbb{R}, \quad t \mapsto \frac{\arccos \frac{1}{t}}{\sqrt{1 - \frac{1}{t^2}}}$$

is strictly increasing. Indeed, we have

$$g'(t) = \frac{1}{t^2(1 - \frac{1}{t^2})} - \frac{\arccos \frac{1}{t}}{t^3(1 - \frac{1}{t^2})^{\frac{3}{2}}} = \frac{1}{t^2(1 - \frac{1}{t^2})}\left(1 - \frac{\arccos \frac{1}{t}}{t\sqrt{1 - \frac{1}{t^2}}}\right)$$
$$= \frac{1}{t^2 - 1}\left(1 - \frac{\arccos \frac{1}{t}}{\sqrt{t^2 - 1}}\right)$$

and [39, Theorem 2 for $b = 1/2$] gives

$$\arccos s < 2\frac{\sqrt{1 - s}}{\sqrt{1 + s}}$$

for $s \in (0, 1)$ so that

$$\frac{\arccos\frac{1}{t}}{\sqrt{t^2-1}} = \frac{\arccos\frac{1}{t}}{t\sqrt{1-\frac{1}{t}}\sqrt{1+\frac{1}{t}}} < \frac{2}{t\sqrt{1+\frac{1}{t}}} = \frac{2}{\sqrt{t^2+t}} < 1$$

holds for $t > 1$, which implies that g' is a positive function. Moreover, g' can be continuously extended by $g'(1) = \frac{1}{3}$ and is decreasing, which implies

$$g'(t) \leq \frac{1}{3} \tag{3.6}$$

for $t > 1$.

Noting that Lemma 3.2 and the convergence $\iota_k(x) \to \iota(x)$ imply that $\left|\frac{x}{\iota_k(x)}\right| < 1$ holds for sufficiently large k, we may combine the identity (3.5) with $\iota_k(x) < \iota(x)$ and the monotonicity properties stated above to deduce $F_k(\iota_k, x) < G(\iota_k, x) < G(\iota, x)$ and hence

$$0 \leq \beta_k(x). \tag{3.7}$$

Next, we estimate $u_k(0)$: Recall that $\iota(0) = \pi$ and therefore

$$u_k(0) = \frac{j_{0,k}}{k} - \pi,$$

so Proposition 2.1(iii) yields

$$-\frac{\pi}{4k} \leq u_k(0) \leq -\frac{\pi}{4k} + \frac{1}{8\pi k(k-\frac{1}{4})}. \tag{3.8}$$

In particular, it follows that $u_k(0)$ is negative for $k \in \mathbb{N}$ so the fact that $u_k(x) = u_k(0)\exp\left(\int_0^x \beta_k(t)\,dt\right)$ and the estimate (3.7) yield

$$u_k(x) \leq u_k(0)$$

for $x > 0$. This implies

$$u_k(x) \leq -\frac{\pi}{4k} + \frac{1}{8\pi k(k-\frac{1}{4})}$$

for $x > 0$ and hence the upper bound stated in the claim.

It remains to prove the lower bound. To this end, we employ arguments inspired by [5] and first note that

$$\sinh(x) \leq x + x^3$$

holds for $x \in (0, 1)$, which implies

$$\frac{\sin\left(\frac{t}{k}\right)}{\frac{t}{k}} \leq 1 + \frac{1}{k^{\frac{4}{3}}} \tag{3.9}$$

for $k \in \mathbb{N}$ and $0 < t \leq t_k := k^{\frac{1}{3}}$. In the following, we fix $x > 0$ and let $y > x$. Then the monotonicity of K_0 and (3.9) yield

$$F_k(y, x) \geq \int_0^{t_k} K_0\left(t\left(1 + \frac{1}{k^{\frac{4}{3}}}\right)\right) e^{-\frac{xt}{y}}\,dt$$

and therefore

$$\begin{aligned} &F_k(y,x) - G(y,x) \\ &\geq \int_0^{t_k} \left[K_0\left(t\left(1+\frac{1}{k^{\frac{4}{3}}}\right)\right) - K_0(t)\right] e^{-\frac{xt}{y}}\,dt - \int_{t_k}^{\infty} K_0(t) e^{-\frac{xt}{y}}\,dt. \end{aligned} \tag{3.10}$$

From

$$K_0(t) \leq K_{\frac{1}{2}}(t) = \sqrt{\frac{\pi}{2t}} e^{-t}$$

we then find

$$\int_{t_k}^{\infty} K_0(t) e^{-\frac{xt}{y}}\,dt \leq \sqrt{\frac{\pi}{2t_k}} e^{-t_k} \int_0^{\infty} e^{-\frac{xt}{y}}\,dt = \sqrt{\frac{\pi}{2t_k}} e^{-t_k} \frac{y}{x}.$$

For any $y_0 > x$ and $\delta \in (0,1)$, we thus find $k_0 \in \mathbb{N}$ such that

$$\int_{t_k}^{\infty} K_0(t) e^{-\frac{xt}{y}}\,dt \leq \frac{\delta}{k^{\frac{4}{3}}} \tag{3.11}$$

holds for $|y - y_0| < y_0 - x$ and $k \geq k_0$.

In order to estimate the other term in (3.10), we note that for $t \in \mathbb{R}$ there exists $\xi_k \in \left(t, t + \frac{t}{k^{\frac{4}{3}}}\right)$ such that

$$K_0\left(t\left(1+\frac{1}{k^{\frac{4}{3}}}\right)\right) - K_0(t) = K_0'(\xi_k)\frac{1}{k^{\frac{4}{3}}} = -K_1(\xi_k)\frac{t}{k^{\frac{4}{3}}} \geq -K_1(t)\frac{t}{k^{\frac{4}{3}}}.$$

This implies

$$\begin{aligned} \int_0^{t_k} \left[K_0\left(t\left(1+\frac{1}{k}\right)\right) - K_0(t)\right] e^{-\frac{xt}{y}}\,dt &\geq -\frac{1}{k^{\frac{4}{3}}} \int_0^{t_k} K_1(t) t e^{-\frac{xt}{y}}\,dt \\ &\geq -\frac{1}{k^{\frac{4}{3}}} \int_0^{\infty} K_1(t) t e^{-\frac{xt}{y}}\,dt, \end{aligned}$$

where [38, p. 388] gives

$$\int_0^{\infty} K_1(t) t e^{-\frac{xt}{y}}\,dt \leq \int_0^{\infty} K_1(t) t\,dt = \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{\pi}{2}.$$

Combined with (3.11), it thus follows that for any $x > 0$, $y_0 > x$ and $\delta \in (0,1)$ there exists $k_0 \in \mathbb{N}$ such that

$$F_k(y,x) - G(y,x) \geq -\left(\frac{\pi}{2} + \delta\right)\frac{1}{k^{\frac{4}{3}}}$$

holds for $k \geq k_0$ and $|y - y_0| < y_0 - x$.

We now proceed by taking $y_0 = \iota(x)$ and note that there exists $k_0' \in \mathbb{N}$ such that $|\iota_k(x) - \iota(x)| < \iota(x) - x$ holds for $k \geq k_0'$. Combined with (3.6), we then conclude that for given $\delta \in (0,1)$ we can find $k_0 \in \mathbb{N}$ such that

$$\begin{aligned} F_k(\iota_k, x) - G(\iota, x) &= F_k(\iota_k, x) - G(\iota_k, x) + G(\iota_k, x) - G(\iota, x) \\ &\geq -\left(\frac{\pi}{2} + \delta\right)\frac{1}{k^{\frac{4}{3}}} - \max_{\xi \in (\iota_k(x), \iota(x))} \frac{dG}{dy}(\xi, x)|\iota_k(x) - \iota(x)| \\ &\geq -\left(\frac{\pi}{2} + \delta\right)\frac{1}{k^{\frac{4}{3}}} - \frac{1}{3}|\iota_k(x) - \iota(x)| \end{aligned}$$

holds for $k \geq k_0$. It follows that

$$\beta_k(x) = \frac{F_k(\iota_k, x) - G(\iota, x)}{\iota_k(x) - \iota(x)} \leq \frac{1}{3} + \left(\frac{\pi}{2} + \delta\right) \frac{1}{k^{\frac{4}{3}}} \frac{1}{|\iota_k(x) - \iota(x)|}$$

and since $|\iota_k(x) - \iota(x)| = |u_k(x)| \geq (\frac{\pi}{4} - \delta)\frac{1}{k}$ holds for sufficiently large k we therefore have

$$\beta_k(x) \leq \frac{1}{3} + \frac{\frac{\pi}{2} + \delta}{\frac{\pi}{4} - \delta} \frac{1}{k^{\frac{1}{3}}}.$$

Consequently, we may choose k_0 such that

$$\beta_k(x) \leq \frac{1}{3} + \varepsilon$$

holds for $k \geq k_0$. Overall, this yields

$$1 \leq \exp\left(\int_0^x \beta_k(t)\, dt\right) \leq \exp\left(\left(\frac{1}{3} + \varepsilon\right) x\right)$$

for $k \geq k_0$. Recalling (3.8) and

$$u_k(x) = u_k(0) \exp\left(\int_0^x \beta_k(t)\, dt\right),$$

the claim thus follows.

□

Remark 3.4 Lemma 3.3 improves the bound obtained in [15, Theorem 2.1] as follows: For any $\varepsilon, \nu > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$j_{\nu,k} < k\, \iota\left(\frac{\nu}{k}\right) - (1 - \varepsilon)\frac{\pi}{4}$$

holds for $k \geq k_0$.

4 Spectral characterization

For $\alpha > 1$ recall the operator

$$L_\alpha = -\Delta + \alpha^2 \partial_\theta^2.$$

If $\varphi \in H_0^1(\mathbf{B})$ is an eigenfunction of $-\Delta$ corresponding to the eigenvalue $j_{\ell,k}^2$, then it follows from the representation (2.1) that φ is also an eigenfunction of L_α with

$$L_\alpha \varphi = (j_{\ell,k}^2 - \alpha^2 \ell^2)\varphi.$$

Since the eigenfunctions of $-\Delta$ constitute an orthonormal basis of $L^2(\mathbf{B})$, we find that the Dirichlet eigenvalues of L_α are given by

$$\left\{ j_{\ell,k}^2 - \alpha^2 \ell^2 :\ \ell \in \mathbb{N}_0,\ k \in \mathbb{N} \right\}.$$

In the following, we wish to study this set in more detail. The following result already shows a stark contrast to the case $\alpha \in [0, 1]$.

Proposition 4.1 *Let $\alpha > 1$. Then the spectrum of the operator $L_\alpha = -\Delta + \alpha^2 \partial_\theta^2$ is unbounded from above and below.*

Proof For $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ we write

$$j_{\ell,k}^2 - \alpha^2\ell^2 = (j_{\ell,k} + \alpha\ell)\ell\left(\frac{j_{\ell,k}}{\ell} - \alpha\right)$$

and note that Corollary 2.2 implies

$$1 - \alpha + \frac{\left(\frac{3\pi}{8}(4k-2)\right)^{\frac{2}{3}}}{2^{\frac{1}{3}}}\ell^{-\frac{2}{3}} < \frac{j_{\ell,k}}{\ell} - \alpha < 1 - \alpha + \frac{\left(\frac{3\pi}{2}k\right)^{\frac{2}{3}}}{2^{\frac{1}{3}}}\ell^{-\frac{2}{3}} + \frac{3}{20}\left(\frac{3\pi}{2}k\right)^{\frac{4}{3}}\frac{2^{\frac{1}{3}}}{\ell^{\frac{4}{3}}}. \tag{4.1}$$

If we choose sequences $(\ell_i)_i$, $(k_i)_i$, such that $\frac{\ell_i}{k_i} \to \infty$, this readily implies $j_{\ell_i,k_i}^2 - \alpha^2\ell_i^2 \to -\infty$, whereas sequences such that $\frac{\ell_i}{k_i} \to 0$ yield $j_{\ell_i,k_i}^2 - \alpha^2\ell_i^2 \to \infty$ and thus the claim. □

In particular, this proves the first part of Theorem 1.1. As noted in the introduction, it is not clear whether the spectrum of L_α only consists of isolated points. Indeed, note that Lemma 3.3 suggests that certain subsequences of $j_{\ell,k} - \alpha\ell$ may converge and it is unclear if there exists a subsequence that even converges to zero. In particular, the spectrum of the operator could even be dense in $\mathbb{R}$.

This is excluded by the second part of Theorem 1.1 which we restate as follows.

Theorem 4.2 *There exists a sequence $(\alpha_n)_n \subset (1,\infty)$ such that the following properties hold for $n \in \mathbb{N}$:*

(i) *The spectrum of L_{α_n} consists of eigenvalues with finite multiplicity.*
(ii) *There exists $c_n > 0$ such that for each $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ we either have $j_{\ell,k}^2 - \alpha_n^2\ell^2 = 0$ or*

$$|j_{\ell,k}^2 - \alpha_n^2\ell^2| \geq c_n j_{\ell,k}. \tag{4.2}$$

(iii) *The spectrum of L_{α_n} has no finite accumulation points.*

Proof We set $\sigma_n := \frac{1}{n}$ and $\alpha_n := \frac{\iota(\sigma_n)}{\sigma_n}$, where the function ι is given by Theorem 3.1. It then suffices to show that there exists $n_0 \in \mathbb{N}$ such that properties (i)-(iii) hold for $n \geq n_0$. In the following, we fix $n \in \mathbb{N}$ and assume that there exists $\Lambda \in \mathbb{R}$ and increasing sequences $(\ell_i)_i$, $(k_i)_i$ such that $j_{\ell_i,k_i}^2 - \alpha_n^2\ell_i^2 \to \Lambda$ as $i \to \infty$. Note that the case of an eigenvalue with infinite multiplicity, i.e., $j_{\ell_i,k_i}^2 - \alpha_n^2\ell_i^2 = \Lambda$ for all i, is included here. The identity $j_{\ell,k}^2 - \alpha_n^2\ell^2 = (j_{\ell,k} + \alpha_n\ell)\ell\left(\frac{j_{\ell,k}}{\ell} - \alpha_n\right)$ then implies that we must have

$$\frac{j_{\ell_i,k_i}}{\ell_i} \to \alpha_n. \tag{4.3}$$

Our goal is to show that such sequences can only converge of order $\frac{1}{l_i}$, which will allow us to derive a suitable contradiction.

Firstly, the estimate (4.1) implies that there must exist $\sigma \in (0,\infty)$ such that $\frac{\ell_i}{k_i} \to \sigma$. We now claim that for any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that

$$(1-\varepsilon)\frac{j_{\sigma k_i,k_i}}{\sigma k_i} < \frac{j_{\ell_i,k_i}}{\ell_i} < (1+\varepsilon)\frac{j_{\sigma k_i,k_i}}{\sigma k_i} \qquad \text{for } i \geq i_0. \tag{4.4}$$

To this end, we first assume that $\ell_i < \sigma k_i$ holds. Then Proposition 2.1(ii) implies

$$\frac{j_{\ell_i,k_i}}{\ell_i} > \frac{j_{\sigma k_i,k_i}}{\sigma k_i}$$

and, in particular, the lower bound. Moreover, the fact that the function $\nu \mapsto j_{\nu,k}$ is increasing for fixed k yields

$$\frac{j_{\ell_i,k_i}}{\ell_i} \le \frac{j_{\sigma k_i,k_i}}{\ell_i} = \frac{\sigma k_i}{\ell_i}\frac{j_{\sigma k_i,k_i}}{\sigma k_i}.$$

Noting that $\frac{\sigma k_i}{\ell_i} \to 1$ as $i \to \infty$, we conclude that for any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that

$$\frac{j_{\ell_i,k_i}}{\ell_i} < (1+\varepsilon)\frac{j_{\sigma k_i,k_i}}{\sigma k_i}$$

holds for $i \ge i_0$ with $\ell_i < \sigma k_i$. The case $\ell_i \ge \sigma k_i$ can be treated analogously.

Overall, (4.4) implies

$$(1-\varepsilon)\frac{\iota(\sigma)}{\sigma} \le \liminf_{n\to\infty} \frac{j_{\ell_i,k_i}}{\ell_i} \le \limsup_{n\to\infty} \frac{j_{\ell_i,k_i}}{\ell_i} \le (1+\varepsilon)\frac{\iota(\sigma)}{\sigma} \tag{4.5}$$

for arbitrary $\varepsilon > 0$, with the function ι given by Theorem 3.1. In particular, (4.3) then yields

$$\frac{\iota(\sigma)}{\sigma} = \lim_{i\to\infty} \frac{j_{\ell_i,k_i}}{\ell_i} = \alpha_n$$

and Lemma 3.2 thus implies that we must have $\sigma = \sigma_n$ due to our choice of α_n. In particular, it follows that $\frac{\ell_i}{k_i} \to \sigma_n$.

We now distinguish two cases:

Case 1 There exists $i_0 \in \mathbb{N}$ such that $\frac{\ell_i}{k_i} \ge \sigma_n$ holds for $i \ge i_0$.
In this case, Proposition 2.1(ii) implies

$$\frac{j_{\ell_i,k_i}}{\ell_i} - \alpha_n \le \frac{j_{\sigma_n k_i,k_i}}{\sigma_n k_i} - \alpha_n = \frac{1}{\sigma_n}\left(\frac{j_{\sigma_n k_i,k_i}}{k_i} - \iota(\sigma_n)\right),$$

so that Lemma 3.3 yields

$$\frac{j_{\ell_i,k_i}}{\ell_i} - \alpha_n \le -\frac{\pi}{8\sigma_n k_i}$$

for $i \ge i_0$, after possibly enlarging i_0. In particular, this implies

$$|j_{\ell_i,k_i} - \alpha_n \ell_i| = \ell_i \left|\frac{j_{\ell_i,k_i}}{\ell_i} - \alpha_n\right| \ge \frac{\pi \ell_i}{8\sigma_n k_i} \ge \frac{\pi}{8}$$

for $i \ge i_0$ and therefore $\liminf_{i\to\infty} |j_{\ell,k} - \alpha_n \ell_i| \ge \frac{\pi}{8}$.

Case 2 There exists $i_0 \in \mathbb{N}$ such that $\frac{\ell_i}{k_i} < \sigma_n$ holds for $i \ge i_0$.
We may write $\ell_i = \sigma_n k_i - \delta_i$ with $\delta_i > 0$ satisfying $\frac{\delta_i}{k_i} \to 0$ as $i \to \infty$. Then

$$\begin{aligned} j_{\ell_i,k_i} - \alpha_n \ell_i &= j_{(\sigma_n k_i - \delta_i),k_i} - \alpha_n(\sigma_n k_i - \delta_i) \\ &= (j_{\sigma_n k_i,k_i} - \alpha_n \sigma_n k_i) + (j_{(\sigma_n k_i-\delta_i),k_i} - j_{\sigma_n k_i,k_i}) + \alpha_n \delta_i \\ &= (j_{\sigma_n k_i,k_i} - \alpha_n \sigma_n k_i) + R_{n,i}\delta_i, \end{aligned}$$

where we have set

$$R_{n,i} := \alpha_n - \frac{j_{\sigma_n k_i,k_i} - j_{\sigma_n k_i-\delta_i,k_i}}{\delta_i}.$$

By Lemma 3.3 we may further enlarge i_0 to ensure that

$$j_{\sigma_n k_i, k_i} - \alpha_n \sigma_n k_i \geq -\frac{\pi}{4} e^{\sigma_n/3}$$

holds for $i \geq i_0$.

Next, Proposition 2.1(iv) gives

$$\frac{j_{\sigma_n k_i, k_i} - j_{\sigma_n k_i - \delta_i, k_i}}{\delta_i} \in \left(1, \frac{\pi}{2}\right)$$

and hence

$$\liminf_{i\to\infty} R_{n,i} \geq \alpha_n - \frac{\pi}{2}.$$

Since $\alpha_n = \frac{\iota(\sigma_n)}{\sigma_n} \to \infty$ as $n \to \infty$ by Lemma 3.2, this term is positive for sufficiently large n, and it therefore follows that

$$\liminf_{i\to\infty} (j_{\ell_i, k_i} - \alpha_n \ell_i) \geq -\frac{\pi}{4} e^{\sigma_n/3} + \left(\alpha_n - \frac{\pi}{2}\right) \inf_{i\in\mathbb{N}} \delta_i.$$

In order to show that the right hand side is positive, we recall that $\sigma_n = \frac{1}{n}$ and therefore the fact that $\ell_i = \sigma_n k_i - \delta_i$ must be a natural number implies $\delta_i = \frac{n'}{n}$ for some $n' \in \mathbb{N}$ and, in particular, $\inf_i \delta_i = \frac{1}{n}$. Moreover, by Lemma 3.2 the associated $\alpha_n = \frac{\iota(\sigma_n)}{\sigma_n}$ is uniquely determined by the equation

$$\pi n = \frac{\pi}{\sigma_n} = \sqrt{\alpha_n^2 - 1} - \left(\frac{\pi}{2} - \arcsin \frac{1}{\alpha_n}\right).$$

Since the right hand side is strictly increasing in α_n and we have

$$\frac{\sqrt{n^2-1} - \left(\frac{\pi}{2} - \arcsin \frac{1}{n}\right)}{n} = \sqrt{1 - \frac{1}{n^2}} + \frac{1}{n} \arcsin \frac{1}{n} - \frac{\pi}{2n} \to 1 < \pi$$

as $n \to \infty$, there must exist $n_0 \in \mathbb{N}$ such that $\alpha_n > n$ holds for $n \geq n_0$. We thus have

$$\begin{aligned} -\frac{\pi}{4} e^{\sigma_n/3} + \left(\alpha_n - \frac{\pi}{2}\right) \inf_i \delta_i &\geq -\frac{\pi}{4} e^{\sigma_n/3} + \frac{1}{n}\left(n - \frac{\pi}{2}\right) \\ &= 1 - \pi \left(\frac{1}{2n} + \frac{e^{\frac{1}{3n}}}{4}\right) \to 1 - \frac{\pi}{4} > 0 \end{aligned}$$

as $n \to \infty$. We conclude that after possibly further enlarging n_0,

$$\kappa_n := -\frac{\pi}{4} e^{\sigma_n/3} + \left(\alpha_n - \frac{\pi}{2}\right) \inf_i \delta_i > 0$$

holds for $n \geq n_0$.

Since we may always pass to a subsequence such that one of these two cases holds, we overall find that

$$\liminf_{i\to\infty} |j_{\ell,k} - \alpha_n \ell_i| \geq \min\left\{\kappa_n, \frac{\pi}{4}\right\} > 0 \tag{4.6}$$

for any sequences $(\ell_i)_i, (k_i)_i$ such that $\frac{\ell_i}{k_i} \to \sigma_n = \frac{1}{n}$, provided $n \geq n_0$. In particular, it follows that $j_{\ell,k}^2 - \alpha_n^2 \ell^2 = (j_{\ell,k} - \alpha_n \ell)(j_{\ell,k} + \alpha_n \ell)$ cannot converge to Λ, which implies (i) and (iii).

In order to complete the proof, we now consider an arbitrary pair of increasing sequences $(\ell_i)_i$, $(k_i)_i$ and distinguish different cases: If $\frac{\ell_i}{k_i} \to 0$ or $\frac{\ell_i}{k_i} \to \infty$, (4.1) implies

$$\liminf_{i\to\infty} |j_{\ell_i,k_i} - \alpha_n \ell_i| \geq \alpha_n - 1 > 0.$$

If $\frac{\ell_i}{k_i} \to \sigma_n = \frac{1}{n}$, we find that (4.6) holds as shown above. In the remaining case, we may pass to a subsequence such that $\frac{\ell_i}{k_i} \to \sigma^*$ for some $\sigma^* > 0$, $\sigma^* \neq \sigma_n$. Repeating the arguments used to obtain (4.4) and (4.5), it then easily follows that

$$\lim_{i\to\infty} \frac{j_{\ell_i,k_i}}{\ell_i} = \frac{\iota(\sigma^*)}{\sigma^*} \neq \alpha_n$$

and therefore

$$\liminf_{i\to\infty} |j_{\ell_i,k_i} - \alpha_n \ell_i| \geq \liminf_{i\to\infty} \ell_i \left| \frac{j_{\ell_i,k_i}}{\ell_i} - \alpha_n \right| = \infty.$$

Overall, this implies

$$\gamma_n := \lim_{N\to\infty} \inf_{\ell,k\geq N} |j_{\ell,k} - \alpha_n \ell| > 0$$

for $n \geq n_0$. Consequently, taking $N_0 \in \mathbb{N}$ such that $\inf_{\ell,k\geq N} |j_{\ell,k} - \alpha_n \ell| > \frac{\gamma_n}{2}$ holds for $N \geq N_0$ and setting

$$c_n := \min \left\{ \frac{\gamma_n}{2}, \inf_{\substack{\ell,k\leq N_0 \\ j_{\ell,k}\neq\alpha_n\ell}} |j_{\ell,k} - \alpha_n \ell| \right\} > 0$$

yields

$$|j_{\ell,k}^2 - \alpha_n^2 \ell^2| = |j_{\ell,k} - \alpha_n \ell||j_{\ell,k} + \alpha_n \ell| \geq c_n j_{\ell,k}$$

as claimed in (ii). This completes the proof. □

Remark 4.3 (i) The sequence $(\alpha_n)_n$ can be characterized further by noting that

$$\pi n = \sqrt{\alpha_n^2 - 1} - \left(\frac{\pi}{2} - \arcsin \frac{1}{\alpha_n} \right)$$

implies

$$\alpha_n^2 = 1 + \left(\pi n + \frac{\pi}{2} - \arcsin \frac{1}{\alpha_n} \right)^2.$$

Since $\arcsin \frac{1}{\alpha_n} = O(n^{-2})$, this implies

$$\alpha_n^2 \approx 1 + \left(\pi n + \frac{\pi}{2} \right)^2.$$

(ii) The methods used above can be further extended to include some additional values of α. If we let $\sigma = \frac{m}{n}$ with $m, n \in \mathbb{N}$, we find that $\inf_i \delta_i = \frac{1}{n}$ and similar arguments as above then lead to the condition

$$0 < \sqrt{\frac{1}{n^2} + \frac{\pi^2}{m^2}} - \pi \left(\frac{1}{2n} + \frac{e^{\frac{m}{3n}}}{4} \right).$$

As $n \to \infty$, we find that this holds for $m = 1, 2, 3$.

Moreover, we note that numerical computations imply that the result should also hold for $\sigma = 1, 2, 3$.

5 Variational characterization of ground states

We now return to solutions of (1.4). Setting

$$L_{\alpha,m} := -\Delta + \alpha^2 \partial_\theta^2 + m$$

for $\alpha > 1$, $m \in \mathbb{R}$, our first goal is to find a suitable domain for the quadratic form

$$u \mapsto \langle L_{\alpha,m} u, u \rangle_{L^2(\mathbf{B})} = \int_{\mathbf{B}} (L_{\alpha,m} u) u \, dx = \int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2 \right) \, dx.$$

In order to simplify the notation, we set

$$\begin{aligned}
\mathcal{I}^+_{\alpha,m} &:= \left\{ (\ell, k) \in \mathbb{N}_0 \times \mathbb{N} : j^2_{\ell,k} - \alpha^2 \ell^2 + m > 0 \right\} \\
\mathcal{I}^0_{\alpha,m} &:= \left\{ (\ell, k) \in \mathbb{N}_0 \times \mathbb{N} : j^2_{\ell,k} - \alpha^2 \ell^2 + m = 0 \right\} \\
\mathcal{I}^-_{\alpha,m} &:= \left\{ (\ell, k) \in \mathbb{N}_0 \times \mathbb{N} : j^2_{\ell,k} - \alpha^2 \ell^2 + m < 0 \right\}
\end{aligned}$$

for $\alpha > 1$, $m \in \mathbb{R}$, i.e., the index sets of positive, zero and negative eigenvalues, respectively. Instead of restricting ourselves to the sequence $(\alpha_n)_n$ given by Theorem 1.1, we consider

$$\mathcal{A} := \left\{ \alpha > 1 : \; |\mathcal{I}^0_{\alpha,0}| < \infty \text{ and } \min_{(\ell,k) \notin \mathcal{I}^0_{\alpha,0}} |j_{\ell,k} - \alpha \ell| > 0 \right\}.$$

In particular, $\mathcal{A}$ contains the sequence $(\alpha_n)_n$ and is therefore nonempty and unbounded. Moreover, writing $j^2_{\ell,k} - \alpha^2 \ell^2 = (j_{\ell,k} + \alpha \ell)(j_{\ell,k} - \alpha \ell)$ we find that for any $\alpha \in \mathcal{A}$ there exists $c_\alpha > 0$ such that

$$|j^2_{\ell,k} - \alpha^2 \ell^2| \geq c_\alpha j_{\ell,k}$$

holds for $(\ell, k) \notin \mathcal{I}^0_{\alpha,0}$.

Lemma 5.1 *Let $\alpha \in \mathcal{A}$ and $m \in \mathbb{R}$. Then $\mathcal{I}^0_{\alpha,m}$ is finite and there exists $c_m > 0$ such that any $(\ell, k) \notin \mathcal{I}^0_{\alpha,m}$ satisfy*

$$|j^2_{\ell,k} - \alpha^2 \ell^2 + m| \geq c_m j_{\ell,k}.$$

In particular, the spectrum of $L_{\alpha,m}$ has no finite accumulation points.

Proof Let $c_\alpha > 0$ be given as above. We first note that

$$|j^2_{\ell,k} - \alpha^2 \ell^2 + m| = (j_{\ell,k} + \alpha \ell) \left| j_{\ell,k} - \alpha \ell + \frac{m}{j_{\ell,k} + \alpha \ell} \right|,$$

so the fact that $\mathcal{I}^0_{\alpha,0}$ is finite by assumption implies that $\mathcal{I}^0_{\alpha,m}$ is finite as well. Moreover, there exist $\ell_0, k_0 \in \mathbb{N}$ such that

$$|j^2_{\ell,k} - \alpha_n^2 \ell^2 + m| \geq (j_{\ell,k} + \alpha \ell) \frac{c_\alpha}{2}$$

holds for all $(\ell, k) \notin \mathcal{I}^0_{\alpha,m}$ with $\ell \geq \ell_0, k \geq k_0$. Setting

$$c_m := \min\left\{\frac{c_\alpha}{2}, \min_{\substack{(\ell,k)\notin\mathcal{I}^0_{\alpha,m}\\ \ell\leq\ell_0,k\leq k_0}} \left| j_{\ell,k} - \alpha\ell + \frac{m}{j_{\ell,k}+\alpha\ell}\right|\right\} > 0$$

then completes the proof. □

Next, we recall the eigenfunctions $\varphi_{\ell,k}, \psi_{\ell,k}$ given in (2.1) and set

$$E_{\alpha,m} := \left\{ u \in L^2(\mathbf{B}) : \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} \left| j_{\ell,k}^2 - \alpha^2\ell^2 + m\right| \left(|\langle u, \varphi_{\ell,k}\rangle|^2 + |\langle u, \psi_{\ell,k}\rangle|^2\right) < \infty \right\}$$

for $\alpha \in \mathcal{A}$, $m \in \mathbb{R}$ and endow $E_{\alpha,m}$ with the scalar product

$$\begin{aligned}\langle u, v\rangle_{\alpha,m} :=& \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} \left| j_{\ell,k}^2 - \alpha^2\ell^2 + m\right| \left(\langle u, \varphi_{\ell,k}\rangle\langle v, \varphi_{\ell,k}\rangle + \langle u, \psi_{\ell,k}\rangle\langle v, \psi_{\ell,k}\rangle\right)\\ &+ \sum_{(\ell,k)\in\mathcal{I}^0_{\alpha,m}} \left(\langle u, \varphi_{\ell,k}\rangle\langle v, \varphi_{\ell,k}\rangle + \langle u, \psi_{\ell,k}\rangle\langle v, \psi_{\ell,k}\rangle\right).\end{aligned}$$

In the following, $\|\cdot\|_{\alpha,m}$ denotes the norm induced by $\langle\cdot,\cdot\rangle_{\alpha,m}$.

Remark 5.2 For fixed $\alpha \in \mathcal{A}$, the norm $\|\cdot\|_{\alpha,m}$ is equivalent to $\|\cdot\|_{\alpha,0}$ and $E_{\alpha,m} = E_{\alpha,0}$, i.e., the spaces are equal as sets. Nonetheless, the use of an m-dependent scalar product is useful for the variational methods we will employ below.

We now consider the following decomposition associated to the eigenspaces of positive, zero and negative eigenvalues of $L_{\alpha,m}$, respectively:

$$\begin{aligned}
E^+_{\alpha,m} &:= \left\{ u \in E_{\alpha,m} : \int_{\mathbf{B}} u\,\varphi_{\ell,k}\,dx = \int_{\mathbf{B}} u\,\psi_{\ell,k}\,dx = 0 \text{ for } (\ell,k) \in \mathcal{I}^0_{\alpha,m} \cup \mathcal{I}^-_{\alpha,m}\right\}\\
E^0_{\alpha,m} &:= \left\{ u \in E_{\alpha,m} : \int_{\mathbf{B}} u\,\varphi_{\ell,k}\,dx = \int_{\mathbf{B}} u\,\psi_{\ell,k}\,dx = 0 \text{ for } (\ell,k) \in \mathcal{I}^+_{\alpha,m} \cup \mathcal{I}^-_{\alpha,m}\right\}\\
E^-_{\alpha,m} &:= \left\{ u \in E_{\alpha,m} : \int_{\mathbf{B}} u\,\varphi_{\ell,k}\,dx = \int_{\mathbf{B}} u\,\psi_{\ell,k}\,dx = 0 \text{ for } (\ell,k) \in \mathcal{I}^+_{\alpha,m} \cup \mathcal{I}^0_{\alpha,m}\right\}
\end{aligned}$$

so that, in particular,

$$E_{\alpha,m} = E^+_{\alpha,m} \oplus E^0_{\alpha,m} \oplus E^-_{\alpha,m} = E^+_{\alpha,m} \oplus F_{\alpha,m},$$

where we have set $F_{\alpha,m} := E^0_{\alpha,m} \oplus E^-_{\alpha,m}$. In the following, we will routinely write

$$u = u^+ + u^0 + u^-$$

where $u^+ \in E^+_{\alpha,m}$, $u^0 \in E^0_{\alpha,m}$, $u^- \in E^-_{\alpha,m}$ are uniquely determined. The use of the norm $\|\cdot\|_{\alpha,m}$ allows us to write

$$\langle L_{\alpha,m}u, u\rangle_{L^2(\mathbf{B})} = \int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\,dx = \|u^+\|^2_{\alpha,m} - \|u^-\|^2_{\alpha,m}.$$

Importantly, $E_{\alpha,m}$ has the following embedding properties:

Proposition 5.3 *Let $p \in (2,4)$, $\alpha \in \mathcal{A}$ and $m \in \mathbb{R}$. Then $E_{\alpha,m} \subset L^p(\mathbf{B})$ and the embedding*

$$E_{\alpha,m} \hookrightarrow L^p(\mathbf{B})$$

is compact.

Proof Because of the compact embedding $H^{\frac{1}{2}}(\mathbf{B}) \hookrightarrow L^p(\mathbf{B})$ it is enough to show that the embedding

$$E_{\alpha,m} \hookrightarrow H^{\frac{1}{2}}(\mathbf{B})$$

is well-defined and continuous. We first note that it suffices to consider $u \in E^+_{\alpha,m} \oplus E^-_{\alpha,m}$, since the space $E^0_{\alpha,m}$ is finite–dimensional and only contains smooth functions. By Lemma 5.1, there exists $c > 0$ such that

$$|j^2_{\ell,k} - \alpha^2\ell^2 + m| \geq c j_{\ell,k}$$

holds for $(\ell,k) \notin \mathcal{I}^0_{\alpha,m}$. This implies

$$\begin{aligned}
\|u\|^2_{H^{\frac{1}{2}}} &= \sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} j_{\ell,k}\left(|\langle u,\varphi_{\ell,k}\rangle|^2 + |\langle u,\psi_{\ell,k}\rangle|^2\right)\\
&\leq \frac{1}{c}\sum_{\ell=0}^{\infty}\sum_{k=1}^{\infty} |j^2_{\ell,k} - \alpha^2\ell^2 + m|\left(|\langle u,\varphi_{\ell,k}\rangle|^2 + |\langle u,\psi_{\ell,k}\rangle|^2\right)\\
&= \frac{1}{c}\|u\|^2_{\alpha,m}
\end{aligned}$$

and thus the claim. □

Remark 5.4 It is natural to ask for the optimal $q > 2$ such that the preceding proposition holds for $p \in (2,q)$. We conjecture that $q = 10$ due to two observations:

Firstly, $q = 10$ appears in the degenerate elliptic case $\alpha = 1$ treated in [19] as the critical exponent for Sobolev-type embeddings for the associated degenerate operator. Secondly, this exponent also appears in a Pohožaev-type identity in [22] with respect to related semilinear problems involving the Tricomi operator.

In particular, the map

$$I_p : E_{\alpha,m} \to \mathbb{R}, \qquad I_p(u) := \frac{1}{p}\int_{\mathbf{B}} |u|^p\,dx = \frac{1}{p}\|u\|^p_p$$

is well-defined and continuous for $p \in (2,4)$. We note the following properties corresponding to the conditions of Theorem 35 in [36].

Lemma 5.5 *Let $\alpha \in \mathcal{A}$, $m \in \mathbb{R}$ and $p \in (2,4)$. Then the following properties hold:*

(i) $\frac{1}{2}I'_p(u)u > I_p(u) > 0$ *for all $u \not\equiv 0$ and I_p is weakly lower semicontinuous.*
(ii) $I'_p(u) = o(\|u\|_{\alpha,m})$ *as $u \to 0$.*
(iii) $\frac{I_p(su)}{s^2} \to \infty$ *uniformly in u on weakly compact subsets of $E_{\alpha,m} \setminus \{0\}$ as $s \to \infty$.*
(iv) I'_p *is a compact map.*

Proof The properties (i),(ii) and (iv) follow from routine computations and Proposition 5.3, while (iii) has essentially been proved in [36, Theorem 16], though we can give a slightly simpler argument in this case:

Let $W \subset E_{\alpha,m} \setminus \{0\}$ be a weakly compact subset. We claim that there exists $c > 0$ such that $\|u\|_p \geq c$ holds for $u \in W$. Indeed, if this was false, there would exist a sequence $(u_n)_n \subset W$ such that $u_n \to 0$ in $L^p(\mathbf{B})$. The weak compactness of W and Proposition 5.3 would then imply $u_n \rightharpoonup 0$, contradicting the fact that $0 \notin W$. We thus have

$$\frac{I_p(su)}{s^2} = \frac{s^{p-2}}{p}\|u\|_p^p \geq \frac{c^p}{p}s^{p-2}$$

and clearly the right hand side goes to infinity uniformly as $s \to \infty$. □

In the following, we always assume that $p \in (2, 4)$ is fixed and consider the energy functional $\Phi_{\alpha,m} : E_{\alpha,m} \to \mathbb{R}$ given by

$$\begin{aligned}\Phi_{\alpha,m}(u) &:= \frac{1}{2}\|u^+\|_{\alpha,m}^2 - \frac{1}{2}\|u^-\|_{\alpha,m} - I_p(u)\\ &= \frac{1}{2}\int_{\mathbf{B}} \left(|\nabla u|^2 - \alpha^2|\partial_\theta u|^2 + mu^2\right)\,dx - \frac{1}{p}\int_{\mathbf{B}} |u|^p\,dx.\end{aligned}$$

In particular, any critical point $u \in E_{\alpha,m}$ of $\Phi_{\alpha,m}$ satisfies

$$\int_{\mathbf{B}} |u|^{p-2}u\,\varphi\,dx = \langle u^+, \varphi\rangle_{\alpha,m} - \langle u^-, \varphi\rangle_{\alpha,m} = \int_{\mathbf{B}} u\,L_{\alpha,m}\varphi\,dx$$

and can thus be interpreted as a weak solution of (1.4). As outlined in the introduction, we will now characterize ground states of $\Phi_{\alpha,m}$ by considering the generalized Nehari manifold

$$\mathcal{N}_{\alpha,m} := \left\{u \in E_{\alpha,m} \setminus F_{\alpha,m} :\ \Phi'_{\alpha,m}(u)u = 0 \text{ and } \Phi'_{\alpha,m}(u)v = 0 \text{ for all } v \in F_{\alpha,m}\right\}.$$

In particular, $\mathcal{N}_{\alpha,m}$ contains all nontrivial critical points of Φ. Consequently, the value

$$c_{\alpha,m} := \inf_{u\in\mathcal{N}_{\alpha,m}} \Phi_{\alpha,m}(u)$$

is the ground state energy in the sense that any critical point $u \in E_{\alpha,m} \setminus \{0\}$ of $\Phi_{\alpha,m}$ satisfies $\Phi_{\alpha,m}(u) \geq c_{\alpha,m}$. This motivates the following definition.

Definition 5.6 Let $\alpha \in \mathcal{A}$, $m \in \mathbb{R}$ and $p \in (2, 4)$. We call a function $u \in E_{\alpha,m}$ a **ground state solution** of (1.4), if u is a critical point of $\Phi_{\alpha,m}$ and satisfies $\Phi_{\alpha,m}(u) = c_{\alpha,m}$.

In order to show that ground state solutions exist, we wish to verify that $\Phi_{\alpha,m}$ satisfies condition (B_2) from [36]. To this end, we let $u \in E_{\alpha,m} \setminus F_{\alpha,m}$ and consider

$$\widehat{E}_{\alpha,m}(u) := \{tu + w :\ t \geq 0,\ w \in F_{\alpha,m}\} = \mathbb{R}^+u \oplus F_{\alpha,m}.$$

Importantly, $u \in \mathcal{N}_{\alpha,m}$ if and only if u is a critical point of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}(u)}$. Moreover, we have $\widehat{E}_{\alpha,m}(u) = \widehat{E}_{\alpha,m}(tu^+)$ for all $t \geq 0$, $u \in E_{\alpha,m} \setminus F_{\alpha,m}$, hence when considering $\widehat{E}_{\alpha,m}(u)$ we may always assume $u \in E^+_{\alpha,m}$. This will be useful in the following.

Lemma 5.7 *For each $u \in E_{\alpha,m} \setminus F_{\alpha,m}$ there exists a unique nontrivial critical point $\hat{m}(u)$ of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}}$. Moreover, $\hat{m}(u)$ is the unique global maximum of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}}$.*

Proof The following argument is essentially taken from [36, Proposition 39]. Without loss of generality we may assume $u \in E^+_{\alpha,m}$ and $\|u\|_{\alpha,m} = 1$.

Claim 1: There exists $R > 0$ such that $\Phi_{\alpha,m}(v) \leq 0$ holds for $v \in \widehat{E}_{\alpha,m}$ and $\|v\|_{\alpha,m} \geq R$. Indeed, if this was false there would exist a sequence $(v_n)_n \subset \widehat{E}_{\alpha,m}(u)$ such that $\|v_n\|_{\alpha,m} \to$

∞ and $\Phi_{\alpha,m}(v_n) > 0$. Setting $w_n := \frac{v_n}{\|v_n\|_{\alpha,m}}$ we may pass to a weakly convergent subsequence and note that

$$0 < \frac{\Phi_{\alpha,m}(v_n)}{\|v_n\|^2_{\alpha,m}} = \frac{1}{2}\|w_n^+\|^2_{\alpha,m} - \frac{1}{2}\|w_n^-\|^2_{\alpha,m} - \frac{1}{p}\frac{\big\|\|v_n\|_{\alpha,m}w_n\big\|_p^p}{\|v_n\|^2_{\alpha,m}}$$
$$\leq \|w_n\|^2_{\alpha,m} - \frac{I(|\|v_n\|_{\alpha,m}w_n)}{\|v_n\|^2_{\alpha,m}}$$

so that Lemma 5.5(iii) implies $0 < \frac{\Phi_{\alpha,m}(v_n)}{\|v_n\|^2_{\alpha,m}} \to -\infty$ if the weak limit is nonzero. Hence we must have $w_n \rightharpoonup 0$. Moreover, the inequality above also implies $\|w_n^+\|_{\alpha,m} \geq \|w_n^-\|_{\alpha,m}$. If $w_n^+ \to 0$, the latter also implies $w_n^- \to 0$ and therefore

$$\|w_n^0\|^2_{\alpha,m} = 1 - \|w_n^+\|_{\alpha,m} - \|w_n^-\|^2_{\alpha,m} \to 1.$$

The fact that $E^0_{\alpha,m}$ is finite-dimensional then implies that w_n^0 converges to a nontrivial function, which contradicts $w_n \rightharpoonup 0$. Hence w_n^+ cannot converge to zero and we may therefore pass to a subsequence such that $\|w_n^+\|_{\alpha,m} \geq \gamma$ holds from some $\gamma > 0$ and all n. However, by definition of $\widehat{E}_{\alpha,m}(u)$ we must have $w_n^+ = u\|w_n^+\|_{\alpha,m}$ and therefore there exists $c > 0$ such that $w_n^+ \to cu$ holds after passing to a subsequence, contradicting $w_n \rightharpoonup 0$. This proves Claim 1.

Next, we note that Lemma 5.5 yields $\Phi_{\alpha,m}(tu) = \frac{t^2}{2} + o(t^2)$ as $t \to 0$ and therefore

$$\sup_{\widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m} > 0.$$

Now Claim 1 implies that any maximizing sequence $(v_n)_n \subset \widehat{E}_{\alpha,m}(u)$ must remain bounded, so we may assume $v_n \rightharpoonup v$ after passing to a subsequence. Moreover, recalling that

$$\Phi_{\alpha,m}(v_n) = \frac{\|v_n^+\|^2_{\alpha,m}}{2} - \frac{\|v_n^-\|^2_{\alpha,m}}{2} - I_p(v_n),$$

we can use that v_n^+ is a multiple of u, while the norm $\|\cdot\|_{\alpha,m}$ and I_p are weakly lower semicontinuous on $E_{\alpha,m}$, making $\Phi_{\alpha,m}$ weakly upper semicontinuous on $\widehat{E}_{\alpha,m}(u)$. It thus follows that $\sup_{\widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m}$ is attained by a critical point u_0 of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}(u)}$. Noting that $\sup_{t\geq 0} \Phi_{\alpha,m}(tu) > 0$ since $u \in E^+_{\alpha,m}$, it follows that $u_0 \in \mathcal{N}_{\alpha,m}$.

It remains to prove that this is the only critical point of $\Phi_{\alpha,m}\big|_{\widehat{E}_{\alpha,m}(u)}$. To this end, we let $w \in E_{\alpha,m}$ such that $u_0 + w \in \widehat{E}_{\alpha,m}(u)$. Since $\widehat{E}_{\alpha,m}(u) = \widehat{E}_{\alpha,m}(u_0)$, there exists $s \geq -1$ such that $u_0 + w = (1+s)u_0 + v$ for some $v \in F_{\alpha,m}$. Setting

$$B(v_1, v_2) := \int_{\mathbf{B}} \left(\nabla v_1 \cdot \nabla v_2 - \alpha^2(\partial_\theta v_1)(\partial_\theta v_2) + m v_1 v_2\right) dx$$
$$= \langle v_1^+, v_2^+\rangle_{\alpha,m} - \langle v_1^-, v_2^-\rangle_{\alpha,m}$$

we then have

$$\Phi_{\alpha,m}(u_0 + w) - \Phi_{\alpha,m}(u_0) = \frac{1}{2}\left(B((1+s)u_0 + v, (1+s)u_0 + v) - B(u_0, u_0)\right)$$
$$- I_p((1+s)u_0 + v) + I_p(u_0)$$
$$= -\frac{\|v^-\|^2_{\alpha,m}}{2} + B\left(u_0, s\left(\frac{s}{2} - 1\right)u_0 + (1+s)v\right)$$
$$- I_p((1+s)u_0 + v) + I_p(u_0),$$

where the fact that $\Phi'_{\alpha,m}(u_0)(\cdot) = B(u_0, \cdot) - I'_p(u_0)(\cdot) = 0$ then implies

$$\begin{aligned} &B\left(u_0, s\left(\frac{s}{2}-1\right)u_0 + (1+s)v\right) - I_p((1+s)u_0+v) + I_p(u_0) \\ &\quad = I'_p(u_0)\left(s\left(\frac{s}{2}-1\right)u_0 + (1+s)v\right) - I_p((1+s)u_0+v) + I_p(u_0) \\ &\quad = \int_{\mathbf{B}} \left(|u_0|^{p-2}u_0\left(s\left(\frac{s}{2}-1\right)u_0 + (1+s)v\right) - \frac{1}{p}|(1+s)u_0+v|^p + \frac{1}{p}|u_0|^p\right) dx \\ &\quad < 0 \end{aligned}$$

by [35, Lemma 2.2]. □

We can then give the following existence result.

Proposition 5.8 *Let $\alpha \in \mathcal{A}$, $m \in \mathbb{R}$ and $p \in (2, 4)$. Then $c_{\alpha,m}$ is positive and attained by a critical point of $\Phi_{\alpha,m}$. In particular, (1.4) thus has a ground state solution.*

Moreover,

$$c_{\alpha,m} = \inf_{w \in E_{\alpha,m} \setminus F_{\alpha,m}} \max_{w \in \widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m}(w)$$

holds.

Proof Note that Lemma 5.5 and Lemma 5.7 imply that $\Phi_{\alpha,m}$ satisfies the conditions of [36, Theorem 35]. □

In particular, this implies Theorem 1.2(i). Notably, this minimax characterization of $c_{\alpha,m}$ will allow us to compare the ground state energy to the minimal energy among radial solutions, which we estimate in the following.

Lemma 5.9 *Let $p > 2$ and $m > -\lambda_1$, where $\lambda_1 > 0$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\mathbf{B}$. Then there exists a unique positive radial solution $u_m \in H^1_{0,rad}(\mathbf{B})$ of (1.4), i.e., satisfying*

$$\begin{cases} -\Delta u + mu = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B}. \end{cases}$$

Moreover, there exists $c > 0$ such that

$$\beta_m^{rad} := \Phi_{\alpha,m}(u_m) \geq cm^{\frac{2}{p-2}}$$

holds for all $\alpha > 1$ and $m \geq 0$.

Proof We consider the functional

$$\begin{aligned} J_m &: H^1_{0,rad}(\mathbf{B}) \to \mathbb{R} \\ J_m(u) &:= \frac{1}{2}\int_{\mathbf{B}} \left(|\nabla u|^2 + mu^2\right) dx - \frac{1}{p}\int_{\mathbf{B}} |u|^p\, dx \end{aligned}$$

which satisfies $J_m(u) = \Phi_{\alpha,m}(u)$ for every $u \in H^1_{0,rad}(\mathbf{B})$ and $\alpha > 1$. We then consider the classical Nehari manifold

$$\mathcal{N}_m^{rad} := \left\{u \in H^1_{0,rad}(\mathbf{B}) \setminus \{0\} :\ J'_m(u)u = 0\right\}.$$

Clearly, any nontrivial radial critical point u of $\Phi_{\alpha,m}$ is contained in $\mathcal{N}_m^{rad}$. Moreover, the map

$$(0,\infty) \to \mathbb{R}, \quad t \mapsto J_m(tu)$$

attains a unique maximum $t_u > 0$ for each $u \in H^1_{0,rad}(\mathbf{B}) \setminus \{0\}$ and simple computations yield

$$J_m(t_u u) = \sup_{t\geq 0} J_m(tu) = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\int_{\mathbf{B}} \left(|\nabla u|^2 + mu^2\right)\,dx}{\left(\int_{\mathbf{B}} |u|^p\,dx\right)^{\frac{2}{p}}}\right)^{\frac{p}{p-2}}$$

and t_u is the unique value $t > 0$ such that $tu \in \mathcal{N}_m$. It can be shown that

$$\beta_m^{rad} := \inf_{u\in\mathcal{N}_m^{rad}} J_m(u)$$

is a critical value of J_m, see e.g. [36]. Moreover, the principle of symmetric criticality (see e.g. [30]) shows that β_m^{rad} is in fact a critical value of $\Phi_{\alpha,m}$ and attained by a unique positive radial function u_m. We note that the uniqueness of u_m is a classical result due to McLeod and Serrin [25], Kwong [20], and Kwong and Li [21] (see also references in [12]). This proves the first part of the lemma.

Next, we note that the characterization above gives

$$\begin{aligned}\beta_m^{rad} &= \inf_{u\in H^1_{0,rad}(\mathbf{B})\setminus\{0\}} \sup_{t\geq 0} J_m(tu)\\ &= \inf_{u\in H^1_{0,rad}(\mathbf{B})\setminus\{0\}} \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{\int_{\mathbf{B}} \left(|\nabla u|^2 + mu^2\right)\,dx}{\left(\int_{\mathbf{B}} |u|^p\,dx\right)^{\frac{2}{p}}}\right)^{\frac{p}{p-2}}.\end{aligned} \tag{5.1}$$

In the following, we assume $m > 0$ and let $B_{\sqrt{m}}$ denote the ball of radius $\sqrt{m}$ centered at the origin. We then consider the function $v_m \in H^1_0(B_{\sqrt{m}})$ given by

$$v_m(x) = m^{-\frac{1}{p-2}} u_m\left(\frac{x}{\sqrt{m}}\right).$$

Then

$$\begin{aligned}\frac{\int_{\mathbf{B}} \left(|\nabla u_m|^2 + mu_m^2\right)\,dx}{\left(\int_{\mathbf{B}} |u_m^2|^p\,dx\right)^{\frac{2}{p}}} &= m^{\frac{2}{p}} \frac{\int_{B_{\sqrt{m}}} \left(|\nabla v_m|^2 + v_m^2\right)dx}{\left(\int_{B_{\sqrt{m}}} |v_m|^p\,dx\right)^{\frac{2}{p}}}\\ &\geq m^{\frac{2}{p}} \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right)dx}{\left(\int_{\mathbb{R}^N} |v|^p\,dx\right)^{\frac{2}{p}}}.\end{aligned}$$

Setting

$$C_p := \inf_{v\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right)dx}{\left(\int_{\mathbb{R}^N} |v|^p\,dx\right)^{\frac{2}{p}}} > 0$$

we thus have

$$\frac{\int_{\mathbf{B}} \left(|\nabla u_m|^2 + mu_m^2\right)\,dx}{\left(\int_{\mathbf{B}} |u_m|^p\,dx\right)^{\frac{2}{p}}} \geq C_p m^{\frac{2}{p}}.$$

Therefore (5.1) implies

$$\beta_m^{rad} \geq \left(\frac{1}{2} - \frac{1}{p}\right) \left(C_p m^{\frac{2}{p}}\right)^{\frac{p}{p-2}}$$

and hence the claim. □

We will compare the previous estimate for the radial energy with suitable estimates for $c_{\alpha,m}$, starting with the following result.

Lemma 5.10 *Let $p \in (2,4)$ and $\alpha \in \mathcal{A}$. Then*

$$c_{\alpha,m} \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \inf_{(\ell,k)\in\mathcal{I}^+_{\alpha,m}} \left(j^2_{\ell,k} - \alpha^2\ell^2 + m\right)^{\frac{p}{p-2}}$$

holds for $m \in \mathbb{R}$.

Proof By Lemma 5.1, there exist $\ell_0, k_0 \in \mathbb{N}$ such that

$$\left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right) = \inf_{(\ell,k)\in\mathcal{I}^+_{\alpha,m}} \left(j^2_{\ell,k} - \alpha^2\ell^2 + m\right)$$

and we set

$$u_0 := \varphi_{\ell_0,k_0} \in E^+_{\alpha,m}.$$

For any $t \geq 0$ and $v \in F_{\alpha,m}$ it then holds that $\int_{\mathbf{B}} u_0 v\, dx = 0$ and therefore

$$\begin{aligned}\|tu_0 + v\|_p^p &\geq |\mathbf{B}|^{1-\frac{p}{2}} \|tu_0 + v\|_2^p = |\mathbf{B}|^{1-\frac{p}{2}} \left(\|tu_0\|_2^2 + \|v\|_2^2\right)^{\frac{p}{2}} \\ &\geq t^p |\mathbf{B}|^{1-\frac{p}{2}} \|u_0\|_2^p = t^p |\mathbf{B}|^{1-\frac{p}{2}}.\end{aligned}$$

This yields

$$\begin{aligned}\Phi_{\alpha,m}(tu_0 + v) &\leq \frac{t^2}{2}\left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right) - \frac{1}{p}\|tu_0 + v\|_p^p \\ &\leq \frac{t^2}{2}\left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right) - \frac{t^p}{p}|\mathbf{B}|^{1-\frac{p}{2}}.\end{aligned}$$

A straightforward computation shows that the right hand side attains a unique global maximum in

$$t^* = \left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right)^{\frac{1}{p-2}} |\mathbf{B}|^{\frac{1}{2}}$$

and therefore

$$\Phi_{\alpha,m}(tu_0 + v) \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right)^{\frac{p}{p-2}}.$$

In particular, this gives

$$\max_{w\in\widehat{E}_{\alpha,m}(u_0)} \Phi_{\alpha,m}(w) \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right)^{\frac{p}{p-2}}$$

and Proposition 5.8 then finally implies

$$c_{\alpha,m} = \inf_{w\in E_{\alpha,m}\setminus F_{\alpha,m}} \max_{w\in\widehat{E}_{\alpha,m}(u)} \Phi_{\alpha,m}(w) \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| \left(j^2_{\ell_0,k_0} - \alpha^2\ell_0^2 + m\right)^{\frac{p}{p-2}}$$

as claimed. □

The previous results allow us to deduce the existence of nonradial ground states whenever

$$\left(\frac{1}{2}-\frac{1}{p}\right)|\mathbf{B}| \inf_{(\ell,k)\in\mathcal{I}^+_{\alpha,m}} \left(j^2_{\ell,k}-\alpha^2\ell^2+m\right)^{\frac{p}{p-2}} < \beta^{rad}_m$$

holds. To this end, we estimate the growth of the left hand side as $m\to\infty$.

Proposition 5.11 *Let $\alpha\in\mathcal{A}$. Then there exist constants $C>0$, $m_0>0$ such that*

$$\inf_{(\ell,k)\in\mathcal{I}^+_{\alpha,m}} \left(j^2_{\ell,k}-\alpha^2\ell^2+m\right) \le Cm^{\frac{1}{2}}$$

holds for $m>m_0$.

Proof By Proposition 2.1 we have

$$\ell+\frac{|a_1|}{2^{\frac{1}{3}}}\ell^{\frac{1}{3}} < j_{\ell,1} < \ell+\frac{|a_1|}{2^{\frac{1}{3}}}\ell^{\frac{1}{3}}+\frac{3}{20}|a_1|^2\frac{2^{\frac{1}{3}}}{\ell^{\frac{1}{3}}}, \tag{5.2}$$

where a_1 denotes the first negative zero of the Airy function $\mathrm{Ai}(x)$. In particular, noting that $j^2_{\ell,1}-\alpha^2\ell^2=(j_{\ell,1}+\alpha\ell)(j_{\ell,1}-\alpha\ell)$ and $j_{\ell,1}-\alpha\ell<(1-\alpha)\ell+\frac{|a_1|}{2^{\frac{1}{3}}}\ell^{\frac{1}{3}}+\frac{3}{20}|a_1|^2\frac{2^{\frac{1}{3}}}{\ell^{\frac{1}{3}}}$ we find that there exists $\ell_0\in\mathbb{N}$ such that the map

$$\ell\mapsto j^2_{\ell,1}-\alpha^2\ell^2$$

is strictly decreasing for $\ell\ge\ell_0$.

Taking $m_0>\alpha^2\ell_0^2-j^2_{\ell_0,1}$ we thus find that for any $m>m_0$ there exists $\ell\ge\ell_0$ such that

$$m\in\left(\alpha^2\ell^2-j^2_{\ell,1},\alpha^2(\ell+1)^2-j^2_{\ell+1,1}\right].$$

In the following, we fix such m and ℓ and note that since $j_{\ell,1}<j_{\ell+1,1}$, we have

$$\begin{aligned}0<j^2_{\ell,1}-\alpha^2\ell^2-(j^2_{\ell+1,1}-\alpha^2(\ell+1)^2) &= j^2_{\ell,1}-j^2_{\ell+1,1}+\alpha^2\left((\ell+1)^2-\ell^2\right)\\ &\le 2\alpha^2\ell+\alpha^2\end{aligned}$$

for $\ell\ge\ell_0$, and therefore

$$0<j^2_{\ell,1}-\alpha^2\ell^2+m\le 2\alpha^2\ell+\alpha^2.$$

Importantly, (5.2) implies that there exists $C=C(\alpha)>0$ independent of m such that

$$2\alpha^2\ell+\alpha^2\le C\left(\alpha^2\ell^2-j^2_{\ell,1}\right)^{\frac{1}{2}}$$

holds for $\ell\ge\ell_0$, after possibly enlarging ℓ_0. Ultimately, we thus find that

$$0<j^2_{\ell,1}-\alpha^2\ell^2+m\le C\left(\alpha^2\ell^2-j^2_{\ell,1}\right)^{\frac{1}{2}}\le Cm^{\frac{1}{2}}$$

holds. Since C was independent of m, this completes the proof. □

Theorem 1.2(ii) is now a direct consequence of the following more general result.

Theorem 5.12 *Let $\alpha\in\mathcal{A}$ and $p\in(2,4)$ be fixed. Then there exists $m_0>0$ such that the ground states of (1.4) are nonradial for $m>m_0$.*

Proof Lemma 5.10 and Proposition 5.11 imply that there exist $C > 0$, $m_0 > 0$ such that

$$c_{\alpha,m} \leq \left(\frac{1}{2} - \frac{1}{p}\right) |\mathbf{B}| C m^{\frac{p}{2(p-2)}}$$

holds for $m > m_0$. On the other hand, Lemma 5.9 gives

$$\beta_m^{rad} \geq c m^{\frac{2}{p-2}}$$

with a constant $c > 0$ independent of m. Noting that the assumption $p < 4$ implies $\frac{p}{2(p-2)} < \frac{2}{p-2}$, it follows that

$$c_{\alpha,m} < \beta_m^{rad}$$

holds for $m > m_0$, after possibly enlarging m_0. □

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Appendix A: Complex-valued solutions

Throughout this section we consider complex-valued function spaces and assume that $p > 2$ is fixed. In this case, the eigenspaces

$$V_k := \left\{u \in H_0^1(\mathbf{B}, \mathbb{C}) : \partial_\theta u = iku\right\}$$

are nonempty for $k \in \mathbb{N}$. This observation can be used to find complex-valued solutions of (1.4) as stated in the following.

Theorem A.1 *Let $\alpha > 1$, $m > 0$ and $k \in \mathbb{N}$ be chosen such that*

$$m - \alpha^2 k^2 > -\lambda_1, \tag{A.1}$$

where $\lambda_1 > 0$ denotes the first Dirichlet eigenvalue of $-\Delta$ on $\mathbf{B}$. Then there exists a weak solution $u \in V_k$ of (1.4). In particular, this solution is nonradial.

We point out that the solutions found in the preceding theorem cannot be real-valued and are thus distinct from the solutions found in Theorem 1.2.

Proof Inspired by [37], the proof is based on a constrained minimization argument for the functional

$$J_{\alpha,m} : H_0^1(\mathbf{B}, \mathbb{C}) \to \mathbb{R},$$
$$J_{\alpha,m}(u) := \frac{1}{2} \int |\nabla u|^2 - \alpha^2 |\partial_\theta u|^2 + m u^2 \, dx.$$

Importantly, for $u \in V_k$ we have

$$J_{\alpha,m}(u) = \frac{1}{2} \int |\nabla u|^2 + (m - \alpha^2 k^2) u^2 \, dx$$

and our goal is to minimize $J_{\alpha,m}$ on V_k subject to the constraint

$$I(u) := \|u\|_p^p = 1.$$

To this end, we let $(u_n)_n \subset V_k$ be a constrained minimizing sequence, i.e., $I(u_n) = 1$ for all n and

$$\lim_{n\to\infty} J_{\alpha,m}(u_n) = \min_{\substack{u \in V_k \\ I(u)=1}} J_{\alpha,m}(u).$$

Note that V_k is a closed subspace of $H_0^1(\mathbf{B}, \mathbb{C})$ and, by assumption, there exist $c, C > 0$ such that

$$c\|u\|^2_{H_0^1(\mathbf{B})} \le J_{\alpha,m}(u) \le C\|u\|^2_{H^1(\mathbf{B})}$$

holds for $u \in V_k$, which implies that the sequence $(u_n)_n$ remains bounded in $H_0^1(\mathbf{B}, \mathbb{C})$ and we may pass to a weakly convergent subsequence with a weak limit $u_0 \in V_k$. The compact embedding $H_0^1(\mathbf{B}, \mathbb{C}) \hookrightarrow L^p(\mathbf{B}, \mathbb{C})$ then implies $I(u_0) = 1$ whereas weak lower semicontinuity yields $J_{\alpha,m}(u_0) \le \liminf J_{\alpha,m}(u_n)$, i.e., u_0 is a minimizer of $J_{\alpha,m}$ subject to the constraint $I(u_0) = 1$.

The minimization property then implies that there exists a Lagrange multiplier $K_0 \in \mathbb{R}$ such that

$$\int \nabla u_0 \cdot \nabla \varphi + (m - \alpha^2 k^2) u_0 \varphi \, dx = K_0 \int |u_0|^{p-2} u_0 \varphi \, dx \tag{A.2}$$

holds for $\varphi \in V_k$. Taking $\varphi = u_0$, the condition (A.1) then implies that K_0 must be positive. We now set

$$E : H_0^1(\mathbf{B}, \mathbb{C}) \to \mathbb{R}, \qquad E(u) := J_{\alpha,m}(u) - K_0 I(u),$$

so that, in particular, u_0 is a nontrivial critical point of $E\big|_{V_k}$.

For $t \in \mathbb{R}$ we then consider the action

$$g_t : H_0^1(\mathbf{B}, \mathbb{C}) \to H_0^1(\mathbf{B}, \mathbb{C}), \qquad [g_t u](x) = e^{-ikt} u(R_t(x)),$$

where R_t was defined in (1.3). Note that g_t is an isometry on $H_0^1(\mathbf{B}, \mathbb{C})$ and $L^p(\mathbf{B})$ so that E is invariant with respect to g_t. Moreover, this defines a group action on $H_0^1(\mathbf{B}, \mathbb{C})$ and we have

$$V_k = \{u \in H_0^1(\mathbf{B}, \mathbb{C}) : g_t u = u\}.$$

The principle of symmetric criticality (see e.g. [30]) then implies that u_0 is also a critical point of E on $H_0^1(\mathbf{B}, \mathbb{C})$ or, equivalently, (A.2) holds for all $\varphi \in H_0^1(\mathbf{B}, \mathbb{C})$. But this means that $K_0^{\frac{1}{p-2}} u_0$ is a weak solution of (1.4). □

By construction, the solutions found above are contained in the eigenspaces of the operator ∂_θ, i.e., for any such solution u there exists $k \in \mathbb{N}$ such that $u \in V_k$ and therefore $\partial_\theta u = iku$. However, this implies that $|u|$ is radial.

In the following, we briefly sketch how our methods can be used to find complex-valued solutions u of (1.1) (which are not real-valued) such that the modulus $|u|$ is also nonradial.

To this end, we combine the ansatz (1.2) for rotating solutions with a standing wave ansatz, i.e.,

$$v(t,x) = e^{i\mu t} u(R_t(x))$$

with R_t given by (1.3) and $\mu > 0$. This reduces (1.1) to the modified problem

$$\begin{cases} -\Delta u + \alpha^2 \partial_\theta^2 u + 2i\mu\partial_\theta u + (m-\mu^2)u = |u|^{p-2}u & \text{in } \mathbf{B} \\ u = 0 & \text{on } \partial\mathbf{B}. \end{cases} \tag{A.3}$$

Here, the eigenvalues of the operator

$$L_{\alpha,m,\mu}u := -\Delta u + \alpha^2\partial_\theta^2 u + 2i\mu\partial_\theta u + (m-\mu^2)u$$

are given by

$$j_{\ell,k}^2 - \alpha^2\ell^2 \pm 2\mu\ell + (m-\mu^2)$$

and the associated eigenfunctions are given by

$$\varphi_{\ell,k}^{\pm}(r,\theta) := e^{\pm i\ell\theta} J_\ell(j_{\ell,k}r), \qquad \ell \in \mathbb{N}_0, k \in \mathbb{N}.$$

This readily implies the following analogue to Lemma 5.1:

Lemma A.2 *Let the sequence* $(\alpha_n)_n \subset (1,\infty)$ *be given by* Theorem 4.2. *Then for any* $n \in \mathbb{N}$ *and* $m \geq 0$ *there exist* $c_{n,m}$, $\mu_n > 0$ *with the following property:*

If $|\mu| \leq \mu_n$ *and* ℓ, k *are such that* $j_{\ell,k}^2 - \alpha^2\ell^2 - 2\mu\ell + (m-\mu^2) \neq 0$ *holds, we have*

$$|j_{\ell,k}^2 - \alpha^2\ell^2 \pm 2\mu\ell + (m-\mu^2)| \geq c_{n,m} j_{\ell,k}.$$

Proof Note that

$$j_{\ell,k}^2 - \alpha^2\ell^2 \pm 2\mu\ell = (j_{\ell,k} + \alpha\ell)\left(j_{\ell,k} - \alpha\ell \pm \frac{2\mu\ell}{j_{\ell,k}+\alpha\ell}\right)$$

and for $\alpha = \alpha_n$ Theorem 4.2 then implies

$$\left|j_{\ell,k} - \alpha_n\ell \pm \frac{2\mu\ell}{j_{\ell,k}+\alpha_n\ell}\right| \geq c_n - \mu\frac{2l}{j_{\ell,k}+\alpha_n\ell} \geq c_n - \frac{2\mu}{1+\alpha_n}$$

for sufficiently large ℓ, k. Setting

$$\mu_n := \frac{1+\alpha_n}{2} c_n,$$

we thus find that

$$\lim_{N\to\infty} \inf_{\ell,k\geq N} \left|j_{\ell,k} - \alpha_n\ell \pm \frac{2\mu\ell}{j_{\ell,k}+\alpha_n\ell}\right| > 0$$

for $\mu < \mu_n$. □

Repeating the arguments of Sect. 5 ultimately gives the following result:

Theorem A.3 *Let* $p \in (2,4)$. *Then there exists a sequence* $(\alpha_n)_n \subset (1,\infty)$ *with the following properties:*

(i) *For each* $n \in \mathbb{N}$ *the problem* (A.3) *has a ground state solution.*
(ii) *For each* $n \in \mathbb{N}$ *there exists* $m_n > 0$ *such that any ground state* u (A.3) *with* $\alpha = \alpha_n$ *and* $m > m_n$ *has a nonradial modulus, i.e.,* $|u|$ *is nonradial.*

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