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Calculus of Variations

Ricci flow does not preserve positive sectional curvature in dimension four

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Abstract

We find examples of cohomogeneity one metrics on S^4 and $\mathbb{C}P^2$ with positive sectional curvature that lose this property when evolved via Ricci flow. These metrics are arbitrarily small perturbations of Grove–Ziller metrics with flat planes that become instantly negatively curved under Ricci flow.

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1 Introduction

The Ricci flow $\frac{\partial}{\partial t}\mathrm{g}(t) = -2\,\mathrm{Ric}_{\mathrm{g}(t)}$ of Riemannian metrics on a smooth manifold is an evolution equation that continues to drive a wide range of breakthroughs in Geometric Analysis, see e.g. [4] for a survey. One of the keys to using Ricci flow is to control how the curvature of $\mathrm{g}(t)$ evolves; in particular, which curvature conditions of the original metric $\mathrm{g}(0)$ are preserved. Our main result establishes that, in dimension $n = 4$, positive sectional curvature ($\sec > 0$) is *not* among them:

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Theorem A *There exist smooth Riemannian metrics with* $\sec > 0$ *on* S^4 *and* $\mathbb{C}P^2$ *that evolve under the Ricci flow to metrics with sectional curvatures of mixed sign.*

In contrast, $\sec > 0$ is preserved on closed manifolds of dimension $n \leq 3$, by the seminal work of Hamilton [14]. Moreover, it was previously known [15] that $\mathrm{Ric} > 0$ is not preserved in dimension $n = 4$, even among Kähler metrics, but these examples do not have $\sec > 0$. Although Theorem A does not readily extend to all $n > 4$, there are examples of homogeneous metrics on flag manifolds of dimensions 6, 12, and 24 with $\sec > 0$ that lose that property when evolved via Ricci flow, see [1, 9, 10]. A state-of-the-art discussion of Ricci flow invariant curvature conditions can be found in [5], see also Remark 5.1.

Theorem A builds on our earlier result [6] that certain metrics with $\sec \geq 0$, introduced by Grove and Ziller [12] in a much broader context (see Sect. 2.1.1), immediately acquire negatively curved planes on S^4 and $\mathbb{C}P^2$, when evolved under Ricci flow. In light of the appropriate continuous dependence of Ricci flow on its initial data [3], the metrics in Theorem A are obtained by means of:

Theorem B *Every Grove–Ziller metric on* S^4 *or* $\mathbb{C}P^2$ *is the limit (in* C^∞*-topology) of cohomogeneity one metrics with* $\sec > 0$.

In full generality, the problem of perturbing $\sec \geq 0$ to $\sec > 0$ is notoriously difficult, see e.g. [20, Prob. 2]. Aside from clearly being unobstructed on S^4 and $\mathbb{C}P^2$, the deformation problem is facilitated here by the presence of natural directions for perturbation, given by the round metric and the Fubini–Study metric, respectively. Indeed, we deform $\sec \geq 0$ into $\sec > 0$ in Theorem B by linearly interpolating lengths of Killing vector fields for the $\mathsf{SO}(3)$-action which is isometric for both the Grove–Ziller metric g_0 and the standard metric g_1 on these spaces. The resulting $\mathsf{SO}(3)$-invariant metrics g_s, $s \in [0, 1]$, are smooth and have $\sec > 0$ for all sufficiently small $s > 0$. For a lower-dimensional illustration, consider the T^2-action on $S^3 \subset \mathbb{C}^2$ via $(e^{i\theta_1}, e^{i\theta_2}) \cdot (z, w) = \big(e^{i\theta_1} z, e^{i\theta_2} w\big)$, and invariant metrics

$$\mathrm{g} = \mathrm{d}r^2 + \varphi(r)^2\, \mathrm{d}\theta_1^2 + \xi(r)^2\, \mathrm{d}\theta_2^2, \quad 0 < r < \tfrac{\pi}{2},$$

written along the geodesic segment $\gamma(r) = (\sin r, \cos r)$. The functions φ and ξ encode the g-lengths of the Killing fields $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$ respectively, and must satisfy certain smoothness conditions at the endpoints $r = 0$ and $r = \frac{\pi}{2}$. The unit round metric g_1 is given by setting φ and ξ to be $\varphi_1(r) = \sin r$ and $\xi_1(r) = \cos r$, while a Grove–Ziller metric g_0 corresponds to concave monotone functions φ_0 and ξ_0 that plateau at a constant value $b > 0$ for at least half of $\big[0, \frac{\pi}{2}\big]$. The curvature operator of g is easily seen to be diagonal, with eigenvalues $-\varphi''/\varphi$, $-\xi''/\xi$, and $-\varphi'\xi'/\varphi\xi$, see e.g. [16, Sect. 4.2.4], so it has $\sec \geq 0$ if and only if φ and ξ are concave and monotone, and $\sec > 0$ if and only if they are *strictly* concave and monotone. Thus,

$$\varphi_s = (1-s)\,\varphi_0 + s\,\varphi_1 \quad \text{and} \quad \xi_s = (1-s)\,\xi_0 + s\,\xi_1$$

give rise to metrics g_s deforming g_0 to have $\sec > 0$ for $s > 0$. It turns out that a similar approach works for proving Theorem B, with the addition of a third (nowhere vanishing) function ψ, to deal with $\mathsf{SO}(3)$-invariant metrics on 4-manifolds. The biggest challenge is verifying that these metrics have $\sec > 0$, since that is no longer equivalent to positive-definiteness of the curvature operator if $n \geq 4$. To overcome this difficulty, we use a much simpler algebraic characterization of $\sec > 0$ in dimension $n = 4$, given by the Finsler–Thorpe trick (Proposition 2.2).

Motivated by the above, it is natural to ask whether the set of cohomogeneity one metrics with $\sec \geq 0$ on a given closed manifold coincides with the closure (say, in C^2-topology) of the set of such metrics with $\sec > 0$, if the latter is nonempty. In contrast to Theorem B, there is some evidence to suggest that Grove–Ziller metrics on certain 7-manifolds cannot be perturbed to have $\sec > 0$, see [21, Sect. 4].

This paper is organized as follows. Background material on cohomogeneity one manifolds and the Finsler–Thorpe trick in dimension 4 is presented in Sect. 2. The smoothness conditions and curvature operator of $\mathsf{SO}(3)$-invariant metrics on S^4 and $\mathbb{C}P^2$ are discussed in Sect. 3. Sect. 4 contains the proof of Theorem B, focusing mainly on the case of S^4, since the proof for $\mathbb{C}P^2$ is mostly analogous. Finally, Theorem A is proved in Sect. 5.

2 Preliminaries

2.1 Cohomogeneity one

We briefly discuss the geometry of cohomogeneity one manifolds to fix notations, see [2, 6, 12, 13, 19, 21] for details.

A cohomogeneity one manifold is a Riemannian manifold (M, g) endowed with an isometric action by a Lie group G, such that the orbit space M/G is one-dimensional. Let $\pi : M \to M/\mathsf{G}$ be the projection map. Throughout, we assume $M/\mathsf{G} = [0, L]$ is a closed interval, and the nonprincipal orbits $B_- = \pi^{-1}(0)$ and $B_+ = \pi^{-1}(L)$ are *singular orbits*. In other words, $B_\pm$ are smooth submanifolds of dimension strictly smaller than the principal orbits $\pi^{-1}(r)$, $r \in (0, L)$, which are smooth hypersurfaces of M. Fix $x_- \in B_-$, and consider a minimal geodesic $\gamma(r)$ in M joining x_- to B_+, meeting it at $x_+ = \gamma(L)$; that is, γ is a horizontal lift of $[0, L]$ to M. Denote by $\mathsf{K}_\pm$ the isotropy group at $x_\pm$, and by H the isotropy at $\gamma(r)$, for $r \in (0, L)$. By the Slice Theorem, given $r^\pm_{\max} > 0$ so that $r^+_{\max} + r^-_{\max} = L$, the tubular neighborhoods $D(B_-) = \pi^{-1}\left(\left[0, r^-_{\max}\right]\right)$ and $D(B_+) = \pi^{-1}\left(\left[L - r^+_{\max}, L\right]\right)$ of the singular orbits are disk bundles over B_- and B_+. Let $D^{l_\pm+1}$ be the normal disks to $B_\pm$ at $x_\pm$. Then $\mathsf{K}_\pm$ acts transitively on the boundary $\partial D^{l_\pm+1}$, with isotropy H, so $\partial D^{l_\pm+1} = S^{l_\pm} = \mathsf{K}_\pm/\mathsf{H}$, and the $\mathsf{K}_\pm$-action on $\partial D^{l_\pm+1}$ extends to a $\mathsf{K}_\pm$-action on all of $D^{l_\pm+1}$. Moreover, there are equivariant diffeomorphisms $D(B_\pm) \cong \mathsf{G} \times_{\mathsf{K}_\pm} D^{l_\pm+1}$, and $M \cong D(B_-) \cup D(B_+)$, where the latter is given by gluing these disk bundles along their common boundary $\partial D(B_\pm) \cong \mathsf{G} \times_{\mathsf{K}_\pm} S^{l_\pm} \cong \mathsf{G}/\mathsf{H}$. In this situation, one associates to M the *group diagram*

$$\mathsf{H} \subset \{\mathsf{K}_-, \mathsf{K}_+\} \subset \mathsf{G}.$$

Conversely, given a group diagram as above, where $\mathsf{K}_\pm/\mathsf{H}$ are spheres, there exists a cohomogeneity one manifold M given as the union of the above disk bundles.

Fix a bi-invariant metric Q on the Lie algebra $\mathfrak{g}$ of G, and set $\mathfrak{n} = \mathfrak{h}^\perp$, where $\mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra of H. Identifying $\mathfrak{n} \cong T_{\gamma(r)}(\mathsf{G}/\mathsf{H})$ for each $0 < r < L$ via action fields $X \mapsto X^*_{\gamma(r)}$, any G-invariant metric on M can be written as

$$\mathrm{g} = \mathrm{d}r^2 + \mathrm{g}_r, \quad 0 < r < L, \tag{2.1}$$

along the geodesic $\gamma(r)$, where g_r is a 1-parameter family of left-invariant metrics on G/H, i.e., of $\mathrm{Ad}(\mathsf{H})$-invariant metrics on $\mathfrak{n}$. As $r \searrow 0$ and $r \nearrow L$, the metrics g_r degenerate, according to how $\mathsf{G}(\gamma(r)) \cong \mathsf{G}/\mathsf{H}$ collapse to $B_\pm = \mathsf{G}/\mathsf{K}_\pm$. Namely, they satisfy *smoothness conditions* that characterize when a tensor defined by means of (2.1) on $M \setminus (B_- \cup B_+) \cong (0, L) \times \mathsf{G}/\mathsf{H}$ extends smoothly to all of M, see [19].

2.1.1 Grove–Ziller metrics

If both singular orbits $B_\pm$ of a cohomogeneity one manifold M have codimension two, then M can be endowed with a new G-invariant metric g_{GZ} with $\sec \geq 0$, as shown in the celebrated work of Grove and Ziller [12, Thm. 2.6]. We now describe this construction, building metrics with $\sec \geq 0$ on each disk bundle $D(B_\pm)$ that restrict to a fixed product metric $\mathrm{d}r^2 + b^2 Q|_{\mathfrak{n}}$ near $\partial D(B_\pm) \cong \mathsf{G}/\mathsf{H}$, so that these two pieces can be isometrically glued together.

Consider one such disk bundle $D(B)$ at a time, say over a singular orbit $B = \mathsf{G}/\mathsf{K}$, and let $\mathfrak{k}$ be the Lie algebra of K. Set $\mathfrak{m} = \mathfrak{k}^\perp$ and $\mathfrak{p} = \mathfrak{h}^\perp \cap \mathfrak{k}$, so that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p} \oplus \mathfrak{h}$ is a Q-orthogonal direct sum. Since $\mathfrak{p}$ is 1-dimensional, the metric $Q_{a,b}$ on G, given by

$$Q_{a,b}|_{\mathfrak{m}} := b^2\, Q|_{\mathfrak{m}}, \qquad Q_{a,b}|_{\mathfrak{p}} := ab^2\, Q|_{\mathfrak{p}}, \qquad Q_{a,b}|_{\mathfrak{h}} := b^2\, Q|_{\mathfrak{h}},$$

has $\sec \geq 0$ whenever $0 < a \leq \frac{4}{3}$ and $b > 0$, see [12, Prop. 2.4] or [8, Lemma 3.2]. Fix $1 < a \leq \frac{4}{3}$, and let $r_{\max} > 0$ be such that

$$y := \tfrac{\rho\sqrt{a}}{\sqrt{a-1}} \quad \text{satisfies} \quad y < r_{\max}, \tag{2.2}$$

where $\rho = \rho(b)$ is the radius of the circle(s) K/H endowed with the metric $b^2\, Q|_{\mathfrak{p}}$. Then, we can find a smooth nondecreasing function $f\colon\ [0, r_{\max}] \to \mathbb{R}$ and some $0 < r_0 < r_{\max}$, with $f(0) = 0$, $f'(0) = 1$, $f^{(2n)}(0) = 0$ for all $n \in \mathbb{N}$, $f''(r) \leq 0$ for all $r \in [0, r_{\max}]$, $f^{(3)}(r) > 0$ for all $r \in [0, r_0)$, and $f(r) \equiv y$ for all $r \in [r_0, r_{\max}]$. The rotationally symmetric metric $\mathrm{g}_{D^2} = \mathrm{d}r^2 + f(r)^2 \mathrm{d}\theta^2$, $0 < r \leq r_{\max}$, on the punctured disk $D^2 \setminus \{0\}$ extends to a smooth metric g_{D^2} on D^2 with $\sec \geq 0$ that, near $\partial D^2 = \{r = r_{\max}\}$, is isometric to a round cylinder $[r_0, r_{\max}] \times S^1(y)$ of radius y. Thus, the product manifold $(\mathsf{G} \times D^2,\, Q_{a,b} + \mathrm{g}_{D^2})$ has $\sec \geq 0$, and so does the orbit space $D(B) \cong \mathsf{G} \times_{\mathsf{K}} D^2$ of the K-action on $\mathsf{G} \times D^2$, when endowed with the metric g_{GZ} that makes the projection map $\Pi\colon\ (\mathsf{G} \times D^2,\, Q_{a,b} + \mathrm{g}_{D^2}) \to (\mathsf{G} \times_{\mathsf{K}} D^2,\, \mathrm{g}_{\mathrm{GZ}})$ a Riemannian submersion. Writing this metric g_{GZ} in the form (2.1), we have

$$\mathrm{g}_{\mathrm{GZ}} = \mathrm{d}r^2 + b^2\, Q|_{\mathfrak{m}} + \tfrac{f(r)^2 a}{f(r)^2 + a\rho^2} b^2\, Q|_{\mathfrak{p}}, \quad 0 < r \leq r_{\max}, \tag{2.3}$$

see e.g. [12, Lemma 2.1, Rem. 2.7] or [8, Lemma 3.1 (ii)]. In particular, $\mathrm{g}_{\mathrm{GZ}} = \mathrm{d}r^2 + b^2\, Q|_{\mathfrak{n}}$ for all $r \in [r_0, r_{\max}]$, since $\frac{f(r)^2 a}{f(r)^2 + a\rho^2} \equiv 1$ for all such r; hence $(D(B), \mathrm{g}_{\mathrm{GZ}})$ is isometric to the prescribed product metric near $\partial D(B) \cong \mathsf{G}/\mathsf{H}$.

This construction can be performed on each disk bundle $D(B_\pm)$ with the same $b > 0$, provided $r_{\max}^\pm > 0$ are chosen sufficiently large so that (2.2) holds for the corresponding radii $\rho_\pm(b)$ of the circles $\mathsf{K}_\pm/\mathsf{H}$ endowed with the metric $b^2\, Q|_{\mathfrak{p}_\pm}$. Gluing these two disk bundles together, we obtain the desired G-invariant metric g_{GZ} with $\sec \geq 0$ on $M \cong D(B_-) \cup D(B_+)$ and $M/\mathsf{G} = [0, L]$, where $L = r_{\max}^+ + r_{\max}^-$. Although it is natural to pick the same (largest) value for $r_{\max}^\pm$, so that the gluing occurs at $r = \frac{L}{2}$, it is convenient to not impose this restriction. Note that

$$L = r_{\max}^+ + r_{\max}^- > \tfrac{\sqrt{a}}{\sqrt{a-1}} \big(\rho_+(b) + \rho_-(b)\big), \tag{2.4}$$

if the gluing interface $\partial D(B_\pm)$ is isometric to $(\mathsf{G}/\mathsf{H},\, b^2 Q|_{\mathfrak{n}})$. Conversely, given $1 < a \leq \frac{4}{3}$, $b > 0$, and L satisfying (2.4), there exists a Grove–Ziller metric on M with gluing interface $(\mathsf{G}/\mathsf{H},\, b^2 Q|_{\mathfrak{n}})$, induced by $Q_{a,b} + \mathrm{g}_{D^2}$, and with $M/\mathsf{G} = [0, L]$.

Remark 2.1 Although this is not a requirement in the original Grove–Ziller construction, we assume that $f^{(3)}(r) > 0$ on $[0, r_0)$, hence the curvature of (D^2, g_{D^2}) is monotonically decreasing for $r \in [0, r_0)$. As a consequence, for each $0 < r_* < r_0$, there is a constant $c > 0$, depending on r_*, so that $\sec_{g_{D^2}} \geq c$ for all $r \in [0, r_*]$.

2.2 Finsler–Thorpe trick

In order to verify $\sec > 0$ on Riemannian 4-manifolds, we shall use a result that became known in the Geometric Analysis community as *Thorpe's trick*, attributed to Thorpe [18], but that actually follows from much earlier work of Finsler [11], and is often referred to as *Finsler's Lemma* in Convex Algebraic Geometry. This rather multifaceted result is also known as the *S-lemma*, or *S-procedure*, in the mathematical optimization and control literature, see e.g. [17]. Details and other geometric perspectives can be found in [7].

Let $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^n) \subset \mathrm{Sym}^2(\wedge^2\mathbb{R}^n)$ be the subspace of symmetric endomorphisms $R\colon \wedge^2 \mathbb{R}^n \to \wedge^2\mathbb{R}^n$ that satisfy the first Bianchi identity. These objects are called *algebraic curvature operators*, and serve as pointwise models for the curvature operators of Riemannian n-manifolds. For instance, $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^n)$ is said to have $\sec \geq 0$, respectively $\sec > 0$, if the restriction of the quadratic form $\langle R(\sigma), \sigma\rangle$ to the oriented Grassmannian $\mathrm{Gr}_2^+(\mathbb{R}^n) \subset \wedge^2\mathbb{R}^n$ of 2-planes is nonnegative, respectively positive. A Riemannian manifold (M^n, g) has $\sec \geq 0$, or $\sec > 0$, if and only if its curvature operator $R_p \in \mathrm{Sym}^2_b(\wedge^2 T_pM)$ has $\sec \geq 0$, or $\sec > 0$, for all $p \in M$.

The orthogonal complement to $\mathrm{Sym}^2_b(\wedge^2\mathbb{R}^n)$ is identified with $\wedge^4\mathbb{R}^n$; so, if $n = 4$, it is 1-dimensional, and spanned by the Hodge star operator $*$. Since $\sigma \in \wedge^2\mathbb{R}^4$ satisfies $\sigma \wedge \sigma = 0$ if and only if $\langle *\sigma, \sigma\rangle = 0$, the quadric defined by $*$ in $\wedge^2\mathbb{R}^4$ is precisely the Plücker embedding $\mathrm{Gr}_2^+(\mathbb{R}^4) \subset \wedge^2\mathbb{R}^4$. As shown by Finsler [11], a quadratic form $\langle R(\sigma), \sigma\rangle$ is nonnegative when restricted to the quadric $\langle *\sigma, \sigma\rangle = 0$ if and only if some linear combination of R and $*$ is positive-semidefinite, yielding:

Proposition 2.2 *(Finsler–Thorpe trick) Let* $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ *be an algebraic curvature operator. Then* R *has* $\sec \geq 0$*, respectively* $\sec > 0$*, if and only if there exists* $\tau \in \mathbb{R}$ *such that* $R + \tau * \succeq 0$*, respectively* $R + \tau * \succ 0$.

Remark 2.3 For a given $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ with $\sec \geq 0$, the set of $\tau \in \mathbb{R}$ such that $R + \tau * \succeq 0$ is a closed interval $[\tau_{\min}, \tau_{\max}]$, which degenerates to a single point, i.e., $\tau_{\min} = \tau_{\max}$, if and only if R does not have $\sec > 0$, see [7, Prop. 3.1]

The equivalences given by Finsler–Thorpe's trick offer substantial computational advantages to test for $\sec \geq 0$ or $\sec > 0$, see the discussion in [7, Sect. 5.4].

3 Cohomogeneity one structure of S^4 and $\mathbb{C}P^2$

Both S^4 and $\mathbb{C}P^2$ admit a cohomogeneity one action by $\mathsf{G} = \mathsf{SO}(3)$ as we now recall, see [6, Sect. 3] and [21, Sect. 2] for details. The G-action on S^4 is the restriction to the unit sphere of the $\mathsf{SO}(3)$-action by conjugation on the space of symmetric traceless 3×3 real matrices, while the G-action on $\mathbb{C}P^2$ is a subaction of the transitive $\mathsf{SU}(3)$-action. The corresponding orbit spaces are $S^4/\mathsf{G} = \left[0, \frac{\pi}{3}\right]$ and $\mathbb{C}P^2/\mathsf{G} = \left[0, \frac{\pi}{4}\right]$, endowing S^4 with the round metric with $\sec \equiv 1$, and $\mathbb{C}P^2$ with the Fubini–Study metric with $1 \leq \sec \leq 4$. Their group diagrams are as follows:

$$S^4: \quad \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \mathsf{S}(\mathsf{O}(1)\mathsf{O}(1)\mathsf{O}(1)) \subset \{\mathsf{S}(\mathsf{O}(1)\mathsf{O}(2)), \mathsf{S}(\mathsf{O}(2)\mathsf{O}(1))\} \subset \mathsf{SO}(3),$$
$$\mathbb{C}P^2: \quad \mathbb{Z}_2 \cong \langle \mathrm{diag}(-1, -1, 1)\rangle \subset \{\mathsf{S}(\mathsf{O}(1)\mathsf{O}(2)), \mathsf{SO}(2)_{1,2}\} \subset \mathsf{SO}(3),$$

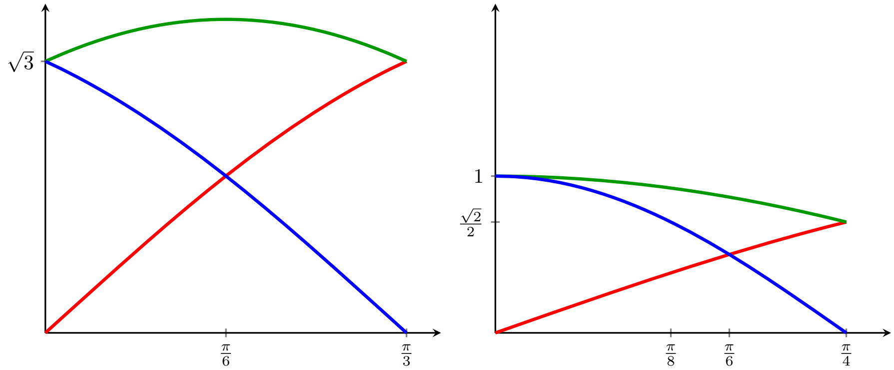

Fig. 1 Graphs of φ_1, ψ_1, ξ_1, for S^4 (*left*) and $\mathbb{C}P^2$ (*right*)

according to an appropriate choice of minimal geodesic $\gamma(r)$, $r \in [0, L]$, see [6, Sect. 3]. In both cases, since H is discrete, $\mathfrak{n} \cong \mathfrak{g} = \mathfrak{so}(3)$. We henceforth fix Q to be the bi-invariant metric such that $\{E_{23}, E_{31}, E_{12}\}$ is a Q-orthonormal basis of $\mathfrak{so}(3)$, where E_{ij} is the skew-symmetric 3×3 matrix with a $+1$ in the (i, j) entry, a -1 in the (j, i) entry, and zeros in the remaining entries. The 1-dimensional subspaces $\mathfrak{n}_k = \mathrm{span}(E_{ij})$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, are pairwise inequivalent for the adjoint action of H in the case of S^4, while $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are equivalent in the case of $\mathbb{C}P^2$, but neither is equivalent to $\mathfrak{n}_3$.

Collectively denoting S^4 and $\mathbb{C}P^2$ with the above cohomogeneity one structures by M^4, we consider *diagonal* G-invariant metrics g on M^4, i.e., metrics of the form

$$\mathrm{g} = \mathrm{d}r^2 + \varphi(r)^2\, Q|_{\mathfrak{n}_1} + \psi(r)^2\, Q|_{\mathfrak{n}_2} + \xi(r)^2\, Q|_{\mathfrak{n}_3}, \quad 0 < r < L, \tag{3.1}$$

where $L = \frac{\pi}{3}$ or $L = \frac{\pi}{4}$ according to whether $M^4 = S^4$ or $M^4 = \mathbb{C}P^2$, cf. (2.1). Note that every G-invariant metric on S^4 is of the above form, i.e., $\mathfrak{n}_k$ are pairwise orthogonal, but $\mathfrak{n}_1$ and $\mathfrak{n}_2$ need not be orthogonal for all G-invariant metrics on $\mathbb{C}P^2$, i.e., the off-diagonal term $\mathrm{g}(E_{23}^*, E_{31}^*)$ need not vanish identically. The standard metric on M^4, with curvatures normalized as above, is obtained setting φ, ψ, ξ to

$$\begin{aligned} S^4: &\quad \varphi_1(r) = 2\sin r, \quad \psi_1(r) = \sqrt{3}\cos r + \sin r, \quad \xi_1(r) = \sqrt{3}\cos r - \sin r, \\ \mathbb{C}P^2: &\quad \varphi_1(r) = \sin r, \quad \psi_1(r) = \cos r, \quad \xi_1(r) = \cos 2r, \end{aligned} \tag{3.2}$$

see Fig. 1 for their graphs.

3.1 Smoothness

The conditions required of φ, ψ, ξ for the metric g in (3.1), which is defined on the open dense set $M^4 \setminus (B_- \cup B_+) \cong (0, L) \times \mathsf{G}/\mathsf{H}$, to extend smoothly to all of M^4 can be extracted from [19] as follows:

Proposition 3.1 *Let φ, ψ, ξ be smooth positive functions on $(0, L)$ which extend smoothly to $r = 0$ and $r = L$. Then, the G-invariant metric (3.1) on $M^4 \setminus (B_- \cup B_+)$ extends to a smooth metric on M^4 if and only if φ, ψ, ξ satisfy the following, where ϕ_k are smooth, $z = L - r$, and $\varepsilon > 0$ is small:*

M^4	Smoothness conditions on φ, ψ, ξ
S^4 $L=\frac{\pi}{3}$	(i) $\varphi(0)=0,\ \varphi'(0)=2,\ \varphi^{(2n)}(0)=0,$ for all $n\geq 1$, (ii) $\psi(r)^2+\xi(r)^2=\phi_1(r^2),$ for all $r\in[0,\varepsilon)$, (iii) $\psi(r)^2-\xi(r)^2=r\,\phi_2(r^2),$ for all $r\in[0,\varepsilon)$, (iv) $\xi(L)=0,\ \xi'(L)=-2,\ \xi^{(2n)}(L)=0,$ for all $n\geq 1$, (v) $\psi(z)^2+\varphi(z)^2=\phi_3(z^2),$ for all $z\in[0,\varepsilon)$, (vi) $\psi(z)^2-\varphi(z)^2=z\,\phi_4(z^2),$ for all $z\in[0,\varepsilon)$.
$\mathbb{C}P^2$ $L=\frac{\pi}{4}$	(i) $\varphi(0)=0,\ \varphi'(0)=1,\ \varphi^{(2n)}(0)=0,$ for all $n\geq 1$, (ii) $\psi(r)^2+\xi(r)^2=\phi_5(r^2),$ for all $r\in[0,\varepsilon)$, (iii) $\psi(r)^2-\xi(r)^2=r^2\,\phi_6(r^2),$ for all $r\in[0,\varepsilon)$, (iv) $\xi(L)=0,\ \xi'(L)=-2,\ \xi^{(2n)}(L)=0,$ for all $n\geq 1$, (v) $\psi(z)^2+\varphi(z)^2=\phi_7(z^2),$ for all $z\in[0,\varepsilon)$, (vi) $\psi(z)^2-\varphi(z)^2=z\,\phi_8(z^2),$ for all $z\in[0,\varepsilon)$.

Proof By [19, Thm. 2], the metric g in (3.1) extends smoothly to all of M^4 if and only if its components satisfy certain functional equations determined from the equivariant geometry of M^4. These equations can be obtained following the discussion in [19, Sect. 3.1, 3.2].

For simplicity, we only analyze the equations corresponding to smoothness at the singular orbit B_- in the case $M^4 = S^4$, i.e., conditions (i), (ii), and (iii). Equation (4) in [19] implies that smoothness in the direction $\mathfrak{p} = \mathrm{span}(E_{23})$ is equivalent to $\varphi(r)^2 = a_1^2 r^2 + r^4\phi(r^2)$, $r \in [0, \varepsilon)$, where ϕ is smooth and $a_1 = |\mathsf{L}\cap\mathsf{H}|$, for $\mathsf{L} = \{\exp(\theta E_{23}) : 0 \leq \theta \leq 2\pi\}$. A simple computation shows that $a_1 = 2$, so the above functional equation is equivalent to (i) by routine Taylor series arguments. From [19, Lemma 5], smoothness of g on $\mathfrak{m} = \mathrm{span}(E_{12}, E_{31})$ is equivalent to

$$\begin{bmatrix}\psi(r)^2 & 0\\ 0 & \xi(r)^2\end{bmatrix} = \begin{bmatrix}\phi_1(r^2) & 0\\ 0 & \phi_1(r^2)\end{bmatrix} + r^{2d/a_1}\begin{bmatrix}\phi_2(r^2) & 0\\ 0 & -\phi_2(r^2)\end{bmatrix}, \quad r\in[0,\varepsilon),$$

where ϕ_1, ϕ_2 are smooth, and d is the speed with which $\mathsf{L} \cong \mathsf{S}^1$ acts by rotations on $\mathfrak{m}$. Another simple computation gives $d = 1$, so the above yields (ii) and (iii). □

Remark 3.2 Since the isotropy groups $\mathsf{K}_\pm$ for the G-action on S^4 are conjugate, the smoothness conditions at the endpoints $r = 0$ and $r = L$ can be obtained from one another by interchanging the roles of φ and ξ. Furthermore, just as the round metric (3.2), all metrics we consider on S^4 have the following additional symmetries:

$$\varphi(r) = \xi\,(L-r)\,, \quad \text{and} \quad \psi(r) = \psi\,(L-r)\,, \quad \text{for all } 0 \leq r \leq L. \tag{3.3}$$

However, metrics on $\mathbb{C}P^2$ do not have any of these features or extra symmetries, as $\mathsf{K}_\pm$ are not conjugate, and, in general $\varphi(r) \neq \xi\,(L-r)$ and $\psi(r) \neq \psi\,(L-r)$.

3.2 Curvature

Computing the curvature operator of the G-invariant metric (3.1) on M^4, with the formulae in [13, Prop. 1.12], one obtains the following:

Proposition 3.3 *Let $\{e_i\}_{i=0}^3$ be the* g*-orthonormal frame along the geodesic $\gamma(r)$, $0 < r < L$, given by $e_0 = \gamma'(r)$, $e_1 = \frac{1}{\varphi(r)}E_{23}^*$, $e_2 = \frac{1}{\psi(r)}E_{31}^*$, $e_3 = \frac{1}{\xi(r)}E_{12}^*$, i.e., e_0 is the unit horizontal direction and $\{e_1, e_2, e_3\}$ are unit Killing vector fields. In the basis $\mathcal{B} := \{e_2\wedge e_3,\ e_0\wedge e_1,\ e_3\wedge$*

$e_1, e_0 \wedge e_2, e_1 \wedge e_2, e_0 \wedge e_3\}$, *the curvature operator* $R\colon \wedge^2 T_{\gamma(r)}M^4 \to \wedge^2 T_{\gamma(r)}M^4$, $0 < r < L$, *is block diagonal, that is,* $R = \operatorname{diag}(R_1, R_2, R_3)$, *with* 2×2 *blocks given as follows:*

$$R_1 = \begin{bmatrix} \frac{\psi^4+\xi^4-\varphi^4+2(\xi^2-\varphi^2)(\varphi^2-\psi^2)}{4\varphi^2\psi^2\xi^2} - \frac{\psi'\xi'}{\psi\xi} & \frac{\psi'(\psi^2+\varphi^2-\xi^2)}{2\varphi\psi^2\xi} + \frac{\xi'(\xi^2+\varphi^2-\psi^2)}{2\varphi\psi\xi^2} - \frac{\varphi'}{\psi\xi} \\ \frac{\psi'(\psi^2+\varphi^2-\xi^2)}{2\varphi\psi^2\xi} + \frac{\xi'(\xi^2+\varphi^2-\psi^2)}{2\varphi\psi\xi^2} - \frac{\varphi'}{\psi\xi} & -\frac{\varphi''}{\varphi} \end{bmatrix},$$

$$R_2 = \begin{bmatrix} \frac{\varphi^4+\xi^4-\psi^4+2(\varphi^2-\psi^2)(\psi^2-\xi^2)}{4\varphi^2\psi^2\xi^2} - \frac{\varphi'\xi'}{\varphi\xi} & \frac{\varphi'(\varphi^2+\psi^2-\xi^2)}{2\varphi^2\psi\xi} + \frac{\xi'(\xi^2+\psi^2-\varphi^2)}{2\varphi\psi\xi^2} - \frac{\psi'}{\varphi\xi} \\ \frac{\varphi'(\varphi^2+\psi^2-\xi^2)}{2\varphi^2\psi\xi} + \frac{\xi'(\xi^2+\psi^2-\varphi^2)}{2\varphi\psi\xi^2} - \frac{\psi'}{\varphi\xi} & -\frac{\psi''}{\psi} \end{bmatrix},$$

$$R_3 = \begin{bmatrix} \frac{\varphi^4+\psi^4-\xi^4+2(\psi^2-\xi^2)(\xi^2-\varphi^2)}{4\varphi^2\psi^2\xi^2} - \frac{\varphi'\psi'}{\varphi\psi} & \frac{\varphi'(\varphi^2+\xi^2-\psi^2)}{2\varphi^2\psi\xi} + \frac{\psi'(\psi^2+\xi^2-\varphi^2)}{2\varphi\psi^2\xi} - \frac{\xi'}{\varphi\psi} \\ \frac{\varphi'(\varphi^2+\xi^2-\psi^2)}{2\varphi^2\psi\xi} + \frac{\psi'(\psi^2+\xi^2-\varphi^2)}{2\varphi\psi^2\xi} - \frac{\xi'}{\varphi\psi} & -\frac{\xi''}{\xi} \end{bmatrix}.$$

The Hodge star operator $*$ is also clearly block diagonal in the basis $\mathcal{B}$, namely,

$$* = \operatorname{diag}(H, H, H), \quad \text{where} \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{3.4}$$

Thus, by the Finsler–Thorpe trick (Proposition 2.2), such $R = \operatorname{diag}(R_1, R_2, R_3)$ as in Proposition 3.3 has $\sec \geq 0$, respectively $\sec > 0$, if and only if there exists $\tau(r)$ such that $R_i + \tau H \succeq 0$ for $i = 1, 2, 3$, respectively $R_i + \tau H \succ 0$ for $i = 1, 2, 3$.

Remark 3.4 Diagonal entries in R_i are sectional curvatures $\sec(e_i \wedge e_j) = R_{ijij}$ of coordinate planes, while off-diagonal entries are R_{ijkl}, with i, j, k, l all distinct, so the Finsler–Thorpe trick states that $\sec \geq 0$ and $\sec > 0$ are respectively equivalent to the existence of τ such that all $R_{ijij} R_{klkl} - (R_{ijkl} + \tau)^2$ are ≥ 0 and > 0.

To illustrate the above, note that setting φ, ψ, ξ to be the functions in (3.2) that correspond to the standard metrics in S^4 and $\mathbb{C}P^2$, the blocks R_i become constant:

$$\begin{aligned} S^4: &\qquad R_1 = R_2 = R_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \mathbb{C}P^2: &\qquad R_1 = R_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}. \end{aligned} \tag{3.5}$$

In particular, τ can be chosen constant, and $R + \tau * \succeq 0$ if and only if $\tau \in [-1, 1]$ for S^4, and $\tau \in [0, 2]$ for $\mathbb{C}P^2$, and $R + \tau * \succ 0$ if and only if τ is in the open intervals.

Similarly, the curvature of a Grove–Ziller metric with gluing interface $\partial D(B_\pm)$ isometric to $(\mathsf{G}/\mathsf{H}, b^2 Q|_{\mathfrak{n}})$ and $L = r_{\max}^+ + r_{\max}^-$ can be computed by setting φ, ψ, ξ instead to be the functions that make (3.1) match with (2.3), namely (see Fig. 2)

$$\begin{aligned} \varphi(r) &= \begin{cases} \frac{f(r)\, b \sqrt{a}}{\sqrt{f(r)^2 + a\rho^2}}, & \text{if } r \in \left(0, r_{\max}^-\right], \text{ where } \rho = \rho_-(b),\ f = f_-, \\ b, & \text{if } r \in \left[r_{\max}^-, L\right), \end{cases} \\ \psi(r) &\equiv b, \\ \xi(r) &= \begin{cases} b, & \text{if } r \in \left(0, r_{\max}^-\right], \\ \frac{f(L-r)\, b \sqrt{a}}{\sqrt{f(L-r)^2 + a\rho^2}}, & \text{if } r \in \left[r_{\max}^-, L\right), \text{ where } \rho = \rho_+(b),\ f = f_+, \end{cases} \end{aligned} \tag{3.6}$$

as $\mathfrak{m} = \mathfrak{n}_2 \oplus \mathfrak{n}_3$ and $\mathfrak{p} = \mathfrak{n}_1$ for the disk bundle $D(B_-)$, but φ and ξ switch roles on the disk bundle $D(B_+)$, in which $\mathfrak{m} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ and $\mathfrak{p} = \mathfrak{n}_3$. Recall that $f(r) \equiv \frac{\sqrt{a}\,\rho}{\sqrt{a-1}}$ for $r_0 \leq r \leq r_{\max}$

on each of $D(B_\pm)$, so, in a neighborhood of the gluing interface $r = r^-_{\max} = L - r^+_{\max}$, the functions $\varphi = \psi = \xi$ are all constant and equal to b.

In what follows, to simplify the exposition, we shall work with φ, ψ, ξ only on the interval $\big(0, r^-_{\max}\big]$, which, at least on S^4, determines their values for all $0 < r < L$ by setting $r^+_{\max} = r^-_{\max}$ and imposing the additional symmetries (3.3), see Remark 3.2.

Straightforward computations using Proposition 3.3 imply the following:

Proposition 3.5 *The curvature operator of the Grove–Ziller metric (2.3); i.e., the metric (3.1) with φ, ψ, ξ as in (3.6), for $r \in \big(0, r^-_{\max}\big]$, is $R = \operatorname{diag}(R_1, R_2, R_3)$, with:*

$$R_1 = \begin{bmatrix} \frac{4b^2-3\varphi^2}{4b^4} & -\frac{\varphi'}{b^2} \\ -\frac{\varphi'}{b^2} & -\frac{\varphi''}{\varphi} \end{bmatrix}, \quad R_2 = R_3 = \begin{bmatrix} \frac{\varphi^2}{4b^4} & \frac{\varphi'}{2b^2} \\ \frac{\varphi'}{2b^2} & 0 \end{bmatrix}.$$

In particular, $R + \tau * \succeq 0$ if and only if $\tau = -\frac{\varphi'}{2b^2}$.

Indeed, it is easy to verify that $\tau = -\frac{\varphi'}{2b^2}$ is the *only* function $\tau(r)$, $r \in \big(0, r^-_{\max}\big]$, such that $R + \tau * \succeq 0$. Namely, for such r, we have that $[R_i + \tau H]_{22} \equiv 0$ for both $i = 2, 3$, and hence $\det(R_2 + \tau H) = -\left(\frac{\varphi'}{2b^2} + \tau\right)^2 \geq 0$. This pointwise uniqueness of τ corresponds to the presence of flat planes for the Grove–Ziller metric at every point $\gamma(r)$; e.g., $\sec(e_0 \wedge e_2) \equiv 0$ for all r. It is interesting to observe how this (forceful) choice of τ stemming from $R_i + \tau H \succeq 0$, $i = 2, 3$, also satisfies $R_1 + \tau H \succeq 0$, i.e., how the expression for φ in (3.6) ensures $\det(R_1 + \tau H) = \big(\frac{4b^2-3\varphi^2}{4b^4}\big)\big(-\frac{\varphi''}{\varphi}\big) - \big(\frac{3\varphi'}{2b^2}\big)^2 \geq 0$.

Lemma 3.6 *The function $\varphi(r)$ in the Grove–Ziller metric (2.3), given by (3.6) for $r \in \big(0, r^-_{\max}\big]$, satisfies $(4b^2 - 3\varphi^2)(-\varphi'') - 9\varphi\varphi'^2 \geq 0$ for all $r \in \big(0, r^-_{\max}\big]$.*

Proof Solving for $f(r)$ in (3.6), we find $f(r) = \frac{\varphi(r)\rho\sqrt{a}}{\sqrt{ab^2-\varphi(r)^2}}$; in particular, we have that $\varphi(r) < \sqrt{a}\, b$. Differentiating twice, it follows that:

$$f'' = \frac{a^{3/2}b^2\rho}{(ab^2 - \varphi^2)^{5/2}}\big(\varphi''(ab^2 - \varphi^2) + 3\varphi\varphi'^2\big). \tag{3.7}$$

Since $f'' \leq 0$, we have $\varphi''(ab^2 - \varphi^2) + 3\varphi\varphi'^2 \leq 0$, so $(3ab^2 - 3\varphi^2)(-\varphi'') - 9\varphi\varphi'^2 \geq 0$, which implies the desired differential inequality since $a \leq \frac{4}{3}$. ◻

4 Positively curved metrics near Grove–Ziller metrics

In this section, we prove Theorem B in the Introduction, perturbing arbitrary Grove–Ziller metrics with $\sec \geq 0$ on S^4 and $\mathbb{C}P^2$ into cohomogeneity one metrics that we show have $\sec > 0$ via the Finsler–Thorpe trick (Proposition 2.2).

4.1 Metric perturbation

Let M^4 be either S^4 or $\mathbb{C}P^2$, with the cohomogeneity one action of $\mathsf{G} = \mathsf{SO}(3)$ from the previous section. Given a Grove–Ziller metric g_{GZ} on M^4 with gluing interface isometric to $(\mathsf{G}/\mathsf{H}, b^2 Q|_{\mathfrak{n}})$, we have that the length of the circle(s) $\mathsf{K}_\pm/\mathsf{H}$ endowed with the metric $b^2\, Q|_{\mathfrak{p}_\pm}$ is $\rho_\pm(b) = b/|(\mathsf{K}_\pm)_0 \cap \mathsf{H}|$, where K_0 is the identity component of K. From the group

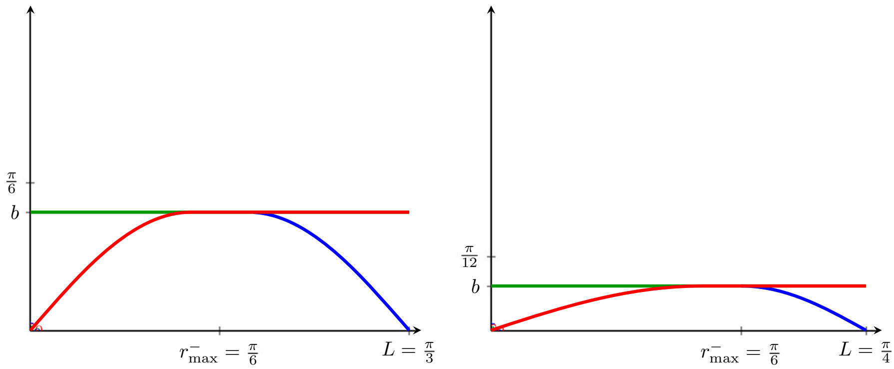

Fig. 2 Graphs of φ_0, ψ_0, ξ_0, for S^4 (*left*) and $\mathbb{C}P^2$ (*right*), cf. (3.6). The upper bound on b and $r^-_{\max} = \frac{\pi}{6}$ follow from (4.1)

diagrams, we compute $|(\mathsf{K}_\pm)_0 \cap \mathsf{H}|$ and obtain $\rho_\pm(b) = b/2$ if $M^4 = S^4$, while $\rho_-(b) = b$ and $\rho_+(b) = b/2$ if $M^4 = \mathbb{C}P^2$. Thus, by (2.4), the length L of the orbit space $M/\mathsf{G} = [0, L]$ satisfies $L > \frac{\sqrt{a}}{\sqrt{a-1}}\, b$ if $M^4 = S^4$, and $L > \frac{3\sqrt{a}}{2\sqrt{a-1}}\, b$ if $M^4 = \mathbb{C}P^2$. Rescaling $(M^4, \mathrm{g}_{\mathrm{GZ}})$ so that $L = \frac{\pi}{3}$ if $M^4 = S^4$, and $L = \frac{\pi}{4}$ if $M^4 = \mathbb{C}P^2$, we obtain a Grove–Ziller metric g_0 homothetic to g_{GZ}, with standardized L, and whose parameters a and b satisfy

$$b < \tfrac{\pi}{3} \tfrac{\sqrt{a-1}}{\sqrt{a}} \text{ if } M^4 = S^4, \quad \text{and} \quad b < \tfrac{\pi}{6} \tfrac{\sqrt{a-1}}{\sqrt{a}} \text{ if } M^4 = \mathbb{C}P^2. \tag{4.1}$$

Using (2.2), it follows that $r^\pm_{\max} = \frac{\pi}{6}$ for $M^4 = S^4$, while $r^-_{\max} = \frac{\pi}{6}$ and $r^+_{\max} = \frac{\pi}{12}$ for $M^4 = \mathbb{C}P^2$. Note that $\varphi_1(r) = \xi_1(r)$ precisely at these values of $r = r^-_{\max}$.

Writing g_0 in the form (3.1) we obtain the functions φ, ψ, ξ in (3.6), which we decorate with the subindex $_0$, i.e., φ_0, ψ_0, ξ_0. Similarly, let g_1 be the standard metric on M^4, and use a subindex $_1$ to decorate the φ, ψ, ξ given in (3.2). Now, define:

$$\begin{aligned} \varphi_s(r) &:= (1-s)\varphi_0(r) + s\,\varphi_1(r), \\ \psi_s(r) &:= (1-s)\psi_0(r) + s\,\psi_1(r), \qquad r \in [0, L]\,, \\ \xi_s(r) &:= (1-s)\,\xi_0(r) + s\,\xi_1(r), \end{aligned} \tag{4.2}$$

i.e., linearly interpolate from φ_0, ψ_0, ξ_0 to φ_1, ψ_1, ξ_1, and set g_s, $s \in [0, 1]$, to be

$$\mathrm{g}_s := \mathrm{d}r^2 + \varphi_s(r)^2\, Q|_{\mathfrak{n}_1} + \psi_s(r)^2\, Q|_{\mathfrak{n}_2} + \xi_s(r)^2\, Q|_{\mathfrak{n}_3}, \quad 0 < r < L. \tag{4.3}$$

The functions (4.2) can be visualized as affine homotopies between Figs. 1 and 2.

It is a straightforward consequence of Proposition 3.1 that g_s are smooth metrics:

Lemma 4.1 *The G-invariant metrics g_s, $s \in [0, 1]$, defined on $M^4 \setminus (B_- \cup B_+)$ by (4.3), extend to smooth metrics on M^4, which we also denote by g_s, $s \in [0, 1]$.*

Proof For simplicity, we focus on the case $M^4 = S^4$, and the case $M^4 = \mathbb{C}P^2$ is left to the reader. The metrics g_s are clearly smooth away from the singular orbits, which correspond to $r = 0$ and $r = L$. In light of Remark 3.2, it suffices to check the smoothness conditions (i)–(iii) in Proposition 3.1, i.e., those regarding $r = 0$.

First, since $\varphi_s^{(k)}(r) = (1-s)\varphi_0^{(k)}(r) + s\,\varphi_1^{(k)}(r)$ for all $k \geq 0$, it is clear that φ_s satisfies (i), as both φ_0 and φ_1 do. Second, if $r \in \left[0, r^-_{\max}\right]$, then $\psi_0(r) = \xi_0(r) = b$, cf. (3.6), so

$\psi_s(r) = (1-s)b + s\,\psi_1(r)$ and $\xi_s(r) = (1-s)b + s\,\xi_1(r)$, and thus:

$$\begin{aligned}\psi_s(r)^2 + \xi_s(r)^2 &= 2(1-s)^2b^2 + 2s(1-s)b(\psi_1(r)+\xi_1(r)) + s^2\left(\psi_1(r)^2+\xi_1(r)^2\right)\\ &= 2(1-s)^2b^2 + 4s(1-s)b\sqrt{3}\cos r + s^2\phi_1(r^2) = \widetilde{\phi}_1(r^2),\\ \psi_s(r)^2 - \xi_s(r)^2 &= 2s(1-s)b\,(\psi_1(r)-\xi_1(r)) + s^2\left(\psi_1(r)^2-\xi_1(r)^2\right)\\ &= 2s(1-s)b\,(-2\sin r) + s^2\,r\,\phi_2(r^2) = r\,\widetilde{\phi}_2(r^2),\end{aligned}$$

where $\widetilde{\phi}_k$, $k=1,2$, are smooth functions, hence (ii) and (iii) are also satisfied. □

Let us introduce functions $\Delta_\varphi, \Delta_\psi, \Delta_\xi$ of r so that (4.2) can be written as

$$\varphi_s = \varphi_0 + s\,\Delta_\varphi, \quad \psi_s = \psi_0 + s\,\Delta_\psi, \quad \xi_s = \xi_0 + s\,\Delta_\xi, \tag{4.4}$$

i.e., $\Delta_\varphi(r) := \varphi_1(r) - \varphi_0(r)$, and similarly for Δ_ψ and Δ_ξ. Note that each of these functions is smooth up to $r=0$ and $r=L$; in particular, bounded on $[0,L]$. In the sequel, we take the point of view (4.4) that φ_s, ψ_s, ξ_s are perturbations of φ_0, ψ_0, ξ_0.

4.2 Regularity of perturbation

By (4.3), Lemma 4.1, and Proposition 3.3, each entry of the curvature operator matrix R_s of g_s along $\gamma(r)$ is a smooth function

$$\frac{P(\varphi_s,\,\psi_s,\,\xi_s,\,\varphi_s',\,\psi_s',\,\xi_s',\,\varphi_s'',\,\psi_s'',\,\xi_s'')}{\varphi_s^2\,\psi_s^2\,\xi_s^2}, \tag{4.5}$$

where P is a polynomial. Note that the g_s-orthonormal basis on which the matrix R_s is being written varies smoothly with s. The singularities in (4.5) at $r=0$ and $r=L$, due to $\varphi_s(0)=0$ and $\xi_s(L)=0$, are removable as a consequence of Lemma 4.1. This corresponds to the fact that also P vanishes to the appropriate order because φ_s, ψ_s, ξ_s satisfy the required smoothness conditions. Moreover, these smoothness conditions imply that (4.5) equals

$$\frac{P(\varphi_s,\,\psi_s,\,\xi_s,\,\varphi_s',\,\psi_s',\,\xi_s',\,\varphi_s'',\,\psi_s'',\,\xi_s'')}{\varphi_0^2\,\psi_0^2\,\xi_0^2} + Q(s,r)\,s, \tag{4.6}$$

where Q is continuous. Furthermore, by (4.4), the numerator above can be written as a polynomial $\widetilde{P}$ in the parameter s, the functions φ_0, ψ_0, ξ_0 and their first and second derivatives, and the functions $\Delta_\varphi, \Delta_\psi, \Delta_\xi$ and their first and second derivatives (indicated as ... below). Thus, (4.6) and hence (4.5) are equal to

$$\frac{\widetilde{P}(s,\,\varphi_0,\,\psi_0,\,\xi_0,\dots,\Delta_\varphi,\,\Delta_\psi,\,\Delta_\xi,\dots)}{\varphi_0^2\,\psi_0^2\,\xi_0^2} + Q(s,r)\,s. \tag{4.7}$$

In particular, the dependence of the above on s is polynomial in the first term, and smooth on the second. Expanding in s, we have

$$\widetilde{P}(s,\,\varphi_0,\,\psi_0,\,\xi_0,\dots,\Delta_\varphi,\Delta_\psi,\Delta_\xi,\dots) = \sum_{n=0}^{d}\widetilde{P}_n(\varphi_0,\psi_0,\xi_0,\dots,\Delta_\varphi,\Delta_\psi,\Delta_\xi,\dots)\,s^n,$$

where $\widetilde{P}_n$ are polynomials. Each coefficient in this sum is a smooth function of r that vanishes at $r=0$ and $r=L$ in such way that the limits of (4.7) as $r\searrow 0$ and $r\nearrow L$ are both finite, so the corresponding coefficients in (4.7) extend to smooth (hence bounded) functions on $[0,L]$. Thus, $\widetilde{P}(s,\,\varphi_0,\,\psi_0,\,\xi_0,\dots,\Delta_\varphi,\,\Delta_\psi,\,\Delta_\xi,\dots)/\varphi_0^2\,\psi_0^2\,\xi_0^2$ can be regarded as a polynomial in the variable s whose coefficients are *continuous* functions of r. We will implicitly (and repeatedly) use this fact in what follows.

Notation *We use $O(s^n)$, respectively $O(r^m)$, to denote any functions of the form $s^n\ F(s,r)$, respectively $r^m\ F(s,r)$, where $F\colon [0,1]\times[0,L]\to\mathbb{R}$ is* bounded.

4.3 Positive curvature on S^4

To simplify the exposition, we shall focus primarily on the case $M^4 = S^4$, in which $r^{\pm}_{\max} = \frac{L}{2} = \frac{\pi}{6}$ and it suffices to verify $\sec > 0$ along the geodesic segment $\gamma(r)$ with $r \in \big[0, r^{-}_{\max}\big]$ due to the additional symmetries (3.3), cf. Remark 3.2.

Let $R_s = \operatorname{diag}\big((R_s)_1, (R_s)_2, (R_s)_3\big)$ be the curvature operator of (S^4, g_s) along $\gamma(r)$, given by Proposition 3.3, where φ, ψ, ξ are set to be φ_s, ψ_s, ξ_s defined in (4.2). As discussed above, R_s, $s \in [0,1]$, extends smoothly to $r = 0$, and this extension (as well as its entries) will be denoted by the same symbol(s). Clearly, R_0 is the curvature operator of the Grove–Ziller metric g_0, so $R_0 + \tau_0 * \succeq 0$ for all $r \in \big[0, r^{-}_{\max}\big]$, where $\tau_0 := -\frac{\varphi_0'}{2b^2}$, see Proposition 3.5. The proof of Theorem B hinges on the next:

Claim 4.2 *If $s > 0$ is sufficiently small, then $R_s + \tau_s * \succ 0$ for all $r \in \big[0, r^{-}_{\max}\big]$, with*

$$\tau_s(r) := \tau_0(r) + \frac{2(\sqrt{3}-b)}{b^3}\, s = -\frac{\varphi_0'(r)}{2b^2} + \frac{2(\sqrt{3}-b)}{b^3}\, s. \tag{4.8}$$

We begin the journey towards Claim 4.2 observing that certain diagonal entries of R_s, which are sectional curvatures with respect to g_s, are positive for all $s \in (0,1]$.

Proposition 4.3 *For all $s \in (0,1]$ and $r \in \big[0, r^{-}_{\max}\big]$, the following hold:*

*(*i*)* $[(R_s)_i]_{22} = \sec_{\mathrm{g}_s}(e_0 \wedge e_i) > 0$ *for* $1 \le i \le 3$*;*
*(*ii*)* $[(R_s)_1]_{11} = \sec_{\mathrm{g}_s}(e_2 \wedge e_3) > 0$.

Proof As the round metric g_1 has $\sec \equiv 1$, we have $\varphi_1''(r) < 0$, $\psi_1''(r) < 0$, $\xi_1''(r) < 0$ by Proposition 3.3, cf. (3.2) and (3.5). Thus $\varphi_s''(r) < 0$, $\psi_s''(r) < 0$, $\xi_s''(r) < 0$ for all $s \in (0,1]$ and $r \in \big[0, r^{-}_{\max}\big]$, which implies, by Proposition 3.3, that $\sec_{\mathrm{g}_s}(e_0 \wedge e_i) > 0$, for $i = 2, 3$. In the case of $\sec_{\mathrm{g}_s}(e_0 \wedge e_1)$, a further argument is required at $r = 0$. Namely, using the smoothness conditions, we see that if $s \in (0,1]$, then

$$\lim_{r\searrow 0} \sec_{\mathrm{g}_s}(e_0\wedge e_1)(r) = (1-s)\sec_{\mathrm{g}_0}(e_0\wedge e_1)(0) + s\sec_{\mathrm{g}_1}(e_0\wedge e_1)(0) > 0,$$

where $(e_0 \wedge e_1)(r)$ denotes the 2-plane in $T_{\gamma(r)}S^4$ spanned by e_0 and e_1, which concludes the proof of (i). Regarding (ii), if $s \in (0,1]$ and $r \in \big(0, r^{-}_{\max}\big]$, then

$$\varphi_s \le \xi_s < \psi_s, \quad \xi_s' < 0, \quad \psi_s' \ge 0,$$

which implies that

$$\begin{aligned}
\sec_{\mathrm{g}_s}(e_2\wedge e_3) &= \frac{\psi_s^4 + \xi_s^4 - \varphi_s^4 + 2(\xi_s^2 - \varphi_s^2)(\varphi_s^2 - \psi_s^2)}{4\,\varphi_s^2\,\psi_s^2\,\xi_s^2} - \frac{\psi_s'\xi_s'}{\psi_s\xi_s} \\
&= \frac{(\xi_s^2-\psi_s^2)^2}{4\,\varphi_s^2\,\psi_s^2\,\xi_s^2} + \frac{2\psi_s^2 - \varphi_s^2}{4\,\psi_s^2\,\xi_s^2} + \frac{\xi_s^2-\varphi_s^2}{2\,\psi_s^2\,\xi_s^2} - \frac{\psi_s'\xi_s'}{\psi_s\xi_s} \ge \frac{b^2}{4\psi_s^2\,\xi_s^2},
\end{aligned}$$

since $2\psi_s^2 - \varphi_s^2 \ge \psi_s^2$ and $\psi_s \ge \psi_0 \equiv b$ is uniformly bounded from below. □

Let us introduce functions η_i, μ_i, ν_i, $i = 1, 2, 3$, such that the blocks of the curvature operator $R_s = \operatorname{diag}\big((R_s)_1, (R_s)_2, (R_s)_3\big)$ of g_s can be written as a perturbation

$$(R_s)_i = (R_0)_i + \begin{bmatrix} \eta_i(s,r) & \mu_i(s,r) \\ \mu_i(s,r) & \nu_i(s,r) \end{bmatrix}, \qquad i = 1, 2, 3, \tag{4.9}$$

of the blocks of the curvature operator $R_0 = \operatorname{diag}\big((R_0)_1, (R_0)_2, (R_0)_3\big)$ of the Grove–Ziller metric g_0. Recall that, for $r \in \big(0, r_{\max}^-\big]$, these blocks $(R_0)_i$ are computed in Proposition 3.5, setting $\varphi = \varphi_0$, i.e., φ is given by (3.6). Clearly, each of η_i, μ_i, ν_i is $O(s^n)$ for some $n \geq 1$.

4.3.1 First block

We first analyze the block $i = 1$ of the matrices R_s and $R_s + \tau_s *$.

Proposition 4.4 *For all $r \in \big[0, r_{\max}^-\big]$, the entries of $(R_s)_1$ satisfy:*

$$\eta_1(s,r) = \left(\frac{3\varphi_0}{2b^5}(\varphi_0(\Delta_\psi + \Delta_\xi) - b\Delta_\varphi) - \frac{\Delta_\psi + \Delta_\xi}{b^3}\right) s + O(s^2),$$

$$\mu_1(s,r) = \left(\frac{\varphi_0(\psi_1' + \xi_1')}{2b^3} - \frac{\Delta_\varphi'}{b^2} + \frac{\varphi_0'}{b^3}(\Delta_\psi + \Delta_\xi)\right) s + O(s^2),$$

$$\nu_1(s,r) = \left(\frac{-\varphi_1''\varphi_0 + \varphi_0''\varphi_1}{\varphi_0^2}\right) s + O(s^2).$$

Proof First, let us consider η_1. From Proposition 3.3,

$$\begin{aligned}[(R_s)_1]_{11} &= \frac{\psi_s^4 + \xi_s^4 - \varphi_s^4 + 2(\xi_s^2 - \varphi_s^2)(\varphi_s^2 - \psi_s^2)}{4\varphi_s^2\,\psi_s^2\,\xi_s^2} - \frac{\psi_s'\xi_s'}{\psi_s\xi_s} \\ &= \frac{(\xi_s^2 - \psi_s^2)^2}{4\varphi_s^2\,\psi_s^2\,\xi_s^2} - \frac{3\varphi_s^2}{4\psi_s^2\,\xi_s^2} + \frac{\xi_s^2 + \psi_s^2}{2\psi_s^2\,\xi_s^2} - \frac{\psi_s'\xi_s'}{\psi_s\xi_s}.\end{aligned}$$

We analyze these four terms separately using (4.4), as follows

$$-\frac{3\varphi_s^2}{4\psi_s^2\,\xi_s^2} = -\frac{3\varphi_0^2}{4b^4} - \frac{3\varphi_0}{2b^5}(b\Delta_\varphi - \varphi_0(\Delta_\psi + \Delta_\xi))s + O(s^2),$$

$$\frac{\xi_s^2 + \psi_s^2}{2\psi_s^2\,\xi_s^2} = \frac{1}{b^2} - \frac{\Delta_\psi + \Delta_\xi}{b^3}s + O(s^2), \quad \frac{(\xi_s^2 - \psi_s^2)^2}{4\varphi_s^2\,\psi_s^2\,\xi_s^2} = O(s^2), \quad -\frac{\psi_s'\xi_s'}{\psi_s\xi_s} = O(s^2).$$

Therefore, adding the above together, we find:

$$[(R_s)_1]_{11} = \frac{4b^2 - 3\varphi_0^2}{4b^4} + \left(\frac{3\varphi_0}{2b^5}(\varphi_0(\Delta_\psi + \Delta_\xi) - b\Delta_\varphi) - \frac{\Delta_\psi + \Delta_\xi}{b^3}\right)s + O(s^2),$$

which establishes the claimed expansion of $\eta_1(s,r) = [(R_s)_1]_{11} - \frac{4b^2 - 3\varphi_0^2}{4b^4}$, cf. (4.9).

Next, consider μ_1. From Proposition 3.3,

$$\begin{aligned}[(R_s)_1]_{12} &= \frac{\xi_s'(\xi_s^2 + \varphi_s^2 - \psi_s^2)}{2\varphi_s\,\psi_s\,\xi_s^2} + \frac{\psi_s'(\varphi_s^2 + \psi_s^2 - \xi_s^2)}{2\varphi_s\,\psi_s^2\,\xi_s} - \frac{\varphi_s'}{\psi_s\,\xi_s} \\ &= \frac{(\xi_s^2 - \psi_s^2)(\xi_s'\psi_s - \psi_s'\xi_s)}{2\varphi_s\,\psi_s^2\,\xi_s^2} + \frac{\varphi_s(\xi_s'\psi_s + \psi_s'\xi_s)}{2\psi_s^2\,\xi_s^2} - \frac{\varphi_s'}{\psi_s\,\xi_s}.\end{aligned}$$

We analyze these three terms separately, using (4.4), as before:

$$\frac{(\xi_s^2-\psi_s^2)(\xi_s'\psi_s-\psi_s'\xi_s)}{2\varphi_s\,\psi_s^2\,\xi_s^2}=O(s^2),\quad \frac{\varphi_s(\xi_s'\psi_s+\psi_s'\xi_s)}{2\psi_s^2\,\xi_s^2}=\frac{\varphi_0(\psi_1'+\xi_1')}{2b^3}s+O(s^2),$$
$$-\frac{\varphi_s'}{\psi_s\,\xi_s}=-\frac{\varphi_0'}{b^2}+\left(\frac{\varphi_0'(\Delta_\psi+\Delta_\xi)}{b^3}-\frac{\Delta_\varphi'}{b^2}\right)s+O(s^2).$$

Thus, adding the above, we have:

$$[(R_s)_1]_{12}=-\frac{\varphi_0'}{b^2}+\left(\frac{\varphi_0(\psi_1'+\xi_1')}{2b^3}-\frac{\Delta_\varphi'}{b^2}+\frac{\varphi_0'(\Delta_\psi+\Delta_\xi)}{b^3}\right)s+O(s^2),$$

which establishes the claimed expansion of $\mu_1(s,r)=[(R_s)_1]_{12}+\frac{\varphi_0'}{b^2}$, cf. (4.9).

Finally, let us consider ν_1. From Proposition 3.3, we have:

$$[(R_s)_1]_{22}=-\frac{\varphi_s''}{\varphi_s}=-\frac{\varphi_0''}{\varphi_0}+\left(\frac{-\varphi_1''\varphi_0+\varphi_0''\varphi_1}{\varphi_0^2}\right)s+O(s^2),$$

which establishes the claimed expansion of $\nu_1(s,r)=[(R_s)_1]_{22}+\frac{\varphi_0''}{\varphi_0}$, cf. (4.9). □

Proposition 4.5 *If $s>0$ is sufficiently small, then the matrix*

$$(R_s)_1+\tau_s H=\begin{bmatrix}\frac{4b^2-3\varphi_0^2}{4b^4}+\eta_1(s,r) & -\frac{3\varphi_0'}{2b^2}+\mu_1(s,r)+\frac{2(\sqrt{3}-b)}{b^3}s\\ -\frac{3\varphi_0'}{2b^2}+\mu_1(s,r)+\frac{2(\sqrt{3}-b)}{b^3}s & -\frac{\varphi_0''}{\varphi_0}+\nu_1(s,r)\end{bmatrix}$$

is positive-definite for all $r\in[0,r_{\max}^-]$.

Proof The expression above for $(R_s)_1+\tau_s H$ follows from Proposition 3.5, as well as (3.4), (4.8), and (4.9). From Proposition 4.3 (ii), we know that $[(R_s)_1]_{11}>0$ for all $s\in(0,1]$ and $r\in[0,r_{\max}^-]$. So, by Sylvester's criterion, it suffices to show that if $s>0$ is sufficiently small, then the following is positive:

$$\begin{aligned}\det\big((R_s)_1+\tau_s H\big)=&\left(\frac{4b^2-3\varphi_0^2}{4b^4}\right)\left(-\frac{\varphi_0''}{\varphi_0}\right)-\left(\frac{3\varphi_0'}{2b^2}\right)^2-\frac{\varphi_0''}{\varphi_0}\,\eta_1(s,r)\\&+\frac{4b^2-3\varphi_0^2}{4b^4}\,\nu_1(s,r)+\frac{3\varphi_0'}{b^2}\left(\mu_1(s,r)+\frac{2(\sqrt{3}-b)}{b^3}s\right)\\&+\eta_1(s,r)\,\nu_1(s,r)-\left(\mu_1(s,r)+\frac{2(\sqrt{3}-b)}{b^3}s\right)^2.\end{aligned}$$

By Proposition 4.4, we have $\det\big((R_s)_1 + \tau_s H\big) = A(r) + B(r)\,s + O(s^2)$, where

$$
\begin{aligned}
A(r) &:= \left(\frac{4b^2 - 3\varphi_0^2}{4b^4}\right)\left(-\frac{\varphi_0''}{\varphi_0}\right) - \left(\frac{3\varphi_0'}{2b^2}\right)^2,\\
B(r) &:= \left(-\frac{\varphi_0''}{\varphi_0}\right)\left(\frac{3\varphi_0}{2b^5}(\varphi_0(\Delta_\psi + \Delta_\xi) - b\Delta_\varphi) - \frac{\Delta_\psi + \Delta_\xi}{b^3}\right)\\
&\quad + \left(\frac{4b^2 - 3\varphi_0^2}{4b^4}\right)\left(\frac{-\varphi_1''\varphi_0 + \varphi_0''\varphi_1}{\varphi_0^2}\right)\\
&\quad + \frac{3\varphi_0'}{b^2}\left(\frac{\varphi_0(\psi_1' + \xi_1')}{2b^3} - \frac{\Delta_\varphi'}{b^2} + \frac{\varphi_0'}{b^3}(\Delta_\psi + \Delta_\xi) + \frac{2(\sqrt{3} - b)}{b^3}\right).
\end{aligned}
$$

Note that $A(r) \geq 0$ if $r \in \big[0, r_{\max}^-\big]$ by Lemma 3.6, but $A(r) \equiv 0$ near $r = r_{\max}^-$. We claim that there exist $0 < r_* < r_{\max}^-$ and constants $\alpha > 0$ and $\beta > 0$ such that

$$
\begin{aligned}
A(r) &\geq \alpha > 0 \text{ for all } 0 \leq r \leq r_*,\\
B(r) &\geq \beta > 0 \text{ for all } r_* \leq r \leq r_{\max}^-,
\end{aligned}
\tag{4.10}
$$

from which it clearly follows that $\det\big((R_s)_1 + \tau_s H\big) > 0$ for all $r \in \big[0, r_{\max}^-\big]$ and sufficiently small $s > 0$, as desired. Recall that there exists $0 < r_0 < r_{\max}^-$ so that:

- for all $r \in (0, r_0)$, we have $\varphi_0'(r) > 0$ and $\varphi_0''(r) < 0$,
- for all $r \in \big[r_0, r_{\max}^-\big]$, we have $\varphi_0(r) = b$, and hence $\varphi_0'(r) = \varphi_0''(r) = 0$,

cf. (3.6) and the Grove–Ziller construction (Section 2.1.1). Moreover, for all $\varepsilon > 0$, there exists $0 < r_* < r_0$, such that for $r \in \big[r_*, r_{\max}^-\big]$, we have:

$$
0 \leq \varphi_0'(r) < \varepsilon, \quad 0 \leq -\varphi_0''(r) < \varepsilon, \quad \text{and} \quad b - \varepsilon < \varphi_0(r) \leq b, \tag{4.11}
$$

and these inequalities are strict on $[r_*, r_0)$. Thus, choosing $\varepsilon > 0$ sufficiently small, we have that for all $r \in \big[r_*, r_{\max}^-\big]$,

$$
\frac{-\varphi_1''\varphi_0 + \varphi_0''\varphi_1}{\varphi_0^2} = \frac{(2\sin r)(\varphi_0 + \varphi_0'')}{\varphi_0^2} \geq \frac{(2\sin r)(b - 2\varepsilon)}{b^2} > \frac{1}{4b}.
$$

Furthermore, by continuity, the following are uniformly bounded on $r \in \big[r_*, r_{\max}^-\big]$,

$$
\begin{aligned}
&\left|-\frac{1}{\varphi_0}\left(\frac{3\varphi_0}{2b^5}(\varphi_0(\Delta_\psi + \Delta_\xi) - b\Delta_\varphi) - \frac{\Delta_\psi + \Delta_\xi}{b^3}\right)\right| < C_1,\\
&\left|\frac{3}{b^2}\left(\frac{\varphi_0(\psi_1' + \xi_1')}{2b^3} - \frac{\Delta_\varphi'}{b^2} + \frac{\varphi_0'(\Delta_\psi + \Delta_\xi)}{b^3} + \frac{2(\sqrt{3} - b)}{b^3}\right)\right| < C_2,
\end{aligned}
$$

where C_1 and C_2 are constants independent of r_*; and $\left(\frac{4b^2 - 3\varphi_0^2}{4b^4}\right) \geq \frac{1}{4b^2}$ by (4.11). Putting the above together, and making $\varepsilon > 0$ even smaller if needed, we conclude

$$
B(r) > -\varepsilon\, C_1 + \tfrac{1}{16b^3} - \varepsilon\, C_2 = \tfrac{1}{16b^3} - \varepsilon\,(C_1 + C_2) > \beta > 0
$$

for all $r \in \big[r_*, r_{\max}^-\big]$, where, e.g., $\beta = \frac{1}{32b^3}$. Finally, in order to prove the inequality regarding $A(r)$ in (4.10), recall there exists $c > 0$ such that $\sec_{g_{D^2}} \geq c > 0$ for all $r \in [0, r_*]$, by

Remark 2.1. From (3.7), in the proof of Lemma 3.6, we have that

$$\sec_{g_{D^2}} = -\frac{f''}{f} = \frac{ab^2}{(ab^2-\varphi_0^2)^2}\,\frac{(-\varphi_0'')(ab^2-\varphi_0^2)-3\varphi_0\varphi_0'^2}{\varphi_0},$$

from which it follows that

$$\frac{3(ab^2-\varphi_0^2)^2}{ab^2}\,\sec_{g_{D^2}} = 3\left(-\frac{\varphi_0''}{\varphi_0}\right)(ab^2-\varphi_0^2) - 9\varphi_0'^2 \le \left(-\frac{\varphi_0''}{\varphi_0}\right)(4b^2-3\varphi_0^2)-9\varphi_0'^2,$$

because $1 < a \le \frac{4}{3}$. Therefore, as $\varphi_0(r) < \sqrt{a}\, b$ for all r, there exists $\alpha > 0$ so that

$$A(r) \ge \frac{3}{4}\,\frac{(ab^2-\varphi_0^2)^2}{ab^2}\,\sec_{g_{D^2}} \ge \frac{3}{4}\,\frac{(ab^2-\varphi_0^2)^2}{ab^2}\,c > \alpha > 0, \quad \text{for all } r \in [0, r_*].$$

□

4.3.2 Remaining blocks

We now handle the remaining blocks $i = 2, 3$.

Proposition 4.6 *For all* $r \in \left[0, r_{\max}^-\right]$, *the entries of* $(R_s)_i$, *for* $i = 2, 3$, *satisfy:*

$$\eta_i(s,r) = \left(\tfrac{\sqrt{3}}{b} + O(r)\right)s + O(s^2),\ \mu_i(s,r) = \left(-\tfrac{2(\sqrt{3}-b)}{b^3} + O(r)\right)s + O(s^2),$$
$$\nu_i(s,r) = \left(\tfrac{\sqrt{3}}{b} + O(r)\right)s + O(s^2).$$

Proof First, let us consider η_2. From Proposition 3.3,

$$[(R_s)_2]_{11} = \frac{\varphi_s^2}{4\psi_s^2\,\xi_s^2} + \frac{\psi_s^2-\xi_s^2}{2\psi_s^2\,\xi_s^2} + \frac{\xi_s^4+2\xi_s^2\psi_s^2-3\psi_s^4-4\varphi_s\psi_s^2\xi_s\varphi_s'\xi_s'}{4\varphi_s^2\,\psi_s^2\,\xi_s^2}.$$

We analyze these three terms separately using (4.4). The first two satisfy

$$\frac{\varphi_s^2}{4\psi_s^2\,\xi_s^2} = \frac{\varphi_0^2}{4b^4} + s\,O(r^2) + O(s^2), \quad \text{and} \quad \frac{\psi_s^2-\xi_s^2}{2\psi_s^2\,\xi_s^2} = s\,O(r) + O(s^2),$$

while the third satisfies

$$\frac{\xi_s^4+2\psi_s^2\xi_s^2-3\psi_s^4-4\varphi_s\psi_s^2\xi_s\varphi_s'\xi_s'}{4\varphi_s^2\,\psi_s^2\,\xi_s^2} = \frac{2(\Delta_\xi-\Delta_\psi)-\varphi_0\varphi_0'\Delta_\xi'}{b\varphi_0^2}\,s + O(s^2)$$
$$= \left(\tfrac{\sqrt{3}}{b} + O(r)\right)s + O(s^2),$$

since $\lim\limits_{r\searrow 0} \frac{2(\Delta_\xi-\Delta_\psi)-\varphi_0\varphi_0'\Delta_\xi'}{\varphi_0^2} = \sqrt{3}$, by L'Hôpital's rule and Proposition 3.1 (i).

Altogether, the above yields $[(R_s)_2]_{11} = \frac{\varphi_0^2}{4b^4} + \left(\frac{\sqrt{3}}{b} + O(r)\right)s + O(s^2)$, and hence establishes the claimed expansion of $\eta_2(s,r) = [(R_s)_2]_{11} - \frac{\varphi_0^2}{4b^4}$, cf. (4.9).

Second, the proof that η_3 has the same expansion as η_2 is similar. Namely,

$$[(R_s)_3]_{11} = \frac{\varphi_s^2}{4\psi_s^2\,\xi_s^2} + \frac{(\psi_s^2-\xi_s^2)(\psi_s^2+3\xi_s^2-2\varphi_s^2)-4\varphi_s\psi_s\xi_s^2\varphi_s'\psi_s'}{4\varphi_s^2\,\psi_s^2\,\xi_s^2},$$

where the first term was already considered above, and the second term satisfies

$$\frac{(\psi_s^2-\xi_s^2)(\psi_s^2+3\xi_s^2-2\varphi_s^2)-4\varphi_s\psi_s\xi_s^2\varphi_s'\psi_s'}{4\varphi_s^2\,\psi_s^2\,\xi_s^2}=\left(\frac{\sqrt{3}}{b}+O(r)\right)s+O(s^2),$$

by similar considerations involving L'Hôpital's rule and Proposition 3.1 (i). Thus, $\eta_3(s,r)=[(R_s)_3]_{11}-\frac{\varphi_0^2}{4b^4}=\left(\frac{\sqrt{3}}{b}+O(r)\right)s+O(s^2)$, cf. (4.9).

Next, consider μ_2. From Proposition 3.3,

$$[(R_s)_2]_{12}=\frac{\varphi_s'}{2\psi_s\,\xi_s}+\frac{\varphi_s'\xi_s(\psi_s^2-\xi_s^2)+\varphi_s\xi_s'(\xi_s^2+\psi_s^2-\varphi_s^2)-2\varphi_s\psi_s\xi_s\psi_s'}{2\varphi_s^2\,\psi_s\,\xi_s^2}$$

The first term above satisfies

$$\frac{\varphi_s'}{2\xi_s\psi_s}=\frac{\varphi_0'}{2b^2}+\left(O(r^2)-\frac{2(\sqrt{3}-b)}{b^3}\right)s+O(s^2),$$

while the second satisfies

$$\frac{\varphi_s'\xi_s(\psi_s^2-\xi_s^2)+\varphi_s\xi_s'(\xi_s^2+\psi_s^2-\varphi_s^2)-2\varphi_s\psi_s\xi_s\psi_s'}{2\varphi_s^2\,\psi_s\,\xi_s^2}=s\,O(r)+O(s^2).$$

So, $\mu_2(s,r)=[(R_s)_2]_{12}-\frac{\varphi_0'}{2b^2}=\left(-\frac{2(\sqrt{3}-b)}{b^3}+O(r)\right)s+O(s^2)$, cf. (4.9). The proof that μ_3 has the same expansion as μ_2 is analogous, and left to the reader.

Finally, let us consider ν_2 and ν_3. From Proposition 3.3 and (4.9), we have

$$\nu_2(s,r)=[(R_s)_2]_{22}=-\frac{\psi_s''}{\psi_s}\quad\text{and}\quad\nu_3(s,r)=[(R_s)_3]_{22}=-\frac{\xi_s''}{\xi_s}.$$

By (4.4), we have $\psi_s''=\Delta_\psi''\,s=\psi_1''\,s$ and $\xi_s''=\Delta_\xi''\,s=\xi_1''\,s$, so

$$\nu_2(s,r)=\left(\frac{\sqrt{3}}{b}+O(r)\right)s+O(s^2),\quad\text{and}\quad\nu_3(s,r)=\left(\frac{\sqrt{3}}{b}+O(r)\right)s+O(s^2).$$

□

Proposition 4.7 *If $s>0$ is sufficiently small, then the matrices*

$$(R_s)_i+\tau_s H=\begin{bmatrix}\frac{\varphi_0^2}{4b^4}+\eta_i(s,r) & \mu_i(s,r)+\frac{2(\sqrt{3}-b)}{b^3}s\\ \mu_i(s,r)+\frac{2(\sqrt{3}-b)}{b^3}s & \nu_i(s,r)\end{bmatrix},\quad i=2,3, \tag{4.12}$$

are positive-definite for all $r\in\left[0,r_{\max}^-\right]$.

Proof The expression (4.12) for $(R_s)_i+\tau_s H$, $i=2,3$, follows from Proposition 3.5, as well as (3.4), (4.8), and (4.9). First, consider the (1, 1)-entry of these matrices:

$$[(R_s)_i]_{11}=\frac{\varphi_0^2}{4b^4}+\left(\frac{\sqrt{3}}{b}+O(r)\right)s+O(s^2),\quad\text{for } i=2,3,$$

cf. Proposition 4.6. Since $\varphi_0(r)>0$ away from $r=0$, and the $O(s)$ part of the above is uniformly positive near $r=0$, it follows that $[(R_s)_i]_{11}>0$ for all $r\in\left[0,r_{\max}^-\right]$ and $i=2,3$, provided $s>0$ is sufficiently small.

Second, let us analyze the determinant of (4.12). By Proposition 4.6,

$$\eta_i(s,r)\nu_i(s,r)=\left(\frac{3}{b^2}+O(r)\right)s^2+O(s^3),$$

$$\mu_i(s,r)+\frac{2(\sqrt{3}-b)}{b^3}s=s\,O(r)+O(s^2).$$

Thus, using that $\nu_i(s,r) = [(R_s)_i]_{22}$, for $i = 2, 3$, we have:

$$\begin{aligned}\det\big((R_s)_i + \tau_s H\big) &= \nu_i(s,r)\,\frac{\varphi_0^2}{4b^4} + \Big(\frac{3}{b^2} + O(r)\Big)\,s^2 + O(s^3)\\ &= [(R_s)_i]_{22}\,\frac{\varphi_0^2}{4b^4} + \Big(\frac{3}{b^2} + O(r)\Big)\,s^2 + O(s^3).\end{aligned}$$

By Proposition 4.3 (i), the $O(s)$ part of the above is positive for $r \in \big(0, r_{\max}^-\big]$, but vanishes at $r = 0$, as $\varphi_0(0) = 0$. Since the $O(s^2)$ part has a positive limit as $r \searrow 0$, we have that $\det\big((R_s)_i + \tau_s H\big) > 0$ for all $r \in \big[0, r_{\max}^-\big]$ and $i = 2, 3$, if $s > 0$ is sufficiently small. Positive-definiteness now follows from Sylvester's criterion. □

The above Proposition 4.5 and 4.7 imply Claim 4.2, since $R_s + \tau_s *$ is block diagonal with blocks $(R_s)_i + \tau_s H$, $i = 1, 2, 3$, see Proposition 3.3 and (3.4). In turn, Claim 4.2 and the Finsler–Thorpe trick (Proposition 2.2) imply that $\sec_{g_s} > 0$ for sufficiently small $s > 0$. This proves Theorem B for $M^4 = S^4$; since, if the original Grove–Ziller metric g_{GZ} was rescaled as $g_0 = \lambda^2\, g_{GZ}$ to standardize $L = \frac{\pi}{3}$, then $\lambda^{-2}\, g_s$ has sec > 0 and is arbitrarily C^∞-close to g_{GZ} for $s > 0$ sufficiently small.

4.4 Positive curvature on $\mathbb{C}P^2$

We now briefly discuss the proof of Theorem B for $M^4 = \mathbb{C}P^2$. Recall that, in this case, $L = \frac{\pi}{4}$, with $r_{\max}^- = \frac{\pi}{6}$ and $r_{\max}^+ = \frac{\pi}{12}$. Differently from S^4, for $M^4 = \mathbb{C}P^2$, the situation on the intervals $[0, r_{\max}^-] = \big[0, \frac{\pi}{6}\big]$ and $[r_{\max}^-, L] = \big[\frac{\pi}{6}, \frac{\pi}{4}\big]$ has to be analyzed separately, cf. Remark 3.2.

Denoting by R_0 the curvature operator of the Grove–Ziller metric g_0 on $\mathbb{C}P^2$, the function $\tau_0\colon [0, L] \to \mathbb{R}$ so that $R_0 + \tau_0 * \succeq 0$ for all $r \in [0, L]$ is given by

$$\tau_0(r) = \begin{cases} -\frac{\varphi_0'(r)}{2b^2}, & \text{if } r \in \big[0, r_{\max}^-\big],\\ -\frac{\xi_0'(r)}{2b^2}, & \text{if } r \in \big[r_{\max}^-, L\big],\end{cases}$$

cf. Proposition 3.5. Note that $\varphi_0' = \xi_0' = 0$ near $r = r_{\max}^-$. The proof of Theorem B follows in the same way as in the case $M^4 = S^4$ above, replacing Claim 4.2 with:

Claim 4.8 *If $s > 0$ is sufficiently small, then $R_s + \tau_s * \succ 0$ for all $r \in [0, L]$, where*

$$\tau_s(r) := \begin{cases} -\frac{\varphi_0'(r)}{2b^2} + \Big(\frac{3}{2b} + \frac{1-b}{b^3}\Big)\,s, & \text{if } r \in \big[0, r_{\max}^-\big],\\ -\frac{\xi_0'(r)}{2b^2} + \frac{\sqrt{2}-2b}{b^3}\,s, & \text{if } r \in \big(r_{\max}^-, L\big].\end{cases}$$

Remark 4.9 Similarly to (4.8) in Claim 4.2, the above function τ_s is obtained from τ_0 by adding a locally constant multiple of s. This $O(s)$ perturbation is not constant as in the case of $M^4 = S^4$, and, as a result, $\tau_s(r)$ is *discontinuous* at $r = r_{\max}^-$ for all $s > 0$. Nevertheless, the application of the Finsler–Thorpe trick (Proposition 2.2) is pointwise and no regularity is needed. A posteriori, a continuous function $\widetilde{\tau}_s(r)$ such that $R_s + \widetilde{\tau}_s * \succ 0$ for all sufficiently small $s > 0$ can be chosen, e.g., as the midpoint $\widetilde{\tau}_s(r) = \frac{1}{2}(\tau_{\min} + \tau_{\max})$ of $[\tau_{\min}, \tau_{\max}]$ for each $r \in [0, L]$, see Remark 2.3.

The proof of Claim 4.8 follows the same template from Claim 4.2, relying on expansions in s of the functions η_i, μ_i, ν_i, cf. (4.9). The statement of Proposition 4.4, regarding $i = 1$ and $r \in [0, r_{\max}^-]$, holds *tout court* for $\mathbb{C}P^2$, since the smoothness conditions of φ, ψ, ξ at $r = 0$ are not used in the proof. The case of $i = 3$ and $r \in [r_{\max}^-, L]$ is analogous. The replacement for Proposition 4.6 is the following:

Proposition 4.10 *For $r \in [0, r_{\max}^-]$, the entries of $(R_s)_i$, $i = 2, 3$, satisfy:*

$$\begin{aligned}
&\eta_2(s,r) = \left(\tfrac{1}{b} + O(r)\right)s + O(s^2),\ \mu_2(s,r) = \left(-\tfrac{3}{2b} - \tfrac{1-b}{b^3} + O(r)\right)s + O(s^2),\\
&\nu_3(s,r) = \left(\tfrac{4}{b} + O(r)\right)s + O(s^2),\\
&\eta_3(s,r) = \left(\tfrac{4}{b} + O(r)\right)s + O(s^2),\ \mu_3(s,r) = \left(\tfrac{3}{2b} - \tfrac{1-b}{b^3} + O(r)\right)s + O(s^2),\\
&\nu_2(s,r) = \left(\tfrac{1}{b} + O(r)\right)s + O(s^2).
\end{aligned}$$

For $r \in [r_{\max}^-, L]$, setting $z = L - r$, the entries of $(R_s)_i$, $i = 1, 2$, satisfy:

$$\begin{aligned}
&\eta_i(s,z) = \left(\tfrac{1}{b\sqrt{2}} + O(z)\right)s + O(s^2),\ \mu_i(s,z) = \left(-\tfrac{\sqrt{2}-2b}{b^3} + O(z)\right)s + O(s^2),\\
&\nu_i(s,z) = \left(\tfrac{1}{b\sqrt{2}} + O(z)\right)s + O(s^2).
\end{aligned}$$

The proof of Proposition 4.10 is totally analogous to that of Proposition 4.6; noting that, in terms of $z = L - r \in [0, r_{\max}^+]$, the functions φ_1, ψ_1, ξ_1 are:

$$\varphi_1(z) = \tfrac{1}{\sqrt{2}}(\cos z - \sin z), \quad \psi_1(z) = \tfrac{1}{\sqrt{2}}(\cos z + \sin z), \quad \xi_1(z) = \sin 2z.$$

Finally, similarly to Proposition 4.5 and 4.7, it can be shown that $(R_s)_i + \tau_s H$, $i = 1, 2, 3$, are positive-definite for all $r \in [0, L]$ and $s > 0$ sufficiently small, which proves Claim 4.8 (and hence Theorem B) for $\mathbb{C}P^2$. Details are left to the reader.

5 Positive turns negative

In this section, we prove Theorem A, using the fact that Grove–Ziller metrics on S^4 and $\mathbb{C}P^2$ immediately acquire negatively curved planes under Ricci flow [6], together with Theorem B, and continuous dependence on initial data [3].

Proof of Theorem A Let M^4 be either S^4 or $\mathbb{C}P^2$, and consider the 1-parameter family of metrics g_s on M^4, defined in (4.3), such that g_0 is a Grove–Ziller metric and g_1 is either the round metric or the Fubini–Study metric, accordingly. From Lemma 4.1, the metrics g_s are smooth, and it is evident from (4.2) and (4.3) that, for all $k \geq 0$ and $0 < \alpha < 1$, there exists a constant $\lambda_{k,\alpha} > 0$ such that

$$\|g_s - g_0\|_{C^{k,\alpha}} \leq \lambda_{k,\alpha}\, s, \quad \text{for all } 0 \leq s \leq 1, \tag{5.1}$$

where $\|\cdot\|_{C^{k,\alpha}}$ denotes the Hölder norm on sections of the bundle $E = \mathrm{Sym}^2 TM^4$ with respect to a fixed background metric. For $0 \leq s \leq 1$, let $g_s(t)$, $0 \leq t < T(g_s)$, be the maximal solution to Ricci flow starting at $g_s(0) = g_s$, where $0 < T(g_s) \leq +\infty$ denotes the maximal (smooth) existence time of the flow. For all $0 \leq s \leq 1$ and $0 \leq t < T(g_s)$, we have that $g_s(t) \in C^\infty(E)$, so $g_s(t)$ is in the proper closed subspace $h^{k,\alpha}(E) \subset C^{k,\alpha}(E)$ for all $k \geq 0$ and $0 < \alpha < 1$, in the notation of [3].

From the main theorem in [6], there exist a 2-plane σ tangent to M^4 and $t_0 > 0$ such that $\sec_{g_0}(\sigma) = 0$ and $\sec_{g_0(t)}(\sigma) < 0$ for all $0 < t < t_0$. Fix $0 < t_* < t_0$, and let $\delta > 0$ be such that $\sec_g(\sigma) < 0$ for all metrics g with $\|g - g_0(t_*)\|_{C^{2,\alpha}} < \delta$. By the continuous dependence of Ricci flow on initial data [3, Thm A], there exist constants $\mathrm{r} > 0$ and $C > 0$, depending only on t_0 and g_0, such that, if $\|g_s - g_0\|_{C^{4,\alpha}} \leq \mathrm{r}$, then $T(g_s) \geq t_0$

and $\|g_s(t) - g_0(t)\|_{C^{2,\alpha}} \leq C\|g_s - g_0\|_{C^{4,\alpha}}$ for all $t \in [0, t_0]$. Together with (5.1), this yields that if $0 \leq s \leq r/\lambda_{4,\alpha}$, then

$$\|g_s(t) - g_0(t)\|_{C^{2,\alpha}} \leq C\,\|g_s - g_0\|_{C^{4,\alpha}} \leq C\,\lambda_{4,\alpha}\, s, \quad \text{for all } 0 \leq t \leq t_0.$$

Thus, $\|g_s(t_*) - g_0(t_*)\|_{C^{2,\alpha}} < \delta$ and so $\sec_{g_s(t_*)}(\sigma) < 0$, for all $0 \leq s < \delta/(C\,\lambda_{4,\alpha})$, while $g_s = g_s(0)$ has $\sec > 0$ if $s > 0$ is sufficiently small, by Theorem B. □

Remark 5.1 The curvature operators $R(t)\colon \wedge^2 TM \to \wedge^2 TM$ of metrics $g(t)$ on M^n evolving under Ricci flow satisfy the PDE $\frac{\partial}{\partial t} R = \Delta R + 2Q(R)$, where $Q(R)$ depends quadratically on R. By Hamilton's Maximum Principle, if an $\mathsf{O}(n)$-invariant cone $\mathcal{C} \subset \mathrm{Sym}^2_b(\wedge^2 TM)$ is preserved by the ODE $\frac{d}{dt} R = 2Q(R)$, then it is also preserved by the above PDE. It was previously known that the cone $\mathcal{C}_{\sec>0}$ of curvature operators with $\sec > 0$ is *not* preserved under the above ODE on R in dimensions $n \geq 4$, since it is easy to find $R_0 \in \partial\mathcal{C}_{\sec>0}$ with $Q(R_0)$ pointing outside of $\mathcal{C}_{\sec>0}$. Nevertheless, this observation alone does not imply the existence of metrics realizing such a family of curvature operators on some closed n-manifold, thus evolving under Ricci flow and losing $\sec > 0$, as the above metrics $g_s(t)$ do.

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