Calc. Var. (2022) 61:135
https://doi.org/10.1007/s00526-022-02236-1

Calculus of Variations

L^∞-truncation of closed differential forms

Stefan Schiffer[1]

Received: 26 November 2021 / Accepted: 5 April 2022 / Published online: 13 May 2022

Abstract

In this paper, we prove that for each closed differential form $u \in L^1(\mathbb{R}^N; (\mathbb{R}^N)^* \wedge ... \wedge (\mathbb{R}^N)^*)$, which is almost in L^∞ in the sense that

$$\int_{\{y \in \mathbb{R}^N : |u(y)| \geq L\}} |u(y)| dy < \varepsilon$$

for some $L > 0$ and a small $\varepsilon > 0$, we may find a closed differential form v, such that $\|u - v\|_{L^1}$ is again small, and v is, in addition, in L^∞ with a bound on its L^∞ norm depending only on N and L. In particular, the set $\{v \neq u\}$ has measure at most $CL^{-1}\varepsilon$. As an application of this theorem, we are able to prove that the $\mathcal{A}$-p-quasiconvex hull of a set does not depend on p. Furthermore, we can prove a classification theorem for $\mathcal{A}$-∞-Young measures.

Mathematics Subject Classification 49J45 · 26B25

1 Introduction

1.1 $\mathcal{A}$-free truncations

An interesting question in the calculus of variations and real analysis is the following: Consider a linear differential operator $\mathcal{A}\colon C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ of first order with constant coefficients, and a bounded sequence of functions $u_n \in L^1(\mathbb{R}^N, \mathbb{R}^d)$ which satisfy $\mathcal{A}u_n = 0$ in the sense of distributions and are close to a bounded set in L^∞, i.e.

$$\lim_{n \to \infty} \int_{\{x \in \mathbb{R}^N : |u_n(x)| \geq L\}} |u_n| \,\mathrm{d}x = 0 \tag{1.1}$$

for some $L > 0$. Does there exist a sequence of functions v_n, such that $\mathcal{A}v_n = 0$, $\|v_n\|_{L^\infty} \leq CL$ and $(u_n - v_n) \to 0$ in measure (in L^1)?

This question was answered first by ZHANG in [38] for sequences of gradients ($u_n = \nabla w_n$), i.e. for the operator $\mathcal{A} = \mathrm{curl}$, which assigns to a function $u\colon \mathbb{R}^N \to \mathbb{R}^N$ the skew-symmetric

Communicated by L. Szekelyhidi.

✉ Stefan Schiffer
schiffer@iam.uni-bonn.de

1 Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany

$(N \times N)$-matrix with entries $\partial_i u_j - \partial_j u_i$. ZHANG'S proof, which builds on the works of LIU [22] and ACERBI- FUSCO [1], proceeds as follows. Denote by Mf the Hardy–Littlewood maximal function of $f \in L^1_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^d)$ and let $u_n = \nabla w_n$. The estimate (1.1) implies that the sets $X^n = \{M(\nabla w_n) \geq L'\}$ have small measure for large n. One then uses (cf. [1]) that

$$|w_n(x) - w_n(y)| \leq CL'|x - y|', \quad x, y \in \mathbb{R}^N \backslash X^n, \tag{1.2}$$

i.e. w_n is Lipschitz continuous on $\mathbb{R}^N \backslash X^n$. The fact that Lipschitz continuous functions on closed subsets of $\mathbb{R}^N$ can be extended to Lipschitz continuous functions on $\mathbb{R}^N$ with the same Lipschitz constant [15] yields the result.

In this paper, we show that the answer to the previously formulated question is also positive for sequences of differential forms and $\mathcal{A} = d$, the operator of exterior differentiation.

Let us denote by Λ^r the r-fold wedge product of the dual space $(\mathbb{R}^N)^*$ of $\mathbb{R}^N$ and by $d \colon C^\infty(\mathbb{R}^N, \Lambda^r) \to C^\infty(\mathbb{R}^N, \Lambda^{r+1})$ the exterior derivative w.r.t. the standard Euclidean geometry on $\mathbb{R}^N$.

Theorem 1.1 (L^∞-truncation of differential forms) *Suppose that we have a sequence $u_n \in L^1(\mathbb{R}^N, \Lambda^r)$ with $du_n = 0$ (in the sense of distributions), and that there exists an $L > 0$ such that*

$$\int_{\{y \in \mathbb{R}^N : |u_n(y)| > L\}} |u_n(y)| \, dy \longrightarrow 0 \quad \text{as } n \to \infty. \tag{1.3}$$

There exists a constant $C_1 = C_1(N, r)$ and a sequence $v_n \in L^\infty(\mathbb{R}^N, \Lambda^r)$ with $dv_n = 0$ and

(i) $\|v_n\|_{L^\infty(\mathbb{R}^N, \Lambda^r)} \leq C_1 L$*;*
(ii) $\|v_n - u_n\|_{L^1(\mathbb{R}^N, \Lambda^r)} \to 0$ *as* $n \to \infty$*;*
(iii) $|\{y \in \mathbb{R}^N : v_n(y) \neq u_n(y)\}| \to 0$.

An analogous version of Theorem 1.1 holds if $\mathbb{R}^N$ is replaced by the N-torus T_N (cf. Theorem 5.1) or by an open Lipschitz set Ω and functions u with zero boundary data (cf. Propostion 5.4). Moreover, the result immediately extends to $\mathbb{R}^m$-valued forms by taking truncations coordinatewise (cf. Proposition 5.5).

In particular, the result of Theorem 1.1 includes a positive answer to the question previously raised for the differential operator $\mathcal{A} = \text{div}$ after suitable identifications of Λ^{N-1} and Λ^N with $\mathbb{R}^N$ and $\mathbb{R}$, respectively.

One key ingredient in the proofs is a version of the Acerbi-Fusco estimate (1.2) for simplices rather than pairs of points in Lemma 3.1. For the estimate, let us consider $\omega \in C^2_c(\mathbb{R}^N, \Lambda^r)$ with $d\omega = 0$ and let D be a simplex with vertices $x_1, ..., x_{r+1}$ and a normal vector $\nu^r \in \mathbb{R}^N \wedge ... \wedge \mathbb{R}^N$ (cf. Sect. 2.3 for the precise definition). Assume that $M\omega(x_i) \leq L$ for $i = 1, ..., r + 1$. Then

$$\left| \int_D \omega(\nu^r) \right| \leq C(N)L \sup_{1 \leq i, j \leq r+1} |x_i - x_j|^r = C(N)L\text{diam}(D)^r. \tag{1.4}$$

The second ingredient is a geometric version of the Whitney extension theorem, which may be of independent interest, cf. Sect. 4.

Combining (1.4) and the extension theorem, one easily obtains the assertion for smooth closed forms. The general case follows by a standard approximation argument.

Before turning to an application of the truncation result, let us also mention that in Theorem 1.1 the hard part is to get the convergence in 1.1) just from the rather weak assumption (1.3). A version of Theorem 1.1 has been seen for a stronger assumption on the smallness of the sequence in [14]. Regarding solenoidal Lipschitz truncations [4, 5], meaning $W^{1,1}$-$W^{1,\infty}$-truncations instead of L^1-L^∞, the smallness corresponding to (1.3) is also assumed to be slightly different from the present setting.

Moreover, in the setting $\mathcal{A} = \text{curl}$, the statement of Theorem 1.1 can be further improved as follows. If K is a compact, convex set and $u_n \to K$ in L^1, we can even get a sequence v_n, such that the L^∞-norm of $\text{dist}(v_n, K)$ converges to 0, cf. [25]. In contrast, Theorem 1.1 only implies an L^∞-bound on v_n and convergence in measure to K. MÜLLER'S technique does not rely directly to a curl-free truncation, but on a Lipschitz truncation. It then uses suitable cut-offs and mollifications. The author does not see any obvious obstruction, why this technique should not work, if we replace the Lipschitz truncation by a general truncation statement on any potential instead of ∇. To keep the paper at a reasonable length, we however focus on $\mathcal{A}$-free truncations.

1.2 $\mathcal{A}$-∞ Young measures

Truncation results like the result by ZHANG or Theorem 1.1 have immediate applications in the calculus of variations. In particular, they provide characterisations of the $\mathcal{A}$-quasiconvex hulls of sets (cf. Sect. 6.1) and the set of Young-measures generated by sequences satisfying $\mathcal{A}u_n = 0$. For a precise definition of $\mathcal{A}$-Young measures we refer to Sect. 6 and [11].

The classical result for Young measures generated by sequences of gradients (i.e. sequences of functions u_n satisfying $\text{curl}\, u_n = 0$) goes back to KINDERLEHRER and PEDREGAL [17, 18]. Here, we show the natural counterpart of their characterisation result, whenever the operator $\mathcal{A}$ admits the following L^∞-truncation result:

We say that $\mathcal{A}$ satisfies the property (ZL) if for all sequences $u_n \in L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$, such that there exists an $L > 0$ with

$$\int_{\{y \in T_N :\, |u_n(y)| > L\}} |u_n(y)|\, \mathrm{d}y \longrightarrow 0 \quad \text{as } n \to \infty,$$

there exists a $C = C(\mathcal{A})$ and a sequence $v_n \in L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

(i) $\|v_n\|_{L^\infty(T_N,\mathbb{R}^d)} \leq CL$;
(ii) $\|v_n - u_n\|_{L^1(T_N,\mathbb{R}^d)} \to 0$ as $n \to \infty$.

By ZHANG [38], the property (ZL) holds for $\mathcal{A} = \text{curl}$ and a version of Theorem 1.1 shows this for $\mathcal{A} = d$ (Corollary 5.2). Further examples are shortly discussed in Example 6.2.

For the characterisation of Young measures, recall that $\text{spt}\, \nu$ denotes the support of a (signed) Radon measure $\nu \in \mathcal{M}(\mathbb{R}^d)$, and for $f \in C_c(\mathbb{R}^d)$

$$\langle \nu, f \rangle := \int_{\mathbb{R}^d} f\, \mathrm{d}\mu.$$

If the property (ZL) holds for some differential operator $\mathcal{A}$, then one is able to prove the following statement.

Theorem 1.2 (Classification of $\mathcal{A}$-∞-Young measures) *Let $\mathcal{A}$ satisfy (ZL). A weak$*$ measurable map $\nu : T_N \to \mathcal{M}(\mathbb{R}^d)$ is an $\mathcal{A}$-∞-Young measure if and only if $\nu_x \geq 0$ a.e. and there exists $K \subset \mathbb{R}^d$ compact and $u \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with*

(i) $\text{spt}\, \nu_x \subset K$ *for a.e.* $x \in T_N$;
(ii) $\langle \nu_x, id \rangle = u(x)$ *for a.e.* $x \in T_N$;

(iii) $\langle \nu_x, f\rangle \geq f(\langle \nu_x, id\rangle)$ *for a.e.* $x \in T_N$ *and all continuous and* $\mathcal{A}$*-quasiconvex* $f : \mathbb{R}^d \to \mathbb{R}$ *i.e.* $f \in C(\mathbb{R}^d)$*, such that for all* $\psi \in C^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$

$$f\left(\int_{T_N} \psi(x)\,\mathrm{d}x\right) \leq \int f(\psi(x))\,\mathrm{d}x.$$

For further reference to classification of $\mathcal{A}$-p-Young measures for $p < \infty$, let us shortly refer to [2, 11, 12, 19, 20].

1.3 Outline

We close the introduction with a brief outline of the paper. In Sect. 2, we introduce some notation, recall some basic facts from multilinear algebra, the theory of differential forms and Young measures. We prove the key estimate (1.4) in Sect. 3. Section 4 is devoted to the proof of the geometric Whitney extension theorem. In Sect. 5, the proof of the truncation result (and its local and periodic variant) is given. Section 6 discusses the applications to $\mathcal{A}$-quasiconvex hulls and $\mathcal{A}$-Young measures. The proofs of the theorems closely follow the arguments in [17] and are discussed in the last Sect. 6.3.

2 Preliminary results

2.1 Notation

We consider an open and bounded Lipschitz set $\Omega \subset \mathbb{R}^N$ and denote by T_N the N-dimensional torus, which arises from identifying faces of $[0, 1]^N$. We may identify functions $f : T_N \to \mathbb{R}^d$ with $\mathbb{Z}^N$-periodic functions $\tilde{f} : \mathbb{R}^N \to \mathbb{R}^d$, and vice versa. We write $B_\rho(x)$ to denote the ball with radius ρ and centre x. Denote by $\mathcal{L}^N$ the Lebesgue measure and, for a set $X \subset \mathbb{R}^N$,

$$|X| := \mathcal{L}^N(X).$$

For a measure μ on $\mathbb{R}^N$ and a μ-measurable set $A \subset \mathbb{R}^N$ with $0 < \mu(A) < \infty$ define the average integral of a μ-measurable function f via

$$\fint_A f\,\mathrm{d}\mu = \frac{1}{\mu(A)} \int_A f\,\mathrm{d}\mu.$$

For $k \in \mathbb{N}$ write $[k] = \{1, ..., k\}$. For a normed vector space V we denote by V^* the dual space of V.

Define the space Λ^r as the r-fold wedge product of $(\mathbb{R}^N)^*$, i.e.

$$\Lambda^r = \underbrace{(\mathbb{R}^N)^* \wedge ... \wedge (\mathbb{R}^N)^*}_{r \text{ copies}}$$

and similarly the space Λ_r as the r-fold wedge product of $\mathbb{R}^N$. Then Λ^r and Λ_r are finite-dimensional vector spaces. For $\mathbb{R}^N$ denote by $\{e_i\}_{i\in[N]}$ the standard basis and by $\cdot$ the standard scalar product. For $(\mathbb{R}^N)^*$ denote by $\theta_1, ..., \theta_N$ the corresponding dual basis of $(\mathbb{R}^N)^*$, i.e. θ_i is the map $y \mapsto y \cdot e_i$. For $k \in I_r := \{l \in [N]^r : l_1 < l_2 < ... < l_r\}$ the vectors

$$e^{k,r} = e_{k_1} \wedge e_{k_2} \wedge ... \wedge e_{k_r} \tag{2.1}$$

form a basis of Λ_r. Denote by $\cdot^r$ the scalar product with respect to this basis, i.e. for $k, l \in I_r$

$$e^{k,r} \cdot^r e^{l,r} = \begin{cases} 1 & k = l, \\ 0 & k \neq l. \end{cases}$$

This also provides us with a suitable norm on Λ_r, which we denote by $\|\cdot\|_{\Lambda_r}$. Similarly, using the standard basis of $(\mathbb{R}^n)^*$, we define a basis $\theta^{k,r}$ and a norm $\|\cdot\|_{\Lambda^r}$. Also note that for $0 \leq s \leq r$ there exists (up to sign) a natural map $\Lambda^r \times \Lambda_s \mapsto \Lambda^{r-s}$ (the interior product), as Λ^s is the dual space of Λ_s and $\Lambda^r = \Lambda^s \wedge \Lambda^{r-s}$. In particular, in the special case $s = 1$ for $h_1, ..., h_r \in \mathbb{R}^{N*}$ and $y \in \mathbb{R}^N$

$$(h_1 \wedge .... \wedge h_r)(y) = \sum_{i=1}^{r} (-1)^{i-1} h_i(y) h_1 \wedge ... \wedge h_{i-1} \wedge h_{i+1} ... \wedge h_r. \tag{2.2}$$

In the case $s = r$ and for $h_1, ..., h_r \in (\mathbb{R}^N)^*$ and $y_1, ..., y_r \in \mathbb{R}^N$

$$(h_1 \wedge .... \wedge h_r)(y_1 \wedge ... \wedge y_r) = \sum_{\sigma \in S_r} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{r} h_i(y_{\sigma(i)}) \right), \tag{2.3}$$

where S_r denotes the group of permutations of $\{1, ..., r\}$. (2.3) also gives us a representation of the map $\Lambda^r \times \Lambda_s \mapsto \Lambda^{r-s}$ as for $h \in \Lambda^r$, $x \in \Lambda_s$ we may consider the element of $\Lambda^{r-s} = (\Lambda_{r-s})^*$ defined by

$$z \longmapsto h(x \wedge z), \quad z \in \Lambda_{r-s}.$$

Let us shortly remark that this notation is slightly different to the usual notation for interior products.

Moreover, note that the space Λ^N is isomorphic to $\mathbb{R}$ via the map I^N defined by

$$a\, \theta_1 \wedge ... \wedge \theta_N \longmapsto a \in \mathbb{R}.$$

2.2 Differential forms

In the following, we will define all objects for an open set $\Omega \subset \mathbb{R}^N$, but these definitions are also valid for $\mathbb{R}^N$ and T_N respectively.

We call a map $f \in L^1_{\mathrm{loc}}(\Omega, \Lambda^r)$ an r-**differential form** on Ω. We define the space

$$\Gamma = \bigcup_{r \in \mathbb{N}} C^\infty(\Omega, \Lambda^r).$$

It is well-known (c.f [7, 9]) that there exists a linear map $d \colon \Gamma \mapsto \Gamma$, called the **exterior derivative** with the following properties

(i) $d^2 = d \circ d = 0$,
(ii) d maps $C^\infty(\Omega, \Lambda^r)$ into $C^\infty(\Omega, \Lambda^{r+1})$,
(iii) We have the **Leibniz rule**: If $\alpha \in C^\infty(\Omega, \Lambda^r)$ and $\beta \in C^\infty(\Omega, \Lambda^s)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta, \tag{2.4}$$

(iv) $d : C^\infty(\Omega, \Lambda^0) \to C^\infty(\Omega, \Lambda^1)$ is the gradient via the identification $\Lambda^0 = \mathbb{R}$, $\Lambda^1 = (\mathbb{R}^N)^* \cong \mathbb{R}^N$.

We sometimes write d_x to indicate that this derivative is taken in terms of a space variable $x \in \mathbb{R}^N$. This map d has the following representation in terms of the standard coordinates (cf. [9]). Let $\omega \in C^\infty(\Omega, \Lambda^r)$, which, for some $a_k \in C^\infty(\Omega, \mathbb{R})$, can be written as

$$\omega(y) = \sum_{k \in I_r} a_k(y)\theta^{k,r}.$$

Then

$$d\omega(y) = \sum_{k \in I_r} \sum_{l \in [N]} \partial_l a_k(y)\theta_l \wedge \theta^{k,r}. \tag{2.5}$$

Remark 2.1 For a fixed $r \in \{0, ..., N-1\}$ we can identify $d\colon C^\infty(\Omega, \Lambda^r) \mapsto C^\infty(\Omega, \Lambda^{r+1})$ with some well-known differential operator $\mathcal{A}$. By definition, for $r = 0$, d can be identified with the gradient. For $r = 1$, after a suitable identification of Λ^2 with $\mathbb{R}^{N\times N}_{skew}$, $d = \operatorname{curl}$, which is the differential operator mapping $u \in C^\infty(\Omega, \mathbb{R}^N)$ to $\operatorname{curl} u \in C^\infty(\Omega, \mathbb{R}^{N\times N}_{skew})$ defined by

$$(\operatorname{curl} u)_{lk} = \partial_l u_k - \partial_k u_l.$$

If $r = N-1$, after identifying Λ^{N-1} with $\mathbb{R}^N$ and Λ^N with $\mathbb{R}$, the differential operator d becomes the divergence of a vector field which is defined for $u \in C^\infty(\Omega, \mathbb{R}^N)$ by

$$\operatorname{div} u = \sum_{k=1}^N \partial_k u_k.$$

Lemma 2.2 *We have the following product rules for* d*:*

(i) Let $\omega \in C^1(\Omega, \Lambda^1)$*,* $z \in \mathbb{R}^N = \Lambda_1$*. Then*

$$d_y\big(\omega(y)(y-z)\big) = \nabla_y\omega(y)\cdot(y-z) + \omega(y), \tag{2.6}$$

where we define $\nabla_y\omega(y)\cdot(y-z) \in C(\Omega, \Lambda^1)$ *as follows:*
If $\omega = \sum_{i=1}^N \omega_i\theta_i$ *and* $(y-z) = \sum_{i=1}^N (y-z)_i e_i$*, then*

$$\nabla_y\omega(y)\cdot(y-z) := \sum_{l=1}^N \sum_{i=1}^N \partial_l\omega_i(y)(y-z)_i\theta_l.$$

(ii) There is a linear bounded map $D^{1,r} \in \operatorname{Lin}((\Lambda^r \times \mathbb{R}^N) \times \mathbb{R}^N, \Lambda^r)$ *such that for* $\omega \in C^1(\Omega, \Lambda^r)$*,* $z \in \mathbb{R}^N$ *we have*

$$d_y\Big(\omega(y)(y-z)\Big) = D^{1,r}\Big(\nabla\omega(y), (y-z)\Big) + \omega(y). \tag{2.7}$$

(iii) There is a linear and bounded map $D^{s,r} \in \operatorname{Lin}((\Lambda^r \times \mathbb{R}^N) \times \Lambda_s, \Lambda^{r-s})$ *such that for* $\omega \in C^1(\Omega, \Lambda^r)$*,* $z \in \mathbb{R}^N$*,* $z_2 \in \Lambda_{s-1}$

$$d_y\Big(\omega(y)((y-z)\wedge z_2)\Big) = D^{s,r}\Big(\nabla_y\omega(y), (y-z)\wedge z_2\Big) + (-1)^{s-1}\omega(y)(z_2). \tag{2.8}$$

Proof (i) simply follows from a calculation, i.e., if as mentioned

$$\omega(y) = \sum_{i=1}^N \omega_i(y)\theta_i \quad \text{and } (y-z) = \sum_{i=1}^N (y-z)_i e_i,$$

then

$$d(\omega(y)(y-z)) = \sum_{l=1}^{N} \partial_l(\omega(y)(y-z))\theta_l$$
$$= \sum_{i,l=1}^{N} \partial_l \omega_i(y)(y-z)_i \theta_l + \sum_{l=1}^{N} \omega_l(y)\theta_l,$$

which is what we claimed. (ii) then follows from (i) and using (2.2). Likewise, (iii) then follows from ii). □

Definition 2.3 For $\omega \in L^1_{loc}(\Omega, \Lambda^r)$ and $u \in L^1_{\text{loc}}(\Omega, \Lambda^{r+1})$ we say that $d\omega = u$ in the sense of distributions if for all $\varphi \in C_c^\infty(\Omega, \Lambda^{N-r-1})$ we have

$$\int_\Omega d\varphi \wedge \omega = (-1)^{N-r} \int_\Omega \varphi \wedge u.$$

Note that this definition is equivalent to the following formula: For all $\varphi \in C_c^\infty(\Omega, \Lambda^s)$ with $0 \le s \le N - r - 1$ and all $\theta \in \Lambda^{N-r-s-1}$ we have

$$(-1)^{r+1} \int_\Omega \omega \wedge d\varphi \wedge \theta = - \int_\Omega u \wedge \varphi \wedge \theta.$$

2.3 Stokes' theorem on simplices

We want to establish a suitable notion of Stokes' theorem for differential forms on simplices. Let $1 \le r \le N$ and $x_1, ..., x_{r+1} \in \mathbb{R}^N$. Define the simplex $\text{Sim}(x_1, ..., x_{r+1})$ as the convex hull of $x_1, ..., x_{r+1}$. We call this simplex degenerate, if its dimension is strictly less than r.

For $i \in \{1, ..., r+1\}$ consider $\text{Sim}(x_1, ...x_{i-1}, x_{i+1}, ..., x_{r+1}) =: \text{Sim}^i(x_1, ...x_{r+1})$. This is an $(r-1)$ dimensional face of $\text{Sim}(x_1, ..., x_{r+1})$ and a subset of the boundary of the manifold $\text{Sim}(x_1, ..., x_{r+1})$, which, for simplicity, will be denoted by $\partial\,\text{Sim}(x_1, ..., x_{r+1})$. Suppose first that we are given the simplex

$$\{\lambda \in [0,1]^r : \sum_{i=1}^{r} \lambda_i \le 1\} \times \{0\}^{N-r} = \text{Sim}(0, e_1, ..., e_r) \subset \mathbb{R}^r \times \{0\}^{N-r} \subset \mathbb{R}^N.$$

Then the classical version of Stokes' theorem on oriented manifolds reads that for every differential form $\tilde{\omega} \in C^1(\mathbb{R}^r \times \{0\}^{N-r}, \mathbb{R}^r \wedge ... \wedge \mathbb{R}^r)$ -$\mathbb{R}^r$ is the corresponding tangential space of the manifold $\text{Sim}(0, e_1, ..., e_r)$- we have

$$\int_{\text{Sim}(0,e_1,...,e_r)} d\tilde{\omega}(y)\, \mathrm{d}\mathcal{H}^r(y) = \int_{\partial^*\,\text{Sim}(0,e_1,...,e_r)} \tilde{\omega}(y) \wedge \nu(y)\, \mathrm{d}\mathcal{H}^{r-1}(y). \tag{2.9}$$

In (2.9), $\nu(y)$ denotes the outer normal unit vector at $y \in \partial^*\,\text{Sim}(0, e_1, ...e_r)$ and ∂^* is the reduced boundary of the simplex, where this outer normal exists (the interior of all $(r-1)$-dimensional faces). In our case, we are given a differential form with the underlying space being $\mathbb{R}^N$ and not $\mathbb{R}^r$ (the tangential space of the manifold/simplex), hence we can modify (2.9) to get for $\omega \in C^1(\mathbb{R}^N, \Lambda^{r-1})$

$$\int_{\mathrm{Sim}(0,e_1,\ldots,e_r)} d\omega(y)(e_1 \wedge \ldots \wedge e_r)\,\mathrm{d}\mathcal{H}^r(y)$$
$$= \sum_{i=1}^{r}(-1)^i \int_{\mathrm{Sim}(0,\ldots,e_{i-1},e_{i+1},\ldots,e_r)} \omega(y)(e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots \wedge e_r)$$
$$+ \int_{\mathrm{Sim}(e_1,\ldots,e_r)} 2^{-r/2}\omega(y)((e_2 - e_1) \wedge (e_3 - e_2) \wedge \ldots \wedge (e_r - e_{r-1})). \tag{2.10}$$

Let us write for simplicity that for $x_1, \ldots, x_{r+1} \in \mathbb{R}^N$

$$\nu^r(x_1, \ldots, x_{r+1}) = ((x_2 - x_1) \wedge (x_3 - x_2) \wedge \ldots \wedge (x_{r+1} - x_r)) \in \Lambda_r.$$

The map ν^r has the following properties:

(i) ν^r is alternating, i.e. for a permutation $\sigma \in S_r$:

$$\nu^r(y_1, \ldots, y_{r+1}) = \mathrm{sgn}(\sigma)\nu^r(y_{\sigma(1)}, \ldots, y_{\sigma(r+1)}).$$

(ii) We have the relation

$$\|\nu^r(y_1, \ldots, y_{r+1})\|_{\Lambda_r} = r\mathcal{H}^r(\mathrm{Sim}(y_1, \ldots, y_{r+1})).$$

A linear change of coordinates from $\mathrm{Sim}(0, e_1, .., e_r)$ to $\mathrm{Sim}(x_1, \ldots, x_{r+1})$ leads from (2.10) to the following: For $\omega \in C^\infty(\mathbb{R}^N, \Lambda^{r-1})$ and $x_1, \ldots x_{r+1} \in \mathbb{R}^N$

$$\frac{1}{r} \fint_{\mathrm{Sim}(x_1,\ldots,x_{r+1})} d\omega(y)(\nu^r(x_1, \ldots, x_{r+1}))\,\mathrm{d}\mathcal{H}^r(y) \tag{2.11}$$
$$= \sum_{i=1}^{r+1} \frac{(-1)^i}{r-1} \fint_{\mathrm{Sim}^i(x_1,\ldots x_{r+1})} \omega(y)(\nu^{r-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_{r+1}))\,\mathrm{d}\mathcal{H}^{r-1}(y),$$

2.4 The maximal function

The Hardy–Littlewood maximal function for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^d)$ is defined by

$$Mu(x) = \sup_{R>0} \fint_{B_R(x)} |u(y)|\,\mathrm{d}y.$$

Again, we can also define the maximal function for functions on the torus using the identification with periodic functions.

Proposition 2.4 (Properties of the maximal function) (cf. [31]) *M is sublinear, i.e. $M(u + v)(y) \leq Mu(y) + Mv(y)$ for all $u, v \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^d)$ and $y \in \mathbb{R}^N$. Moreover, $M : L^p(\mathbb{R}^N, \mathbb{R}^d) \to L^p(\mathbb{R}^N, \mathbb{R})$ is bounded for $1 < p \leq \infty$ and bounded from L^1 to $L^{1,\infty}$. In particular, this means that for $1 \leq p < \infty$*

$$|\{Mu > \lambda\}| \leq C_p \lambda^{-p} \|u\|^p_{L^p(\mathbb{R}^N,\mathbb{R}^d)}.$$

If $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^d)$ is a $\mathbb{Z}^N$-periodic function, i.e. $u \in L^p(T_N, \mathbb{R}^d)$, then

$$|\{Mu > \lambda\} \cap [0, 1]^N| \leq C_p \lambda^{-p} \|u\|^p_{L^p([0,1]^N,\mathbb{R}^d)}.$$

3 A geometric estimate for closed differential forms

In this section we prove a key lemma for our main theorem.

Lemma 3.1 *There exists a constant $C = C(N, r)$ such that for all $\omega \in C^1(\mathbb{R}^N, \Lambda^r)$, $\lambda > 0$ with $d\omega = 0$ and $x_1, ..., x_{r+1} \in \{M\omega \le \lambda\}$ we have*

$$\left| \fint_{\mathrm{Sim}(x_1,...,x_{r+1})} \omega(\nu^r(x_1, ..., x_{r+1})) \right| \le C\lambda \max_{1\le i,j\le r+1} |x_i - x_j|^r$$

This lemma can be seen as a natural analogue of Lipschitz continuity on the set where the maximal function is small. In particular, it has been proven (for example in [1]) that for $u \in W^{1,1}_{loc}(\mathbb{R}^N, \mathbb{R}^m)$ and for $y_1, y_2 \in \{M\nabla u(x) \le L\}$

$$\left| \int_0^1 \nabla u(ty_1 + (1-t)y_2) \cdot (y_1 - y_2)\, \mathrm{d}t \right| = |u(y_1) - u(y_2)| \le CL|y_1 - y_2|.$$

Hence, one should view Lemma 3.1 as a generalisation of this result.

Proof For simplicity write $|\omega| := \|\omega\|_{\Lambda^r}$. Recall that

$$\|\nu^r(x_1, ..., x_{r+1})\|_{\Lambda_r} = r\mathcal{H}^r(\mathrm{Sim}(x_1, ..., x_{r+1})) \le C \max_{1\le i,j\le r+1} |x_i - x_j|^r.$$

It suffices to show that there exists $z \in \mathbb{R}^N$ such that

$$\sum_{i=1}^{r+1} \int_{\mathrm{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} |\omega|\, \mathrm{d}\mathcal{H}^r(y) \le C\lambda \max_{1\le i,j\le r+1} |x_i - x_j|^r. \tag{3.1}$$

Indeed, to see that (3.1) is enough, note that

$$\sum_{i=1}^{r+1} \fint_{\mathrm{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} \omega(\nu^r(x_1, ...x_{i-1}, z, x_{i+1}, ...))\, \mathrm{d}\mathcal{H}^r(y) \tag{3.2}$$
$$= \fint_{\mathrm{Sim}(x_1,...,x_{r+1})} \omega(\nu^r(x_1, ..., x_{r+1}))\, \mathrm{d}\mathcal{H}^r(y).$$

and

$$\fint_{\mathrm{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} \omega(\nu^r(x_1, ...x_{i-1}, z, x_{i+1}, ...))\, \mathrm{d}\mathcal{H}^r(y)$$
$$\le \frac{1}{r} \int_{\mathrm{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} |\omega|\, \mathrm{d}\mathcal{H}^r(y).$$

The equation (3.2) can be verified by Stokes' theorem (2.11), using that boundary terms with a simplex with vertex z cancel out on the left-hand side of (3.2) (Fig. 1).

We now prove (3.1). W.l.o.g. $R = \max_{i,j\in[r+1]} |x_i - x_j| = |x_1 - x_2|$. Note that there exists a dimensional constant C_1 such that

$$|B_R(x_1) \cap B_R(x_2)| \ge C_1 R^N.$$

First, consider $x_1, ..., x_r \in B_R(x_1)$. For $z \in B_R(x_1)$ define $E(z)$ to be the r-dimensional hyperplane going through $x_1, ..., x_r$ and z. This is well-defined if z is not in the $(r-1)$ dimensional hyperplane F going through $x_1, ..., x_r$. Note that for $z, \tilde{z} \notin F$

$$z \in E(\tilde{z}) \Leftrightarrow \tilde{z} \in E(z). \tag{3.3}$$

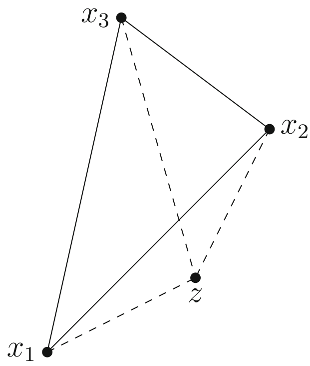

Fig. 1 Illustration of (3.2) for $r = 2$. The integrals on the dashed 1-dimensional faces cancel out in (3.2) after applying Stokes' theorem

As $M\omega(x_1) \leq \lambda$, we know that

$$\int_{B_R(x_1)} |\omega|(z)\,\mathrm{d}z \leq \lambda b_N R^N,$$

where b_N is the volume of the N-dimensional unit ball $B_1(0)$. As $\mathcal{H}^r(E(z) \cap B_R(x_1)) = b_r R^r$, it also follows by Fubini and (3.3)

$$\int_{B_R(x_1)} \int_{E(z)\cap B_R(x_1)} |\omega|(y)\,\mathrm{d}\mathcal{H}^r(y)\,\mathrm{d}z \leq \lambda b_N b_r R^{N+r}.$$

Using that $\mathrm{Sim}(x_1, ..., x_r, z) \subset E(z) \cap B_R(x_1)$, we conclude that for $\mu > 0$

$$\left|\left\{z \in B_R(x_1) \colon \left|\int_{\mathrm{Sim}(x_1,...,x_r,z)} |\omega|(y)\,\mathrm{d}y\right| \geq \mu\right\}\right| \leq \frac{\lambda b_r b_N R^{N+r}}{\mu}. \tag{3.4}$$

Choose now $\mu^* = 2(r+1)b_r b_N R^r \lambda C_1^{-1}$. Plugging this into (3.4), we see that the measure of this set is smaller than $R^N(2(r+1))^{-1}$. Repeating this procedure for all $(r-1)$-dimensional faces of $\mathrm{Sim}(x_1, ..., x_{r+1})$, we get that for $i > 1$

$$\left|\left\{z \in B_R(x_1) \colon \left|\int_{\mathrm{Sim}(x_1,...,x_{i-1},z,x_{i+1},...)} |\omega|(y)\,\mathrm{d}\mathcal{H}^r(y)\right| \geq \mu^*\right\}\right| \leq \frac{C_1 R^N}{2(r+1)},$$

and for $i = 1$

$$\left|\left\{z \in B_R(x_2) \colon \left|\int_{\mathrm{Sim}(z,x_2,...x_{r+1})} |\omega|(y)\,\mathrm{d}\mathcal{H}^r(y)\right| \geq \mu^*\right\}\right| \leq \frac{C_1 R^N}{2(r+1)}.$$

Hence, there exists $z \in B_R(x_1) \cap B_R(x_2)$ such that all the summands of (3.1) are smaller than $\mu^* = ((2(r+1))b_r b_N C_1^{-1})R^r\lambda$, i.e.

$$\sum_{i=1}^{r+1} \int_{\mathrm{Sim}(x_1,...x_{i-1},z,x_{i+1},...)} |\omega|\,\mathrm{d}\mathcal{H}^r(y) \leq \left(2(r+1)^2 b_r b_N C_1^{-1}\right)\lambda \max_{1\leq i,j\leq r+1} |x_i - x_j|^r.$$

This is what we wanted to prove. □

4 A Whitney-type extension theorem

First, let us recall the following Lipschitz extension theorem.

Theorem 4.1 (Lipschitz extension theorem) *Let $X \subset \mathbb{R}^N$ be a closed set and $u \in C(X, \mathbb{R}^d)$ such that*

$$|u(x) - u(y)| \leq L|x - y|. \tag{4.1}$$

Then there exists a function $v \in C(\mathbb{R}^N, \mathbb{R}^d)$ with $v_{|X} = u$ and such that v is Lipschitz on $\mathbb{R}^N$ with Lipschitz constant at most $C(N)L$ (i.e. the Lipschitz constant does not depend on X).

Of course, there are several ways to prove such a theorem, even with $C(N) = 1$ [15]. However, WHITNEY'S proof [36] plays with the geometry of $\mathbb{R}^N$ quite nicely. Similar geometric ideas lies behind our proof for closed differential forms. First, let us define an analogue of (4.1).

Suppose that X is a closed subset of $\mathbb{R}^N$, such that $X^C = \mathbb{R}^N \setminus X$ is bounded and $|\partial X| = 0$. Let $u \in C_c^{\infty}(\mathbb{R}^N, \Lambda^r)$ with $du = 0$. Let $L > 0$ be such that $\|u\|_{L^{\infty}(X)} \leq L$ and that for all $x_1, ..., x_{r+1} \in X$ we have

$$\left| \fint_{\mathrm{Sim}(x_1,...,x_{r+1})} u(y)(v^r(x_1, ..., x_{r+1}))\, \mathrm{d}y \right| \leq L \max |x_i - x_j|^r. \tag{4.2}$$

Lemma 4.2 *(Whitney-type extension theorem) There exists a constant $C = C(N, r)$ such that for all $u \in C_c^{\infty}(\mathbb{R}^N, \Lambda^r)$ and X meeting the requirements above there exists $v \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \Lambda^r)$ with*

(i) $dv = 0$ in the sense of distributions;
(ii) $v(y) = u(y)$ for all $y \in X$;
(iii) $\|v\|_{L^{\infty}} \leq CL$.

Remark 4.3 The constant C does not depend on the choice of u or X, it is only important that the pair (u, X) satisfies (4.2). The assumption that X^C is bounded makes the proof easier, but may be dropped. It is not clear, whether the assumption that $|\partial X| = 0$ is necessary for the statement to hold or not.

Remark 4.4 As one can see in the proof, the assumption $u \in C_c^{\infty}(\mathbb{R}^N, \Lambda^r)$ can be weakened to $u \in C_c^2(\mathbb{R}^N, \Lambda^r)$, as we only need the first two derivatives of u. However, it is important to remember that we cannot prove Lemma 4.2 for the even weaker assumption $u \in L^1_{\mathrm{loc}}$, as (4.2) is not well-defined.

For the proof we follow the classical approach by Whitney with a few little twists. First, we will define the extension in (4.4). Then we prove that v satisfies properties (i)–(iii). (ii) and (iii) are quite easy to see from the definition of v, however it is hard to verify that (i) holds. On the one hand, we show that the strong derivative of v exists almost everywhere, namely in $\mathbb{R}^N \setminus \partial X$ and that $dv = 0$ almost everywhere, where we use the assumption that the boundary of X is a null-set. On the other hand, we then prove that the distributional derivative dv is in fact also an L^1 function, yielding that $dv = 0$ in the sense of distributions.

We now start with the definition of the extension. Let us recall (cf. [31]) that for $X \subset \mathbb{R}^N$ closed we can find a collection of pairwise disjoint open cubes $\{Q_i^*\}_{i \in \mathbb{N}}$ such that

- Q_i^* are open dyadic cubes;
- $\cup_{i \in \mathbb{N}} \bar{Q}_i^* = X^C$;
- $\mathrm{dist}(Q_i^*, X) \leq l(Q_i^*) \leq 4\, \mathrm{dist}(Q_i^*, X)$,; where $l(Q_i^*)$ denotes the sidelength of the cube.

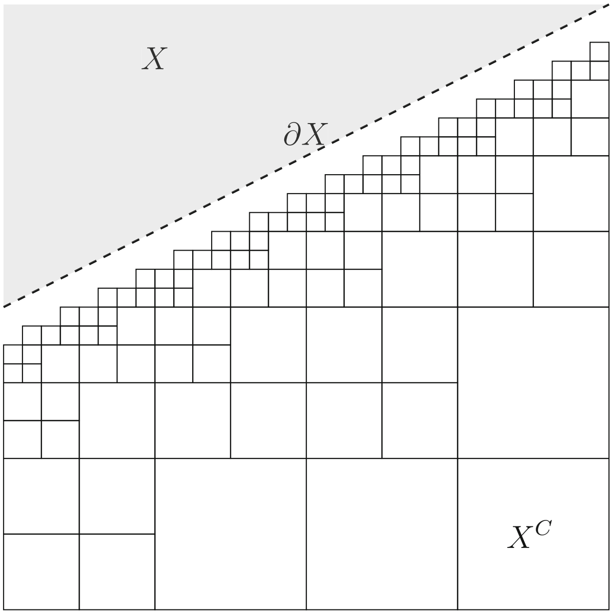

Fig. 2 A collection of cubes Q_j^* near the boundary (up to a certain size)

Choose $0 < \varepsilon < 1/4$ and define another collection of cubes by $Q_i = (1+\varepsilon)Q_i^*$ (cube with the same center and sidelength $(1+\varepsilon)l(Q_i^*)$). Then

- $\cup_{i\in\mathbb{N}} Q_i = X^C$;
- For all $i \in \mathbb{N}$, the number of cubes Q_j such that $Q_i \cap Q_j \neq \emptyset$ is bounded by a dimensional constant $C(N)$;
- In particular, all $x \in \mathbb{R}^N$ are only contained in at most $C(N)$ cubes Q_i;
- The distance to the boundary is again comparable to the sidelength, i.e.

$$1/2\,\mathrm{dist}(Q_i, X) \leq l(Q_i) \leq 8\,\mathrm{dist}(Q_i, X).$$

Note that if X is $\mathbb{Z}^N$-periodic, then also Q_i can be chosen to be $\mathbb{Z}^N$ periodic (initially, we have a collection of dyadic cubes) (Fig. 2).

Now consider $\varphi \in C_c^\infty((-1-\varepsilon, 1+\varepsilon)^N, [0,\infty))$ with $\varphi = 1$ on $(-1,1)^N$. We can rescale φ such that we obtain functions $\varphi_j^* \in C_c^\infty(Q_j)$ with $\varphi_j^* = 1$ on Q_j^*. Define the partition of unity on X^C by

$$\varphi_j = \frac{\varphi_j^*}{\sum_{i\in\mathbb{N}} \varphi_i^*}.$$

Note that $0 \leq \varphi_j \leq 1$ and that there exists a constant $C > 0$ such that for all $j \in \mathbb{N}$

$$|\nabla\varphi_j| \leq C/8\, l(Q_j)^{-1} \leq C\,\mathrm{dist}(Q_j, X)^{-1}.$$

For each cube Q_i, we may find an $x \in X$ such that $\mathrm{dist}(Q_i, x) = \mathrm{dist}(Q_i, X)$. Denote this x by x_i. For a multiindex $I = (i_1, ..., i_{r+1}) \in \mathbb{N}^{r+1}$, define

$$G(x_{i_1}, ..., x_{i_{r+1}}) = G(I) := \fint_{\mathrm{Sim}(x_{i_1},...,x_{i_{r+1}})} u(y)\,\mathrm{d}y.$$

We now define the differential form $\alpha \in L^1(\mathbb{R}^N, \Lambda^r)$ by

$$\alpha(y) := \sum_{I \in \mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))). \tag{4.3}$$

Note that in this setting $G(I)(\nu^r(...)) \in \mathbb{R} = \Lambda^0$.

We claim that the function $v \in L^1_{\text{loc}}(\mathbb{R}^N, \Lambda^r)$ given by

$$v(y) := \begin{cases} u(y) & y \in X, \\ (-1)^r \alpha(y) & y \in X^C \end{cases} \tag{4.4}$$

is the function satisfying all the properties of Lemma 4.2.

Lemma 4.5 *The differential form α defined in (4.3) satisfies $\alpha \in L^1(X^C, \Lambda^r)$ and the sum in (4.3) converges pointwise and in L^1.*

Proof Pointwise convergence is clear, as for fixed $y \in X^C$ only finitely many summands are nonzero in a neighbourhood of y (φ_i is only nonzero in Q_i and any point is only covered by at most $C(N)$ cubes). For L^1 convergence fix some $i_1 \in \mathbb{N}$. Note that there are at most $C(N)^r$ summands in $i_2, ..., i_{r+1}$, which are nonzero, as Q_{i_1} only intersects with $C(N)$ other cubes. Furthermore, note that for all i_l with $Q_{i_l} \cap Q_{i_1} \neq \emptyset$

$$\|d\varphi_{i_l}(y)\|_{\Lambda^1} \leq C \operatorname{dist}(y, X)^{-1} \leq Cl(Q_{i_1})^{-1}.$$

Moreover, we can bound ν^r by

$$\|\nu^r(x_{i_1}, ..., x_{i_{r+1}})\|_{\Lambda_r} \leq \max_{a,b \in \{i_1,...,i_{r+1}\}} |x_a - x_b|^r \leq Cl(Q_{i_1})^r.$$

Hence, we can bound the L^∞-norm of a nonzero summand of (4.3) by $C\|u\|_{L^\infty}$, as $|G(I)| \leq \|u\|_{L^\infty}$. As the support of the summand is contained in Q_{i_1}, we have that its L^1 norm is bounded by

$$C\|u\|_{L^\infty}|Q_{i_1}|.$$

Remember that any point in X^C is covered by only $C(N)$ cubes, such that the sum of $|Q_i|$ is bounded by $C(N)|X^C|$. Hence, the sum in (4.3) converges absolutely in L^1 and its L^1 norm is bounded by $C(N)^{r+1}C\|u\|_{L^\infty}|X^C|$. □

Lemma 4.6 *The function v is strongly differentiable almost everywhere and satisfies $dv(y) = 0$ for all $y \in \mathbb{R}^N \setminus \partial X$.*

Proof Note that $u \in C_c^\infty(\mathbb{R}^N, \Lambda^r)$ and hence v is strongly differentiable in $X \setminus \partial X$. Furthermore, the sum in (4.3) is a finite sum in a neighbourhood of y for all $y \in X^C$. As the summands are also C^∞, the sum is C^∞ in the interior of X^C.

By assumption $du = 0$, hence it remains to prove that $d\alpha(y) = 0$ for all $y \in X^C$. Note that in a neighbourhood of $y \in X^C$ again only finitely many summands are nonzero. Using that $d^2 = 0$ and the Leibniz rule, we get

$$d\alpha(y) = \sum_{I \in \mathbb{N}^{r+1}} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y)(G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))). \tag{4.5}$$

Observe that this term does not converge in L^1 and hence this identity is only valid pointwise.

Pick some $j \in \mathbb{N}$ such that $y \in Q_j$. As all φ_i sum up to 1 in X^C, we have

$$d\varphi_j(y) = -\sum_{I \in \mathbb{N}\setminus\{j\}} d\varphi_i(y).$$

Replace $d\varphi_j$ in the sum in (4.5) by $-\sum_{I \in \mathbb{N}\setminus\{j\}} d\varphi_i(y)$. Recall that $\nu^r(x_1, ..., x_{r+1}) = 0$ if $x_l = x_{l'}$ for some $l \neq l'$. Hence,

$$\begin{aligned}
d\alpha(y) &= \sum_{I \in \mathbb{N}^{r+1}} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y) \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \\
&= \sum_{I \in (\mathbb{N}\setminus\{j\})^{r+1}} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y) \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \\
&\quad + \sum_{l=1}^{r+1} \sum_{I \in \mathbb{N}^{r+1}:\, i_l = j} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y) \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \\
&= \sum_{I \in (\mathbb{N}\setminus\{j\})^{r+1}} d\varphi_{i_1}(y) \wedge ... \wedge d\varphi_{i_{r+1}}(y) \wedge (G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \\
&\quad - \sum_{l=1}^{r+1} \sum_{I \in (\mathbb{N}\setminus\{j\})^{r+1}} d\varphi_{i_1(y)} \wedge ... \wedge d\varphi_{i_{r+1}}(y) \\
&\qquad\qquad \wedge (G(x_{i_1}, ...x_{i_{l-1}}, x_j, x_{i_{l+1}}, ...)(\nu^r(x_{i_1}, ...x_{i_{l-1}}, x_j, x_{i_{l+1}}, ...))).
\end{aligned}$$

We apply Stokes' theorem (2.11) to the r-form u and the simplex with vertices $x_j, x_{i_1}, ..., x_{i_{r+1}}$, use that $du = 0$ and conclude that this term is 0, i.e.

$$\begin{aligned}
&G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}})) - \sum_{l=1}^{r+1} G(x_{i_1}, ..., x_{i_{l-1}}, x_j, x_{i_{l+1}}, ...)(\nu^r(x_{i_1}, ..., x_j, x_{i_{l+1}}, ...)) \\
&\quad = -\frac{r-1}{r} \fint_{\mathrm{Sim}(x_j, x_{i_1}, ..., x_{i_{r+1}})} du(y)(\nu^{r+1}(x_j, x_{i_1}, ..., x_{i_{r+1}}))\, \mathrm{d}\mathcal{H}^r(y) = 0.
\end{aligned}$$

Hence, the pointwise derivative equals 0 almost everywhere. □

It is important to note that the sum (4.3) in the definition of α converges in L^1, but in general does not converge in $W^{1,1}$, and thus we have no information on the behaviour at the boundary of X^C. However, it suffices to show that the distribution dv for v given by (4.4) is actually an L^1 function. If $dv \in L^1$, we can conclude with Lemma 4.6 that $dv = 0$ in the sense of distributions.

Lemma 4.7 *The distributional exterior derivative of v defined in (4.4) satisfies $dv \in L^1(\mathbb{R}^N, \Lambda^{r+1})$, i.e. there exists an L^1 function $h \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ such that for all $\psi \in C_c^\infty(\mathbb{R}^N, \Lambda^{N-r-1})$*

$$(-1)^r \int_{X^C} \alpha \wedge d\psi + \int_X u \wedge d\psi = \int_{\mathbb{R}^N} h \wedge \psi.$$

Proof Consider

$$\int_{X^C} \alpha(y) \wedge d\psi(y)\, \mathrm{d}y.$$

In view of the definition of α, this expression is given by:

$$\int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}}(G(I)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \wedge d\psi \ \mathrm{dy} = (*).$$

We use the splitting $G(I) = (G(I) - u(\cdot)) + u(\cdot)$ and write $(*)$ as

$$\begin{aligned}(*) &= \int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge ((G(I) - u(\cdot))(\nu^r(x_{i_1}, ..., x_{i_{r+1}})) \wedge d\psi \\ &+ \int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))) \wedge d\psi \\ &= \text{(I)} + \text{(II)}.\end{aligned} \tag{4.6}$$

Note that (I) defines a distribution given by an L^1 function. Indeed, the sum

$$\varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge ((G(I) - u(y))(\nu^r(x_{i_1}, ..., x_{i_{r+1}}))$$

converges in $W^{1,1}(\mathbb{R}^N, \Lambda^{r+1})$. To see this, one can repeat the proof of Lemma 4.5 and use that there are additional factors in the estimate of the norms. For this, note that if $z \in Q_{i_1}$

$$\|G(I) - u(z)\|_{\Lambda^r} \le Cl(Q_i)\|\nabla u\|_{L^\infty}$$

and

$$\|\nabla(G(I) - u(\cdot))(z)\|_{\Lambda^r} \le C\|\nabla u\|_{L^\infty}.$$

One gets improved regularity and may integrate by parts to eliminate the derivative of ψ.

Term (II) is not so easy to handle. We prove the following claims:

Claim 1 *Let* $1 \le s \le r$ *and* $I' = (i_s, ..., i_{r+1}) \in \mathbb{N}^{r-s+2}$. *There exists* $h_s \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ *such that*

$$\begin{aligned}&\int_{X^C} \sum_{I' \in \mathbb{N}^{r-s+2}} \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(x_{i_s}, ..., x_{i_{r+1}}))) \wedge d\psi \\ &\quad = \int_{X^C} h_s \wedge \psi \\ &\quad - \int_{X^C} \sum_{I' \in \mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s}(x_{i_{s+1}}, ..., x_{i_{r+1}})) \wedge d\psi.\end{aligned} \tag{4.7}$$

Here we use the notation that $\nu^0(x_{i_{r+1}}) = 1 \in \Lambda_0 = \mathbb{R}$.

Claim 2 *There is* $\tilde{h} \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ *such that*

$$\begin{aligned}&\int_{\mathbb{R}^N} \sum_{I' \in \mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^r(x_{i_1}, ..., x_{i_{r+1})}))) \wedge d\psi \\ &\quad = \int_{X^C} \tilde{h} \wedge \psi + (-1)^r \int_{X^C} u \wedge d\psi.\end{aligned} \tag{4.8}$$

Note that Claim 2 follows from Claim 1 by an inductive argument. The domain of integration in (4.8) can be replaced by X^C as well, as all φ_{i_j} are supported in X^C.

First, let us conclude the proof under the assumption that Claim 1 holds true. Using (4.6) and Claim 2 we see that there is an $h \in L^1(\mathbb{R}^N, \mathbb{R}^d)$ such that

$$\int_{X^C} \alpha \wedge d\psi = \int_{\mathbb{R}^N} h \wedge \psi + (-1)^r \int_{X^C} u \wedge d\psi.$$

Recall that $du = 0$ in the sense of distributions and therefore

$$-\int_{X^C} u \wedge d\psi = \int_X u \wedge d\psi.$$

We conclude that there exists an L^1 function $h \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ such that

$$\int_{X^C} \alpha \wedge d\psi + (-1)^r \int_X u \wedge d\psi = \int_{\mathbb{R}^N} h \wedge \psi.$$

Thus, dv is an L^1 function.

It remains to prove Claim 1. Note that

$$\nu^{r-s+1}(x_{i_s}, ..., x_{i_{r+1}}) = \sum_{j=s}^{r+1} \nu^{r-s+1}(x_{i_s}, ..., x_{i_{j-1}}, y, x_{i_{j+1}}, ..., x_{i_{r+1}}). \tag{4.9}$$

This can be verified using that the wedge product is alternating and explicitly writing the right-hand side of (4.9).

Using this identity, we may split the right-hand side of (4.7) (denoted by (III)), i.e.

$$\begin{aligned}
\text{(III)} = \sum_{j=s+1}^{r+1} \int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r-s+2}} & \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \\
& \wedge (u(\cdot)(\nu^{r-s+1}(x_{i_s}, ..., x_{i_{j-1}}, y, x_{i_{j+1}}, ..., x_{i_{r+1}}))) \wedge d\psi \\
+ \int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r-s+2}} & \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(y, x_{i_{s+1}}, ..., x_{i_{r+1}}))) \wedge d\psi \\
= \text{(IIIa)} + \text{(IIIb)}.&
\end{aligned}$$

Arguing as in Lemma 4.5, we see that the sum

$$\sum_{I \in \mathbb{N}^{r-s+2}} \varphi_{i_s} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(x_{i_s}, ..., x_{i_{j-1}}, y, x_{i_{j+1}}, ..., x_{i_{r+1}})))$$

is in fact convergent in L^1. Moreover, the index i_j only appears once in this sum. Recall that for $y \in X^C$

$$\sum_{i_s \in \mathbb{N}} d\varphi_{i_s}(y) = 0.$$

Thus,

$$\text{(IIIa)} = 0.$$

For (IIIb) note that $\sum_{i_1 \in \mathbb{N}} \varphi_{i_s} = 1_{X^C}$ and, by the same argument as for (IIIa), we can write

$$\text{(IIIb)} = \int_{X^C} \sum_{I \in \mathbb{N}^{r-s+1}} d\varphi_{i_{s+1}} \wedge ... \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(y, x_{i_{s+1}}, ..., x_{i_{r+1}}))) \wedge d\psi.$$

We can now integrate by parts to eliminate the exterior derivative in front of $\varphi_{i_{s+1}}$. Applying Lemma 2.2, using $d^2 = 0$, the Leibniz rule and the fact that $\varphi_{i_j} \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$

$$
\begin{aligned}
&(-1)^{r-s+1}(\text{IIIb}) \\
&= \int_{X^C} \sum_{I \in \mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge \ldots \wedge d\varphi_{i_{r+1}} \wedge d(u(\cdot)(\nu^{r-s+1}(y, x_{i_{s+1}}, \ldots, x_{i_{r+1}}))) \wedge d\psi \\
&= \int_{X^C} \sum_{I \in \mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge \ldots \wedge d\varphi_{i_{r+1}} \\
&\qquad\qquad \wedge D^{r-s+1,r}(\nabla u(\cdot), (\nu^{r-s+1}(y, x_{i_{s+1}}, \ldots, x_{i_{r+1}}))) \wedge d\psi \\
&+ \quad (-1)^{(r-s)} \int_{X^C} \sum_{I \in \mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge \ldots \wedge d\varphi_{i_{r+1}} \\
&\qquad\qquad \wedge u(\cdot)(\nu^{r-s}(x_{i_{s+1}}, \ldots, x_{i_{r+1}}))) \wedge d\psi
\end{aligned}
$$

Arguing similarly to Lemma 4.5 and as for term (I), we can show that

$$
\sum_{I \in \mathbb{N}^r} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge \ldots \wedge \varphi_{i_{r+1}} \wedge D^{r-s+1,r}(\nabla u(\cdot), (\nu^{r-s+1}(y, x_{i_{s+1}}, \ldots, x_{i_{r+1}}))) \in W^{1,1}(\mathbb{R}^N, \Lambda^r),
$$

and that this sum is convergent in $W^{1,1}$. Hence, we have shown that there exists $h_s \in L^1(\mathbb{R}^N, \Lambda^{r+1})$ such that

$$
\begin{aligned}
(\text{III}) &= \int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r-s+2}} \varphi_{i_s} d\varphi_{i_{s+1}} \wedge \ldots \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s+1}(x_{i_1}, \ldots, x_{i_{r+1}}))) \wedge d\psi \\
&= \int_{\mathbb{R}^N} h_s \wedge \psi \\
&\quad - \int_{\mathbb{R}^N} \sum_{I \in \mathbb{N}^{r-s+1}} \varphi_{i_{s+1}} d\varphi_{i_{s+2}} \wedge \ldots \wedge d\varphi_{i_{r+1}} \wedge (u(\cdot)(\nu^{r-s}(x_{i_{s+1}}, \ldots, x_{i_{r+1}}))) \wedge d\psi
\end{aligned}
\tag{4.10}
$$

Hence, Claim 1 holds, completing the proof of Lemma 4.7 □

This proves Lemma 4.2. The property that

$$
dv = 0 \quad \text{in the sense of distributions}
$$

follows from Lemma 4.6 and Lemma 4.7. By definition, $v = u$ on X. Finally, we can bound the L^∞-norm of v by CL, as in the definition of α

$$
\sum_{I \in \mathbb{N}^{r+1}} \varphi_{i_1} d\varphi_{i_2} \wedge \ldots \wedge d\varphi_{i_{r+1}} \wedge (G(I)(\nu^r(x_{i_1}, \ldots, x_{i_{r+1}})))
$$

every summand can be bounded by CL due to (4.2) and the estimate $|d\varphi_j| \le C\,\mathrm{dist}(Q_j, X)^{-1}$. Again, we get the L^∞ bound, as only finitely many summands are nonzero for every $y \in X^C$.

With slight modifications one is able to prove the following variants.

Corollary 4.8 *Let $u \in C^\infty(\mathbb{R}^N, \Lambda^r)$ with $du = 0$, let $L > 0$, and let $X \subset \mathbb{R}^N$ be a nonempty closed set such that $\|u\|_{L^\infty(X)} \le L$ and for all $x_1, \ldots, x_{r+1} \in X$ we have*

$$
\left| \fint_{\mathrm{Sim}(x_1,\ldots,x_{r+1})} u(y)(\nu^r(x_1, \ldots, x_{r+1}))\, \mathrm{d}y \right| \le L \max |x_i - x_j|^r.
$$

Suppose further that $|\partial X| = 0$.

There exists a constant $C = C(N, r)$ such that for all $u \in C^\infty(\mathbb{R}^N, \Lambda^r)$ and X meeting these requirements there exists $v \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \Lambda^r)$ with

(i) $dv = 0$ *in the sense of distributions;*
(ii) $v(y) = u(y)$ *for all* $y \in X$*;*
(iii) $\|v\|_{L^\infty} \le CL$.

This statement is proven in the same way as Lemma 4.2, but all the statements are only true locally (e.g. the L^1 bounds on α are replaced by bounds in $L^1_{\mathrm{loc}}(X^C, \Lambda^r)$).

If we choose u and X to be $\mathbb{Z}^N$ periodic we get a suitable statement for the torus.

Corollary 4.9 *Let* $u \in C^\infty(T_N, \Lambda^r)$ *with* $du = 0$*, let* $L > 0$*, and let* $X \subset \mathbb{R}^N$ *be a nonempty, closed,* $\mathbb{Z}^N$*-periodic set (which can be viewed as a subset of* T_N*) such that* $\|u\|_{L^\infty(X)} \le L$ *and for all* $x_1, \dots, x_{r+1} \in X$ *we have*

$$\left| \fint_{\mathrm{Sim}(x_1,\dots,x_{r+1})} \tilde{u}(y)(v^r(x_1, \dots, x_{r+1}))\,\mathrm{d}y \right| \le L \max |x_i - x_j|^r,$$

where $\tilde{u} \in C^\infty(\mathbb{R}^N, \Lambda^r)$ *is the* $\mathbb{Z}^N$*-periodic representative of* u*. Suppose further that* $|\partial X| = 0$.

There exists a constant $C = C(N, r)$ *such that for all* $u \in C^\infty(T_N, \Lambda^r)$ *and* X *meeting these requirements there exists* $v \in L^1(T_N, \Lambda^r)$ *with*

(i) $dv = 0$ *in the sense of distributions;*
(ii) $v(y) = u(y)$ *for all* $y \in X \subset T_N$*;*
(iii) $\|v\|_{L^\infty} \le CL$.

As mentioned before, we can choose the cubes Q_j to be rescaled dyadic cubes. As the set X is periodic, the set of cubes (and hence also the partition of unity) and their projection points may also be chosen to be $\mathbb{Z}^N$-periodic. By definition then also the extension will be $\mathbb{Z}^N$-periodic.

5 L^∞-truncation

Now we prove the main result of this paper on the L^∞-truncation of closed forms.

Theorem 5.1 (L^∞-truncation of differential forms) *There exist constats* $C_1, C_2 > 0$ *such that for all* $u \in L^1(T_N, \Lambda^r)$ *with* $du = 0$ *and all* $L > 0$ *there exists* $v \in L^\infty(T_N, \Lambda^r)$ *with* $dv = 0$ *and*

(i) $\|v\|_{L^\infty(T_N,\Lambda^r)} \le C_1 L$*;*
(ii) $|\{y \in T_N : v(y) \ne u(y)\}| \le \dfrac{C_2}{L} \displaystyle\int_{\{y \in T_N :\, |u(y)| > L\}} |u(y)|\,\mathrm{d}y$*;*
(iii) $\|v - u\|_{L^1(T_N,\Lambda^r)} \le C_2 \displaystyle\int_{\{y \in T_N :\, |u(y)| > L\}} |u(y)|\,\mathrm{d}y$.

Given the Whitney-type extension obtained in Lemma 4.9 and Lemma 4.2 combined with Lemma 3.1, the proof now roughly follows Zhang's proof for Lipschitz truncation in [38]. First, we prove the statement in the case that v is smooth directly using our extension theorem for the set $X = \{Mu \le L\}$. After calculations similar to [38] we are able to show that this extension satisfies the properties of Theorem 5.1. Afterwards, we prove the statement for $u \in L^1(T_N, \Lambda^r)$ by a standard density argument.

Proof First, suppose that $u \in C^\infty(T_N, \Lambda^r)$. For $\lambda > 0$ define the set

$$X_\lambda = \{y \in T_N : Mu(y) \leq \lambda\}.$$

Choose $2L \leq \lambda \leq 3L$ such that $|\partial X_\lambda| = 0$. Then, by Lemma 3.1 and the extension Lemma 4.9, there exists a $v \in L^1(T_N, \Lambda^r)$ with

1 $\{y \in T_N : v(y) \neq u(y)\} \subset X_\lambda^C$.
2 $\|v\|_{L^\infty} \leq C\lambda$.
3 $dv = 0$ in the sense of distributions.

We need to show that

$$\|v - u\|_{L^1(T_N,\Lambda^r)} \leq C_2 \int_{\{y:\, |u(y)|>L\}} |u(y)|\, \mathrm{d}y \tag{5.1}$$

and that

$$|\{y \in T_N : v(y) \neq u(y)\}| \leq \frac{C_2}{L} \int_{\{y:\, |u(y)|>L\}} |u(y)|\, \mathrm{d}y. \tag{5.2}$$

Indeed, (5.1) follows from (5.2), as $\{v \neq u\} \subset X_\lambda^C$ and thus

$$\begin{aligned}
\int_{T_N} |v(y) - u(y)|\, \mathrm{d}y &= \int_{X_\lambda^C} |v(y) - u(y)|\, \mathrm{d}y \\
&\leq \int_{\{Mu \geq \lambda\}} |u(y)|\, \mathrm{d}y + \int_{\{Mu \geq \lambda\}} |v(y)|\, \mathrm{d}y \\
&\leq \int_{\{|u| \geq \lambda\}} |u(y)|\, \mathrm{d}y + 2CL|\{Mu \geq \lambda\}|.
\end{aligned}$$

Thus, it suffices to prove (5.2).

To this end, define the function $h\colon \Lambda^r \to \mathbb{R}$ by

$$h(z) = \begin{cases} 0 & \text{if } |z| < L, \\ |z| - L & \text{if } |z| \geq L. \end{cases}$$

Let $y \in \{Mu > \mu\}$ for $\mu \in \mathbb{R}$. Then there exists an $R > 0$ such that

$$⨍_{B_R(y)} |u(z)|\, \mathrm{d}z > \mu.$$

Thus,

$$\begin{aligned}
M(h(u))(y) &\geq ⨍_{B_R(y)} |h(u)(z)|\, \mathrm{d}z \\
&= \frac{1}{|B_R(y)|} \int_{B_R(y)\cap\{u \geq L\}} |u(z)| - L\, \mathrm{d}z \\
&\geq ⨍_{B_R(y)} |u(z)|\, \mathrm{d}z - \frac{1}{|B_R(y)|} \int_{B_R(y)\cap\{u \leq L\}} |u(z)|\, \mathrm{d}z \\
&\quad - \frac{1}{|B_R(y)|} \int_{B_R(y)\cap\{|u| \geq L\}} L\, \mathrm{d}z \\
&\geq \mu - L.
\end{aligned}$$

Thus, $\{y \in T_N : Mu > \mu\} \subset \{y \in T_N : Mh(u)(y) > \mu - L\}$.

Using the weak-L^1 estimate for the maximal function (Proposition 2.4), we get

$$\begin{aligned} |\{y \in T_N : Mu(y) \geq \lambda\}| &\leq |\{y \in T_N : Mh(u) \geq \lambda - L\}| \\ &\leq \frac{1}{\lambda - L} C \int_{T_N} |h(u)(z)| \,\mathrm{d}z \\ &\leq \frac{C}{L} \int_{T_N \cap \{|u| \geq L\}} |u(z)| \,\mathrm{d}z. \end{aligned} \tag{5.3}$$

This is what we wanted to show. Note that the proof only uses $u \in C^\infty(T_N, \Lambda^r)$ to define v and nowhere else, hence estimate (5.3) is valid for all $u \in L^1(T_N, \Lambda^r)$.

For general $u \in L^1(T_N, \Lambda^r)$, one may consider a sequence $u_n \in C^\infty(T_N, \Lambda^r)$ with $du_n = 0$ and $u_n \to u$ in L^1 and pointwise almost everywhere. This sequence can be easily constructed by convolving with standard mollifiers.

Observe that for $\lambda > 0$

$$\begin{aligned} \int_{\{|u_n| \geq 2\lambda\}} |u_n| \,\mathrm{d}y &\leq \int_{\{|u_n - u| \geq |u|\} \cap \{|u_n| \geq 2\lambda\}} |u_n| \,\mathrm{d}y + \int_{\{|u_n - u| \leq |u|\} \cap \{|u_n| \geq 2\lambda\}} |u_n| \,\mathrm{d}y \\ &\leq 2 \int_{\{|u| \geq \lambda\}} |u| \,\mathrm{d}y + 2\|u_n - u\|_{L^1}. \end{aligned} \tag{5.4}$$

Furthermore, we use the subadditivity of the maximal function and see that for all $y \in T_N$

$$Mu_n(y) \leq Mu(y) + M(u - u_n)(y).$$

Thus,

$$\{y \in T_N : Mu_n(y) \geq 2\lambda\} \subset \{y \in T_N : Mu(y) \geq \lambda\} \cup \{y \in T_N : M(u - u_n)(y) \geq \lambda\}.$$

Using the weak-L^1estimate for the maximal function (Proposition 2.4) we see that

$$|\{y \in T_N : Mu(y) \leq \lambda\} \cap \{y \in T_N : Mu_n(y) \geq 2\lambda\}| \longrightarrow 0 \quad \text{as } n \to \infty. \tag{5.5}$$

Choose some $\lambda \in (4L, 6L)$ such that for all $n \in \mathbb{N}$ $|\partial\{y \in T_N : Mu_n(y) \geq 2\lambda\}| = 0$. Then extend like in the first part of the proof to get a sequence v_n with $dv_n = 0$ and

(a) $\|v_n\|_{L^\infty(T_N, \Lambda^r)} \leq 2C_1\lambda$;
(b) $|\{y \in T_N : v_n(y) \neq u_n(y)|\} \leq \dfrac{C_2}{2\lambda} \displaystyle\int_{\{y :\, |u_n(y)| > 2\lambda\}} |u_n(y)| \,\mathrm{d}y$;
(c) $\|v_n - u_n\|_{L^1(T_N, \Lambda^r)} \leq C_2 \displaystyle\int_{\{y :\, |u_n(y)| > 2\lambda\}} |u_n(y)| \,\mathrm{d}y$.

Letting $n \to \infty$, by a) this sequence converges, up to extraction of a subsequence, weakly$*$ to some $v \in L^\infty(T_N, \Lambda^r)$. The weak$*$-convergence implies $dv = 0$. Moreover, by construction, the set $\{y \in T_N : v_n \neq u_n\}$ is contained in the set $\{y \in T_N : Mu_n(y) \geq 2\lambda\}$. As $u_n \to u$ pointwise a.e. and in L^1, we get using (5.5) that $v = u$ on the set $\{y \in T_N : Mu(y) \leq \lambda\}$. (If v_n converges to u in measure on a set A and v_n weakly to some v, then $v = u$ on A.)

Hence, v defined as the weak$*$ limit of v_n satisifies

(i) $\|v\|_{L^\infty(T_N, \Lambda^r)} \leq C_1\lambda \leq 6C_1L$;
(ii) using (5.3) and $v = u$ on $\{y \in T_N : Mu(y) \leq \lambda\}$

$$|\{y \in T_N : u(y) \neq v(y)\}| \leq \frac{C_2}{L} \int_{\{y \in T_N :\, |u(y)| > L\}} |u(y)| \,\mathrm{d}y;$$

(iii) using triangle inequality and $v_n - u_n \to 0$ in L^1, one obtains

$$\|v-u\|_{L^1(T_N,\Lambda^r)} \le C_2 \int_{\{y\in T_N:\, |u(y)|>L\}} |u(y)|\,\mathrm{d}y.$$

Hence, v meets the requirements of Theorem 5.1. □

Corollary 5.2 (L^∞-truncation for sequences) *Suppose that we have a sequence $u_n \in L^1(\mathbb{R}^N, \Lambda^r)$ with $du_n = 0$, and that there exists $L > 0$ such that*

$$\int_{\{y\in T_N:\, |u_n(y)|>L\}} |u_n(y)|\,\mathrm{d}y \longrightarrow 0 \quad \text{as } n\to\infty.$$

There exists a $C_1 = C_1(N, r)$ and a sequence $v_n \in L^1(T_N, \Lambda^r)$ with $dv_n = 0$ and

(a) $\|v_n\|_{L^\infty(T_N,\Lambda^r)} \le C_1 L$;
(b) $\|v_n - u_n\|_{L^1(T_N,\Lambda^r)} \to 0$ *as* $n\to\infty$;
(c) $|\{y\in T_N:\, v_n(y)\neq u_n(y)\}| \to 0$.

This directly follows by applying Theorem 5.1.

The proof of Theorem 5.1 also works if L^1 is replaced by L^p for $1 < p < \infty$. Furthermore, we do not need to restrict us to periodic functions on $\mathbb{R}^N$, the statement is also valid for non-periodic functions.

Proposition 5.3 *Let $1 \le p < \infty$. There exist constants $C_1, C_2 > 0$, such that, for all $u \in L^p(\mathbb{R}^N, \Lambda^r)$ with $du = 0$ and all $L > 0$, there exists $v \in L^p(\mathbb{R}^N, \Lambda^r)$ with $dv = 0$ and*

(i) $\|v\|_{L^\infty(\mathbb{R}^N,\Lambda^r)} \le C_1 L$;
(ii) $|\{y\in\mathbb{R}^N:\, v(y)\neq u(y)\}| \le \dfrac{C_2}{L^p}\displaystyle\int_{\{y\in\mathbb{R}^N:\, |u(y)|>L\}} |u(y)|^p\,\mathrm{d}y$;
(iii) $\|v-u\|^p_{L^p(\mathbb{R}^N,\Lambda^r)} \le C_2 \displaystyle\int_{\{y\in\mathbb{R}^N:\, |u(y)|>L\}} |u(y)|^p\,\mathrm{d}y$.

As described, the proof is pretty much the same as for Theorem 5.1. We may also want to truncate closed forms supported on an open bounded subset $\Omega \subset \mathbb{R}^N$ (cf. [4, 5]). This is possible, but we may lose the property, that they are supported in this subset. Let us, for simplicity, consider balls $\Omega = B_\rho(0)$ and, after rescaling, $\rho = 1$.

Proposition 5.4 *Let $1 \le p < \infty$. There exist constants $C_1, C_2 > 0$ such that, for all $u \in L^p(\mathbb{R}^N, \Lambda^r)$ with $du = 0$ and $\mathrm{spt}(u) \subset B_1(0)$ and all $L > 0$, there exists $v \in L^p(\mathbb{R}^N, \Lambda^r)$ with $dv = 0$ and*

(i) $\|v\|_{L^\infty(\mathbb{R}^N,\Lambda^r)} \le C_1 L$;
(ii) $|\{y\in\mathbb{R}^N:\, v(y)\neq u(y)\}| \le \dfrac{C_2}{L^p}\displaystyle\int_{\{y\in\mathbb{R}^N:\, |u(y)|>L\}} |u(y)|^p\,\mathrm{d}y$;
(iii) $\|v-u\|^p_{L^p(\mathbb{R}^N,\Lambda^r)} \le C_2 \displaystyle\int_{\{y\in\mathbb{R}^N:\, |u(y)|>L\}} |u(y)|^p\,\mathrm{d}y$;
(iv) $\mathrm{spt}(v) \subset B_R(0)$, *where R only depends on the L^p-norm of u and on L.*

Again, this proof is very similar to the proof of Theorem 5.1. Property 5.4) comes from the fact that if a function u is supported in $B_1(0)$, then its maximal function $Mu(y)$ decays fast as $y \to \infty$. Regarding the construction made in Sect. 4 and Lemma 3.1, it is not clear, how to avoid the rather weak statement 5.4), i.e. we cannot directly deal with arbitrary boundary values and need to modify the truncation.

Let us mention that this result also holds for vector-valued differential forms, i.e. $u \in L^p(\mathbb{R}^N, \Lambda^r \times \mathbb{R}^m)$, where the exterior derivative is taken componentwise.

Proposition 5.5 *(Vector-valued forms on the torus) There exist constants $C_1, C_2 > 0$ such that, for all $u \in L^1(T_N, \Lambda^r \times \mathbb{R}^m)$ with $du = 0$ and all $L > 0$, there exists $v \in L^1(T_N, \Lambda^r \times \mathbb{R}^m)$ with $dv = 0$ and*

i) $\|v\|_{L^\infty(T_N,\Lambda^r\times\mathbb{R}^m)} \leq C_1 L$;

ii) $|\{y \in T_N : v(y) \neq u(y)\}| \leq \frac{C_2}{L} \int_{\{y\in T_N :\, |u(y)|>L\}} |u(y)|\, \mathrm{d}y$;

iii) $\|v - u\|_{L^1(T_N,\Lambda^r\times\mathbb{R}^m)} \leq C_2 \int_{\{y\in T_N :\, |u(y)|>L\}} |u(y)|\, \mathrm{d}y$.

This statement follows directly from the proof of Theorem 5.1 by simply truncating every component of u. Likewise, similar statements as in Propositions 5.2, 5.3 and 5.3 follow for vector-valued differential forms.

6 Applications to $\mathcal{A}$-quasiconvexity and Young measures

In the following, we consider a linear and homogeneous differential operator of first order, i.e. we are given $\mathcal{A} : C^\infty(\mathbb{R}^N, \mathbb{R}^d) \to C^\infty(\mathbb{R}^N, \mathbb{R}^l)$ of the form

$$\mathcal{A}u = \sum_{k=1}^{N} A_k \partial_k u,$$

where $A_k : \mathbb{R}^d \to \mathbb{R}^l$ are linear maps. We call a continuous function $f : \mathbb{R}^d \to \mathbb{R}$ $\mathcal{A}$-quasiconvex if for all $\varphi \in C^\infty(T_N, \mathbb{R}^d)$ with $\int_{T_N} \varphi(y)\, \mathrm{d}y = 0$ and $\mathcal{A}\varphi = 0$, and for all $x \in \mathbb{R}^d$ then the following version of Jensen's inequality

$$f(x) \leq \int_{T_N} f(x + \varphi(y))\, \mathrm{d}y \tag{6.1}$$

holds true. Fonseca and Müller showed that [11], if the constant rank condition seen below holds, then $\mathcal{A}$-quasiconvexity is a necessary and sufficient condition for weak$*$ lower-semicontinuity of the functional $I : L^\infty(\Omega, \mathbb{R}^d) \to [0, \infty)$ defined by

$$I(u) = \begin{cases} \int_\Omega f(u(y))\, \mathrm{d}y & \mathcal{A}u = 0 \\ \infty & \text{else.} \end{cases}$$

Define the symbol $\mathbb{A} \colon \mathbb{R}^N \backslash \{0\} \to \mathrm{Lin}(\mathbb{R}^d, \mathbb{R}^l)$ of the operator $\mathcal{A}$ by

$$\mathbb{A}(\xi) = \sum_{k=1}^{N} \xi_k A_k.$$

The operator $\mathcal{A}$ is said to satisfy the **constant rank property** (cf. [27]) if for some fixed $r \in \{0, ..., d\}$ and all $\xi \in \mathbb{S}^{N-1} = \{\xi \in \mathbb{R}^N : |\xi| = 1\}$

$$\dim(\ker \mathbb{A}(\xi)) = r.$$

We call a homogeneous differential operator $\mathcal{B} : C^\infty(T_N, \mathbb{R}^m) \to C^\infty(T_N, \mathbb{R}^d)$, which is not necessarily of order one, the potential of $\mathcal{A}$ if

$$\mathrm{Im}\, \mathbb{B}(\xi) = \ker \mathbb{A}(\xi),$$

i.e. if $\psi = \hat{u}(\lambda)e^{-2\pi i x\cdot\lambda}$ for $\lambda \in \mathbb{R}^N \setminus \{0\}$, then $\mathcal{A}\psi = 0$ if and only if there is $\hat{w}(\lambda)$, such that $\psi = \mathcal{B}(\hat{w}(\lambda)e^{-2\pi i x\cdot\lambda})$. Recently, Raiţa showed that $\mathcal{A}$ has such a potential if and only if $\mathcal{A}$ satisfies the constant rank property ([28]). In the following, we always assume that $\mathcal{A}$ satisfies the constant rank property and that $\mathcal{B}$ is the potential of $\mathcal{A}$.

Definition 6.1 We say that $\mathcal{A}$ satisfies the property (ZL) if for all sequences $u_n \in L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that there exists an $L > 0$ with

$$\int_{\{y\in T_N :\, |u_n(y)|>L\}} |u_n(y)|\, \mathrm{d}y \longrightarrow 0 \quad \text{as } n \to \infty,$$

there exists a $C = C(\mathcal{A})$ and a sequence $v_n \in L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

i) $\|v_n\|_{L^\infty(T_N,\mathbb{R}^d)} \leq C_1 L$;
ii) $\|v_n - u_n\|_{L^1(T_N,\mathbb{R}^d)} \to 0$ as $n \to \infty$.

Our goal now is to show that (ZL) implies further properties for the operator $\mathcal{A}$. We first look at a few examples.

Example 6.2 (a) As shown by Zhang [38], the operator $\mathcal{A} = \operatorname{curl}$ has the property (ZL). This is shown by using that its potential is the operator $\mathcal{B} = \nabla$. In fact, most of the applications here have been shown for $\mathcal{B} = \nabla$ relying on (ZL), but can be reformulated for $\mathcal{A}$ satisfying (ZL).

(b) Let $W^k = (\mathbb{R}^N \otimes ... \otimes \mathbb{R}^N)_{\text{sym}} \subset (\mathbb{R}^N)^k$. We may identify $u \in C^\infty(T_N, W^k)$ with $\tilde{u} \in C^\infty(T_N, (\mathbb{R}^N)^k)$ and define the operator

$$\operatorname{curl}^{(k)} \colon C^\infty(T_N, W^k) \to C^\infty(T_N, (\mathbb{R}^N)^{k-1} \times \Lambda^2)$$

as taking the curl on the last component of $\tilde{u}$, i.e. for $I \in [N]^{k-1}$

$$(\operatorname{curl}^{(k)} u)_I = 1/2 \sum_{i,j\in\mathbb{N}} \partial_i \tilde{u}_{Ij} - \partial_j \tilde{u}_{Ii} e_i \wedge e_j$$

Note that this operator has the potential $\nabla^k : C^\infty(\mathbb{R}^N, \mathbb{R}) \to C^\infty(\mathbb{R}^N, W^k)$ (cf. [23]). To the best of the author's knowledge the proof of the property (ZL) is in this setting not written down anywhere explicitly, but basically combining the works [1, 13, 31, 38] yields the result.

(c) In this work, it has been shown that the exterior derivative d satisfies the property (ZL). The most prominent example is $\mathcal{A} = \operatorname{div}$.

(d) The result is also true, if we consider matrix-valued functions instead (cf. Proposition 5.4). For example, (ZL) also holds if we consider $\operatorname{div} : C^\infty(\mathbb{R}^N, \mathbb{R}^{N\times M}) \to C^\infty(\mathbb{R}^N, \mathbb{R}^M)$, where

$$\operatorname{div}_i u(x) = \sum_{j=1}^{N} \partial_j u_{ji}(x).$$

(e) Likewise, let $\mathcal{A}_1 \colon C^\infty(T_N, \mathbb{R}^{d_1}) \to C^\infty(T_N, \mathbb{R}^{l_1})$ and $\mathcal{A}_2 \colon C^\infty(T_N, \mathbb{R}^{d_2}) \to C^\infty(T_N, \mathbb{R}^{l_2})$ be two differential operators satisfying (ZL). Then also the operator

$$\mathcal{A} : C^\infty(T_N, \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \to C^\infty(T_N, \mathbb{R}^{l_1} \times \mathbb{R}^{l_2})$$

defined componentwise for $u = (u_1, u_2)$ by

$$\mathcal{A}(u_1, u_2) = (\mathcal{A}_1 u_1, \mathcal{A}_2 u_2)$$

satisfies the property (ZL). The truncation is again done separately in the two components. The most prominent example, which is also covered by the result of this paper, is $\mathcal{A}_1 =$ curl and $\mathcal{A}_2 =$ div, which is highly significant in elasticity and in the framework of compensated compactness.

An overview of the results one is able to prove using property (ZL) can be found in the lecture notes [26,Sec. 4] and in the book [29,Sec. 4,7], where they are formulated for the case of (curl)-quasiconvexity.

6.1 $\mathcal{A}$-quasiconvex hulls of compact sets

For $f \in C(\mathbb{R}^d, \mathbb{R})$ we can define the quasiconvex hull of f by (cf. [6, 11])

$$\mathcal{Q}_{\mathcal{A}} f(x) := \inf \left\{ \int_{T_N} f(x + \psi(y))\,\mathrm{d}y \colon \psi \in C^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A},\ \int_{T_N} \psi = 0 \right\}. \quad (6.2)$$

$\mathcal{Q}_{\mathcal{A}} f$ is the largest $\mathcal{A}$-quasiconvex function below f [11].

In view of the separation theorem for convex sets in Banach spaces we define (cf. [8, 33, 34]) the $\mathcal{A}$-quasiconvex hull of a set $K \subset \mathbb{R}^d$ by

$$K_\infty^{\mathcal{A}qc} := \left\{ x \in \mathbb{R}^d \colon\ \forall f \colon \mathbb{R}^d \to \mathbb{R}\ \mathcal{A}\text{-quasiconvex with } f_{|K} \le 0 \text{ we have } f(x) \le 0 \right\},$$

and the $\mathcal{A}$-p-quasiconvex hull for $1 \le p < \infty$ by

$$K_p^{\mathcal{A}qc} := \Big\{ x \in \mathbb{R}^d \colon\ \forall f \colon \mathbb{R}^d \to \mathbb{R}\ \mathcal{A}\text{-quasiconvex with } f_{|K} \le 0 \ \text{ and } \ |f(v)| \le C(1 + |v|^p) \text{ we have } f(x) \le 0 \Big\}.$$

The $\mathcal{A}$-p-quasiconvex hull for $1 \le p < \infty$ can be alternatively defined via

$$K_p^{\mathcal{A}qc*} := \left\{ x \in \mathbb{R}^d \colon (\mathcal{Q}_{\mathcal{A}} \operatorname{dist}^p(\cdot, K))(x) = 0 \right\}.$$

If K is compact, then $K_p^{\mathcal{A}qc} = K_p^{\mathcal{A}qc*}$. Moreover, the spaces $K_p^{\mathcal{A}qc}$ are nested, i.e. $K_q^{\mathcal{A}qc} \subset K_{q'}^{\mathcal{A}qc}$ if $q \le q'$. In [8] it is shown that equality holds for $\mathcal{A}$ being the symmetric divergence of a matrix, K compact and $1 < q, q' < \infty$. The proof can be adapted for different $\mathcal{A}$, but uses the Fourier transform and is not suitable for the cases $p = 1$ and $p = \infty$. Here, the property (ZL) comes into play.

For a compact set K we define the set $K^{\mathcal{A}app}$ (cf. [26]) as the set of all $x \in \mathbb{R}^d$ such that there exists a bounded sequence $u_n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with

$$\operatorname{dist}(x + u_n, K) \longrightarrow 0 \quad \text{in measure, as } n \to \infty.$$

Theorem 6.3 *Suppose that K is compact and $\mathcal{A}$ is an operator satisfying (ZL). Then*

$$K^{\mathcal{A}app} = K_\infty^{\mathcal{A}qc} = \left\{ x \in \mathbb{R}^d \colon \mathcal{Q}_{\mathcal{A}}(\operatorname{dist}(\cdot, K))(x) = 0 \right\}. \quad (6.3)$$

Proof We first prove $K^{\mathcal{A}app} \subset K_\infty^{\mathcal{A}qc}$. Let $x \in K^{\mathcal{A}app}$ and take an arbitrary $\mathcal{A}$-quasiconvex function $f \colon \mathbb{R}^d \to [0, \infty)$ with $f_{|K} = 0$. We claim that then $f(x) = 0$.

Take a sequence u_n from the definition of $K^{\mathcal{A}app}$. As f is continuous and hence locally bounded, $f(x+u_n) \to 0$ in measure and $0 \le f(x+u_n) \le C$. Quasiconvexity and dominated convergence yield

$$f(x) \le \liminf_{n\to\infty} \int_{T_N} f(x + u_n(y))\,\mathrm{d}y = 0.$$

$K_\infty^{\mathcal{A}qc} \subset \left\{x \in \mathbb{R}^d : \mathcal{Q}_\mathcal{A}(\mathrm{dist}(\cdot, K))(x) = 0\right\}$ is clear by definition, as $\mathcal{Q}_\mathcal{A}(\mathrm{dist}(\cdot, K))$ is an admissible separating function.

The proof of the inclusion $\{x \in \mathbb{R}^d : \mathcal{Q}_\mathcal{A}(\mathrm{dist}(\cdot, K))(x) = 0\} \subset K^{\mathcal{A}app}$ uses (ZL). If $\mathcal{Q}_\mathcal{A}(\mathrm{dist}(\cdot, K)) = 0$, then there exists a sequence $\varphi_n \in C^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with $\int_{T_N} \varphi_n = 0$ such that

$$0 = \mathcal{Q}_\mathcal{A}(\mathrm{dist}(\cdot, K))(x) = \lim_{n\to\infty} \int_{T_N} \mathrm{dist}(x + \varphi_n(y), K)\, \mathrm{d}y.$$

As K is compact, there exists $R > 0$ such that $K \subset B(0, R)$. Moreover, as $x \in K_\infty^{\mathcal{A}qc}$, also $x \in B(0, R)$. This implies that

$$\lim_{n\to\infty} \int_{T_N \cap \{|\varphi_n| \geq 6R\}} |\varphi_n|\, \mathrm{d}y = 0.$$

We may apply (ZL) and find a sequence $\psi_n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that

$$\|\varphi_n - \psi_n\|_{L^1(T_N, \mathbb{R}^d)} \longrightarrow 0 \quad \text{as } n \to \infty$$

and

$$\|\psi_n\|_{L^\infty(T_N, \mathbb{R}^d)} \leq CR.$$

Hence, $x \in K^{\mathcal{A}app}$. □

Remark 6.4 Theorem 6.3 shows that for all $1 \leq p < \infty$

$$K^{\mathcal{A}app} = K_\infty^{\mathcal{A}qc} = \left\{x \in \mathbb{R}^d : \mathcal{Q}_\mathcal{A}(\mathrm{dist}(\cdot, K)^p)(x) = 0\right\} = K_p^{\mathcal{A}qc}.$$

This follows directly, as all the sets $K_p^{\mathcal{A}qc}$ are nested and, conversely, all the hulls of the distance functions are admissible f in the definition of $K_\infty^{\mathcal{A}qc}$.

Remark 6.5 Such a kind of theorem is not true for general unbounded closed sets K. As a counterexample one may consider $\mathcal{A} = \mathrm{curl}$ (i.e. usual quasiconvexity) and look at the set of conformal matrices $K = \{\lambda Q : \lambda \in \mathbb{R}^+, Q \in SO(n)\} \subset \mathbb{R}^{n\times n}$. If $n \geq 2$ is even, by [24], there exists a quasiconvex function $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ with $F(x) = 0 \Leftrightarrow x \in K$ and

$$0 \leq F(A) \leq C(1 + |A|^{n/2}).$$

On the other hand, let $n \geq 4$ be even and $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ be a rank-one convex function with $F_{|K} = 0$ and for some $p < n/2$

$$0 \leq F(A) \leq C(1 + |A|^p).$$

Then $F = 0$ by [37].

A reason for the nice behaviour of compact sets is that for such sets all distance functions are coercive, i.e.

$$\mathrm{dist}(v, K)^p \geq |v|^p - C,$$

which is obviously not true for unbounded sets. Coercivity of a function is often needed for relaxation results (c.f [6]).

6.2 $\mathcal{A}$-∞ Young measures

We consider $\mathcal{M}(\mathbb{R}^d)$ the set of signed Radon measures with finite mass. Note that this is the dual space of $C_c(\mathbb{R}^d)$ with the dual pairing

$$\langle \mu, f\rangle = \int_{\mathbb{R}^d} f(y)\,\mathrm{d}\mu(y).$$

For a measurable set $E \subset \mathbb{R}^N$ we call $\mu : E \to \mathcal{M}(\mathbb{R}^d)$ weak$*$ measurable if the map

$$x \longmapsto \langle \mu_x, f\rangle$$

is measurable for all $f \in C_c(\mathbb{R}^d)$. Later, we may consider the space $L^\infty_w(E, \mathcal{M}(\mathbb{R}^d))$, which is the space of all weakly measurable maps such that $\operatorname{spt}\mu_x \subset B(0, R)$ for some $R > 0$ and for a.e. $x \in E$. This space is equipped with the topology $\nu^n \overset{*}{\rightharpoonup} \nu$ iff $\forall f \in C_0(\mathbb{R}^d)$

$$\langle \nu^n_x, f\rangle \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle \text{ in } L^\infty(E).$$

Remark 6.6 The topology of $L^\infty_w(E, \mathcal{M}(\mathbb{R}^d))$ is metrisable on bounded sets. In this setting, we call a set $X \subset L^\infty_w(E, \mathcal{M}(\mathbb{R}^d))$ bounded, if

1 There is $R > 0$, such that for all $\mu \in X$ the measure μ_x is supported in $B(0, R)$ for almost every $x \in E$;
2 There is $C > 0$, such that for all $\mu \in X$ the mass $\|\mu_x\|_{\mathcal{M}(\mathbb{R}^d)} \leq C$ for almost every $x \in E$.

Note that ν^n supported on $B(0, R)$ converges to ν if and only if for all $f \in C(\bar{B}(0, R))$ and all $g \in L^1(E)$

$$\int_E \langle \nu^n_x, f\rangle g(x)\,\mathrm{d}x \longrightarrow \int_E \langle \nu_x, f\rangle g(x)\,\mathrm{d}x.$$

If ν^n is bounded, then this equation holds for all f, g if and only if it holds for dense subsets of $C(\bar{B}(0, R))$ and $L^1(E)$. As these spaces are separable, we may consider a countable dense subset $(f_k, g_k)_{k\in\mathbb{N}}$ of $C(\bar{B}(0, R)) \times L^1(E)$ and the pseudo-metric

$$d_k(\nu, \mu) = \left| \int_E \langle \nu_x - \mu_x, f_k\rangle g_k(x)\,\mathrm{d}x \right|,$$

and then define the metric

$$d(\nu, \mu) = \sum_{k\in\mathbb{N}} 2^{-k} \frac{d_k(\nu, \mu)}{1 + d_k(\nu, \mu)}.$$

Let us now recall the Fundamental Theorem of Young measures(cf. [3, 32]).

Proposition 6.7 (Fundamental Theorem of Young measures) *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and $u_j : E \to \mathbb{R}^d$ a sequence of measurable functions. There exists a subsequence u_{j_k} and a weak$*$ measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ such that the following properties hold:*

(i) $\nu_x \geq 0$ *and* $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} 1\,\mathrm{d}\nu_x \leq 1$*;*
(ii) $\forall f \in C_0(\mathbb{R}^d)$ *define* $\bar{f}(x) = \langle \nu_x, f\rangle$*.Then* $f(u_{j_k}) \overset{*}{\rightharpoonup} \bar{f}$ *in* $L^\infty(E)$*;*
(iii) If $K \subset \mathbb{R}^d$ *is compact, then* $\operatorname{spt}\nu_x \subset K$ *if* $\operatorname{dist}(u_{j_k}, K) \to 0$ *in measure;*

(iv) It holds

$$\|\nu_x\|_{\mathcal{M}(\mathbb{R}^d)} = 1 \textit{ for a.e. } x \in E \tag{6.4}$$

if and only if

$$\lim_{M\to\infty} \sup_{k\in\mathbb{N}} \left|\{|u_{j_k}| \geq M\}\right| = 0;$$

(v) If (6.4) holds, then for all $A \subset E$ measurable and for all $f \in C(\mathbb{R}^d)$ such that $f(u_{j_k})$ is relatively weakly compact in $L^1(A)$, also

$$f(u_{j_k}) \rightharpoonup \bar{f} \text{ in } L^1(A);$$

(vi) If (6.4) holds, then (iii) holds with equivalence.

We call such a map $\nu : E \to \mathcal{M}(\mathbb{R}^d)$ the Young measure generated by the sequence u_{j_k}. One may show that every weak$*$ measurable map $E \to \mathcal{M}(\mathbb{R}^d)$ satisfying (i) is generated by some sequence u_{j_k}.

Remark 6.8 If u_k generates a Young measure ν and $v_k \to 0$ in measure (in particular, if $v_k \to 0$ in L^1), then the sequence $(u_k + v_k)$ still generates ν.

If $u : T_N \to \mathbb{R}^d$ is a function, we may consider the oscillating sequence $u_n(x) := u(nx)$. This sequence generates the homogeneous (i.e. $\nu_x = \nu$ a.e.) Young measure ν defined by

$$\langle \nu, f\rangle = \int_{T_N} f(u_n(y))\, \mathrm{d}y.$$

Question 6.9 What happens to the Young measure generated by a sequence u_{j_k} if we impose further conditions on it, for instance $\mathcal{A}u_{j_k} = 0$?

For $1 \leq p < \infty$ we call a sequence $v_j \in L^p(\Omega, \mathbb{R}^d)$ p-equi-integrable if

$$\lim_{\varepsilon\to 0} \sup_{j\in\mathbb{N}} \sup_{E\subset\Omega:\, |E|<\varepsilon} \int_E |v_j(y)|^p\, \mathrm{d}y = 0.$$

Definition 6.10 Let $1 \leq p \leq \infty$. We call a map $\nu : \Omega \to \mathbb{R}^d$ an $\mathcal{A}$-p-Young measure if there exists a p-equi-integrable sequence $\{v_j\} \subset L^p(\Omega, \mathbb{R}^d)$ (for $p = \infty$ a bounded sequence), such that v_j generates ν and satisfies $\mathcal{A}v_j = 0$.

For $1 \leq p < \infty$ the set of $\mathcal{A}$-p Young measures was classified by Fonseca and Müller in [11] and for the special case $\mathcal{A} = \text{curl}$ already in [18].

Proposition 6.11 *Let $1 \leq p < \infty$ and $\mathcal{A}$ be a constant rank operator. A Young-measure $\nu : T_N \to \mathcal{M}(\mathbb{R}^d)$ is an $\mathcal{A}$-p-Young measure if and only if*

(i) $\exists v \in L^p(T_N, \mathbb{R}^d)$ such that $\mathcal{A}v = 0$ and

$$v(x) = \langle \nu_x, \mathrm{id}\rangle = \int_{\mathbb{R}^d} y\, \mathrm{d}\nu_x(y) \textit{ for a.e. } x \in T_N;$$

(ii) $\displaystyle\int_{T_N}\int_{\mathbb{R}^d} |z|^p\, \mathrm{d}\nu_x(z)\, \mathrm{d}x < \infty;$

(iii) for a.e. $x \in T_N$ and all continuous g with $|g(v)| \leq C(1 + |v|^p)$ we have

$$\langle \nu_x, g\rangle \geq \mathcal{Q}_{\mathcal{A}} g(\langle \nu_x, \mathrm{id}\rangle).$$

Recently, there has also been progress for so-called generalized Young measures ($p = 1$ is a special case), cf. [2, 10, 16, 19, 20].

Proposition 6.11 only uses the constant rank property, the property (ZL) is not needed. However, for $p = \infty$ the situation changes. Let us recall the result of Kinderlehrer and Pedregal for $W^{1,\infty}$-Gradient Young measures (cf. [17, 21]), whose proof relies on the validity of (ZL) for curl.

Proposition 6.12 *A weak∗ measurable map* $\nu : \Omega \to \mathcal{M}(\mathbb{R}^{N\times m})$ *is a* curl-∞-*Young measure if and only if* $\nu_x \geq 0$ *a.e. and there exists* $K \subset \mathbb{R}^{N\times m}$ *compact,* $v \in L^\infty(\Omega, \mathbb{R}^{N\times m})$ *such that*

(a) $\operatorname{spt} \nu_x \subset K$ *for a.e.* $x \in \Omega$*;*
(b) $\langle \nu_x, \mathrm{id}\rangle = v(x)$ *for a.e.* $x \in \Omega$*;*
(c) for a.e. $x \in \Omega$ *and all continuous* $g\colon \mathbb{R}^{N\times m} \to \mathbb{R}$ *we have*

$$\langle \nu_x, g\rangle \geq \mathcal{Q}_{\mathrm{curl}} g(\langle \nu_x, \mathrm{id}\rangle).$$

It is possible to state such a theorem in the general setting that $\mathcal{A}$ satisfies (ZL). The proofs from [17] mostly rely on this fact and this general case can be treated in the same fashion with few modifications. We do not give all the details of the proofs, but only the crucial steps where we use

Let us first state the classification theorem for so called homogeneous $\mathcal{A}$-∞-Young measures, i.e. $\mathcal{A}$-∞-Young measures $\nu\colon T_N \to \mathcal{M}(\mathbb{R}^d)$ with the following properties

(i) $\operatorname{spt} \nu_x \subset K$ for a.e. $x \in T_N$ where $K \subset \mathbb{R}^d$ is compact;
(ii) ν is a homogeneous Young measure, i.e. there exists $\nu_0 \in \mathcal{M}(\mathbb{R}^d)$ such that $\nu_x = \nu_0$ for a.e. $x \in T_N$.

Define the set $\mathcal{M}^{\mathcal{A}qc}(K)$ by (cf. [35])

$$\mathcal{M}^{\mathcal{A}qc}(K)=\Big\{\nu \in \mathcal{M}(\mathbb{R}^d)\colon \nu \geq 0,\ \ \operatorname{spt}\nu \subset K,\ \langle \nu, f\rangle \geq f(\langle \nu, \mathrm{id}\rangle)\ \forall f\colon \mathbb{R}^d \to \mathbb{R}\ \mathcal{A}\text{-qc}\Big\}. \tag{6.5}$$

Denote by $H_{\mathcal{A}}(K)$ the set of homogeneous $\mathcal{A}$-∞-Young measures supported on K. We are now able to formulate the classification of these measures (cf. [17,Theorem 5.1.]).

Proposition 6.13 (Characterisation of homogeneous $\mathcal{A}$-∞-Young measures) *Let* $\mathcal{A}$ *satisfy the property (ZL) and* K *be a compact set. Then*

$$H_{\mathcal{A}}(K) = \mathcal{M}^{\mathcal{A}qc}(K).$$

Using this result, one may prove the Characterisation of $\mathcal{A}$-∞-Young measures (c.f [17,Theorem 6.1]).

Proposition 6.14 (Characterisation of $\mathcal{A}$-∞-Young measures) *Suppose that* $\mathcal{A}$ *satisfies the property (ZL). A weak∗ measurable map* $\nu : T_N \to \mathcal{M}(\mathbb{R}^d)$ *is an* $\mathcal{A}$-∞-*Young measure if and only if* $\nu_x \geq 0$ *a.e. and there exists* $K \subset \mathbb{R}^d$ *compact and* $u \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ *with*

(i) $\operatorname{spt} \nu_x \subset K$ *for a.e.* $x \in T_N$.
(ii) $\langle \nu_x, \mathrm{id}\rangle = u$ *for a.e.* $x \in T_N$*,*
(iii) $\langle \nu_x, f\rangle \geq f(\langle \nu_x, \mathrm{id}\rangle)$ *for a.e.* $x \in T_N$ *and all continuous and* $\mathcal{A}$*-quasiconvex* $f : \mathbb{R}^d \to \mathbb{R}$.

As mentioned, the proofs in the case $\mathcal{A}$ = curl can be found in [17, 26, 29]. Let us shortly describe the strategy of the proofs. For Proposition 6.13 one may prove that $H_{\mathcal{A}}(K)$

is weakly compact, that averages of (non-homogeneous) $\mathcal{A}$-infty Young measures are in $H_{\mathcal{A}}(K)$ and that the set $H^x_A(K) = \{\nu \in H_{\mathcal{A}}: \langle \nu, \mathrm{id}\rangle = x\}$ is weak* closed and convex. The characterisation theorem then follows by using Hahn–Banachs separation theorem and showing that any $\mu \in M^{\mathcal{A}qc}$ cannot be separated from $H_{\mathcal{A}}(K)$, i.e. for all $f \in C(K)$ and for all $\mu \in M^{\mathcal{A}qc}(K)$ with $\langle \mu, \mathrm{id}\rangle = 0$

$$\langle \nu, f\rangle \geq 0 \text{ for all } \nu \in H^0_{\mathcal{A}}(K) \Rightarrow \langle \mu, f\rangle \geq 0.$$

Proposition 6.14 then can be shown using Proposition 6.13 and a localisation argument.

6.3 On the proofs of Propositions 6.13 and 6.14

In this section, we present the proof of Proposition 6.13, basing on its counterpart for gradient Young measures in [26]. After that we shortly sketch the proof of 6.14, which is then done by a standard technique of approximation on small cubes.

The property (ZL) is helpful due to the following two observations:

1. If $\nu \in H_{\mathcal{A}}(K)$ is a homogeneous $\mathcal{A}$-∞-Young measure, then by using (ZL) we can find a sequence generating ν with an L^∞-bound only depending on $|K|_\infty := \sup_{y\in K} |y|$ (cf. Lemma 6.16)
2. A Young measure ν is an $\mathcal{A}$-∞-Young measure if there is $v_n \in L^1(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ and $L > 0$ such that

$$\int_{|u_n|\geq L} |u_n|\,\mathrm{d}x \longrightarrow 0 \quad \text{as } n \to \infty.$$

Remark 6.15 Moreover, note that, if a sequence $u_n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ generates a homogeneous Young measure ν, we can find $v_n \in C^\infty_c((0,1)^N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with $\|v_n\|_{L^\infty} \leq C\|u_n\|_{L^\infty}$ and $\|u_n - v_n\|_{L^1} \to 0$. In particular, v_n still generates the same homogeneous Young measure.

To find such a sequence, recall that there is a potential $\mathcal{B}$ of order $k_{\mathcal{B}}$ to the differential operator $\mathcal{A}$. Let us, for simplicity, assume that all u_n have zero average. Then we can write

$$u_n = \mathcal{B}U_n$$

with $\|U_N\|_{W^{k_{\mathcal{B}},q}} \leq C_q\|u_n\|_{L^q} \leq C_q\|u_n\|_{L^\infty}$ for all $1 < q < \infty$ and a constant $C_q > 0$. Let us define

$$U_{n,i,j}(x) = \varphi_j(x) i^{-k_{\mathcal{B}}} U_n(ix), \quad u_{n,i,j}(x) = \mathcal{B}U_{n,i,j}(x),$$

for a suitable sequence of cut-offs $\varphi_j \to 1$ in $L^1((0,1)^N, \mathbb{R})$. Picking suitable subsequences $i(n)$ and $j(n)$ we obtain a sequence $u_{n,i(n),j(n)}$ bounded in L^∞, still generating ν, but with compact support in $(0,1)^N$. Convolution with a standard mollifier gives a sequence v_n that is also in $C^\infty_c((0,1)^N, \mathbb{R}^d)$

Lemma 6.16 (Properties of $H_{\mathcal{A}}(K)$)

(i) If $\nu \in H_{\mathcal{A}}(K)$ with $\langle \nu, \mathrm{id}\rangle = 0$, then there exists a sequence $u_j \in L^\infty(T_N, \mathbb{R}^d)$ such that $\mathcal{A}u_j = 0$, u_j generates ν and $\|u_j\|_{L^\infty(T_N,\mathbb{R}^d)} \leq C \sup_{z\in K} |z| = C|K|_\infty$.

(ii) $H_{\mathcal{A}}(K)$ is weakly$*$ compact in $\mathcal{M}(\mathbb{R}^d)$.

Proof (i) follows from the definition of $H_{\mathcal{A}}(K)$. The uniform bound on the L^∞ norm of u_j can be guaranteed by (ZL) and vi) in Theorem 6.7.

For the weak$*$ compactness note that $H_{\mathcal{A}}(K)$ is contained in the weak$*$ compact set $\mathcal{P}(K)$ of probability measures on K. As the weak$*$ topology is metrisable on $\mathcal{P}(K)$ it suffices to show that $H_{\mathcal{A}}(K)$ is sequentially closed. Hence, we consider a sequence $\nu_k \subset H_{\mathcal{A}}(K)$ with $\nu_k \overset{*}{\rightharpoonup} \nu$ and show that $\nu \in H_{\mathcal{A}}(K)$.

Due to the definition of Young measures, we may find sequences $u_{j,k} \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ such that $u_{j,k}$ generates ν_k for $j \to \infty$. Recall that the topology of generating Young measures is metrisable on bounded set of $L^\infty(T_N, \mathbb{R}^d)$ (c.f. Remark 6.6). We may find a subsequence $u_{j_k,k}$ which generates ν. As we know that $\|u_{j_k,k}\|_{L^\infty} \leq C|K|_\infty$, $\nu \in H_{\mathcal{A}}(K)$ and hence $H_{\mathcal{A}}(K)$ is closed. □

Lemma 6.17 *Let ν be an $\mathcal{A}$-∞-Young measure generated by a bounded sequence $u_k \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$. Then the measure $\bar{\nu}$ defined via duality for all $f \in C_0(\mathbb{R}^d)$ by*

$$\langle \bar{\nu}, f \rangle = \int_{T_N} \langle \nu_x, f \rangle \, \mathrm{d}x$$

is in $H_{\mathcal{A}}(K)$.

Proof For $n \in \mathbb{N}$ define $u_k^n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ by $u_k^n(x) = u_k(nx)$. Then for all $f \in C_0(\mathbb{R}^d)$

$$f(u_k^n) \overset{*}{\rightharpoonup} \int_{T_N} f(u_k) \text{ in } L^\infty(T_N, \mathbb{R}^d) \quad \text{as } n \to \infty.$$

Note that by Theorem 6.7 ii) we also have

$$\int_{T_N} f(u_k(x)) \, \mathrm{d}x \longrightarrow \int_{T_N} \langle \nu_x, f \rangle \, \mathrm{d}x \quad \text{as } k \to \infty.$$

Due to metrisability on bounded sets (Remark 6.6), we can find a subsequence $u_k^{k(n)}$ in $L^\infty(T_N, \mathbb{R}^d)$ such that

$$f(u_k^{n(k)}) \overset{*}{\rightharpoonup} \int_{T_N} \langle \nu_x, f \rangle \, \mathrm{d}x \quad \text{as } k \to \infty.$$

Thus, $\bar{\nu} \in H_{\mathcal{A}}(K)$. □

Lemma 6.18 *Define the set $H_{\mathcal{A}}^x(K) := \{\nu \in H_{\mathcal{A}} : \langle \nu, \mathrm{id} \rangle = x\}$. Then $H_{\mathcal{A}}^x(K)$ is weak$*$ closed and convex.*

Proof Weak$*$-closedness is clear by Lemma 6.16. For convexity, let ν_1, ν_2 be $\mathcal{A}$-∞-Young measures. By an argumentation following Remark 6.15, we can find sequences $v_n \in C_c^\infty((0, \lambda) \times (0, 1)^{N-1}, \mathbb{R}^d)$ and $w_n C_c^\infty((\lambda, 1) \times (0, 1)^{N-1}, \mathbb{R}^d)$ that generate ν_1 and ν_2, respectively. Define

$$u_n = \begin{cases} v_n & \text{in } (0, \lambda) \times (0, 1)^{N-1}, \\ w_n & \text{in } (\lambda, 1) \times (0, 1)^{N-1}. \end{cases}$$

and $u_{n,i}$ via $u_{n,i}(x) = u_n(ix)$. Then proceeding as in Lemma 6.17, picking a suitable subsequence $i(n)$ yields that $u_{n,i(n)}$ generates $\lambda\nu_1 + (1 - \lambda)\nu_2$. □

We proceed with the proof of the characterisation of homogeneous $\mathcal{A}$-∞-Young measures.

Proof of Theorem 6.13 We have that $H_{\mathcal{A}}(K) \subset \mathcal{M}^{\mathcal{A}qc}$ due to the fundamental theorem of Young measures: $\nu \geq 0$ and spt $\nu \subset K$ are clear by i) and iii) of Theorem 6.7. The corresponding inequality follows by $\mathcal{A}$-quasiconvexity, i.e. if $u_n \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ generates the Young measure ν, then

$$\langle \nu, f \rangle = \lim_{n\to\infty} \int_{T_N} f(u_n(y))\,\mathrm{d}y \geq \liminf_{n\to\infty} f\left(\int_{T_N} u_n(y)\,\mathrm{d}y\right) = f(\langle \nu, \mathrm{id}\rangle).$$

To prove $M^{\mathcal{A}qc}(K) \subset H_{\mathcal{A}}(K)$, w.l.o.g. consider a measure such that $\langle \nu, \mathrm{id}\rangle = 0$. We just proved that $H^0_{\mathcal{A}}(K)$ is weak$*$ closed and convex. Remember that $C(K)$ is the dual space of the space of signed Radon measures $\mathcal{M}(K)$ with the weak$*$ topology (see e.g. [30]). Hence, by Hahn-Banach separation theorem, it suffices to show that for all $f \in C(K)$ and all $\mu \in M^{\mathcal{A}qc}(K)$ with $\langle \mu, \mathrm{id}\rangle = 0$

$$\langle \nu, f \rangle \geq 0 \text{ for all } \nu \in H^0_{\mathcal{A}}(K) \quad \Longrightarrow \quad \langle \mu, f \rangle \geq 0.$$

To this end, fix some $f \in C(K)$, consider a continuous extension to $C_0(\mathbb{R}^d)$ and let

$$f_k(x) = f(x) + k\,\mathrm{dist}^2(x, K).$$

We claim that

$$\lim_{k\to\infty} \mathcal{Q}_{\mathcal{A}} f_k(0) \geq 0. \tag{6.6}$$

If we show (6.6), μ satisfies

$$\langle \mu, f \rangle = \langle \mu, f_k \rangle \geq \langle \mu, \mathcal{Q}_{\mathcal{A}} f_k \rangle \geq \mathcal{Q}_{\mathcal{A}} f_k(0),$$

finishing the proof. For the identity $\langle \mu, f \rangle = \langle \mu, f_k \rangle$ recall that μ is supported in K and $\mathrm{dist}(x, K) = 0$ for $x \in K$.

Hence, suppose that (6.6) is wrong. As f_k is strictly increasing, there exists $\delta > 0$ such that

$$\mathcal{Q}_{\mathcal{A}} f_k(0) \leq -2\delta, \quad k \in \mathbb{N}.$$

Using the definition of the $\mathcal{A}$-quasiconvex hull (6.2), we get $u_k \in L^\infty(T_N, \mathbb{R}^d) \cap \ker \mathcal{A}$ with $\int_{T_N} u_k(y)\,\mathrm{d}y = 0$ and

$$\int_{T_N} f_k(u_k(y))\,\mathrm{d}y \leq -\delta. \tag{6.7}$$

We may assume that $u_k \rightharpoonup 0$ in $L^2(T_N, \mathbb{R}^d)$ and also that $\mathrm{dist}^2(u_k, K) \to 0$ in $L^1(T_N)$. By property (ZL), there exists a sequence $v_k \in \ker \mathcal{A}$ bounded in $L^\infty(T_N, \mathbb{R}^d)$ with $\|u_k - v_k\|_{L^1} \to 0$. v_k generates (up to taking subsequences) a Young measure ν with spt $\nu_x \subset K$.

Then for fixed $j \in \mathbb{N}$, using Lemma 6.17 and that $\bar{\nu} \in H_{\mathcal{A}}(K) \subset M^{\mathcal{A}qc}(K)$,

$$\liminf_{k\to\infty} \int_{T_N} f_j(u_k(y))\,\mathrm{d}y \geq \liminf_{k\to\infty} \int_{T_N} f_j(v_k(y))\,\mathrm{d}y = \int_{T_N} \int_{\mathbb{R}^d} f_j\,\mathrm{d}\nu_x\,\mathrm{d}x = \langle \bar{\nu}, f \rangle \geq 0.$$

But this is a contradiction to (6.7), as $f_k \geq f_j$ if $k \geq j$. □

Let us finally outline the strategy of the proof for Proposition 6.14. For details we refer to [17, 26].

Proof of Propostion 6.14 (Sketch) Necessity of condition (i)–(iii) is established by the following argument. (i) and (ii) follow directly from the fact that the Young-measure μ is generated by an $\mathcal{A}$-free sequence that, up to a subsequence, has a weak-$*$-limit u. iii) follows from the lower-semicontinuity statement of FONSECA and MÜLLER [11].

To prove sufficiency of these conditions, one needs to construct a sequence generating the Young-measure ν. Let us suppose that $u = 0$, otherwise we define the Young-measure $\tilde{\nu} = \nu - u$. Then we find a sequence v_n generating $\tilde{\nu}$ and, consequently, $v_n + u$ generates ν.

To find such a sequence one divides T_N into subcubes and approximates ν by maps $\nu_n : T_N \to \mathcal{M}(\mathbb{R}^d)$, which are constant on the subcubes. For each subcube Q one then constructs a sequence $v^Q_{n,m} \in L^\infty(Q, \mathbb{R}^d) \cap \ker \mathcal{A}$, $m \in \mathbb{N}$, that generates ν_n and satisfies $v^Q_{n,m} \in C^\infty_c(Q, \mathbb{R}^d)$. These $v^Q_{n,m}$ then give a sequence $v_{n,m}$ generating ν_n and taking a suitable diagonal sequence one may find a sequence generating ν (cf. [26,Proof of Theorem 4.7]). □

Acknowledgements The author would like to thank Stefan Müller for introducing him to the topic and for insightful discussions. Moreover, the author is very grateful towards the anonymous reviewers for careful reading and their helpful suggestions. The author has been supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the graduate school BIGS of the Hausdorff Center for Mathematics (GZ EXC 59 and 2047/1, Projekt-ID 390685813).

Funding Open Access funding enabled and organized by Projekt DEAL.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Acerbi, E., Fusco, N.: Semicontinuity problems in the calculus of variations. Arch. Rat. Mech. Anal. **86**, 125–145 (1984)
2. Arroyo-Rabasa, A.: Characterization of generalized Young measures generated by A-free measures. Arch. Rat. Mech. Anal. **242**, 235–325 (2021)
3. Ball, J.M.: A version of the fundamental theorem for young measures. In: Rascle, M., Serre, D., Slemrod, M. (eds.) PDEs and Continuum Models of Phase Transitions, pp. 207–215. Springer, Berlin, Heidelberg (1989)
4. Breit, D., Diening, L., Fuchs, M.: Solenoidal Lipschitz truncation and applications in fluid mechanics. J. Differ. Equ. **253**(6), 1910–1942 (2012)
5. Breit, D., Diening, L., Schwarzacher, S.: Solenoidal Lipschitz turnation for parabolic PDEs. Math. Models Methods Appl. Sci. **23**(14), 2671–2700 (2013)
6. Braides, A., Fonseca, I., Leoni, G.: A-quasiconvexity: relaxation and homogenization. ESAIM: Control Optim. Calculus Variat. **5**, 539–577 (2000)
7. Cartan, E.: Sur certaines expressions différentielles et le problème de Pfaff. Annales Scientifiques de l'École Normale Supérieure **16**, 239–332 (1899)
8. Conti, S., Müller, S., Ortiz, M.: Symmetric div-quasiconvexity and the relaxation of static problems. Arch. Ration. Mech. Anal. **235**(2), 841–880 (2020)
9. Conlon, L.: Differentiable Manifolds. Birkhäuser Verlag (2001)
10. De Philippis, G., Rindler, F.: Characterization of generalized young measures generated by symmetric gradients. Arch. Rat. Mech. Anal. **224**, 1087–1125 (2017)

11. Fonseca, I., Müller, S.: A-quasiconvexity, lower-semicontinuity and Young measures. SIAM J. Math. Anal. **30**(6), 1355–1390 (1999)
12. Fonseca, I., Müller, S., Pedregal, P.: Analysis of concentration and oscillation effects generated by gradients. SIAM J. Math. Anal. **29**(3), 736–756 (1998)
13. Francos, G.: Luzin type approximation of functions of bounded variation. http://www.d-scholarship.pitt.edu/7947/ (2011)
14. Gallenmüller, D.: Müller-Zhang truncation for general linear constraints with first or second order potential. Calc. Var. **60**(118) (2021)
15. Kirszbraun, M.D.: Über die zusammenziehende und Lipschitzsche Transformation. Fund. Math. **22**, 77–108 (1934)
16. Kirchheim, Bernd, Kristensen, Jan: On rank one convex functions that are homogeneous of degree one. Arch. Ration. Mech. Anal. **221**(1), 527–558 (2016)
17. Kinderlehrer, D., Pedregal, P.: Characterization of Young measures generated by gradients. Arch. Rat. Mech. Anal. **115**, 329–365 (1991)
18. Kinderlehrer, D., Pedregal, P.: Gradient Young measures generated by sequences in Sobolev spaces. J. Geom. Anal. **4**(1), 59–90 (1994)
19. Kristensen, J., Rindler, F.: Characterization of generalized gradient young measures generated by sequences in W1,1 and BV. Arch. Rat. Mech. Anal. **197**, 539–598 (2012)
20. Kristensen, J., Raiţă, B.: Oscillation and concentration in sequences of pde constrained measures. https://arxiv.org/abs/1912.09190 (2019)
21. Kristensen, J.: Finite functionals and Young measures generated by gradients of Sobolev functions. F. MAT–Report 1994–34, Math. Inst., Technical University of Denmark (1994)
22. Liu, F.C.: A Luzin type property of Sobolev functions. Indiana Univ. Math. J. **26**(4), 645–651 (1977)
23. Meyers, N.: Quasiconvexity and the lower semicontinuity of multiple variational integrals of any order. Trans. Am. Math. Soc. **119**(1), 125–149 (1965)
24. Müller, S., Šverák, V., Yan, B.: Sharp stability results for almost conformal maps in even dimensions. J. Geom. Anal. **9**(4), 671–681 (1999)
25. Müller, S.: A sharp version of Zhang's theorem on truncating sequences of gradients. Trans. Am. Math. Soc. **351**(11), 4585–4597 (1999)
26. Müller, S.: Variational models for microstructure and phase transitions. In: Calculus of Variations and Geometric Evolution Problems. Lecture Notes in Mathematics, pp. 85–210. Springer, Berlin, Heidelberg (1999)
27. Murat, F.: Compacité par compensation: condition necessaire et suffisante de continuité faible sous une hypothése de rang constant. Ann. Sc. Norm. Sup. Pisa **8**, 69–102 (1981)
28. Raiţă, B.: Potentials for A-quasiconvexity. Calc. Var. **58**, 105 (2019)
29. Rindler, F.: Calculus of Variations. Springer, Berlin (2018)
30. Rudin, W.: Functional Analysis. McGraw-Hill, New York (1973)
31. Stein, E.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1971)
32. Tartar, L.: Compensated compactness and applications to partial differential equations. In Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, volume 4, pp. 136–212. Pitman Res. Notes Math (1979)
33. Šverák, V.: On Tartar's conjecture. Ann. Inst. Henry Poincaré **10**(4), 405–412 (1993)
34. Šverák, V.: On the problem of two wells. Microstructure and Phase transitions, pp. 183–189 (1993)
35. Sverák, V.: Lower-semicontinuity of variational integrals and compensated compactness. In: Proceedings of the International Congress of Mathematicians, pp. 1153–1158 (1995)
36. Whitney, H.: Analytic extensions of differentiable functions defined in closed sets. Trans. Am. Math. Soc. **36**, 63–89 (1934)
37. Yan, B.: On rank-one convex and polyconvex conformal energy functions with slow growth. Proc. R. Soc. Edinb. **127**, 651–663 (1997)
38. Zhang, K.: A construction of quasiconvex functions with linear growth at infinity. Annal. S. N. S. Pisa **19**(3), 313–326 (1992)