Calculus of Variations

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Shape optimization of a thermal insulation problem

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Abstract

We study a shape optimization problem involving a solid $K \subset \mathbb{R}^n$ that is maintained at constant temperature and is enveloped by a layer of insulating material Ω which obeys a generalized boundary heat transfer law. We minimize the energy of such configurations among all (K, Ω) with prescribed measure for K and Ω, and no topological or geometrical constraints. In the convection case (corresponding to Robin boundary conditions on $\partial\Omega$) we obtain a full description of minimizers, while for general heat transfer conditions, we prove the existence and regularity of solutions and give a partial description of minimizers.

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1 Introduction

Given a measurable set $K \subseteq \mathbb{R}^n$ along with a Lipschitz open set $\Omega \supset K$, we consider the energy functional

$$E_\Theta(K, \Omega) := \min \left\{ \int_\Omega |\nabla v|^2 \mathrm{d}x + \int_{\partial\Omega} \Theta(v) \mathrm{d}\mathscr{H}^{n-1}, \ v \in H^1(\Omega, [0, 1]) : v = 1 \text{ a.e. on } K \right\} \tag{1}$$

where $\Theta : [0, 1] \to \mathbb{R}_+$ is a rather general nondecreasing function that vanishes at 0. If all data are smooth, meaning Ω is a smooth open set, K is a smooth compact set and $\Theta \in \mathcal{C}^1$, a

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minimizer u satisfies

$$
\begin{cases}
\Delta u = 0 & \Omega \setminus K, \\
-\frac{\partial u}{\partial \nu} = \frac{1}{2}\Theta'(u) & \partial\Omega \cap \{u > 0\} \setminus \partial K, \\
-\frac{\partial u}{\partial \nu} \leq \frac{1}{2}\Theta'(u) & \partial\Omega \cap \{u = 0\}, \\
u \equiv 1 & K.
\end{cases}
\tag{2}
$$

A classical prototype, $\Theta(u) = \beta u^2$ for some $\beta > 0$, corresponds to the so called Robin boundary conditions $\frac{\partial u}{\partial \nu} + \beta u = 0$ for the harmonic function u minimizing the energy, ν being the outward normal at the boundary. The physical motivation which leads us to study the functional $E_\Theta(K, \Omega)$ and related shape optimization problems can be found in optimal design in thermal insulation.

Consider a body K of given constant fixed temperature T_K, surrounded by an insulator A. The temperature distribution T inside the insulator satisfies the classical heat equation (Laplace equation) with the condition that T is continuous across the boundary which separates A and K. If we now assume that the body $\Omega = K \cup A$ is immersed into an environment of fixed temperature T_e, the heat exchange rate across the surface of Ω (between the body and the environment) has to be modeled according to the physical process governing the mechanism. The most common assumption is to assume that the temperature at the boundary of Ω is kept constant (T_e), leading to the so-called Dirichlet boundary conditions, which corresponds to conduction heat transfer. But if we assume that the environment is a fluid (gas or liquid) then, convection heat transfer, radiation heat transfer or even more general laws, have to be taken into account. If for instance, convection heat transfer is the leading mechanism, then the rate of heat flux per unit of surface, across the solid fluid interface, is proportional to $T|_{\partial\Omega} - T_e$ (also known as Newton's law of cooling), $T|_{\partial\Omega}$ being the temperature T at the solid surface. While for radiation heat transfer (according to Stefan-Boltzmann law), the heat flux per unit of surface, across the solid fluid interface, is proportional to $T^4_{\partial\Omega} - T^4_e$.

On the other hand, the heat flux across a solid surface is proportional (Fourier's law) to the normal derivative of the temperature $-\frac{\partial T}{\partial \nu}$.

Assume that $T_K > T_e$ and denote by $u = \frac{T - T_e}{T_K - T_e}$. One can easily check that, up to a constant of proportionality, convection corresponds to

$$
\Theta(u) = u^2.
$$

Radiation, on the other hand, is modeled up to a multiplicative constant by

$$
\Theta(u) = \frac{u^5}{5} + \gamma u^4 + 2\gamma^2 u^3 + 2\gamma^3 u^2,
\tag{3}
$$

where $\gamma = \frac{T_e}{T_K - T_e}$.

Both mechanisms can be taken into account simultaneously, upon considering a linear combination of the previous functions. When considering a quadratic Θ (pure convection heat transfer at the boundary), E_Θ is proportional to the heat loss rate. In general, regardless of the choice of Θ, the energy functional E_Θ can be considered a measure of the goodness of the thermal insulation; the less the energy the better the insulation.

In this article we are interested in the shape optimization problem of both K and Ω of prescribed volume which lead to a minimal energy configuration. For a given set K, when only the geometry of Ω is unknown, the problem has already been considered in the literature in [9, 10, 14], with a different purpose, namely to obtain qualitative information on the free boundary of Ω. Its analysis relies on the fine study of optimal configurations in the framework of free discontinuity problems in SBV.

We search to optimize both K and Ω and the purpose is to understand whether or not an optimal configuration is given by two concentric balls (as common sense suggests). From this perspective, our problem is more of isoperimetric type. As we shall prove, depending on the dissipation law Θ, we can give either full description of the optimal sets as two concentric balls (for instance in the convection case) or some partial answer leading to the same geometric configuration for more general dissipation laws. A striking phenomenon, which has been observed in some specific geometric configurations in [12], and that we have to handle is the following: if only a small amount of insulator is available, then in some cases it is better to not use it ($\Omega = K$).

Let us denote by ω_n the volume of the unit ball and by $M := R^n \omega_n$ the volume of the ball of radius $R \geq 1$. Without restricting generality, we shall fix the measure of K to be ω_n and leave the other parameters free (the general case is obtained by rescaling). Given $M \geq \omega_n$ and $\Lambda > 0$, we are interested in the following minimization problem: find an open set Ω and a relatively closed set K, solutions of

$$\inf_{K \subset \Omega, \; |K| = \omega_n, \; |\Omega| \leq M} E_\Theta(K, \Omega), \tag{4}$$

or in its penalized version

$$\inf_{K \subset \Omega, \; |K| = \omega_n} E_\Theta(K, \Omega) + \Lambda |\Omega|. \tag{5}$$

Throughout the paper we assume that $\Theta : [0, 1] \to \mathbb{R}$ is a lower semicontinuous, nondecreasing function such that $\Theta(0) = 0$. Here are our main results.

Theorem 1 (The convection case) *For $\Theta(u) = \beta u^2$ and $M = R^n \omega_n$, the solutions of problems (4) consist of two concentric balls. The radius of the outer ball equals either R or 1, according to $\min\{E_\Theta(B_1, B_R), E_\Theta(B_1, B_1)\}$ and the associated state function u is radial.*

Theorem 2 *For any $\Lambda > 0$ and any admissible Θ, the solution of problem (5) consists of two concentric balls and the associated state function u is radial.*

We introduce the following hypothesis

$$\inf_{0 < s < 1} \frac{\Theta(s/3)}{\Theta(s)} > 0. \tag{6}$$

Theorem 3 (The general case) *There exists some dimensional constant c_n, such that if Θ satisfies hypothesis (6) and*

$$M < \omega_n + c_n \left(\inf_{0 < s < 1} \frac{\Theta(s/3)}{\Theta(s)} \right)^{2n} \int_0^1 \frac{t^{2n-1} dt}{\Theta(t)^n}, \tag{7}$$

then problem (4) has a solution (K, Ω). If $|\Omega| < M$ then (K, Ω) are two concentric balls. Otherwise, Ω is an open set with rectifiable topological boundary such that $\mathcal{H}^{n-1}(\partial \Omega) < \infty$ and K is relatively closed in Ω with locally finite perimeter in Ω. The temperature $u \in H^1(\Omega)$ is $\mathcal{C}_{loc}^{0, \frac{2}{n+2}}(\Omega)$. In two dimensions, $\partial K \cap \Omega$ is analytic.

If moreover Θ is $\mathcal{C}^1$ in a neighbourhood of 1 and

$$\frac{\Theta'(1)^2}{\Theta(1)} < 4(n-1),$$

then there exists $M > \omega_n$ depending on n and Θ such that the solution of problem (4) is $K = \Omega = B_1$.

Remark 4   Notice that any homogenous functions $\Theta(u) = u^\alpha$, $\alpha > 0$ satisfies the assumption (6). Moreover, if $\Theta(u) = O(u^2)$ then in (7) any value $M \in [\omega_n, +\infty)$ is accepted. In the convection case $\Theta(u) = \beta u^2$ the full picture of the solutions is understood. Other interesting choices of Θ may be given by (3), corresponding to thermal radiation, or $\Theta(u) = cu$ corresponding to a constant heat flux. Moreover, discontinuous functions Θ, with $\Theta(0_+) > 0$, are admissible. This means that one can consider functions like $\Theta(u) = c_1 1_{u>0} + c_2 u^\alpha$ on $[0, 1]$ which models a cost of the highly insulating material on $\partial(A \cup K)$, cost which is proportional to its surface measure.

The function Θ may be extended to $\mathbb{R}$ as a nondecreasing and nonnegative function, and the constraint $1_K \leq v \leq 1$ a.e. in (1) may be relaxed into $v_{|K} \geq 1$ a.e. Whatever the extension of Θ is outside $[0, 1]$, the new problem is equivalent to the original one, by truncation below 0 and above 1.

Clearly, we know many more things on the penalized problem (5) than on the constrained problem (4), in particular that the solutions are always concentric balls. This stills leaves open the following question:

> *Under reasonable hypotheses on Θ, is it true that the solution of* (4) *always consists on two concentric balls?*

The organisation of the paper is as follows.

- In Sect. 2 we discuss the convection case and we prove Theorem 1. The proof follows the strategy of Bossel and Daners for the Faber-Krahn inequality for the Robin Laplacian and is based on the construction of a so called H-function. Up to knowledge of the authors, this is the only case (aside from Faber-Krahn) where this strategy works. However, we point out that this strategy is fully working only in dimension 2, while for $n \geq 3$ it works only for $\beta > n - 2$. This section is mostly independent from the rest of the paper, except for the case $n \geq 3$, $\beta \in (0, n-2]$ which is a consequence of the analysis by free discontinuity techniques, for which we refer to Corollary 21 and Remark 22.
- Section 3 is devoted to the analysis of the existence of relaxed solutions for the constrained problem (4) in the context of a general dissipation functions Θ, and to the regularity of the free boundaries. These results are rather technical and they prepare the Proofs of Theorems 2 and 3. The key idea is that once we know that problem (4) has a sufficiently smooth solution, we can extract qualitative information out of its optimality. We work in the framework of free discontinuity problems, based on the analysis of special functions with bounded variation.
- In Sect. 4 we prove Theorem 2 and in Sect. 5 we prove Theorem 3.

2 The convection case: Proof of Theorem 1

In this section we consider $\Theta(s) = \beta s^2$. In this case, the energy E_Θ is simply denoted by E_β and takes the form

$$E_\beta(K, \Omega) := \inf_{v \in H^1(\Omega), v \geq 1_K} \int_\Omega |\nabla v|^2 \mathrm{d}x + \beta \int_{\partial \Omega} v^2 \mathrm{d}\mathcal{H}^{n-1}. \tag{8}$$

For fixed K, minimizers of the functional $\Omega \to E_\beta(K, \Omega) + \Lambda |\Omega|$ have been studied in [9, 10] and [14], in particular the existence of an optimal set Ω and its regularity.

The claim of Theorem 1 is that the solution of the constrained problem (4) consists of two centered balls for every $M > \omega_n$, and the size of the balls is then given by the study of the

function

$$R \mapsto E_\beta(B_1, B_R).$$

Let us first describe this function in detail. We let

$$\Phi_n(\rho) = \begin{cases} \log(\rho) & \text{if } n = 2 \\ -\frac{1}{(n-2)\rho^{n-2}} & \text{if } n \geq 3, \end{cases}$$

where the sign convention is taken such that Φ_n is always increasing. Relying on the expression of radial harmonic functions $(x \mapsto a + b\Phi_n(|x|))$ and the boundary conditions we obtain that the temperature associated to (B_1, B_R) is given by

$$u(x) = 1 - \frac{\beta(\Phi_n(|x|) - \Phi_n(1))_+}{\Phi'_n(R) + \beta(\Phi_n(R) - \Phi_n(1))}$$

and

$$E_\beta(B_1, B_R) = \frac{\beta \text{Per}(B_1)\Phi'_n(1)}{\Phi'_n(R) + \beta(\Phi_n(R) - \Phi_n(1))}.$$

Notice in particular that

$$\frac{d}{dR} E_\beta(B_1, B_R) \leq 0 \text{ iff } \frac{d}{dR}(\Phi'_n(R) + \beta\Phi_n(R)) \geq 0 \text{ iff } R \geq \frac{n-1}{\beta}.$$

Moreover the extremal values are

$$E_\beta(B_1, B_1) = \beta n\omega_n, \quad \lim_{R \to \infty} E_\beta(B_1, B_R) = (n-2)n\omega_n.$$

In two dimension there are two cases

- if $\beta \geq 1$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ is decreasing.
- if $\beta < 1$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ increases on $[1, \beta^{-1}]$ and decreases on $[\beta^{-1}, +\infty[$, with the existence of a unique $R_\beta > \beta^{-1}$ such that $E_\beta(B_1, B_{R_\beta}) = E_\beta(B_1, B_1)$.

In higher dimension there are three cases

- if $\beta \geq n - 1$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ is decreasing.
- if $n - 1 > \beta > n - 2$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ increases on $[1, \frac{n-1}{\beta}]$, decreases on $[\frac{n-1}{\beta}, +\infty)$, with the existence of a unique $R_\beta > \frac{n-1}{\beta}$ such that $E_\beta(B_1, B_{R_\beta}) = E_\beta(B_1, B_1)$.
- if $\beta \leq n - 2$ then $R \in [1, +\infty[\mapsto E_\beta(B_1, B_R)$ reaches its minimum at $R = 1$.

We postpone the analysis of the last case, and start by proving the result for the first two cases. For this we will need the following.

Lemma 5 *Let $R > 1$, $\beta > 0$, and let u^* be the (unique) minimizer in (8) on (B_1, B_R). Then $\frac{|\nabla u^*|}{u^*} \leq \beta$ on $B_R \setminus B_1$ if and only if*

$$\forall \rho \in [1, R], \ E_\beta(B_1, B_\rho) \geq E_\beta(B_1, B_R).$$

Proof We remind the expression

$$u^*(x) = 1 - \frac{\beta(\Phi_n(|x|) - \Phi_n(1))}{\Phi'_n(R) + \beta(\Phi_n(R) - \Phi_n(1))},$$

so by straightforward computations

$$\frac{|\nabla u^*|}{u^*} \le \beta \text{ iff } \Phi'_n(\rho) + \beta\Phi_n(\rho) \le \Phi'_n(R) + \beta\Phi_n(R) \text{ iff } E_\beta(B_1, B_\rho) \ge E_\beta(B_1, B_R).$$

$\square$

We may now prove the result. The proof relies on the study of the so called H function introduced by Bossel [3], see also [11] and [6], for the study of the Faber-Krahn inequality involving the first eigenvalue of the Laplace operator with Robin boundary conditions.

Proof of Theorem 1 Let (K, Ω) be smooth sets such that $K \subseteq \Omega$, $|K| = \omega_n$, $|\Omega| \le M$ (and R the radius defined by $M = |B_R|$). Let u be the minimizer of $E_\beta(K, \Omega)$ and denote $\Omega_t = \{u > t\}$. We decompose $\partial\Omega_t$ into two disjoint (up to a $\mathcal{H}^{n-1}$-negligible set) sets; $\partial^i\Omega_t = \{u = t\} \cap \Omega$, and $\partial^e\Omega_t = \partial\Omega_t \cap \partial\Omega$. Then for a.e. $t \in [0, 1]$

$$
\begin{aligned}
0 &= \int_{\{t<u<1\}} \frac{\Delta u}{u} dx = \int_{\{t<u<1\}} \left(\nabla \cdot \left(\frac{\nabla u}{u} \right) - \nabla u \cdot \nabla \frac{1}{u} \right) dx \\
&= \int_{\partial\{t<u<1\}} \nu_{\Omega_t} \cdot \frac{\nabla u}{u} d\mathcal{H}^{n-1} + \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} dx \\
&= \int_{\partial K \cap \Omega} |\nabla u| d\mathcal{H}^{n-1} - \int_{\partial^i\Omega_t} \frac{|\nabla u|}{u} d\mathcal{H}^{n-1} \\
&\quad - \beta\mathcal{H}^{n-1}(\partial\{t < u < 1\} \cap \partial\Omega) + \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} dx.
\end{aligned}
$$

Since

$$
\begin{aligned}
E_\beta(K, \Omega) &= \int_{\Omega\setminus K} \nabla \cdot (u\nabla u) dx + \beta \int_{\partial\Omega} u^2 d\mathcal{H}^{n-1} \\
&= \int_{\partial K \cap \Omega} |\nabla u| d\mathcal{H}^{n-1} + \int_{\partial\Omega\setminus\partial K} \left(\beta u^2 + u\partial_\nu u \right) d\mathcal{H}^{n-1} + \int_{\partial K \cap \partial\Omega} \beta u^2 \\
&= \int_{\partial K \cap \Omega} |\nabla u| d\mathcal{H}^{n-1} + \beta\mathcal{H}^{n-1}(\partial K \cap \partial\Omega)
\end{aligned}
$$

then injecting this in the previous equation we obtain

$$E_\beta(K, \Omega) = \beta\mathcal{H}^{n-1}(\partial^e\Omega_t) + \int_{\partial^i\Omega_t} \frac{|\nabla u|}{u} d\mathcal{H}^{n-1} - \int_{\Omega_t} \frac{|\nabla u|^2}{u^2} dx.$$

Let us define, for all $t \in [0, 1]$ and $\phi \ge 0$,

$$H(t, \phi) = \beta\mathcal{H}^{n-1}(\partial^e\Omega_t) + \int_{\partial^i\Omega_t} \phi d\mathcal{H}^{n-1} - \int_{\Omega_t} \phi^2 dx.$$

Lemma 6 *For any nonnegative L^∞ function ϕ, there exists some $t \in]0, 1[$ for which $H(t, \phi) \le E_\beta(K, \Omega)$.*

Proof Let $w = \phi - \frac{|\nabla u|}{u}$, then

$$H(t, \phi) - E_\beta(K, \Omega) = H(t, \phi) - H\left(t, \frac{|\nabla u|}{u} \right)$$

$$= \int_{\partial^i \Omega_t} w \, d\mathcal{H}^{n-1} + \int_{\Omega_t} \left(\left(\frac{|\nabla u|}{u} \right)^2 - \phi^2 \right) dx$$

$$\leq \int_{\partial^i \Omega_t} w \, d\mathcal{H}^{n-1} - 2 \int_{\Omega_t} \frac{|\nabla u|}{u} w \, dx$$

$$= -\frac{1}{t} \frac{d}{dt} \left(t^2 \int_{\Omega_t} \frac{|\nabla u|}{u} w \, dx \right),$$

where the last line is obtained by coarea formula on the level sets of u. So in particular

$$\int_0^1 t (H(t, \phi) - E_\beta(K, \Omega)) dt \leq - \left[t^2 \int_{\Omega_t} \frac{|\nabla u|}{u} w \, dx \right]_{t=0}^{t=1} = 0,$$

which proves the lemma. □

Proof of Theorem 1, *(continuation).* Recall that u^* is the solution on (B_1, B_R), which is radially symmetric and decreasing. Let ϕ be the dearrangement of $\frac{|\nabla u^*|}{u^*}$ on Ω following the level sets of u. To be more precise, for any $x \in \Omega \setminus K$, let $r(x) > 0$ be defined by the formula

$$|B_{r(x)}| = |\{u > u(x)\}|,$$

then we let $\phi(x)$ be the value of $\frac{|\nabla u^*|}{u^*}$ on $\partial B_{r(x)}$.

Let t be chosen as in Lemma 6, so that $H(t, \phi) \leq E_\beta(K, \Omega)$. Assuming that $\frac{|\nabla u^*|}{u^*} \leq \beta$ on $\partial B_{r(x)}$, we have

$$H(t, \phi) = \beta \mathcal{H}^{n-1}(\partial^e \Omega_t) + \int_{\partial^i \Omega_t} \phi \, d\mathcal{H}^{n-1} - \int_{\Omega_t} \phi^2 dx$$

$$\geq \int_{\partial \Omega_t} \left(\frac{|\nabla u^*|}{u^*} \right)_{|\partial B_{r(x)}} d\mathcal{H}^{n-1} - \int_{\Omega_t^*} \frac{|\nabla u^*|^2}{(u^*)^2} dx$$

$$\geq \int_{\partial \Omega_t^*} \frac{|\nabla u^*|}{u^*} d\mathcal{H}^{n-1} - \int_{\Omega_t^*} \frac{|\nabla u^*|^2}{(u^*)^2} dx$$

$$= H^* \left(u^*_{|\partial B_{r(x)}}, |\nabla u^*|/u^* \right)$$

$$= E_\beta(B_1, B_R)$$

The inequalities above rely on the rearrangement properties, the isoperimetric inequality and on the hypothesis $\beta \geq \frac{|\nabla u^*|}{u^*}$.

We conclude by discussing when the assumption $\frac{|\nabla u^*|}{u^*} \leq \beta$ is verified. If $\beta \geq n - 1$ the inequality $\frac{|\nabla u^*|}{u^*} \leq \beta$ is verified since $R \mapsto E_\beta(B_1, B_R)$ is decreasing on $[1, +\infty)$, by Lemma 5.

If instead $n - 2 < \beta < n - 1$, two situations occur. If $M \geq |B_{R(n,\beta)}|$, then Lemma 5 still works and the previous computation applies, with the same result. If $M \in [\omega_n, |B_{R(n,\beta)}|]$ then we may consider u^* the solution on $(B_1, B_{R(n,\beta)})$ restricted to B_R; it verifies $\frac{|\nabla u^*|}{u^*} \leq \beta$ and the same argument applies. We obtain that

$$E_\beta(K, \Omega) \geq E_\beta(B_1, B_{R(n,\beta)}).$$

Since $E_\beta(B_1, B_{R(n,\beta)}) = E_\beta(B_1, B_1)$, we conclude with the minimality of the couple (B_1, B_1).

The case $n \geq 3$, $\beta \leq n - 2$ can not be treated directly with the H function, but is a direct consequence of Corollary 21. Indeed, it is enough to take $\Lambda \rightarrow 0$ in Corollary 21, since $E_\beta(B_1, B_R) \geq E_\beta(B_1, B_1)$ for all $R \geq 1$ in this case. □

Remark 7 Finally we have the following picture for problem (4) with $M = R^n \omega_n$; the minimizer is (B_1, B_r), where r is defined through the following case disjunction.

(a) If $n - 1 \leq \beta$ then $r = R$.
(b) If $n - 2 < \beta < n - 1$, then defining $R(n, \beta) > \frac{n-1}{\beta}$ as the unique non-trivial solution of the equation $E_\Theta(B_1, B_{R(n,\beta)}) = n\beta\omega_n$ we have

  - if $R(n, \beta) > R \geq 1$ then $r = 1$
  - if $R = R(n, \beta)$ then $r = 1$ or $r = R$
  - if $R > R(n, \beta)$ then $r = R$

(c) If $\beta \leq n - 2$ then $r = 1$.

3 Preparatory results: existence and regularity of relaxed solutions

We analyze in this section the existence of a solution of problem (4) for general Θ. Precisely, our purpose is to prove the following.

Proposition 8 *Let $M > \omega_n$ and Θ admissible, such that, for some dimensional constant c_n (that will be specified)*

$$M < \omega_n + c_n \left(\inf_{0 < s < 1} \frac{\Theta(s/3)}{\Theta(s)} \right)^{2n} \int_0^1 \frac{t^{2n-1} dt}{\Theta(t)^n}, \tag{9}$$

Then problem (4) has a (sufficiently regular) solution (K, Ω).

The word "sufficiently" above will be described later, and refers to the regularity that we need in the Proofs of Theorems 2 and 3. The proof of this proposition relies on ideas inspired from the the relaxation strategy in the SBV framework (defined below) as introduced in [6] and from [15, Chapter 29] on the existence proof of minimal clusters for the perimeter. The main difficulty in our case comes from both the generality of the function Θ and from the fact that two measure constraints have to be satisfied simultaneously.

As mentioned previously, the functional space used for our relaxation procedure is the SBV space, on which more information may be found in the books [1, 4]. In few words, for an open set $\Omega \subset \mathbb{R}^n$, $SBV(\Omega)$ is defined as the set of functions $u \in L^1(\Omega)$ such that Du (the differential of u in the sense of distribution) is a Radon measure that decomposes as

$$Du = \nabla u \mathcal{L}^n + (u_+ - u_-)\nu_u \mathcal{H}^{n-1} \lfloor J_u,$$

where $\nabla u \in L^1(\Omega)$ and J_u is the set of jump points, meaning points $x \in \mathbb{R}^n$ such that $y \mapsto u(x + ry)$ converges in $L^1_{\text{loc}}(\mathbb{R}^n)$ as r goes to 0 to

$$u_+ 1_{y:y\cdot\nu_u > 0} + u_- 1_{y:y\cdot\nu_u < 0}$$

for some $u_+ \neq u_- \in \mathbb{R}$, $\nu_u \in \mathbb{S}^{n-1}$. The set J_u turns out to be rectifiable. We will make use of the following compactness result, which can be obtained by combination of [4, Th. 2.3] and [4, Th. 2.12] applied to $\phi(z, w) = \Theta(z) + \Theta(w)$.

Proposition 9 *Let $(u_i)_{i\in\mathbb{N}}$ be a sequence in $SBV_{loc}(\mathbb{R}^n)$ with values in $[0, 1]$ such that for any $R > 0$,*

$$\sup_{i\in\mathbb{N}}\left(\int_{B_R}|\nabla u_i|^2 + \mathcal{H}^{n-1}(J_{u_i}\cap B_R)\right) < \infty$$

then up to the extraction of a subsequence, there is a function $u \in SBV_{loc}(\mathbb{R}^n)$ such that

$$u_i \xrightarrow[a.e.]{} u$$
$$\nabla u_i \xrightarrow[L^2_{loc}]{} \nabla u$$
$$\forall R > 0, \int_{J_u\cap B_R}\left[\Theta(u_+) + \Theta(u_-)\right]d\mathcal{H}^{n-1}$$
$$\leq \liminf_{i\to\infty}\int_{J_{u_i}\cap B_R}\left[\Theta((u_i)_+) + \Theta((u_i)_-)\right]d\mathcal{H}^{n-1}$$

Here is the main strategy.

Step 1. (Relaxation) We change the problem into finding a minimizer for the following problem

$$\inf_{u\in SBV(\mathbb{R}^n),\,|\{u=1\}|\geq\omega_n,\,|\{u>0\}|\leq M}\mathcal{E}_\Theta(u), \tag{10}$$

where

$$\mathcal{E}_\Theta(u) = \int_{\mathbb{R}^n}|\nabla u|^2\mathrm{d}x + \int_{J_u}(\Theta(u_+) + \Theta(u_-))\mathrm{d}\mathcal{H}^{n-1}. \tag{11}$$

This means that we will need afterward to prove that such a relaxed minimizer (meaning a minimizer of the relaxed problem) corresponds to a classical minimizer.

Step 2. (A priori regularity) We prove that we may restrict to considering functions that verify an a priori estimate of the form $u \geq \delta 1_{\{u>0\}}$ for some explicit δ; for Robin boundary conditions this kind of estimate is a cornerstone of the regularity theory. This is the only place where we use the hypothesis (9). Similar estimates may be found in [7, 9] and [10] (see also [2] for a related problem). Moreover, we prove a concentration lemma that says all the support of u - up to a set of measure ϵ - is contained in a union of $C\epsilon^{-n}$ unit cubes for some explicit constant C, and we prove a cut-off lemma that says that one may replace u by 0 outside a large enough cube and lower its energy by a controlled amount.

Step 3. (Existence of a relaxed solution) We rely on Step 2 to prove the existence of a minimizer. This is done by the direct method, considering a minimizing sequence, applying the concentration lemma a first time to translate the minimizing sequence such that it converges to a non-trivial solution, and then applying again the concentration lemma and cut-off lemma for an appropriately small ϵ.

Step 4. (Regularity of the relaxed solution) Finally we prove the regularity of such a relaxed minimizer by using the theory of almost quasiminimizers of the Mumford-Shah functionnal to handle the support of u, and the theory of the regularity of Alt-Caffarelli problem inside Ω.

As mentioned above, we are first interested in the relaxed problem (10). Notice that the generalized energy (11) takes into account "cracks", meaning part of the jump set where u

may be nonzero on both sides. The general energy (11) may also be used to define $E_\Theta(K, \Omega)$ in a more general setting ; for any open set Ω, and any measurable $K \subset \Omega$, we let

$$E_\Theta(K, \Omega) = \inf \left\{ \mathscr{E}_\Theta(u), \ u \in SBV(\mathbb{R}^n, [0, 1]) : |K \setminus \{u = 1\}| \right.$$
$$= |\{u \neq 0\} \setminus \Omega| = \mathscr{H}^{n-1}(J_u \setminus \partial\Omega) = 0 \Big\}$$

When Ω is a smooth open set, this is coherent with the first definition.

Lemma 10 *Assume that* (9) *is verified. There is a constant* $\delta = \delta_{n,\Theta,M} > 0$ *such that for any admissible* u *in* (10), *there is some* $t > \delta$ *verifying* $\mathscr{E}_\Theta(u 1_{\{u > t\}}) \leq \mathscr{E}_\Theta(u)$.

Proof We actually show the stronger result; there is some $t > s > e^{-c_{n,\Theta}M}$ such that

$$\mathscr{E}_\Theta(u 1_{\{s < u < t\}^c}) \leq \mathscr{E}_\Theta(u).$$

Let $\epsilon > 0$, suppose $\mathscr{E}_\Theta(u 1_{\{s < u < t\}^c}) > \mathscr{E}_\Theta(u)$ for every $\epsilon < s < t < 1$. We write

- $\Omega(s, t) = \{s < u \leq t\}$.
- $\gamma(s, t) = \int_{J_u} \left(1_{s < u_+ \leq t} + 1_{s < u_- \leq t} \right) d\mathscr{H}^{n-1}$.
- $h(t) = \mathscr{H}^{n-1}(\{u = t\} \setminus J_u)$.

Our hypothesis becomes that for every $\eta \in]2\epsilon, \frac{2}{3}[$, $t \in]0, \frac{\eta}{2}[$, $\mathscr{E}_\Theta(u 1_{\{\eta - t < u < \eta + t\}^c}) > \mathscr{E}_\Theta(u)$ so

$$\int_{\Omega(\eta - t, \eta + t)} |\nabla u|^2 dx + \Theta\left(\frac{1}{2}\eta\right) \gamma(\eta - t, \eta + t) \leq \Theta\left(\frac{3}{2}\eta\right) (h(\eta - t) + h(\eta + t)).$$
(12)

The proof is based on a lower bound of $\int_{\Omega(\eta - t, \eta + t)} |\nabla u| dx$ and an upper bound of $\int_{\Omega(\eta - t, \eta + t)} |\nabla u|^2 dx$ that are in contradiction when ϵ is small enough.

- For all $t \in]0, \frac{1}{2}\eta[$ we let:

$$f_\eta(t) = \int_{\eta - t}^{\eta + t} h(s) ds = \int_{\Omega(\eta - t, \eta + t)} |\nabla u| dx.$$

f_η is absolutely continuous and

$${}'_\eta(t) = h(\eta - t) + h(\eta + t).$$

Moreover,

Then, $f_\eta(t) \leq |\Omega(\eta - t, \eta + t)|^{\frac{1}{2}} \left(\int_{\Omega(\eta - t, \eta + t)} |\nabla u|^2 dx \right)^{\frac{1}{2}}$

$$\leq C_n \mathrm{Per}(\Omega(\eta - t, \eta + t))^{\frac{n}{2(n-1)}} \left(\Theta\left(\frac{3}{2}\eta\right) (h(\eta - t) + h(\eta + t)) \right)^{\frac{1}{2}}$$

by isoperimetric inequality

$$\leq C_n (h(\eta - t) + \gamma(\eta - t, \eta + t)$$

$$+ h(\eta + t))^{\frac{n}{2(n-1)}} \left(\Theta\left(\frac{3}{2}\eta\right) (h(\eta - t) + h(\eta + t)) \right)^{\frac{1}{2}}$$

$$\leq C_n \Theta\left(\frac{3}{2}\eta\right)^{\frac{2n-1}{2n-2}} \Theta\left(\frac{1}{2}\eta\right)^{-\frac{n}{2n-2}} f'_\eta(t)^{\frac{2n-1}{2n-2}}.$$

Raising to power $\frac{2n-2}{2n-1}$, after summation on $[0, t]$

$$\left(\int_{\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)} |\nabla u| dx\right)^{\frac{1}{2n-1}} \geq c_n t \Theta\left(\frac{3}{2}\eta\right)^{-1} \Theta\left(\frac{1}{2}\eta\right)^{\frac{n}{2n-1}}.$$

And so

$$\int_{\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)} |\nabla u| dx \geq c_n \eta^{2n-1} \Theta\left(\frac{3}{2}\eta\right)^{-(2n-1)} \Theta\left(\frac{1}{2}\eta\right)^{n}.$$

- We let

$$g_\eta(t) = \int_{\Omega(\eta-t, \eta+t)} |\nabla u|^2 dx, \ G_\eta(t) = \int_0^t g_\eta.$$

We begin by finding a upper bound for G_η. We take t in $[0, \eta/2]$, then:

$$G_\eta(t) \leq \int_0^t \Theta\left(\frac{3}{2}\eta\right)(h(\eta-s) + h(\eta+s))ds$$

$$\leq \Theta\left(\frac{3}{2}\eta\right)|\Omega(\eta-t, \eta+t)|^{\frac{1}{2}} G'_\eta(t)^{\frac{1}{2}}.$$

Thus:

$$G'_\eta(t) G_\eta(t)^{-2} \geq \Theta\left(\frac{3}{2}\eta\right)^{-2} \left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|^{-1}.$$

We integrate from t to $2t$ (up to supposing $t < \eta/4$)

$$G_\eta(t)^{-1} \geq t \Theta\left(\frac{3}{2}\eta\right)^{-2} \left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|^{-1}.$$

Thus:

$$G_\eta(t) \leq t^{-1} \Theta\left(\frac{3}{2}\eta\right)^2 \left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|.$$

Since g_η is increasing, then up to supposing $t < \eta/8$

$$g_\eta(t) \leq \frac{1}{t}\int_t^{2t} g_\eta \leq \frac{G_\eta(2t)}{t} \leq t^{-2} \Theta\left(\frac{3}{2}\eta\right)^2 \left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|.$$

Combining the previous inequalities, that are valid for $t \in [0, \eta/8]$, we get

$$c_n \eta^{2n-1} \Theta\left(\frac{3}{2}\eta\right)^{-(2n-1)} \Theta\left(\frac{1}{2}\eta\right)^n \leq \int_{\Omega(\eta-t, \eta+t)} |\nabla u| dx$$

$$\leq |\Omega(\eta-t, \eta+t)|^{\frac{1}{2}} \left(\int_{\Omega(\eta-t, \eta+t)} |\nabla u|^2 dx\right)^{\frac{1}{2}}$$

$$\leq t^{-1} \Theta\left(\frac{3}{2}\eta\right) \left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right|.$$

Taking $t = \eta/8$, we get:

$$\left|\Omega\left(\frac{1}{2}\eta, \frac{3}{2}\eta\right)\right| \geq \frac{c_n \eta^{2n}}{\Theta\left(\frac{3}{2}\eta\right)^{2n} \Theta\left(\frac{1}{2}\eta\right)^{-n}},$$

for a constant c_n that may be made explicit. Now, suppose that the quantity ϵ chosen at the start was of the form 3^{-K} for some $K \in \mathbb{N}^*$, and then by taking $\eta = \frac{2}{3^k}$ for $k = 1, \ldots, K$, we get

$$M - \omega_n \geq |\{u > 3^{-K}\}| \geq \sum_{k=1}^{K} \left| \Omega \left(\frac{1}{3^p}, \frac{1}{3^{p-1}} \right) \right|$$

$$\geq c_n \sum_{k=1}^{K} \frac{3^{-2nk}}{\Theta \left(3^{-(k-1)} \right)^{2n} \Theta \left(3^{-k} \right)^{-n}}$$

$$\geq c_n \left(\inf_{0<s<1} \frac{\Theta(s/3)}{\Theta(s)} \right)^{2n} \int_{3^{-K}}^{1} \frac{t^{2n-1} dt}{\Theta(t)^n}.$$

Thus if the hypothesis (9) in the result is verified, this gives an upper bound on K.   □

With this, we may replace any minimizing sequence (u_i) by a minimising sequence $(u_i 1_{\{u_i > t_i\}})$ such that

$$\inf_{\{u_i > 0\}} (u_i) \geq \delta$$

so we will now suppose that all the functions we consider verify this property.
For any $p \in \mathbb{Z}^n$, let us write $K_p = p + [0, 1]^n$.

Lemma 11 *Let u be admissible for* (10) *and $\delta > 0$ such that $u \geq \delta 1_{\{u>0\}}$. Let $\epsilon > 0$. Then there exists a set $F = \cup_{i \in I} K_i$ where $I \Subset \mathbb{Z}^n$ is such that*

$$|I| \leq C_n M \left(\frac{\Theta(\delta)^{-1} \mathscr{E}_\Theta(u) + M}{\epsilon} \right)^n , \quad |\{u > 0\}\backslash F| \leq \epsilon.$$

Proof We first prove the following technical result. Let $(K_i)_{i \in I}$ be a finite family of unit cubes, $F = \cup_{i \in I} K_i$, then

$$\max_{p \notin I} \left| \{u > 0\} \cap K_p \right| \geq c_n \left(\frac{|\{u > 0\}\backslash \cup_{p \notin I} K_p|}{|\{u > 0\}| + \Theta(\delta)^{-1} \mathscr{E}_\Theta(u)} \right)^n .$$

For any $p \notin I$, we may write

$$|\{u > 0\} \cap K_p| = |\{u > 0\} \cap K_p|^{\frac{1}{n}} |\{u > 0\} \cap K_p|^{1 - \frac{1}{n}}$$

$$\leq C_n \left(\max_{q \notin I} |\{u > 0\} \cap K_p| \right)^{\frac{1}{n}} \left(|\{u > 0\} \cap K_p| + \mathscr{H}^{n-1}(J_u \cap K_p) \right)$$

by the embedding $BV(K_p) \hookrightarrow L^{\frac{n}{n-1}}(K_p)$

$$\leq C_n \left(\max_{q \notin I} |\{u > 0\} \cap K_p| \right)^{\frac{1}{n}} \left(|\{u > 0\} \cap K_p| + \Theta(\delta)^{-1} \mathscr{E}_\Theta(u|K_p) \right).$$

where the last term is defined as

$$\mathscr{E}_\Theta(u|\Omega) := \int_\Omega |\nabla u|^2 dx + \int_{J_u} (\Theta((u 1_\Omega)_+) + \Theta((u 1_\Omega)_-)) d\mathscr{H}^{n-1}$$

And so, by summing in $p \in \mathbb{Z}^n \backslash I$:

$$|\{u > 0\}\backslash \cup_{q \notin I} K_q| \leq C_n \left(\max_{q \notin I} |\{u > 0\} \cap K_p| \right)^{\frac{1}{n}} \left(|\{u > 0\}| + \Theta(\delta)^{-1} \mathscr{E}_\Theta(u) \right),$$

which is the result.

We now construct F by induction starting with $F_0 = \emptyset$ and as long as $|\{u > 0\} \setminus F_k| \geq \epsilon$, we take $I_{k+1} = I_k \cup \{p\}$ where $p \notin F_k$ is chosen with the previous lemma such that $\left|\{u > 0\} \cap K_p\right| \geq c_n \left(\frac{\epsilon}{|\{u>0\}|+\Theta(\delta)^{-1}\mathscr{E}_\Theta(u)}\right)^n$. Suppose that this goes on until a rank N, then

$$M \geq |\{u > 0\}| \geq \sum_{i \in I_N} |\{u > 0\} \cap K_i| \geq c_n N \left(\frac{\epsilon}{|\{u > 0\}| + \Theta(\delta)^{-1}\mathscr{E}_\Theta(u)}\right)^n,$$

so N is bounded uniformly and conclude the proof of the lemma. □

For a closed set F, we shall write $d_F(x) = \inf_{y \in F} |x - y|$.

Lemma 12 *Let u be admissible for* (10) *and $\delta > 0$ such that $u \geq \delta 1_{\{u>0\}}$. Then there exist constants $\tau_{n,\Theta,\delta}$, $C_{n,\Theta,\delta}$ such that for any closed set F there exists some $r \in \left[0, C_{n,\Theta,\delta}|\{u > 0\} \setminus F|^{\frac{1}{n}}\right]$ such that*

$$\mathscr{E}_\Theta(u1_{\{d_F < r\}}) \leq \mathscr{E}_\Theta(u) - \tau_{n,\Theta,\delta}|\{u > 0\} \cap \{d_F > r\}|^{1-\frac{1}{n}}.$$

Proof Let $\Omega_r = \{u > 0\} \cap \{d_F \geq r\}$, and $m(r) = |\Omega_r|$. Suppose that the result we want to prove is not true for $r \in [0, r_1]$ for some $r_1 > 0$, meaning that for a constant $\tau > 0$ that will be chosen later and for any $r \in]0, r_1[$,

$$\mathscr{E}_\Theta(u|\Omega_r) \leq \int_{\partial^*\Omega_r \setminus J_u} \Theta(u) d\mathscr{H}^{n-1} + \tau |\Omega_r|^{1-\frac{1}{n}},$$

where we remind $\mathscr{E}_\Theta(u|\Omega) := \int_\Omega |\nabla u|^2 dx + \int_{J_u} (\Theta((u1_\Omega)_+) + \Theta((u1_\Omega)_-)) d\mathscr{H}^{n-1}$. Now, for any $r \in]0, r_1[$,

$$m(r)^{\frac{n-1}{n}} \leq C_n \mathrm{Per}(\Omega_r) \text{ (by isoperimetry)}$$

$$\leq C_n \Theta(\delta)^{-1} \left(\int_{\partial^*\Omega_r \setminus J_u} \Theta(u) d\mathscr{H}^{n-1} + \int_{\partial^*\Omega_r \cap J_u} (\Theta((u1_{\Omega_r})_-) + \Theta((u1_{\Omega_r})_+)) d\mathscr{H}^{n-1}\right)$$

$$\leq C_n \Theta(\delta)^{-1} \left(\int_{\partial^*\Omega_r \setminus J_u} \Theta(u) d\mathscr{H}^{n-1} + \mathscr{E}_\Theta(u|\Omega_r)\right)$$

$$\leq C_n \Theta(\delta)^{-1} \left(2 \int_{\partial^*\Omega_r \setminus J_u} \Theta(u) d\mathscr{H}^{n-1} + \tau |\Omega_r|^{1-\frac{1}{n}}\right),$$

so for $\tau := \frac{\Theta(\delta)}{2C_n}$ this implies

$$m(r)^{\frac{n-1}{n}} \leq 4C_n \Theta(\delta)^{-1} \int_{\partial^*\Omega_r \setminus J_u} \Theta(u) d\mathscr{H}^{n-1} \leq 4C_n \frac{\Theta(1)}{\Theta(\delta)} \mathscr{H}^{n-1}(\partial^*\Omega_r \setminus J_u).$$

Moreover $\mathscr{H}^{n-1}(\partial^*\Omega_r \setminus J_u) \leq -m'(r)$. So by integrating this for every $r \in]0, r_1[$, we get

$$m(r_1)^{\frac{1}{n}} \leq m(0)^{\frac{1}{n}} - \frac{\Theta(\delta)}{4C_n \Theta(1)} r_1.$$

Since $m(r_1) \geq 0$ this means that necessarily

$$r_1 \leq 4C_n \frac{\Theta(\delta)}{\Theta(1)} |\{u > 0\} \setminus F|^{\frac{1}{n}}.$$

Thus, there is always some $r \in \left[0, 8C_n \frac{\Theta(\delta)}{\Theta(1)} |\{u > 0\} \backslash F|^{\frac{1}{n}}\right]$ such that

$$\mathscr{E}_\Theta(u 1_{\{d_F < r\}}) \leq \mathscr{E}_\Theta(u) - \tau |\{u > 0\} \cap \{d_F > r\}|^{1 - \frac{1}{n}},$$

where $\tau = \frac{\Theta(\delta)}{2C_n}$ as defined earlier. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In all that follows we will need the following lemma for controlled infinitesimal volume exchange between measurable set, that may be found for instance in [15, Lemma 29.13].

Lemma 13 *Let U be a connected open set, $E_1, \ldots, E_N$ be a measurable partition of U such that, for every i, $|E_i \cap U| > 0$. Then there are vector fields X_{ij} with disjoint support such that for any i, j, k,*

$$\int_{E_k} \mathrm{div}(X_{ij}) = \begin{cases} +1 & \text{if } k = i, \\ -1 & \text{if } k = j, \\ 0 & \text{else}. \end{cases}$$

Proof Consider the linear application

$$L : \begin{cases} \mathcal{C}_c^\infty(U) \to \mathbb{R}^N, \\ X \mapsto \left(\int_{E_i} \mathrm{div}(X)\right)_{i=1,\ldots,N}. \end{cases}$$

The range of L is included in $\{(m_1, \ldots, m_N) : \sum_{i=1}^N m_i = 0\}$. If it were not equal to this subspace, there would be a non trivial vector $(a_1, \ldots, a_N)$ independant of $(1, \ldots, 1)$ such that, for any X, $\sum_{i=1}^N a_i \int_{E_i} \mathrm{div}(X) = 0$, meaning that $\nabla \left(\sum_{i=1}^N a_i 1_{E_i}\right) = 0$ in $\mathcal{D}'(U)$, which is a contradiction by the connectedness of U. $\qquad\qquad\qquad\qquad\qquad\square$

Proposition 14 *Under the hypotheses of Proposition 8, there exists a minimizer of $\mathscr{E}_\Theta$ in Problem (10).*

Proof Consider a minimizing sequence (u_i); up to truncation we may suppose that $\inf_{\{u_i > 0\}}(u_i) \geq \delta_{n,\Theta,m,M}$. We may apply Lemma 11 with $\epsilon = m/2$ to find sequences $(p_i^k)_{1 \leq k \leq N, i \geq 0}$ such that $|\{u_i > 0\} \backslash \cup_k K_{p_i^k}| \leq \frac{\omega_n}{2}$. In particular, $|\{u_i = 1\} \cap \cup_k K_{p_i^k}| \geq \frac{\omega_n}{2}$ so for each i there is some k_i such that $|K_{p_i^{k_i}} \cap \{u = 1\}| \geq \frac{\omega_n}{2N}$. Up to a translation we will suppose that $p_i^{k_i} = 0$, such that $K_{p_i^{k_i}} = [0, 1]^n$.

Now, by compactness arguments in SBV, see Theorem 9, we know that up to an extraction (u_i) converges almost everywhere (with lower semicontinuity on the energy) to a non-trivial limit $u \in SBV$, in particular such that $|\{u = 1\}| > 0$ and Fatou's lemma tells us that for any cube K, $|\{u = 1\} \cap K| \geq \limsup_{i \to \infty} |\{u_i = 1\} \cap K|$. We also denote $R := (2M)^{1/n}$ to be such that $|[0, R]^n \cap \{u = 0\}| \geq M$.

Lemma 15 *Up to extraction, either $|\{u > 0\} \cap [0, R]^n| > \limsup_{i \to \infty} |\{u_i > 0\} \cap [0, R]^n|$, which we call the loose case, or*

$$1_{\{u_i = 1\} \cap [0, R]^n} \to 1_{\{u = 1\} \cap [0, R]^n}$$

in the weak-$*$ sense, which we call the saturated case.

Proof Suppose that $|\{u > 0\} \cap [0, R]^n| = \limsup_{i \to \infty} |\{u_i > 0\} \cap [0, R]^n|$. We denote ν a weak limit of the sequence of measures given by the density $1_{\{u_i=1\}}$; by hypothesis $\nu([0, R]^n) = |\{u = 1\} \cap [0, R]^n|$. Moreover for any nonnegative continuous function $\varphi \in C^0([0, R]^n, \mathbb{R})$, by Fatou's lemma,

$$\nu(\varphi) = \lim_{i \to \infty} \int_{[0,R]^n} \varphi 1_{\{u_i=1\}} dx \le \int_{[0,R]^n} \varphi \limsup_{i \to \infty} 1_{\{u_i=1\}} dx \le \int_{[0,R]^n} \varphi 1_{\{u=1\}} dx,$$

which concludes the lemma. □

We now let $\epsilon > 0$ to be a small number (it will be fixed later), and we find a union of $N (= N_{n,\Theta,m,M,\epsilon})$ cube (we also include cubes to cover $[0, R]^n$), denoted F_i, such that $|\{u_i > 0\} \setminus F_i| \le \epsilon$; by applying Lemma 12 to (u_i, F_i), we find a radius $r_i \le C_{n,\Theta,M} |\{u_i > 0\} \setminus F_i|^{\frac{1}{n}}$ such that

$$\mathscr{E}_\Theta(u_i 1_{\{d_{F_i} < r_i\}}) \le \mathscr{E}_\Theta(u_i) - \tau_{n,\Theta,M} |\{u_i > 0\} \cap \{d_{F_i} > r_i\}|^{1-\frac{1}{n}}.$$

We let $v_i = u_i 1_{\{d_F < r_i\}}$, and we now differentiate between the loose and the saturated case.

- Loose case: here we may choose $\epsilon < |\{u = 1\} \cap [0, R]^n| - \limsup_{i \to \infty} |\{u_i = 1\} \cap [0, R]^n|$. Since the support of v_i is on a finite (not depending on i) number of cubes, they may be moved around so that v_i is supported in a compact set. Then by SBV compactness Theorem 9 we obtain that $v_i \to v$, such that $\mathscr{E}_\Theta(v) \le \liminf_{i \to \infty} \mathscr{E}_\Theta(v_i) = \inf_{|\{u=1\}| \ge \omega_n, |\{u>0\}| \le M} \mathscr{E}_\Theta(u)$. Moreover, $|\{v > 0\}| = \lim_{i \to \infty} |\{v_i > 0\}| \le M$ and

$$|\{v = 1\}| \ge |\{u = 1\} \cap [0, R]^n| - \limsup_{i \to \infty} |\{u_i = 1\} \cap [0, R]^n| + \limsup_{i \to \infty} |\{v_i = 1\}|$$

$$\ge |\{u = 1\} \cap [0, R]^n| - \limsup_{i \to \infty} |\{u_i = 1\} \cap [0, R]^n| - \epsilon + \omega_n$$

$$\ge \omega_n,$$

so v is admissible and this proves the result.
- Saturated case: based on the partition $\{u = 0\}$, $\{0 < u < 1\}$, $\{u = 1\}$ of $[0, R]^n$, where the first and last set have positive measure, there exists a vector field $\xi \in C_c^\infty((0, R)^n, \mathbb{R}^n)$ such that

$$\int_{\{u=1\}} \text{div}(\xi) dx = 1, \quad \int_{\{0<u<1\}} \text{div}(\xi) dx = 0, \quad \int_{\{u=0\}} \text{div}(\xi) dx = -1.$$

Moreover notice that $\phi_t(x) := x + t\xi(x)$ is a diffeomorphism with compact support for any small enough t; ϕ_t will be used to regulate the measure of $\{u_i = 1\}$ after truncation. Using the weak convergence of the measure of the supports of the (u_i), we may suppose that for any large enough i and any small enough t (not depending on i)

$$|\{u_i \circ \phi_t^{-1} = 1\}| = \int_{\{u_i=1\}} \det(D\phi_t) dx = \int_{\{u_i=1\}} \left(1 + t\,\text{div}(\xi) + t^2 P_\xi(t)\right) dx$$

$$\ge |\{u_i = 1\}| + \frac{t}{2},$$

$$|\{0 < u_i \circ \phi_t^{-1} < 1\}| = \int_{\{0<u_i<1\}} \det(D\phi_t) dx = \int_{\{0<u_i<1\}} \left(1 + t\,\text{div}(\xi) + t^2 P_\xi(t)\right) dx$$

$$\le |\{0 < u_i < 1\}| + Ct^2.$$

where P_ξ is a polynomial of degree $n - 2$ that depends on ξ.

Now, let $t_i = 8(\omega_n - \{v_i = 1\})$. We know that $t_i = 8|\{u_i > 0\}\backslash\{d_{F_i} > r_i\}| \le 8\epsilon$; when ϵ is small enough, for any $|t| < 8\epsilon$ we know ϕ_t is a diffeomorphism and the estimates above hold for any large enough i. Consider

$$w_i(x) = v_i \circ \phi_{t_i}^{-1}\left(\left[\frac{\omega_n}{\omega_n - \frac{1}{4}t_i}\right]^{1/n} x\right).$$

Then

$$|\{w_i = 1\}| = \frac{\omega_n - \frac{1}{4}t_i}{\omega_n}|\{v_i \circ \phi_{t_i}^{-1} = 1\}| \ge \frac{\omega_n - \frac{1}{4}t_i}{\omega_n}\left(|\{v_i = 1\}| + \frac{1}{2}t_i\right),$$

$$\ge |\{v_i = 1\}| + \frac{t_i}{8} = \omega_n$$

$$|\{0 < w_i < 1\}| = \frac{\omega_n - \frac{1}{4}t_i}{\omega_n}|\{0 < v_i \circ \phi_{t_i}^{-1} < 1\}| \le \frac{\omega_n - \frac{1}{4}t_i}{\omega_n}\left(|\{0 < v_i < 1\}| + Ct_i^2\right)$$

$$\le |\{0 < v_i < 1\}| \le |\{0 < u_i < 1\}|,$$

where in both lines we use that t_i is taken arbitrarily small (less than $8\epsilon > 0$, not depending on i). Thus (w_i) is admissible. Then it may be computed with a similar method that

$$\mathcal{E}_\Theta(w_i) - \mathcal{E}_\Theta(v_i) = \int_{\{\xi \ne 0\}}\left(\left|\left((I + t_i D\xi)^*\right)^{-1}\nabla u_i\right|^2 \det(I_n + t_i D\xi) - |\nabla u|^2\right)dx$$

$$+ \int_{J_u \cap \{\xi \ne 0\}}\left(\Theta(u_i^+) + \Theta(u_i^-)\right)\left(v_{J_{u_i}}^*(I_n + t_i D\xi)v_{J_{u_i}} - 1\right)d\mathcal{H}^{n-1}$$

$$\le C_0 t_i \text{ where } C_0 \text{ does not depend on } i.$$

And then, due to Lemma 12,

$$\mathcal{E}_\Theta(w_i) \le \mathcal{E}_\Theta(u_i) + 8C_0|\{u_i > 0\}\backslash\{d_{F_i} > r_i\}|$$

$$-\tau_{n,\Theta,M}|\{u_i > 0\} \cap \{d_{F_i} > r_i\}|^{1-\frac{1}{n}} \le \mathcal{E}_\Theta(u_i),$$

where the last inequality is due to $|\{u_i > 0\} \cap \{d_{F_i} > r_i\}| \le \epsilon$ and ϵ is chosen small enough (depending on the flow, which was defined before ϵ). So (w_i) is an admissible minimizing sequence that is confined in a disjoint union of N unit cubes, and up to moving these cubes we may suppose that w_i has support in a certain ball B not depending on i. So with the compactness result 9, it converges to a minimizer.   □

Lemma 16 *Let u be a relaxed minimizer of (10). Then J_u is $\mathcal{H}^{n-1}$-essentially closed, and there exists a bounded open set Ω with $\partial\Omega = \overline{J_u}$ and a relatively closed set $K \subset \Omega$ such that (K, Ω) is a solution of (4), associated to the function u, with $u \in H^1(\Omega) \cap \mathcal{C}_{loc}^{0,\frac{2}{n+2}}(\Omega)$ and $K = \{u = 1\}$.*

Proof We assume without loss of generality that $|\{0 < u < 1\}| > 0$; otherwise the minimizer is directly identified by the isoperimetric inequality.

- We first prove that u is an almost quasiminimizer of the Mumford-Shah functionnal, meaning that there are constants $c_u, r_u > 0$ such that for any ball $B_{x,r}$ with $r < r_u$ and any function $v \in SBV(\mathbb{R}^n)$ that differs from u on $B_{x,r}$ only,

$$\int_{B_{x,r}}|\nabla u|^2dx + \Theta(\delta)\mathcal{H}^{n-1}(B_{x,r} \cap J_u)$$

$$\leq \int_{B_{x,r}} |\nabla v|^2 dx + 2\Theta(1)\mathcal{H}^{n-1}(B_{x,r} \cap J_v) + c_u|\{u \neq v\}|. \tag{13}$$

Let U be the union of two balls of arbitrarily small radius δ such that

$$|U \cap \{u = 1\}|, \ |U \cap \{0 < u < 1\}|, \ |U \cap \{u = 0\}| > 0 \tag{14}$$

$$|U^c \cap \{u = 1\}|, \ |U^c \cap \{0 < u < 1\}|, \ |U^c \cap \{u = 0\}| > 0. \tag{15}$$

The second condition (15) is automatic as soon as δ is small enough, and the first condition (14) may be obtained by continuity of $x \mapsto |B_{x,\delta} \cap A|$ for $A = \{u = 1\}, \{0 < u < 1\}, \{u = 0\}$. We may apply Lemma 13 for these three sets in U, thus obtaining three smooth vector fields (X, Y, Z) with support in U that transfer measure between $(\{u = 0\}, \{0 < u < 1\}, \{u = 1\})$, in particular

$$\int_{\{u=1\}} \text{div}(X) = -\int_{\{u=0\}} \text{div}(X) = 1, \ \int_{\{0<u<1\}} \text{div}(X) = 0$$

$$\int_{\{u=1\}} \text{div}(Y) = -\int_{\{u=0\}} \text{div}(Y) = 1, \ \int_{\{0<u<1\}} \text{div}(Y) = 0$$

Write $\phi_t(x) = x + tX(x)$, $\psi_t(x) = x + tY(x)$, and $\Phi_{s,t} = \phi_s \circ \psi_t$. Consider the application

$$G : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (s,t) \mapsto \left(\big| \Phi_{s,t}(\{u = 1\}) \big| - |\{u = 1\}| , \big| \Phi_{s,t}(\{0 < u < 1\}) \big| - |\{0 < u < 1\}| \right). \end{cases}$$

Then G is smooth and $DG(0,0) = I_2$; there exists $\epsilon_0 > 0$ such that G is invertible on B_{ϵ_0} with $\frac{1}{2}|(s,t)| \leq |G(s,t)| \leq 2|(s,t)|$ and $G(B_{\epsilon_0}) \supset B_{\epsilon_0/2}$. Let $r > 0$ and $x \in \mathbb{R}^n$ be such that $|B_{x,r} \cap U| = 0$ and $|B_{x,r}| < \epsilon_0/4$, and $v \in SBV$ be such that $\{u \neq v\} \Subset B_{x,r}$. Up to truncating v from above (by 1, that can only decrease the energy) and from below (by some $t \geq \delta$, by Lemma 10) we assume that $\delta 1_{\{v>0\}} \leq v \leq 1$.

Let $(a,b) = (|\{u = 1\}| - |\{v = 1\}|, |\{0 < u < 1\}| - |\{0 < v < 1\}|)(\in B_{\epsilon_0/2})$, and $(s,t) = G^{-1}_{|B_{\epsilon_0}}(a,b)$. Then $v \circ \Phi_{s,t}^{-1}$ satisfies the measure constraints (ω_n, M), so

$$\mathcal{E}_{\Theta}(u) \leq E_{\Theta}(v \circ \Phi_{s,t}^{-1}).$$

So

$$\int_{B_{x,r}} |\nabla u|^2 dx + \Theta(\delta)\mathcal{H}^{n-1}(B_{x,r} \cap J_u) \leq \int_{B_{x,r}} |\nabla v|^2 dx + 2\Theta(1)\mathcal{H}^{n-1}(B_{x,r} \cap J_v) + R,$$

where

$$R = \int_U \left(\big|(D\Phi_{s,t}^*)^{-1}\nabla u\big|^2 \det(D\Phi_{s,t}) - |\nabla u|^2 \right) dx$$

$$+ \int_{U \cap J_u} \left(v_u^\perp D\Phi_{s,t} v_u^\perp - 1 \right) (\Theta(u_-) + \Theta(u_+)) d\mathcal{H}^{n-1}$$

$$\leq C(|s| + |t|) \leq 2C(|a| + |b|).$$

This proves that u is an almost quasiminimizer at any positive distance of U. Now by our choice of U, the condition (15) allows us to choose similarly a set $U' \subset \mathbb{R}^n \setminus \overline{U}$ that

is a union of two balls of arbitrarily small radii such that

$$|U' \cap \{u = 1\}|, \ |U' \cap \{0 < u < 1\}|, \ |U' \cap \{u = 0\}| > 0$$

and by choosing similarly vector fields X', Y', Z' with support in U' that transit measure between these three sets, u is an almost quasiminimizer at any positive distance of U'. Thus u is an almost quasiminimizer, and this concludes the proof of the estimate (13).

- By [9, Theorem 3.1], u being an almost quasiminimizer of Mumford-Shah implies that $\overline{J_u}$ is essentially closed, meaning $\mathscr{H}^{n-1}(\overline{J_u} \backslash J_u) = 0$. We now let Ω be the union of the bounded connected components of $\mathbb{R}^n \backslash \overline{J_u}$, then $u \in H^1(\Omega)$ and $\partial \Omega = \overline{J_u}$.

- Let us prove Ω is bounded by proving an explicit lower density estimate; indeed, we let $r := r_u$ as defined in the first point and considering $v = u 1_{\mathbb{R}^n \backslash B_{x,\rho}}$ for any $x \in \mathbb{R}^n$ and $\rho \in \,]0, r[$ in (13) we get

$$\int_{B_{x,\rho}} |\nabla u|^2 \mathrm{d}x + \Theta(\delta) \mathscr{H}^{n-1}(B_{x,\rho} \cap J_u)$$
$$\leq 2\Theta(1) \mathscr{H}^{n-1}(\partial B_{x,\rho} \cap \{u > 0\} \backslash J_u) + c_u |B_{x,\rho} \cap \{u > 0\}|.$$

Let $m(\rho) = |\Omega \cap B_{x,\rho}|$, then by the isoperimetric inequality

$$c_n m(\rho)^{1-\frac{1}{n}} \leq \mathscr{H}^{n-1}(\partial \Omega \cap B_\rho) + \mathscr{H}^{n-1}(\Omega \cap \partial B_\rho) \leq \left(1 + 2\frac{\Theta(1)}{\Theta(\delta)}\right) m'(\rho) + c_u m(\rho).$$

So if $|B_{x,r/2} \cap \Omega| > 0$ then by integrating this estimate, $|B_{x,r} \cap \Omega| \geq cr^n$ for a constant c that does not depend on x. Since $|\Omega| \leq M$ then there are at most $N \leq \frac{M}{cr^n}$ points $(x_i)_{i=1,\dots,N}$ such that $|x_i - x_j| \geq r$ for all $i \neq j$ and $|\Omega \cap B(x_i, r/2)| > 0$, meaning that Ω is bounded.

- We prove that u is locally Hölder, which implies the relative closedness of K with $K = \{u = 1\}$. Indeed let $K = \{u = 1\}$ and $B \Subset \Omega$ be a small ball inside Ω such that $|B \cap K|$ and $|B \backslash K|$ are positive (such a ball exists as soon as $|\Omega| > |K|$, otherwise K and $\Omega \backslash K$ would be disconnected, and (K, K) would have strictly lower energy than (K, Ω)). Let $\xi \in C_c^\infty(B, \mathbb{R}^n)$ be such that

$$\int_{B \cap K} \mathrm{div}(\xi) \mathrm{d}x = 1, \ \int_{B \backslash K} \mathrm{div}(\xi) \mathrm{d}x = -1,$$

and let $\phi_t(x) = x + t\xi(x)$ be the associated diffeomorphism for a small enough t. Consider then any ball $B_{x,r} \Subset \Omega$ such that

$$B_{x, r^{\frac{n}{n+2}}} \Subset \Omega \backslash B,$$

we prove that provided r is small enough (depending only on the choice of the flow and the parameters of the problem and not on x) there is a constant $C > 0$ not depending on x and r such that

$$\int_{B_{x,r}} |\nabla u|^2 \mathrm{d}x \leq C r^{\frac{n^2}{n+2}}.$$

This directly implies the local Hölder continuity in $\Omega \backslash B$ and the relative closedness of K using classical integral growth argument (see for instance [13, cor. 3.2]), and the same may be done for another small ball B' that has positive distance from B which conclude the result. Let us now focus on this estimate.

Let $R = r^{\frac{n}{n+2}}$ and h be the harmonic extension of $u_{|\partial B_{x,R}}$ on $B_{x,R}$ (that we extend simply by u outside $B_{x,R}$). We suppose that r is small enough such that $r < \frac{R}{2}$. Notice that h might not be admissible, however $|\{h = 1\}| \geq \omega_n - |B_R|$; we let $t = 2|B_R|$ and for a small enough R, we have

$$|\{h \circ \phi_t^{-1} = 1\}| \geq \omega_n,$$

so $h \circ \phi_t^{-1}$ is admissible. By comparison with the minimizer u we get

$$\int_{B_{x,R}} |\nabla(u - h)|^2 \mathrm{d}x \leq \int_B \left(\left| \left((I + tD\xi)^*\right)^{-1} \nabla u \right|^2 \det(I + tD\xi) - |\nabla u|^2 \right) \mathrm{d}x \leq C R^n.$$

for some constant C that depends on n, $u_{|B}$ and on the flow ξ. Now, using the subharmonicity of $|\nabla h|^2$ on $B_{x,R}$,

$$\int_{B_{x,r}} |\nabla u|^2 \mathrm{d}x \leq 2 \int_{B_{x,r}} |\nabla(u - h)|^2 \mathrm{d}x + 2 \int_{B_{x,r}} |\nabla h|^2 \mathrm{d}x$$

$$\leq C R^n + 2 \left(\frac{r}{R/2} \right)^n \int_{B_{x,R/2}} |\nabla h|^2 \mathrm{d}x$$

$$\leq C R^n + C' \frac{r^n}{R^2} \text{ by Cacciopoli inequality on the second term.}$$

This ends the proof. □

Remark 17 Summarizing our results, we know that Ω is an open set with rectifiable topological boundary such that $\mathscr{H}^{n-1}(\partial \Omega) < \infty$ and that the temperature $u \in H^1(\Omega)$ is $\mathcal{C}_{\mathrm{loc}}^{0, \frac{2}{n+2}}(\Omega)$.

4 The penalized problem: Proof of Theorem 2

In this section we prove Theorem 2. For any $\Lambda > 0$, we denote

$$E_{\Theta,\Lambda}(K, \Omega) = E_\Theta(K, \Omega) + \Lambda |\Omega \setminus K|, \quad \mathscr{E}_{\Theta,\Lambda}(u) = \mathscr{E}_\Theta(u) + \Lambda |\{0 < u < 1\}|.$$

We claim that it is enough to prove the result in the case for a function Θ which satisfies (6) and such that $\Theta(u) = \mathcal{O}_{u \to 0}(u^2)$. Indeed, for any small $\epsilon > 0$ one may replace Θ on $[0, 1]$ with a perturbation that verifies (6), namely

$$\Theta^\epsilon(u) = \min \left((\Theta(1) + \epsilon) \left(\frac{u}{\epsilon} \right)^2, \Theta(u) + \epsilon 1_{(0,1]} \right)$$

and define E_{Θ^ϵ}, $E_{\Theta^\epsilon,\Lambda}$, accordingly. Then we know $E_{\Theta^\epsilon,\Lambda}$ is minimal on some (B_1, B_{R^ϵ}), and that the associated state function has the form

$$u^\epsilon(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 1 - (1 - u^\epsilon(R^\epsilon)) \frac{\Phi_n(|x|) - \Phi_n(1)}{\Phi_n(R^\epsilon) - \Phi_n(1)} & \text{if } 1 \leq |x| \leq R^\epsilon, \\ 0 & \text{if } |x| > R^\epsilon, \end{cases}$$

due to the general form of radial harmonic functions ($x \mapsto a + b\Phi_n(|x|)$). The associated energy takes the form

$$E_{\Theta^\epsilon,\Lambda}(B_1, B_{R^\epsilon}) = \left(1 - u^\epsilon(R^\epsilon)\right)^2 \frac{\mathrm{Per}(B_r)\Phi_n'(1)}{\Phi_n(R^\epsilon) - \Phi_n(1)} + \mathrm{Per}(B_{R^\epsilon})\Theta^\epsilon\left(u^\epsilon(R^\epsilon)\right).$$

R^ϵ is bounded uniformly in ϵ; indeed, due to the penalization term,

$$|B_{R^\epsilon}| \leq \Lambda^{-1} E_{\Theta^\epsilon}(B_1, B_{R^\epsilon}) \leq \Lambda^{-1} E_{\Theta^\epsilon}(B_1, B_1) = \Lambda^{-1}(\Theta(1) + \epsilon)\mathrm{Per}(B_1)$$

so we may suppose without loss of generality that $R^\epsilon \underset{\epsilon \to 0}{\longrightarrow} R$ and $u^\epsilon(R^\epsilon) \underset{\epsilon \to 0}{\longrightarrow} l \in [0, 1]$, and define

$$u(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 1 - (1 - l) \frac{\Phi_n(|x|) - \Phi_n(1)}{\Phi_n(R) - \Phi_n(1)} & \text{if } 1 < |x| < R, \\ 0 & \text{if } |x| > R, \end{cases}$$

Then by lower semicontinuity of Θ and the fact that $\Theta(0) = 0$ we get

$$E_{\Theta,\Lambda}(B_1, B_R) \leq \mathscr{E}_{\Theta,\Lambda}(u) \leq \liminf_{\epsilon \to 0} E_{\Theta^\epsilon,\Lambda}(B_1, B_{R^\epsilon})$$

and then for any admissible v we get by dominated convergence

$$\mathscr{E}_{\Theta,\Lambda}(v) = \lim_{\epsilon \to 0} \mathscr{E}_{\Theta^\epsilon,\Lambda}(v) \geq \liminf_{\epsilon \to 0} E_{\Theta^\epsilon,\Lambda}(B_1, B_{R^\epsilon}) \geq E_{\Theta,\Lambda}(B_1, B_R)$$

This is a similar method as what was used in [8] to handle a similar function with $\Theta(v) \sim 2\beta cv$ near 0 for some constants $c, \beta > 0$.

Lemma 18 *Problem* (5) *has a solution.*

Proof We let $\Theta^\epsilon(u)$ defined above. Notice that

$$\inf_{0 < s < 1} \frac{\Theta^\epsilon(s/3)}{\Theta^\epsilon(s)} \geq \min\left(\frac{1}{9}, \inf_{0 < s < 1} \frac{\Theta(s/3) + \epsilon}{\Theta(s) + \epsilon}\right) \geq \min\left(\frac{1}{9}, \frac{\epsilon}{\Theta(1) + \epsilon}\right) > 0$$

and $\Theta^\epsilon(v) = \mathcal{O}_{s \to 0}(s^2)$ so the hypothesis (7) is automatically verified. For what comes next we drop the ϵ to lighten the notations.

The results of Sect. 3 apply; for any $M \geq \omega_n$, there exists a minimizer (K^M, Ω^M) of problem (4). $M \mapsto E_\Theta(K^M, \Omega^M)$ is nonincreasing, and we prove that it is lower semicontinuous. Indeed, consider $M_i \underset{i \to \infty}{\longrightarrow} M$, and consider the sequence of minimizers (K^{M_i}, Ω^{M_i}), with their state function (u_i). Then proceeding as in the Proof of Proposition 14, we find a modified sequence (w_i) obtained from (u_i) through truncation with uniformly bounded support, such that its limit w verifies $|\{w = 1\}| \geq \omega_n$, $|\{w > 0\}| \leq M$, and $\mathscr{E}_\Theta(w) \leq \liminf_{i \to \infty} \mathscr{E}_\Theta(u_i)$; this prove the lower semi-continuity. Since

$$E_{\Theta,\Lambda}(K^M, \Omega^M) \geq \Lambda(M - \omega_n) \underset{M \to +\infty}{\longrightarrow} +\infty,$$

then there is a (non-necessarily unique) $M_\Lambda > 0$ such that $[\omega_n, +\infty) \ni M \mapsto E_{\Theta,\Lambda}(K^M, \Omega^M)$ is minimal at $M = M_\Lambda$ and $(K^{M_\Lambda}, \Omega^{M_\Lambda})$ is a minimum of $E_{\Theta,\Lambda}$ with $|\Omega| \leq M$. $\qquad\square$

We may now prove the main result of this section. We will say a set A has the center property if any hyperplane that goes through the origin divides A in two parts of same measure. This is in particular the case if A has central symmetry. We will also call a minimizer (K, Ω) "centered" if K has the center property.

Proof of Theorem 2 Consider u a minimizer of $\mathcal{E}_{\Theta,\Lambda}$ with the associated sets $\Omega = \{u > 0\}$, $K = \{u = 1\}$. We remind that we know Ω to be open and $u \in H^1(\Omega)$. When e is a unit vector and $\lambda \in \mathbb{R}$, we will write

$$u_{e,\lambda}^+(x) = \begin{cases} u(x) & \text{if } x \cdot e \geq \lambda \\ u(S_{e,\lambda}(x)) & \text{if } x \cdot e < \lambda \end{cases}, \quad u_{e,\lambda}^-(x) = \begin{cases} u(S_{e,\lambda}(x)) & \text{if } x \cdot e \geq \lambda \\ u(x) & \text{if } x \cdot e < \lambda \end{cases},$$

where $S_{e,\lambda}$ is the reflexion relative to the hyperplane $\{x : x \cdot e = \lambda\}$.

- We first show that there exists a minimizer with the center property by building a minimizer with central symmetry. Indeed, consider u a minimizer associated to (K, Ω), $\lambda_1 \in \mathbb{R}$ such that $\{x : x_1 = \lambda_1\}$ cuts K in half. Then u_{e_1,λ_1}^+ and u_{e_1,λ_1}^- are also minimizers; we may in particular replace u with u_{e_1,λ_1}^+ or u_{e_1,λ_1}^- (the choice does not matter here) to suppose u has a symmetry along $\{x : x_1 = \lambda_1\}$. We do the same for $\{x : x_i = \lambda_i\}$ successively for $i = 2, \ldots, n$; in the end we arrive to a minimizer

$$\tilde{u} = \left(\ldots \left(u_{e_1,\lambda_1}^{\pm} \right)_{e_2,\lambda_2}^{\pm} \ldots \right)_{e_n,\lambda_n}^{\pm},$$

  that is symmetric relative to every S_{e_i,λ_i}. Up to a translation of λ, $\tilde{u}$ is invariant for every $S_{e_i,0}$, and so it is invariant by their composition which is the central symmetry $x \mapsto -x$.
- Suppose that u is a minimizer (with $K = \{u = 1\}$, $\Omega = \{u > 0\}$) where K has the center property (we know such a minimizer exists from the previous point). We prove that the free boundary $\partial K \cap \Omega$ is a union of spherical arcs centered at the origin.

  If $\{0 < u < 1\}$ is empty then the problem is equivalent to the isoperimetric inequality and we are done, so we suppose that it is not. Consider A a connected component of $\{0 < u < 1\}$, which is open by the regularity of minimizers. The set $\partial A \cap \partial K$ is not empty, since it would otherwise mean that $\mathcal{E}_\Theta(u 1_{\mathbb{R}^n \setminus A}) < \mathcal{E}_\Theta(u)$.

  Let $H_e = \{x : x \cdot e = 0\}$ be a hyperplane going through the origin such that $H_e \cap A \neq \emptyset$, then $u_{e,0}^+$ is also a minimizer; in particular this means by analyticity of u in A that $\nabla u_{|H_e}$ is colinear to H_e. Since this is true for every e, then $u_{|A}$ is a radial function (restricted to a set A that may not be radial), and since it is harmonic it has an expression of the form

$$u_{|A}(x) = a_A - b_A \Phi_n(|x|),$$

  where Φ_n is the fundamental solution of the Laplacian (taken with the sign convention that it is increasing). Moreover, since u is 1 on $\partial K \cap \partial A$, we know this set is an arc of circle around 0, and we call its radius r_A. While r_A might depend on the component A, notice that the optimality condition for the Alt-Caffarelli problem near $\partial K \cap \Omega$ is exactly that for a certain constant $\lambda > 0$ that does not depend on A,

$$\forall x \in \partial K \cap \partial \{0 < u < 1\}, \ |\nabla u(x)| = \lambda,$$

  where $\nabla u(x)$ refers in this case to the limit of $\nabla u(y)$ as $y(\in A) \to x$. With $u(x) = 1$ this exactly reduces to

$$\begin{cases} a_A - b_A \Phi_n(r_A) &= 1, \\ -b_A \Phi_n'(r_A) &= \lambda, \end{cases}$$

  so (a_A, b_A) are fully determined by r_A.
- We prove a reflexion lemma that will be useful for successive reflections.

Lemma 19 *Let u be a centered minimizer. Then for any vector e, $u_{e,0}^+$ is also a centered minimizer.*

Proof We let $v = u_{e,0}^+$. If $\{0 < v < 1\}$ has zero measure then as a solution of the isoperimetric problem we know v is the indicator of a ball so we are done; we suppose it is not the case. We may consider A a connected component of $\{0 < v < 1\}$ as previously and $Z \Subset \partial A \cap \partial K$ a small spherical cap that is open with respect to ∂B_{r_A}. Consider another vector f that is completed in an orthogonal basis $f = f_1, f_2, \ldots, f_n$ of $\mathbb{R}^n$. As previously we may reflect v successively as

$$\tilde{v} = \left(\ldots \left(v_{f_1, \lambda_1}^\pm \right)_{f_2, \lambda_2}^\pm \ldots \right)_{f_n, \lambda_n}^\pm,$$

Where the signs are chosen such that v and $\tilde{v}$ coincide on a quadrant that has non-empty intersection with Z. By construction $\tilde{v}$ is centrally symmetric around the point $\lambda = (\lambda_1, \ldots, \lambda_n)$, so Z is a spherical cap both around the origin and λ; this implies that $\lambda = 0$ so in particular $\lambda_1 = 0$, which means that $H_{f,0}$ cuts $\{u_{e,0}^+ = 1\}$ in half; this is what we wanted to prove. $\quad\square$

- Consider a centered minimizer u (it exists due to the first step, since there exists a centrally symmetric minimizer), such that $|\{0 < u < 1\}| > 0$. We construct another centered minimizer v such that for some connected component A of $\{0 < v < 1\}$, $\partial A \cap \partial \{v = 1\}$ contains a centered sphere. We do this by iteration of the following lemma.

Lemma 20 *Let u be a centered minimizer, let A be a connected component of $\{0 < u < 1\}$ and let $\lambda \in]1, 2[$ and $D_{x,\rho}^{\partial B_r} \Subset \partial A \cap \partial K$ be a small spherical disk in the sphere of radius $r := r_A$. Then there exists another centered minimizer v obtained by a finite number of reflexions on u, associated to the sets $(K', \Omega') = (\{v = 1\}, \{v > 0\})$, such that for some connected component A' of $\{0 < v < 1\}$, $D_{x,\lambda\rho}^{\partial B_r} \subset \partial K' \cap \partial A'$.*

Proof We will denote $D(x, \rho)$ the ball relative the the unit sphere. If $\rho > \frac{\pi}{2} r$ then one reflexion along the hyperplane $x^\perp$ is enough; we suppose that $\rho \leq \frac{\pi}{2} r$.

Let $\epsilon = 1 - \frac{\lambda}{2}$. For every $y \in \partial D(x, (1 - \epsilon)\rho)$, the reflexion by the hyperplane going through y and tangent to $\partial D(x, (1 - \epsilon)\rho)$ reflects a neighbourhood of $[x, y]$ on a neighbourhood $N(y)$ of $[x', y]$ for some point x' in the continuation of the geodesic $[xy)$ at distance $\lambda\rho$ from x. By compactness there is some finite set $(y_i)_{i \in I}$ in $\partial D(x, (1 - \epsilon)\rho)$ such that $\cup_{i \in I} N(y_i) \supset D(x, \lambda\rho)$. By iterating the associated reflexions we obtain the result. $\quad\square$

- There exists a centered minimizer u such that $\{u = 1\}$ is a ball. Indeed, consider a minimizer (K, Ω) with a function u as previously, such that there is some $r > 0$ for which $\partial B_r \subset \partial K$. For every such r, the function $u_r = \max(u, 1_{B_r})$ is also a minimizer associated to $(K \cup B_r, \Omega \cup B_r)$ that has the same energy as u, thus $u = 1$ on B_r. In particular this means that u is radially decreasing on the whole component of Ω that contains a small neighbourhood of B_r; let us denote Ω_1 this component and $\Omega_2 = \Omega \backslash \Omega_1$, as well as $u_i = u 1_{\Omega_i}$. Then $u = u_1 + u_2$ where u_1, u_2 have disjoint support and $J_{u_1} \cup J_{u_2} \subset J_u$. If u_2 is not zero this means that u is a disconnected minimizer; let us prove that this is not possible.

Indeed, let $m_i = |\{u_i = 1\}|$ and $v_i(\cdot) = u_i \left(\left(\frac{m_i}{\omega_n} \right)^{1/n} \cdot \right)$. Then $|\{v_i = 1\}| = \omega_n$ and

$$\mathcal{E}_{\Theta,\Lambda}(u_i) = \left(\frac{m_i}{\omega_n} \right)^{1 - \frac{2}{n}} \int_{\mathbb{R}^n} |\nabla v_i|^2 \mathrm{d}x + \left(\frac{m_i}{\omega_n} \right)^{1 - \frac{1}{n}} \int_{J_{v_i}} \Theta(v_i) \mathrm{d}\mathcal{H}^{n-1}$$

$$+\frac{m_i}{\omega_n}\Lambda|\{v_i > 0\}| > \frac{m_i}{\omega_n}\mathscr{E}_{\Theta,\Lambda}(v_i),$$

where the inequality is strict because $\mathscr{E}_\Theta(v_i) > 0$ and $m_i < \omega_n$ (or else one of the v_i is zero, which we supposed is not the case). Now,

$$\mathscr{E}_{\Theta,\Lambda}(u) = \mathscr{E}_{\Theta,\Lambda}(u_1) + \mathscr{E}_{\Theta,\Lambda}(u_2) > \frac{m_1}{\omega_n}\mathscr{E}_{\Theta,\Lambda}(v_1) + \frac{m_2}{\omega_n}\mathscr{E}_{\Theta,\Lambda}(v_2) \geq \inf \mathscr{E}_{\Theta,\Lambda}.$$

This contradicts the fact that u is a minimizer. Thus $u_2 = 0$ and $u = u_1$ with $\{u_1 = 1\} = B_r$, which turns out to be the ball of volume ω_n, this determines r to be 1 and in turns means that u is of the form

$$u(x) = \varphi(|x|)1_\Omega,$$

for some radial function φ that takes the value 1 on $[0, r]$ and some set A that contains B_1, and takes the value $1 - c(\Phi_n(|x|) - \Phi_n(1))$ on Ω for some $c > 0$.

- Consider such a minimizer u associated with the sets (K, Ω), such that $\{u = 1\}$ is the ball B_1, $\{u > 0\}$ contains a neighbourhood of B_1 and $u(x)$ is a radial function $\varphi(|x|)$ restricted to a non-necessarily radial set Ω. Then we prove that for some $R > 0$, $\varphi 1_{B_R}$ is also a minimizer.

First replace Ω with its spherical cap rearrangement Ω^s around the axis $\{te_1, t > 0\}$, meaning that for every $t > 0$, $\Omega^s \cap \partial B_t$ is a spherical cap centered in te_1 with the same $\mathscr{H}^{n-1}$-measure as $\Omega \cap \partial B_t$. Said differently this means that $\mathscr{H}^{n-1}(\Omega \cap \partial B_t) = \mathscr{H}^{n-1}(\Omega^s \cap \partial B_t)$ for all $t > 0$ with $\partial \Omega^s = \{f(e \cdot e_1)e, e \in \mathbb{S}^{n-1}\}$ for some nondecreasing $f : [-1, 1] \to (r, +\infty)$ that is continuous in 1 by openness of Ω^s (it may also be checked afterward that in our case, f is continuous at -1 by minimality although we will not need it). A property of spherical cap rearrangement (see for instance [16, Prop. 3 and Rem. 4]) is that

$$\int_{\partial^*\Omega^s} \varphi(|x|)d\mathscr{H}^{n-1} \leq \int_{\partial^*\Omega} \varphi(|x|)d\mathscr{H}^{n-1}$$

and $|\Omega^s| = |\Omega|$, $\int_{\Omega^s}|\varphi'(|x|)|^2 dx = \int_\Omega |\varphi'(|x|)|^2 dx$. As a consequence, $\varphi(|\cdot|)1_{\Omega^s}$ is still a minimizer. Write $R := f(1)$, for any small $\epsilon > 0$, there is a small enough $\rho > 0$ such that

$$f([1 - \rho, 1]) \subset [R - \epsilon, R].$$

We may iterate Lemma 20 again to obtain a minimizer $(u_\epsilon, K_\epsilon, \Omega_\epsilon)$ such that u_ϵ is still given by the same radial function φ on a non-necessarily radial set, and such that

$$K_\epsilon = B_r, \quad B_{R-\epsilon} \subset \Omega_\epsilon \subset B_R.$$

Since this may be done for any arbitrarily small ϵ, by lower semi-continuity as $\epsilon \to 0$ we obtain that $(\varphi(|\cdot|)1_{B_R}, B_r, B_R)$ is a minimizer.

Notice we could have done the same for $(B_1, B_{F(-e_1)})$; if $F(-e_1) < F(e_1)$ it means we are in a case in which $\arg\min\{R \geq 1 \mapsto E_{\Theta,\Lambda}(B_1, B_R)\}$ is not uniquely defined. □

Corollary 21 *If, additionally to the hypotheses of Theorem 2, Θ is such that $[1, +\infty[\ni R \to E_\Theta(B_1, B_R)$ is minimal at 1, then for any (K, Ω)*

$$E_\Theta(K, \Omega) \geq E_\Theta(B_1, B_1).$$

Proof (B_1, B_1) is always the minimizer of $E_{\Theta,\Lambda}$ as $\Lambda \to 0$, hence the result. □

Remark 22 This covers the case $\beta \leq n - 2$ for $\Theta(u) = \beta u^2$.

5 Proof of Theorem 3

Part of the assertions of the Theorem 3, namely the existence of a solution to problem (4) and its regularity, have already been proved in the preparatory Sect. 3 (see Proposition 8 and Remark 17).

Proof of Theorem 3 In order to complete the Proof of Theorem 3, we give the following.

Lemma 23 *Let (K, Ω) a solution of* (4). *Then K has locally finite perimeter in Ω and $\partial K \cap \Omega$ is analytic when $n = 2$.*

Proof We refer to [5] where a closely related problem is treated. The same arguments apply in our case.   □

Lemma 24 *Let Θ be l.s.c. and nondecreasing. Let (K, Ω) be a solution of* (4) *for $M = R^n \omega_n$. Then either $|\Omega| = M$ or $E_\Theta(K, \Omega) = \inf_{r \in [1, R]} E_\Theta(B_1, B_r)$.*

Proof Suppose there is a minimizer with measure strictly less than M. Roughly speaking, in this case the measure constraint is not saturated so that the problem behaves under many aspects an unconstrained one.

By compactness, there exists (K, Ω) that is the minimizer with the lowest volume, associated to a function u. Indeed, consider (K_i, Ω_i) a sequence of such minimizers, associated with functions (u_i), then reproducing the existence proof in Sect. 3 on the sequence (u_i), we obtain a new minimizer with minimal volume, still denoted (K, Ω).

If $|\Omega| = \omega_n$ then we are done, so we suppose that this is not the case.

Let $\mathcal{H}$ be the set of every hyperplane that cuts K in half. It is not quite identified with $\mathbb{RP}^{n-1}$ since there may be several parallel hyperplanes that cut K in half if K is not connected. It is, however, straightforward that $\mathcal{H}$ is a connected set for the natural topology given by $\{(H, v) \in \mathbb{RP}^{n-1} \times \mathbb{R}^n \text{ s.t. } v \in H^\perp\}$. For every hyperplane H, we write

$$m(H) = \sup \left\{ |H^+ \cap \Omega|, |H^- \cap \Omega| \right\}$$

and we let

$$\mathcal{H}_< = \left\{ H \in \mathcal{H} : m(H) < \frac{M}{2} \right\},$$

$$\mathcal{H}_= = \left\{ H \in \mathcal{H} : m(H) = \frac{M}{2} \right\},$$

$$\mathcal{H}_> = \left\{ H \in \mathcal{H} : m(H) > \frac{M}{2} \right\}.$$

Then $\mathcal{H}_<$ and $\mathcal{H}_>$ are open relatively to $\mathcal{H}$ by the continuity of m. The minimality of the volume of Ω and the fact that $|\Omega| < M$ implies that $\mathcal{H}_=$ is empty (otherwise we could construct a minimizer with stricly lower volume by reflection around an element of $\mathcal{H}_=$).

Finally, $\mathcal{H}_<$ is not empty, as may be seen by considering, for each $\theta \in \frac{\mathbb{R}}{\pi\mathbb{Z}}$, an hyperplane orthogonal to $\cos(\theta)e_1 + \sin(\theta)e_2$ and using intermediate value theorem after making a full turn. By connectedness, this directly implies that $\mathcal{H}_> = \emptyset$ and so $\mathcal{H}_< = \mathcal{H}$. Now, let $H \in \mathcal{H}$, then Ω may be reflected on both sides of H: since the measure of Ω is minimal among minimizers, then

$$|H^- \cap \Omega| = |H^+ \cap \Omega|.$$

Thus, for every hyperplane that cuts K in half, Ω is also cut in half; this means that we can use the same arguments than for the penalized problem in Theorem 2: by successive reflections, we find a solution with minimal volume such that the free boundary of u is ∂B_1. By spherical rearrangement around $\{te_1, t > 0\}$ we obtain a new minimizer (K, Ω). Now, if Ω is not a ball, then $\left|H_{e_1}^+ \cap \Omega\right| > \left|H_{e_1}^- \cap \Omega\right|$, which is a contradiction. □

We prove now the last assertion of Theorem 3. We begin with an a priori estimate that is similar to Lemma 10, only that it is done with in mind the idea to find a lower bound arbitrarily close to 1.

Proposition 25 *Let Θ be l.s.c and nondecreasing. There exists a constant $C_n > 0$ such that, if $\delta \in]0, 1[$ verifies*

$$\delta + C_n \frac{\Theta(1)}{\sqrt{\Theta(\delta)}} (M - \omega_n)^{\frac{1}{2n}} < 1,$$

then for all u such that $|\{u \geq 1\}| \geq \omega_n$, $|\{u > 0\}| \leq M$, there is some $t > \delta$ such that $\mathscr{E}_\Theta(u 1_{\{u > t\}}) \leq \mathscr{E}_\Theta(u)$.

This lemma is applicable in particular when δ is close to 1 and $M - \omega_n$ is small.

Proof This proof is very similar to Lemma 10, so some steps are only briefly described. We actually show the stronger result; there is some $t > s > \delta$ such that

$$\mathscr{E}_\Theta \left(u 1_{\{s < u < t\}^c}\right) \leq \mathscr{E}_\Theta(u).$$

Let $\epsilon = 1 - \delta > 0$, suppose $\mathscr{E}_\Theta(u 1_{\{s < u < t\}^c}) > \mathscr{E}_\Theta(u)$ for every $1 - \epsilon < s < t < 1$. We write

- $\Omega(s, t) = \{s < u \leq t\}$.
- $\gamma(s, t) = \int_{J_u} \left(1_{s < u_+ \leq t} + 1_{s < u_- \leq t}\right) d\mathscr{H}^{n-1}$.
- $h(t) = \mathscr{H}^{n-1}(\{u = t\} \setminus J_u)$.

Let $\eta := 1 - \frac{\epsilon}{2}$, our hypothesis becomes that for every, $t \in]0, \frac{\epsilon}{2}[$, $\mathscr{E}_\Theta(u 1_{\{\eta - t < u < \eta + t\}^c}) > \mathscr{E}_\Theta(u)$ so

$$\int_{\Omega(\eta - t, \eta + t)} |\nabla u|^2 dx + \Theta(1 - \epsilon)\gamma(\eta - t, \eta + t) \leq \Theta(1)(h(\eta - t) + h(\eta + t)). \quad (16)$$

The proof is, as previously, based on a lower bound of $\int_{\Omega(\eta - t, \eta + t)} |\nabla u| dx$ and an upper bound of $\int_{\Omega(\eta - t, \eta + t)} |\nabla u|^2 dx$ that are in contradiction when ϵ is too large.

- For all $t \in]0, \frac{1}{2}\epsilon[$ we let:

$$f(t) = \int_{\eta - t}^{\eta + t} h = \int_{\Omega(\eta - t, \eta + t)} |\nabla u| dx.$$

f is absolutely continuous and

$$f'(t) = h(\eta - t) + h(\eta + t).$$

Moreover,

$$f(t) \leq |\Omega(\eta - t, \eta + t)|^{\frac{1}{2}} \left(\int_{\Omega(\eta - t, \eta + t)} |\nabla u|^2 dx\right)^{\frac{1}{2}}$$

$$\le C_n \mathrm{Per}(\Omega\,(\eta - t, \eta + t))^{\frac{n}{2(n-1)}}\,(\Theta(1)(h\,(\eta - t) + h\,(\eta + t)))^{\frac{1}{2}}$$

by isoperimetric inequality

$$\le C_n \Theta(1)^{\frac{1}{2}}\,(h\,(\eta - t) + \gamma\,(\eta - t, \eta + t) + h\,(\eta + t))^{\frac{n}{2(n-1)}}\,(h\,(\eta - t) + h\,(\eta + t))^{\frac{1}{2}}$$

$$\le C_n \left(\frac{\Theta(1)}{\Theta(1 - \epsilon)}\right)^{\frac{n}{2(n-1)}}\Theta(1)^{\frac{1}{2}}\,f'(t)^{\frac{2n-1}{2n-2}},$$

so:

$$f'(t)\,f(t)^{-\frac{2n-2}{2n-1}} \ge c_n \Theta(1)^{-\frac{n-1}{2n-1}}\left(\frac{\Theta(1)}{\Theta(1 - \epsilon)}\right)^{-\frac{n}{2n-1}}.$$

We integrate on $[0, t]$:

$$\left(\int_{\Omega(\eta - t, \eta + t)}|\nabla u|\mathrm{d}x\right)^{\frac{1}{2n-1}} \ge c_n \Theta(1)^{-\frac{n-1}{2n-1}}\left(\frac{\Theta(1)}{\Theta(1 - \epsilon)}\right)^{-n}t.$$

And so:

$$\int_{\Omega(\eta - t, \eta + t)}|\nabla u|\mathrm{d}x \ge c_n \Theta(1)^{-(n-1)}t^{2n-1}.$$

- We let:

$$g(t) = \int_{\Omega(\eta - t, \eta + t)}|\nabla u|^2\mathrm{d}x,\ \ G(t) = \int_0^t g$$

We begin by finding a upper bound for G. We take t in $[0, \epsilon/2]$, then:

$$G(t) \le \int_0^t \Theta(1)(h(\eta - s) + h(\eta + s))\mathrm{d}s$$

$$\le c_n \Theta(1)|\Omega\,(\eta - t, \eta + t)\,|^{\frac{1}{2}}\,G'(t)^{\frac{1}{2}}.$$

Thus:

$$G'(t)G(t)^{-2} \ge c_n \Theta(1)^{-2}|\Omega\,(1 - \epsilon, 1)\,|^{-1}.$$

We integrate from t to $2t$ (up to supposing $t < \epsilon/4$):

$$G(t)^{-1} \ge c_n t\Theta(1)^{-2}|\Omega\,(1 - \epsilon, 1)\,|^{-1}.$$

Thus:

$$G(t) \le C_n \Theta(1)^2 t^{-1}|\Omega\,(1 - \epsilon, 1)\,|.$$

Since g is increasing, then up to supposing $t < \epsilon/8$:

$$g(t) \le \frac{1}{t}\int_t^{2t}g \le \frac{G(2t)}{t} \le c_n \Theta(1)^2 t^{-2}|\Omega\,(1 - \epsilon, 1)\,|.$$

Combining the previous inequalities, that are valid for $t \in [0, \epsilon/8]$, we get:

$$c_n \Theta(1)^{-(n-1)}\left(\frac{\Theta(1)}{\Theta(1 - \epsilon)}\right)^{-n}t^{2n-1} \le \int_{\Omega(\eta - t, \eta + t)}|\nabla u|\mathrm{d}x$$

$$\le |\Omega\,(\eta - t, \eta + t)\,|^{\frac{1}{2}}\left(\int_{\Omega(\eta - t, \eta + t)}|\nabla u|^2\mathrm{d}x\right)^{\frac{1}{2}}$$

$$\leq C_n \Theta(1) t^{-1} |\Omega(1 - \epsilon, 1)|.$$

Taking $t = \epsilon/8$, and with $|\Omega(1 - \epsilon, 1)| \leq M - \omega_n$, we get:

$$\epsilon \leq C_n \Theta(1) \Theta(1 - \epsilon)^{-\frac{1}{2}} (M - \omega_n)^{\frac{1}{2n}},$$

for a constant $C_n > 0$ that only depends on n; this proves the result. □

Proof of Theorem 3 *(continuation).* Let Θ be l.s.c. and nondecreasing. Suppose moreover that it is of class $\mathcal{C}^1$ near 1 and

$$\frac{\Theta'(1)^2}{\Theta(1)} < 4(n - 1).$$

Then there is some $M_0 > \omega_n$ depending on n, Θ such that, for any $M \in [\omega_n, M_0]$ and for any admissible u with $|\{u \geq 1\}| \geq \omega_n$, $|\{u > 0\}| \leq M$,

$$\mathscr{E}_\Theta(u) \geq E_\Theta(B_1, B_1) (= \Theta(1) n \omega_n).$$

Denote for simplicity $\Omega = \{u > 0\}$, $K = \{u = 1\}$, and let δ be defined as in the previous result (it is always possible when $M - \omega_n$ is small enough), and suppose that it is close enough to 1 such that Θ in $\mathcal{C}^1$ on $[\delta, 1]$. Using the previous lemma, we lose no generality in supposing $u_{|\Omega} \geq \delta$, and $|\Omega| = M$. Then

$$\int_{\partial\Omega} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} + \int_\Omega \Theta'(u) |\nabla u| \mathrm{d}x$$

$$= \int_{\partial\Omega} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} + \int_\delta^1 \Theta'(t) \mathrm{Per}(\{u > t\}; \Omega) \mathrm{d}t$$

$$= \int_{\partial\Omega} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} - \int_\delta^1 \Theta'(t) \mathrm{Per}(\{u > t\}; \partial\Omega) \mathrm{d}t$$

$$+ \int_\delta^1 \Theta'(t) \mathrm{Per}(\{u > t\}) \mathrm{d}t$$

$$= \Theta(\delta) \mathrm{Per}(\Omega) + \int_\delta^1 \Theta'(t) \mathrm{Per}(\{u > t\}) \mathrm{d}t$$

$$\geq \Theta(\delta) \left(\frac{M}{\omega_n}\right)^{1-\frac{1}{n}} \mathrm{Per}(B_1) + \int_\delta^1 \Theta'(t) \mathrm{Per}(B_1) \mathrm{d}t$$

$$= \Theta(1) \mathrm{Per}(B_1) + \Theta(\delta) \mathrm{Per}(B_1) \left(\left(\frac{M}{\omega_n}\right)^{\frac{n-1}{n}} - 1\right).$$

Now,

$$\int_\Omega \Theta'(u) |\nabla u| \mathrm{d}x \leq \int_\Omega \|\Theta'\|_{\infty, [\delta, 1]} |\nabla u| \mathrm{d}x \leq \int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{\|\Theta'\|_{\infty, [\delta, 1]}^2}{4} (M - \omega_n),$$

Thus,

$$\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{\partial\Omega} \Theta(u) \mathrm{d}\mathscr{H}^{n-1} - \Theta(1) \mathrm{Per}(B_1)$$

$$\geq \Theta(\delta) \mathrm{Per}(B_1) \left(\left(\frac{M}{\omega_n}\right)^{\frac{n-1}{n}} - 1\right) - \frac{\|\Theta'\|_{\infty, [\delta, 1]}^2}{4} (M - \omega_n).$$

So we obtain the result as soon as

$$\frac{\|\Theta'\|^2_{\infty,[\delta,1]}}{\Theta(\delta)} \leq 4\mathrm{Per}(B_1)\frac{\left(\frac{M}{\omega_n}\right)^{\frac{n-1}{n}} - 1}{M - \omega_n}.$$

As $M \to \omega_n$, we may take $\delta \to 1$, and this gives the result. □

Remark 26 Conversely, suppose that

$$\frac{\Theta'(1)^2}{\Theta(1)} > 4(n - 1),$$

then for every $R > 1$ close enough to 1, $E_\Theta(B_1, B_R) < E_\Theta(B_1, B_1)$. Indeed, let $\epsilon > 0$ and

$$u_\epsilon(x) = \begin{cases} 1 & (B_1), \\ 1 - \frac{|x|-r}{2}\Theta'(1) & (B_{1+\epsilon} \backslash B_1). \end{cases}$$

Then

$$E_\Theta(B_1, B_{1+\epsilon}) \leq \mathscr{E}_\Theta(u_\epsilon) = n\omega_n(1 + \epsilon)^{n-1}\Theta\left(1 - \frac{1}{2}\Theta'(1)\epsilon\right) + \omega_n\left((1 + \epsilon)^n - r^n)\right)\frac{\Theta'(1)^2}{4}$$

$$= E_\Theta(B_1, B_1) + \left((n - 1)\Theta(1) - \frac{1}{4}\Theta'(1)^2\right)\epsilon\mathrm{Per}(B_1) + o_{\epsilon \to 0}(\epsilon).$$

The first-order term is negative and this proves the converse for a small enough ϵ.

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