Setting the Free Material Design problem through the methods of optimal mass distribution

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Received: 11 November 2020 / Accepted: 13 January 2022 / Published online: 4 March 2022
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Abstract

The paper deals with the Free Material Design (FMD) problem aimed at constructing the least compliant structures from an elastic material, the constitutive field of which plays the role of design variable in the form of a tensor valued measure λ supported in the design domain. Point-wise the constitutive tensor is referred to a given anisotropy class $\mathcal{H}$, while the integral of a cost $c(\lambda)$ is bounded from above. The convex p -homogeneous elastic potential j is parameterized by the constitutive tensor. The work puts forward the existence result and shows that the original problem can be reduced to the Linear Constrained Problem (LCP) known from the theory of optimal mass distribution by G. Bouchitté and G. Buttazzo. A theorem linking solutions of (FMD) and (LCP) allows to effectively solve the original problem. The developed theory encompasses several optimal anisotropy design problems known in the literature as well as it unlocks new ones. By employing the derived optimality conditions we give several analytical examples of optimal designs.

Mathematics Subject Classification 74P05 · 74B99 · 49N99 · 46N10

1 Introduction

Under the term compliance of an elastic structure, the properties of which are characterized by a constitutive tensor field λ , we understand the value of the elastic energy induced by a

Communicated by A. Malchiodi.

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given static load F :

$$C = C(\lambda) := \sup \left\{ \int \langle u, F \rangle - \int j(\lambda, e(u)) : u \in (\mathcal{D}(\mathbb{R}^d))^d \right\}, \quad (1.1)$$

where, for a 4th order elasticity tensor $H \in \mathcal{H}$ and a symmetric 2nd order strain tensor $\xi \in \mathcal{S}^{d \times d}$, function j returns the elastic energy density $j(H, \xi)$. The elasticity or the Hooke tensor H plays the role of a parameter for the Hooke's law of elasticity: $j(H, \cdot)$ is the elastic potential inducing the constitutive law $\sigma \in \partial j(H, \xi)$, where $\sigma \in \mathcal{S}^{d \times d}$ is the stress tensor, and the subdifferential is computed with respect to the second argument. The strain field $e(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ is the symmetric part of the gradient of a vectorial function u representing the displacement field. The load is modelled by a vector valued measure $F \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$. Since the body is assumed not to be kinematically fixed, i.e. there is no Dirichlet condition imposed on u , an extra condition on F must be enforced:

Definition 1.1 The load $F \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$ is said to be *balanced* if the resultant force and the resultant moment are zero, namely:

$$\int F(dx) = 0, \quad \int (x_i F_j(dx) - x_j F_i(dx)) = 0 \quad \forall i, j \in \{1, \dots, d\}.$$

For a chosen cost function $c : \mathcal{H} \rightarrow \mathbb{R}_+$ by *Free Material Design* (FMD) we shall call the problem of finding the constitutive field being a tensor valued measure $\lambda \in \mathcal{M}(\overline{\Omega}; \mathcal{H})$ that minimizes the compliance:

$$C_{\min} = \min \left\{ C(\lambda) : \lambda \in \mathcal{M}(\overline{\Omega}; \mathcal{H}), \int c(\lambda) \leq C_0 \right\} \quad (\text{FMD})$$

In the context of the linear theory of elasticity, i.e. for quadratic potential $j(H, \xi) = \frac{1}{2}\langle H\xi, \xi \rangle$, and with the trace cost $c(H) = \text{Tr } H$, the (FMD) problem has been for the first time considered in [31], where the set $\mathcal{H}$ consisted of all positive semi-definite Hooke tensors. Soon then, in [4,5] the (FMD) was reformulated as a problem in which only one scalar variable is involved: $\mu := \text{Tr } \lambda$. One of the first existence results may be found in [36] where the Hooke field was assumed to be a function, i.e. $\lambda = \mathcal{C} \mathcal{L}^d$, and, to gain compactness in L^∞ , the author imposed an additional local constraint $\text{Tr } \mathcal{C}(x) \leq c_0$. Similar assumption was kept throughout the papers [25,38] or [26].

The present work puts forward a far more general framework where (FMD) is in fact a family of optimal design problems parameterized by: the elastic potential j , the cost function c , and the set of admissible Hooke tensors $\mathcal{H}$. The set $\mathcal{H}$ may be chosen as any closed convex cone contained in the set of positive semi-definite Hooke tensors. The cost function c is any norm on the space of Hooke tensors restricted to $\mathcal{H}$. Finally, for a chosen $p \in (1, \infty)$, the potential $j : \mathcal{H} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ will be subject to the following assumptions:

- for each $H \in \mathcal{H}$ there hold:
 - (H1) the function $j(H, \cdot)$ is real-valued, non-negative and convex on $\mathcal{S}^{d \times d}$;
 - (H2) the function $j(H, \cdot)$ is positively p -homogeneous on $\mathcal{S}^{d \times d}$;
- whilst for each $\xi \in \mathcal{S}^{d \times d}$ there hold:
 - (H3) the function $j(\cdot, \xi)$ is concave and upper semi-continuous on $\mathcal{H}$;
 - (H4) the function $j(\cdot, \xi)$ is one-homogeneous on $\mathcal{H}$;
 - (H5) if $\xi \neq 0$ then there exists $H \in \mathcal{H}$ such that $j(H, \xi) > 0$.

Based on assumptions (H1)–(H5), in Sect. 2 for a fixed function u the properties of the elastic energy functional $\mathcal{M}(\overline{\Omega}; \mathcal{H}) \ni \lambda \mapsto \int j(\lambda, e(u))$ are established, including its upper semi-continuity in the weak-* topology proved in Proposition 2.2. This renders $\mathcal{C} = \mathcal{C}(\lambda)$ convex and weak-* lower semi-continuous, which shall in turn furnish the existence result:

Theorem 1.1 *Assuming that j satisfies (H1)–(H5) the Free Material Design problem (FMD) admits a solution $\check{\lambda}$ once F is balanced.*

In the case when j is quadratic, $c = \text{Tr}$, and $\mathcal{H}$ is the set of all anisotropic Hooke tensors – which shall be referred to as the *Anisotropic Material Design* (AMD) setting – in works [18, 19] the (FMD) problem has been reformulated on the formal level to a pair of mutually dual problems:

$$Z := \sup \left\{ \int \langle u, F \rangle : u \in (\mathcal{D}(\mathbb{R}^d))^d, \rho(e(u)) \leq 1 \text{ in } \overline{\Omega} \right\} \quad (\mathcal{P})$$

$$= \min \left\{ \int \rho^0(\tau) : \tau \in \mathcal{M}(\overline{\Omega}; \mathcal{S}^{d \times d}), -\text{div } \tau = F \right\} \quad (\mathcal{P}^*)$$

where, in the AMD setting only, $\rho = \rho^0 = |\cdot|$ is the Euclidean norm on the space of symmetric matrices. The matrix-valued measure $\tau \in \mathcal{M}(\overline{\Omega}; \mathcal{S}^{d \times d})$ plays a role of the *stress field* that equilibrates the load F or, adopting the language of the *Optimal Transport Problem* (cf. [35]), transports certain parts of F to its other parts. The equilibrium equation $-\text{div } \tau = F$ is intended in the sense of distributions on the whole $\mathbb{R}^d$. The equality $\sup(\mathcal{P}) = \min(\mathcal{P}^*)$ is a result of a standard duality argument, which also furnishes existence in $(\mathcal{P}^*)$.

In this work the general (FMD) problem is proved to be equivalent to the pair of problems $(\mathcal{P})$, $(\mathcal{P}^*)$, starting with the following variational equality:

Theorem 1.2 *The minimum value of compliance in (FMD) equals*

$$C_{\min} = \frac{1}{p' C_0^{p'-1}} Z^{p'} \quad (1.2)$$

where $Z = \sup \mathcal{P} = \inf \mathcal{P}^*$ and $p' = p/(p-1)$ is the Hölder conjugate exponent.

The equality (1.2) depends on $\rho, \rho^0 : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ being mutually polar real closed gauge functions constructed through finite dimensional programs:

$$\frac{1}{p}(\rho(\xi))^p = \max_{H \in \mathcal{H}_1} j(H, \xi), \quad \frac{1}{p'}(\rho^0(\sigma))^{p'} = \min_{H \in \mathcal{H}_1} j^*(H, \sigma), \quad (1.3)$$

where $\mathcal{H}_1$ is the convex compact subset of Hooke tensors $H \in \mathcal{H}$ satisfying the unit cost condition $c(H) \leq 1$; the convex conjugate function j^* is intended with respect to the second argument. Functions ρ, ρ^0 involve all the parameters $\mathcal{H}, j, c$; hence, the pair of gauges encodes in $(\mathcal{P})$, $(\mathcal{P}^*)$ the actual setting of the (FMD) problem.

In parallel to research on the Free Material Design problem the so-called *Mass Optimization Problem* (MOP) was put forth in [8]. In (MOP) we seek an elastic material or “mass” distribution $\mu \in \mathcal{M}_+(\overline{\Omega})$ that minimizes the compliance. In [8] one finds that (MOP) is equivalent to the pair of problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ as well; the authors gave a rigorous proof of the passage. One of the paramount differences between the two design problems, however, is the following: for (MOP) the functions ρ, ρ^0 are data that fully constitute the law of elasticity, whereas here the gauges are to be constructed in accordance with (1.3). The present work essentially adapts and generalizes the methods of (MOP) to provide a rigorous mathematical

framework for the family of (FMD) problems. We shall directly build upon the work [10] where the (MOP) theory is already further developed.

After [10], the two displacement-based maximization problems: the one in (1.1) and the problem $(\mathcal{P})$ require relaxation of the differentiability condition. Based on the compactness result established in [8], solution $\hat{u}$ of a relaxed problem $(\overline{\mathcal{P}})$ is attained on a set of continuous a.e. differentiable functions with $e(\hat{u}) \in L^\infty(\Omega; \mathcal{S}^{d \times d})$. A simple adaptation of the result from [10] proves that the same may be inferred about solution $\check{u}$ of (1.1) provided that $\lambda = \check{\lambda}$ is optimal for the (FMD) problem. Such a solution $\check{u}$ proves to be the displacement field in the optimally designed structure.

The optimal measure $\check{\lambda}$ may be decomposed to $\check{\lambda} = \check{\mathcal{C}}\check{\mu}$ where $\check{\mathcal{C}} \in L^\infty_\mu(\overline{\Omega}; \mathcal{H})$ with $c(\check{\mathcal{C}}) = 1$ $\check{\mu}$ -a.e.; the positive Radon measure $\check{\mu}$ plays the role similar to the mass distribution in [8, 10]. We may speak of a stress $\check{\sigma} \in L^{p'}_\mu(\overline{\Omega}; \mathcal{S}^{d \times d})$ in the optimal body – if the displacement field $\check{u}$ is smooth, the stress $\check{\sigma}$ is the function that satisfies the equilibrium equation and the constitutive law:

$$-\operatorname{div}(\check{\sigma}\check{\mu}) = F, \quad \check{\sigma} \in \partial j(\check{\mathcal{C}}, e(\check{u})) \quad \check{\mu}\text{-a.e.}, \quad (1.4)$$

where the subdifferential is taken with respect to the second argument of j . In general case $\check{\sigma}$ is the solution of the stress-based elasticity minimization problem that is dual to (1.1), cf. (3.9) in Sect. 3.2. The full information about the optimal design is encoded in the quadruple $(\check{u}, \check{\mu}, \check{\sigma}, \check{\mathcal{C}})$, more precisely:

Definition 1.2 We shall say that a quadruple: $\check{u} \in C(\overline{\Omega}; \mathbb{R}^d)$, $\check{\mu} \in \mathcal{M}_+(\overline{\Omega})$, $\check{\sigma} \in L^p_\mu(\overline{\Omega}; \mathcal{S}^{d \times d})$, and $\check{\mathcal{C}} \in L^\infty_\mu(\overline{\Omega}; \mathcal{H})$ solves the (FMD) problem whenever:

- (i) $\check{\lambda} = \check{\mathcal{C}}\check{\mu}$ solves the compliance minimization problem with $c(\check{\mathcal{C}}) = 1$ $\check{\mu}$ -a.e.;
- (ii) $\check{u}$ solves the relaxed displacement-based elasticity problem (1.1) for $\lambda = \check{\lambda}$;
- (iii) $\check{\sigma}$ solves the stress-based elasticity problem (3.9) for $\lambda = \check{\lambda}$.

In works [8, 10] the solution of (MOP) was recast based on the solution of the pair $(\mathcal{P})$, $(\mathcal{P}^*)$ or, as it was there referred to, of the *Linear Constrained Problem* (LCP). One of the main results in the present work generalizes those ideas towards the (FMD) problem. If by $\hat{\tau}$ we denote a solution of $(\mathcal{P}^*)$, we may decompose it to $\hat{\tau} = \hat{\sigma}\hat{\mu}$ where $\hat{\sigma} \in L^\infty_\mu(\overline{\Omega}; \mathcal{S}^{d \times d})$ satisfies $\rho^0(\hat{\sigma}) = 1$ $\hat{\mu}$ -a.e. In (LCP) the information on the Hooke tensor function $\mathcal{C} = \hat{\mathcal{C}}$ is implicit and may be point-wise recovered based on the finite dimensional minimization problem in (1.3): for any $\sigma \in \mathcal{S}^{d \times d}$ by $\mathcal{H}_1(\sigma)$ we shall understand the set of minimizers $H \in \mathcal{H}_1$ of $j^*(H, \sigma)$. The definition of quadruple solving (LCP) may be given:

Definition 1.3 By solution of (LCP) we mean a quadruple: $\hat{u} \in C(\overline{\Omega}; \mathbb{R}^d)$, $\hat{\mu} \in \mathcal{M}_+(\overline{\Omega})$, $\hat{\sigma} \in L^\infty_\mu(\overline{\Omega}; \mathcal{S}^{d \times d})$, $\hat{\mathcal{C}} \in L^\infty_\mu(\overline{\Omega}; \mathcal{H})$ such that:

- (i) $\hat{u}$ solves the relaxed problem $(\overline{\mathcal{P}})$;
- (ii) $\hat{\tau} = \hat{\sigma}\hat{\mu} \in \mathcal{M}(\overline{\Omega}; \mathcal{S}^{d \times d})$ solves $(\mathcal{P}^*)$ with $\rho^0(\hat{\sigma}) = 1$ $\hat{\mu}$ -a.e.;
- (iii) $\hat{\mathcal{C}}$ is any $\hat{\mu}$ -measurable selection of the multifunction $x \mapsto \mathcal{H}_1(\hat{\sigma}(x))$.

The main result, that allows to solve (FMD) through (LCP) may readily be stated:

Theorem 1.3 Assuming that $C_{\min} \in (0, \infty)$ let us choose a quadruple $\hat{u} \in C(\bar{\Omega}; \mathbb{R}^d)$, $\hat{\mu} \in \mathcal{M}_+(\bar{\Omega})$, $\hat{\sigma} \in L^1_{\hat{\mu}}(\bar{\Omega}; \mathcal{S}^{d \times d})$, and $\hat{\mathcal{C}} \in L^1_{\hat{\mu}}(\bar{\Omega}; \mathcal{H})$ and define

$$\check{\mathcal{C}} = \hat{\mathcal{C}}, \quad \check{\mu} = \frac{C_0}{Z} \hat{\mu}, \quad \check{\sigma} = \frac{Z}{C_0} \hat{\sigma}, \quad \check{u} = \left(\frac{Z}{C_0} \right)^{p'/p} \hat{u}. \quad (1.5)$$

Then quadruple $(\hat{u}, \hat{\mu}, \hat{\mathcal{C}}, \hat{\sigma})$ solves (LCP) if and only if $(\check{u}, \check{\mu}, \check{\mathcal{C}}, \check{\sigma})$ solves (FMD).

The proof of Theorem 1.3 is long yet not very technical as it mostly uses variational inequalities and equalities, including the one in Theorem 1.2. The proposed method of solving (FMD), however, cannot be deemed fully universal as long as existence of solution of (LCP) in the sense of Definition 1.3 is not assured. Existence of the optimal triple $\hat{u}, \hat{\mu}, \hat{\sigma}$ was already proved in [8,10]; here the novel question is as follows: does there always exist a $\hat{\mu}$ -measurable function $\hat{\mathcal{C}}$ that is point-wise optimal in the sense that $j^*(\hat{\mathcal{C}}(x), \hat{\sigma}(x)) = \min_{H \in \mathcal{H}_1} j^*(H, \hat{\sigma}(x))$ for $\hat{\mu}$ -a.e. x ? In some particular settings of (FMD) this is straightforward, e.g. for the AMD setting there exists an explicit formula: $H \in \mathcal{H}_1(\sigma) \Leftrightarrow H = \sigma \otimes \sigma$. The general statement is herein provided by the more technical Lemma 3.1 where the multifunction $\mathcal{S}^{d \times d} \ni \sigma \mapsto \mathcal{H}_1(\sigma) \in 2^{\mathcal{H}_1} \setminus \emptyset$ is proved to be upper semi-continuous, cf. [15].

The present paper is organized as follows. In Sect. 2 we study weak-* upper semi-continuity of the energy integral functional $\mathcal{M}(\bar{\Omega}; \mathcal{H}) \ni \lambda \mapsto \int j(\lambda, \epsilon) \in \mathbb{R}_+$ to guarantee existence of solution of (FMD). Section 3 is devoted to equivalence of the (FMD) problem and the abstract (LCP); Theorems 1.2 and 1.3 are proved, which delivers the main method of this work. Section 4 again generalizes the ideas of the work [10]: we give necessary and sufficient conditions for the quadruple $(u, \mu, \sigma, \mathcal{C})$ to solve the (FMD) problem; due to lack of differentiability of u in general we employ the tools of μ -tangential calculus introduced in [12] and developed in [9,10]. Section 5.1 presents a number of examples of settings of (FMD): e.g. by choosing $\mathcal{H}$ to be the set of isotropic Hooke tensors we recover the *Isotropic Material Design* (IMD) problem already considered in [17,21]; cf. also the work on the *Young Modulus Design* in [20]. Moreover, by utilizing the general framework of the herein developed theory, new settings are proposed, including the *Fibrous Material Design* (FibMD) problem that is proved to be equivalent to the renown Michell problem, cf. [28] or [13]. Finally, by employing the newly derived optimality conditions, analytical solutions of a simple design problem are given in Sect. 5.2 for the settings: AMD, IMD, FibMD.

2 Elastic energy as integral functional on the space of tensor valued measures. Existence of solution in the Free Material Design problem

2.1 Formulation of the design problem

Within a d -dimensional ambient space $\mathbb{R}^d$ we shall consider an elastic body: a plate in case of $d = 2$ or a solid for $d = 3$. Let us begin by discussing the convention for the finite dimensional spaces used in this text.

By $\mathcal{S}^{d \times d}$ we shall understand the space of symmetric $d \times d$ matrices. Since $\mathcal{S}^{d \times d}$ is isomorphic to the space of symmetric 2nd-order tensors we shall sometimes refer to its elements as: *strain tensors* $\xi \in \mathcal{S}^{d \times d}$ or *stress tensors* $\sigma \in \mathcal{S}^{d \times d}$. We endow $\mathcal{S}^{d \times d}$ with a scalar product $\langle \xi, \sigma \rangle := \sum_{i,j=1}^d \xi_{ij} \sigma_{ij}$ and the corresponding Euclidean norm $|\xi| := (\langle \xi, \xi \rangle)^{1/2}$. The eigenvalues of $\xi \in \mathcal{S}^{d \times d}$ shall be denoted by $\lambda_i(\xi)$, while the trace will read

Index of notation

General notation (V is any finite dimensional vector space):

$\mathbb{R}_+, \mathbb{R}_-$	the set of non-negative and non-positive real numbers
$\overline{\mathbb{R}}$	the extended real line $[-\infty, \infty]$
$\mathbb{R}^d, \mathcal{S}^{d-1}$	d -dimensional Euclidean space and its unit sphere
$\mathcal{S}^{d \times d}$	the space of symmetric $d \times d$ matrices
$\mathcal{S}_+^{d \times d}, \mathcal{S}_-^{d \times d}$	the cones of positive and negative semi-definite sym. $d \times d$ matrices
$x \otimes y$	the tensor product of vectors $x, y \in \mathbb{R}^d$, a $d \times d$ matrix
$x \overset{s}{\otimes} y$	the symmetric part of $x \otimes y$, element of $\mathcal{S}^{d \times d}$
I	the identity matrix, element of $\mathcal{S}^{d \times d}$
$\mathcal{L}(\mathcal{S}^{d \times d})$	the space of linear symmetric operators from $\mathcal{S}^{d \times d}$ to $\mathcal{S}^{d \times d}$
$\mathcal{L}_+(\mathcal{S}^{d \times d})$	the cone in $\mathcal{L}(\mathcal{S}^{d \times d})$ consisting of positive semi-definite operators H
$H = \sigma \otimes \sigma$	the tensor product of matrices $\sigma \in \mathcal{S}^{d \times d}$, an operator in $\mathcal{L}_+(\mathcal{S}^{d \times d})$
Id	the identity operator in $\mathcal{L}(\mathcal{S}^{d \times d})$
$\langle v_1, v_2 \rangle$	a scalar product for $v_1, v_2 \in V$
$ v $	the Euclidean norm $(\langle v, v \rangle)^{1/2}$ of a vector $v \in V$
$\lambda_i(\sigma), \lambda_i(H)$	the i -th eigenvalue of a matrix $\xi \in \mathcal{S}^{d \times d}$ or of an operator $H \in \mathcal{L}(\mathcal{S}^{d \times d})$
$\text{Tr } \sigma, \text{Tr } H$	the trace being the sum of the respective eigenvalues
$\chi_A : V \rightarrow \overline{\mathbb{R}}$	the indicator function of a set $A \subset V$
$\chi_A^* : V \rightarrow \overline{\mathbb{R}}$	the support function of a set $A \subset V$
$\text{co}(A)$	the convex hull of a set $A \subset V$
Ω	a bounded domain in $\mathbb{R}^d$
$C(\overline{\Omega}; V)$	V -valued continuous functions over a compact set $\overline{\Omega}$
$\ \cdot\ _\infty$	the supremum norm on Ω
$(\mathcal{D}(\mathbb{R}^d))^d$	d copies of the space of smooth functions with compact support in $\mathbb{R}^d$
$(\mathcal{D}'(\mathbb{R}^d))^d$	d copies of the space of distributions on $\mathbb{R}^d$
$e(u)$	the symmetric part of the gradient, i.e. $\frac{1}{2}(\nabla u + (\nabla u)^\top)$
$\mathcal{M}_+(\overline{\Omega})$	positive Radon measures μ on the compact set $\overline{\Omega}$
$\mathcal{L}^d, \mathcal{H}^k, \delta_{x_0}$	the Lebesgue, the k -dimensional Hausdorff, and the Dirac delta measures on $\mathbb{R}^d$
$\mathcal{M}(\overline{\Omega}; V)$	V -valued Radon measures on the compact set $\overline{\Omega}$
$L_\mu^q(\overline{\Omega}; V)$	V -valued μ -measurable functions integrable with an exponent $q \in [1, \infty]$

Notation specific to the topic of the paper:

p, p'	a fixed exponent $p \in (1, \infty)$ and its Hölder conjugate $p' = \frac{p}{p-1}$
$\mathcal{H}$	a closed convex cone contained in $\mathcal{L}_+(\mathcal{S}^{d \times d})$
$c : \mathcal{H} \rightarrow \mathbb{R}_+$	a norm on $\mathcal{L}(\mathcal{S}^{d \times d})$ restricted to $\mathcal{H}$
ξ, σ	strain and stress matrices, elements of $\mathcal{S}^{d \times d}$
$j = j(H, \xi)$	a real non-negative function on the product $\mathcal{H} \times \mathcal{S}^{d \times d}$
$j^* = j^*(H, \sigma)$	the convex conjugate of j with respect to the second argument
$\partial j(H, \xi) \subset \mathcal{S}^{d \times d}$	the subdifferential of j with respect to the second argument
λ	a <i>Hooke field</i> , an element of $\mathcal{M}(\overline{\Omega}; \mathcal{H})$
τ	a <i>stress field</i> , an element of $\mathcal{M}(\overline{\Omega}; \mathcal{S}^{d \times d})$
F	a <i>load field</i> , an element of $\mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$
$\sigma = \frac{d\tau}{d\mu}, \mathcal{C} = \frac{d\lambda}{d\mu}$	the Radon-Nikodym derivatives, elements of $L_\mu^1(\overline{\Omega}; \mathcal{S}^{d \times d}), L_\mu^1(\overline{\Omega}; \mathcal{H})$ resp.
$\text{div } \tau$	the distributional divergence of τ , an element of $(\mathcal{D}'(\mathbb{R}^d))^d$
$\rho, \rho^0 : \mathcal{S}^{d \times d} \rightarrow \overline{\mathbb{R}}$	a closed gauge $\rho = \rho(\xi)$ – a non-negative convex positively one-homogeneous lower semi-continuous function – and its polar $\rho^0 = \rho^0(\sigma)$

$\text{Tr } \xi := \sum_{i=1}^d \xi_{ii} = \sum_{i=1}^d \lambda_i(\xi)$. By a tensor product $x \otimes y$ of vectors $x, y \in \mathbb{R}^d$ we will understand a (possibly asymmetric) $d \times d$ matrix A whose components are $A_{ij} = x_i y_j$; it is the unique matrix that satisfies $Az = \langle y, z \rangle x$ for each $z \in \mathbb{R}^d$. Clearly $x \otimes x \in \mathcal{S}^{d \times d}$ for

any $x \in \mathbb{R}^d$, while $x \overset{s}{\otimes} y \in \mathcal{S}^{d \times d}$ will stand for the symmetrization $\frac{1}{2}(x \otimes y + y \otimes x)$. The identity matrix in $\mathcal{S}^{d \times d}$ will be denoted by $\mathbf{I}$.

The symbol $\mathcal{L}(\mathcal{S}^{d \times d})$ will stand for the space of symmetric linear operators from $\mathcal{S}^{d \times d}$ to $\mathcal{S}^{d \times d}$. We shall work with an abstract form of operators $H \in \mathcal{L}(\mathcal{S}^{d \times d})$, i.e. without invoking their matrix representation. The notion of an eigenvalue of an operator $\lambda_i(H)$ is meaningful for $i \in \{1, \dots, N(d)\}$ where $N(d) = \frac{1}{2}d(d+1)$ is the dimension of $\mathcal{S}^{d \times d}$ and definition of the trace follows: $\text{Tr } H = \sum_{i=1}^{N(d)} \lambda_i(H)$. The scalar product can be defined as $\langle H_1, H_2 \rangle := \frac{1}{2} \text{Tr}(H_1 \circ H_2 + H_2 \circ H_1)$ where $\circ$ stands for the composition of operators. The Euclidean norm reads $|H| := (\langle H, H \rangle)^{1/2} = (\text{Tr}(H \circ H))^{1/2}$. For $\sigma \in \mathcal{S}^{d \times d}$ by a tensor product $H = \sigma \otimes \sigma$ we understand the unique element of $\mathcal{L}(\mathcal{S}^{d \times d})$ satisfying: $H\xi = \langle \xi, \sigma \rangle \sigma$ for any $\xi \in \mathcal{S}^{d \times d}$. By $\text{Id} \in \mathcal{L}(\mathcal{S}^{d \times d})$ we will denote the identity operator, namely $\text{Id } \xi = \xi$ for all $\xi \in \mathcal{S}^{d \times d}$.

In the classical elasticity the anisotropy of the body is point-wise characterized by a *Hooke tensor*: a 4-th order tensor that enjoys certain symmetries and is positive semi-definite. In fact, the set of Hooke tensors is isomorphic to the closed convex cone

$$\mathcal{L}_+(\mathcal{S}^{d \times d}) := \left\{ H \in \mathcal{L}(\mathcal{S}^{d \times d}) : H \text{ is positive semi-definite} \right\}. \quad (2.1)$$

We thus agree that henceforward by Hooke tensors we shall mean the operators $H \in \mathcal{L}_+(\mathcal{S}^{d \times d})$ precisely (slightly abusing the terminology).

In the sequel we will restrict the admissible class of anisotropy by admitting Hooke tensors in a chosen subcone of $\mathcal{L}_+(\mathcal{S}^{d \times d})$, more accurately:

$\mathcal{H}$ is an arbitrary non-trivial closed convex cone contained in $\mathcal{L}_+(\mathcal{S}^{d \times d})$.

We now display some cases of cones $\mathcal{H}$ that will be of interest to us:

Example 2.1 The subcone $\mathcal{H}$ may be chosen so that the condition $H \in \mathcal{H}$ implies a certain type of anisotropy symmetry, for instance $\mathcal{H} = \mathcal{H}_{iso}$ will be the set of isotropic Hooke tensors; we have the characterization

$$\mathcal{H}_{iso} = \left\{ H \in \mathcal{L}_+(\mathcal{S}^{d \times d}) : H = dK \left(\frac{1}{d} \mathbf{I} \otimes \mathbf{I} \right) + 2G \left(\text{Id} - \frac{1}{d} \mathbf{I} \otimes \mathbf{I} \right), K, G \geq 0 \right\}, \quad (2.2)$$

where the non-negative numbers K, G are the so-called *bulk* and *shear moduli*, respectively. In the case of plane elasticity, i.e. $d = 2$, for later purposes we give the relation between the moduli and the pair: Young modulus E , Poisson ratio ν

$$E = 2 \left(\frac{1}{2K} + \frac{1}{2G} \right)^{-1} = \frac{4KG}{K+G}, \quad \nu = \frac{K-G}{K+G}. \quad (2.3)$$

It must be stressed that some symmetry classes generate cones that are non-convex. This is the case with classes that distinguish directions, e.g. orthotropy, cubic symmetry.

Example 2.2 Let us denote by $\mathcal{H}_{axial}$ the set of uni-axial Hooke tensors, i.e.

$$\mathcal{H}_{axial} = \left\{ H \in \mathcal{L}_+(\mathcal{S}^{d \times d}) : H = a (\eta \otimes \eta) \otimes (\eta \otimes \eta), a \geq 0, \eta \in \mathcal{S}^{d-1} \right\}$$

where by $\mathcal{S}^{d-1}$ we mean the unit sphere in $\mathbb{R}^d$. Above $A = \eta \otimes \eta$ is an element of $\mathcal{S}^{d \times d}$ whilst $A \otimes A$ is an operator. Further, we shall abuse notation by writing $\eta \otimes \eta \otimes \eta \otimes \eta := (\eta \otimes \eta) \otimes (\eta \otimes \eta)$. The set $\mathcal{H}_{axial}$ is clearly a cone yet it is non-convex for $d > 1$, and, thus,

a natural step is to consider the smallest closed convex cone containing $\mathcal{H}_{axial}$, i.e. its closed convex hull:

$$\mathcal{H} = \overline{\text{co}(\mathcal{H}_{axial})} = \text{co}(\mathcal{H}_{axial}),$$

where we used the fact that in a finite dimensional space the convex hull of a closed cone is closed. This family of Hooke tensors relates to materials that are made of 1D fibres.

The constitutive law, i.e. the point-wise relation between the stress tensor $\sigma \in \mathcal{S}^{d \times d}$ and the strain tensor $\xi \in \mathcal{S}^{d \times d}$, will be parameterized by the Hooke tensor $H \in \mathcal{H}$; therefore, the elastic energy will depend on two arguments:

$$j : \mathcal{H} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+. \quad (2.4)$$

Throughout the present section we shall assume that, for a chosen exponent $p \in (1, \infty)$, the function j satisfies assumption (H1)–(H4) given in the introduction; let us note that, for the time being, we shall not need the ellipticity assumption (H5). We henceforward agree that the subdifferential $\partial j(H, \xi)$ will be intended with respect to the second variable, and, similarly, we shall understand the convex conjugate j^* . This way the constitutive law $\sigma \in \partial j(H, \xi)$ may be rewritten as the equality $\langle \xi, \sigma \rangle = j(H, \xi) + j^*(H, \sigma)$.

Example 2.3 The simplest case of a function j for $p = 2$ is the one from linear elasticity:

$$j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle.$$

It is easy to see that the assumptions (H1)–(H4) are satisfied by the function above. The assumption (H5) is virtually put on the set $\mathcal{H}$ as it has to contain "enough" Hooke tensors.

In the optimization problem herein considered the Hooke field, being a tensor valued measure $\lambda \in \mathcal{M}(\Omega; \mathcal{H})$, plays a role of the design variable. A natural constraint to impose on λ is to bound its total cost by a constant C_0 ; therefore, we must choose a cost integrand $c : \mathcal{H} \rightarrow \mathbb{R}_+$ that satisfies essential properties: convexity, positive homogeneity, lower semi-continuity on $\mathcal{H}$, and $c(H) = 0 \Leftrightarrow H = 0$. Since $\mathcal{H}$ is a closed convex cone consisting of positively semi-definite tensors, for every non-zero $H \in \mathcal{H}$ we have $-H \notin \mathcal{H}$. Then, it is easily seen that every such function c extends to a norm on the whole space $\mathcal{L}(\mathcal{S}^{d \times d})$. It is thus suitable that

we choose the cost function c as restriction of any norm on $\mathcal{L}(\mathcal{S}^{d \times d})$ to $\mathcal{H}$.

Example 2.4 In the pioneering work on the Free Material Design problem [31] the cost function c was proposed as the trace function, i.e.

$$c(H) = \text{Tr } H = \sum_{i=1}^{N(d)} \lambda_i(H) \quad \forall H \in \mathcal{H}.$$

This is an exceptional example of a cost function c for it is linear on $\mathcal{H}$.

By $\Omega \subset \mathbb{R}^d$ we shall understand a bounded connected open set. According to Radon-Nikodym theorem any measure $\lambda \in \mathcal{M}(\overline{\Omega}; \mathcal{H})$ can be decomposed as follows:

$$\lambda = \mathcal{C} \mu, \quad \mu \in \mathcal{M}_+(\overline{\Omega}), \quad \mathcal{C} \in L^\infty_\mu(\overline{\Omega}; \mathcal{H}), \quad c(\mathcal{C}) = 1 \quad \mu\text{-a.e.}, \quad (2.5)$$

that is μ can be computed as the variation measure $c(\lambda)$, while $\mathcal{C}$ is the Radon-Nikodym derivative $d\lambda/(d c(\lambda))$. The condition that $\mathcal{C}(x) \in \mathcal{H}$ for μ -a.e. x virtually defines

$\mathcal{M}(\overline{\Omega}; \mathcal{H})$ as a subset of the Banach space $\mathcal{M}(\overline{\Omega}; \mathcal{L}(S^{d \times d}))$; since $\mathcal{H}$ is a cone, this definition is meaningful as it does not depend on the norm c .

In (2.5) the information on the Hooke tensor field λ has been split into two: information on the distribution of elastic material μ and information on the anisotropy $\mathcal{C}$. The measure μ , whose support may be a proper subset of $\overline{\Omega}$, determines the topology and shape of the elastic body. Additional geometric features of the design may be point-wise identified via the space tangent to measure T_μ (see e.g. [12]) – measure λ may account for lower dimensional structural elements such as membrane shells, ribs, or bars.

As already announced in the introduction, we state the *Free Material Design* problem (FMD) of designing in a feasible domain $\overline{\Omega}$ an elastic anisotropic body of minimum compliance under a load being vector valued measure $F \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$:

$$C_{\min} = \min \left\{ \mathcal{C}(\lambda) : \lambda \in \mathcal{M}(\overline{\Omega}; \mathcal{H}), \int c(\lambda) \leq C_0 \right\} \quad (\text{FMD})$$

where compliance $\mathcal{C}(\lambda)$ is defined variationally in (1.1).

2.2 Elastic energy as integral functional on the space of tensor valued measures. The existence result

The compliance functional $\mathcal{C} : \mathcal{M}(\overline{\Omega}; \mathcal{H}) \rightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ is a point-wise supremum of a family of functionals $\lambda \mapsto \int \langle u, F \rangle - \int j(\lambda, e(u))$ parameterized by smooth function $u \in (\mathcal{D}(\mathbb{R}^d))^d$, cf. the definition (1.1). It is well established (see e.g. [22]) that to show convexity and weak-* lower semi-continuity of $\mathcal{C}$ it is enough to show this property for each functional in such family or, in this particular case, we have to prove that for any continuous strain field $\epsilon \in C(\overline{\Omega}; S^{d \times d})$ the functional

$$\mathcal{M}(\overline{\Omega}; \mathcal{H}) \ni \lambda \mapsto \int j(\lambda, \epsilon) := \int j(\mathcal{C}(x), \epsilon(x)) \mu(dx) \quad (2.6)$$

is concave and weak-* upper semi-continuous (above we have utilized the decomposition $\lambda = \mathcal{C}\mu$ from (2.5) and one-homogeneity of j with respect to the first argument). Concavity follows directly from assumption (H3), and the present subsection is devoted to proving the upper semi-continuity result. We shall start by proving that the integrand j itself is u.s.c. jointly in both arguments. Beforehand we make an observation:

Remark 2.1 In convex analysis a convex function restricted to a convex subset of a linear space can be equivalently treated as a function defined on the whole space if extended by $+\infty$. Since the real function $j : \mathcal{H} \times S^{d \times d} \rightarrow \mathbb{R}_+$ is concave with respect to the first variable H , we can by analogy speak of an extended real function $j : \mathcal{L}(S^{d \times d}) \times S^{d \times d} \rightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ such that $j(H, \xi) = -\infty$ for any $\xi \in S^{d \times d}$ and any $H \in \mathcal{L}(S^{d \times d}) \setminus \mathcal{H}$. This way, the functional (2.6) may be naturally extended to the whole Banach space of Radon measures $\mathcal{M}(\overline{\Omega}; \mathcal{L}(S^{d \times d}))$. Then, the condition on the energy $\int j(\lambda, \epsilon) = \int j(\mathcal{C}, \epsilon) d\mu > -\infty$ enforces that $\mathcal{C} \in \mathcal{H}$ μ -a.e. and, therefore, λ must lie in the set $\mathcal{M}(\overline{\Omega}; \mathcal{H})$. This is convenient since in the (FMD) problem we may only require that $\lambda \in \mathcal{M}(\overline{\Omega}; \mathcal{L}(S^{d \times d}))$, thus avoiding analysis of the weak-* closedness of $\mathcal{M}(\overline{\Omega}; \mathcal{H})$.

Proposition 2.1 *The function j is upper semi-continuous on the product $\mathcal{L}(S^{d \times d}) \times S^{d \times d}$, i.e. jointly in variables H and ξ .*

Proof We fix a pair $(\check{H}, \check{\xi}) \in \mathcal{H} \times \mathcal{S}^{d \times d}$. Let us take any ball $U \subset \mathcal{L}(\mathcal{S}^{d \times d})$ centred at $\check{H}$ and introduce a compact set $K = \overline{U} \cap \mathcal{H}$. We observe that for any fixed $\xi \in \mathcal{S}^{d \times d}$ the set $\{j(H, \xi) : H \in K\}$ is bounded in $\mathbb{R}$. The zero lower bound follows from non-negativity of $j|_{\mathcal{H}}$, whereas, since $j(\cdot, \xi)$ is real-valued concave and upper semi-continuous on $\mathcal{H}$, it achieves its finite maximum on K . According to [32, Theorem 10.6], the shown point-wise boundedness combined with convexity of every $j(H, \cdot)$ imply that the family of functions $\{j(H, \cdot) : H \in K\}$ is equi-continuous on any bounded subset of $\mathcal{S}^{d \times d}$. Upon fixing $\varepsilon > 0$ we may thus choose $\delta_1 > 0$ such that

$$|j(H, \xi) - j(H, \check{\xi})| < \frac{\varepsilon}{2} \quad \forall \xi \in B(\check{\xi}, \delta_1) \subset \mathcal{S}^{d \times d}, \quad \forall H \in K \subset \mathcal{H},$$

where it must be stressed that K does not depend on ε . Due to the upper semi-continuity of $j(\cdot, \check{\xi})$, we can also choose $\delta_2 > 0$ for which $B(\check{H}, \delta_2) \subset U$ and

$$j(H, \check{\xi}) < j(\check{H}, \check{\xi}) + \frac{\varepsilon}{2} \quad \forall H \in B(\check{H}, \delta_2).$$

For any pair $(H, \xi) \in (B(\check{H}, \delta_2) \cap \mathcal{H}) \times B(\check{\xi}, \delta_1)$ we therefore obtain

$$j(H, \xi) = j(H, \check{\xi}) + (j(H, \xi) - j(H, \check{\xi})) < j(\check{H}, \check{\xi}) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

which proves that j is upper semi-continuous on $\mathcal{H} \times \mathcal{S}^{d \times d}$ being a convex closed subset of $\mathcal{L}(\mathcal{S}^{d \times d}) \times \mathcal{S}^{d \times d}$. Extending j by $-\infty$ guarantees it is u.s.c on $\mathcal{L}(\mathcal{S}^{d \times d}) \times \mathcal{S}^{d \times d}$. $\square$

Instead of proving upper semi-continuity of the concave functional (2.6), we may focus on lower semi-continuity of the convex functional $\lambda \mapsto \int j^-(\lambda, \epsilon)$ where simply $j^- := -j$. Then, the idea is to use the classical Reshetnyak's theorem; however, non-positivity of the integrand j^- requires an additional trick involving minorization of j^- :

Proposition 2.2 *For any $\epsilon \in C(\overline{\Omega}; \mathcal{S}^{d \times d})$ the functional $\lambda \mapsto \int j(\lambda, \epsilon)$ is concave and weakly-* upper semi-continuous on the space $\mathcal{M}(\overline{\Omega}; \mathcal{L}(\mathcal{S}^{d \times d}))$.*

Proof The idea is to show there exists a continuous function $G : \mathcal{S}^{d \times d} \rightarrow (\mathcal{L}(\mathcal{S}^{d \times d}))^* \equiv \mathcal{L}(\mathcal{S}^{d \times d})$ such that for every $\xi \in \mathcal{S}^{d \times d}$ we obtain a minorization $\langle G(\xi), \cdot \rangle \leq j^-(\cdot, \xi)$ on $\mathcal{L}(\mathcal{S}^{d \times d})$ where $j^- := -j$. Once this is established, we define $g : \overline{\Omega} \times \mathcal{L}(\mathcal{S}^{d \times d}) \rightarrow \mathbb{R}$ by

$$g(x, H) := j^-(H, \epsilon(x)) - \langle G(\epsilon(x)), H \rangle.$$

Since G is continuous and j^- is lower semi-continuous jointly on $\mathcal{L}(\mathcal{S}^{d \times d}) \times \mathcal{S}^{d \times d}$ by Proposition 2.1, we see by uniform continuity of ϵ that the function g is lower semi-continuous jointly on $\overline{\Omega} \times \mathcal{L}(\mathcal{S}^{d \times d})$. Then, owing to non-negativity of g and its convexity together with positive one-homogeneity with respect to the second variable, the functional $\lambda \mapsto \int g(x, \lambda(dx))$ is weakly-* lower semi-continuous on $\mathcal{M}(\overline{\Omega}; \mathcal{L}(\mathcal{S}^{d \times d}))$ due to Reshetnyak's theorem, cf. [2, Theorem 2.38]. We observe that for any $\epsilon \in C(\overline{\Omega}; \mathcal{S}^{d \times d})$

$$\int j^-(\lambda, \epsilon) = \int g(x, \lambda(dx)) + \int \langle G(\epsilon(x)), \lambda(dx) \rangle;$$

hence, the functional $\lambda \mapsto \int j^-(\lambda, \epsilon)$ is a sum of a continuous linear functional (the function $G \circ \epsilon : \overline{\Omega} \rightarrow \mathcal{L}(\mathcal{S}^{d \times d})$ is uniformly continuous) and weakly-* lower semi-continuous functional on $\mathcal{M}(\overline{\Omega}; \mathcal{L}(\mathcal{S}^{d \times d}))$, which through equality $\int j(\lambda, \epsilon) = -\int j^-(\lambda, \epsilon)$ furnishes the thesis.

To conclude the proof we must therefore show existence of the function G . First we show that for every $\xi \in S^{d \times d}$ the proper convex l.s.c. function $j^-(\cdot, \xi) : \mathcal{L}(S^{d \times d}) \rightarrow \overline{\mathbb{R}}$ is subdifferentiable at the origin, i.e. $\partial_1 j^-(0, \xi) \neq \emptyset$ where in this proof by $\partial_1 j^-$ we shall understand the subdifferential with respect to the first argument. By [32, Theorem 23.3], the scenario $\partial_1 j^-(0, \xi) = \emptyset$ can occur only if there exists a direction $\Delta H \in \mathcal{L}(S^{d \times d})$ such that the directional derivative with respect to the first argument $j_{\Delta H}^-(0, \xi)$ equals $-\infty$. Since $j^-(\cdot, \xi)$ is positively one-homogeneous, our argument for subdifferentiability at the origin amounts to verifying that $j^-(H, \xi) > -\infty$ for every H in a unit sphere in $\mathcal{L}(S^{d \times d})$. But this is trivial since, by definition, the function j^- does not take the value $-\infty$.

We have thus arrived at a multifunction $\Gamma : S^{d \times d} \ni \xi \mapsto \partial_1 j^-(0, \xi) \in (2^{\mathcal{L}(S^{d \times d})} \setminus \emptyset)$ that is convex-valued and closed-valued. According to [29, Theorem 3.2"], in order to show that there exists a continuous selection G of Γ it suffices to show that Γ is l.s.c. (in the sense of theory of multifunctions). This boils down to proving that the function $(\Delta H, \xi) \mapsto \chi_{\partial_1 j^-(0, \xi)}^*(\Delta H)$ is l.s.c. on $\mathcal{L}(S^{d \times d}) \times S^{d \times d}$ where χ_A^* stands for the support function of a set $A \subset \mathcal{L}(S^{d \times d})$, see e.g. [11, Lemma A2]. If ξ is fixed, then [32, Theorem 23.2] says that $\Delta H \mapsto \chi_{\partial_1 j^-(0, \xi)}^*(\Delta H)$ is the closure (l.s.c. envelope) of the mapping $\Delta H \mapsto f(\Delta H)$ where $f(\Delta H)$ is the directional derivative of $j^-(\cdot, \xi)$ evaluated at $H = 0$ in the direction ΔH . Due to the one-homogeneity of $j^-(\cdot, \xi)$, we have $f(\Delta H) = j^-(\Delta H, \xi)$; therefore, f is l.s.c. owing to (H3). Ultimately $\chi_{\partial_1 j^-(0, \xi)}^*(\Delta H) = j^-(\Delta H, \xi)$, and all amounts to showing lower semi-continuity of $j^- = -j$ on $\mathcal{L}(S^{d \times d}) \times S^{d \times d}$, which is guaranteed by Proposition 2.1 herein. $\square$

The first existence result may readily be proved:

Proposition 2.3 *Assuming that j satisfies (H1)–(H4) the Free Material Design problem (FMD) admits a solution whenever $C_{\min} < \infty$.*

Proof Compliance functional $\mathcal{C}$ is a point-wise supremum of functionals $\lambda \mapsto \int \langle u, F \rangle - \int j(\lambda, e(u))$ that are convex and weakly-* lower semi-continuous due to Proposition 2.2. Therefore $\mathcal{C}$ is itself convex and weakly-* l.s.c. Since c is a norm, minimization in (FMD) runs over a compact subset of $\mathcal{M}(\overline{\Omega}; \mathcal{L}(S^{d \times d}))$ (see Remark 2.1), and the thesis follows by the Direct Method of Calculus of Variations. $\square$

We note that Proposition 2.3 does not employ the ellipticity condition (H5). In Sect. 3 we shall show that, under this additional assumption, finiteness of $C_{\min}$ is equivalent to load F being balanced, which will deliver Theorem 1.1 announced in the introduction.

3 The link between the free material design problem and the linear constrained problem

3.1 The linear constrained problem and the variational equality

We start the passage to the Linear Constrained Problem by showing the variational equality constituting Theorem 1.2 announced in the introduction:

$$C_{\min} = \frac{1}{p' C_0^{p'-1}} Z^{p'} \quad (3.1)$$

where Z is the supremum in the primal problem

$$Z := \sup \left\{ \int \langle u, F \rangle : u \in (\mathcal{D}(\mathbb{R}^d))^d, \rho(e(u)) \leq 1 \text{ in } \overline{\Omega} \right\} \quad (\mathcal{P})$$

With definition (1.1) of $\mathcal{C}(\lambda)$ plugged into (FMD) problem, we arrive at a min-max problem:

$$C_{\min} = \inf_{\substack{\lambda \in \mathcal{M}(\overline{\Omega}; \mathcal{H}), \\ \int c(\lambda) \leq C_0}} \sup_{u \in (\mathcal{D}(\mathbb{R}^d))^d} \left\{ \int \langle u, F \rangle - \int j(\lambda, e(u)) \right\}. \quad (3.2)$$

We shall see that inf and sup above can be swapped, which will allow to formulate a variant of [10, Theorem 1], but first we introduce some additional notions. The function $\bar{j} : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ shall represent the strain energy that is maximal with respect to the admissible anisotropy represented by Hooke tensor $H \in \mathcal{H}$ of a unit c -cost:

$$\bar{j}(\xi) := \sup_{H \in \mathcal{H}_1} j(H, \xi), \quad \mathcal{H}_1 := \left\{ H \in \mathcal{H} : c(H) \leq 1 \right\}. \quad (3.3)$$

As a point-wise supremum of a family of convex functions $\{j(H, \cdot) : H \in \mathcal{H}_1\}$ the function $\bar{j}$ is convex as well. Furthermore, since each $j(H, \cdot)$ is positively p -homogeneous by assumption (H2), the function $\bar{j}$ inherits this property. Next, due to concavity and upper semi-continuity of $j(\cdot, \xi)$ together with compactness of $\mathcal{H}_1$, we see that $\bar{j}(\xi) = \max_{H \in \mathcal{H}_1} j(H, \xi) = j(\bar{H}_\xi, \xi)$ for some $\bar{H}_\xi \in \mathcal{H}_1$, and, in particular, $\bar{j}$ is finite on $\mathcal{S}^{d \times d}$, which, in conjunction with its convexity, renders continuity of $\bar{j}$. It is natural to define

$$\bar{\mathcal{H}}_1(\xi) := \arg \max_{H \in \mathcal{H}_1} j(H, \xi)$$

being a non-empty, convex, and compact subset of $\mathcal{H}_1$ for every $\xi \in \mathcal{S}^{d \times d}$. The short over-bar $\bar{\cdot}$ will be consistently used in the sequel to mark maximization with respect to Hooke tensor and should not be confused with long over-bar $\overline{\cdot}$ denoting e.g. the closure of a set.

We have just showed that $\bar{j}$ is a convex, continuous, and positively p -homogeneous function, and it is well-known (see [32, Corollary 15.3.1]) that it can be written as

$$\bar{j}(\xi) = \frac{1}{p} (\rho(\xi))^p, \quad (3.4)$$

where $\rho : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ is a *closed gauge* – a non-negative, convex, lower semi-continuous, and positively one-homogeneous function. Since ρ is finite valued it is in fact continuous.

We are ready to prove the first link between problems (FMD) and (P) being the equality in Theorem 1.2:

Proof of Theorem 1.2 In (3.2) the set over which the infimum is taken is weakly- $*$ compact. Therefore, by acknowledging Proposition 2.2, we easily verify the assumptions of Ky Fan's min-max theorem (cf. [37, Theorem 2.10.2]), which allows to interchange inf and sup:

$$C_{\min} = \sup_{u \in (\mathcal{D}(\mathbb{R}^d))^d} \left\{ \int \langle u, F \rangle - \bar{J}(e(u)) \right\},$$

where for any continuous stress field $\epsilon \in C(\overline{\Omega}; \mathcal{S}^{d \times d})$ we define

$$\bar{J}(\epsilon) = \sup_{\substack{\lambda \in \mathcal{M}(\overline{\Omega}; \mathcal{H}) \\ \int c(\lambda) \leq C_0}} \int j(\lambda, \epsilon) = \sup_{\substack{\mu \in \mathcal{M}_+(\overline{\Omega}), \int d\mu \leq C_0 \\ \mathcal{C} \in L^\infty_\mu(\overline{\Omega}; \mathcal{H}), c(\mathcal{C})=1}} \int j(\mathcal{C}, \epsilon) d\mu,$$

where we decomposed λ to $\mathcal{C}\mu$ with $c(\mathcal{C}) = 1$ μ -a.e. (the symbol $\bar{J}$ is not to be confused with l.s.c. regularization of a functional J). Further we fix a strain field ϵ . For any pair $\mathcal{C}, \mu$ admissible above we easily find an estimate

$$\int j(\mathcal{C}, \epsilon) d\mu \leq \int \bar{j}(\epsilon) d\mu \leq \|\bar{j}(\epsilon)\|_{\infty} \int d\mu \leq C_0 \|\bar{j}(\epsilon)\|_{\infty},$$

which yields $\bar{J}(\epsilon) \leq C_0 \|\bar{j}(\epsilon)\|_{\infty}$. We shall show that the right hand side of this inequality is attainable for a certain candidate $\bar{\lambda}_{\epsilon}$.

Due to the continuity of $\bar{j}$ and of ϵ , the function $\bar{j}(\epsilon(\cdot))$ is continuous on the compact set $\bar{\Omega}$ as well, and, thus, there exists $\bar{x} \in \bar{\Omega}$ such that $\|\bar{j}(\epsilon)\|_{\infty} = \bar{j}(\epsilon(\bar{x}))$. We put $\bar{\lambda}_{\epsilon} = C_0 \bar{H}_{\epsilon(\bar{x})} \delta_{\bar{x}}$ where $\bar{H}_{\epsilon(\bar{x})}$ is any Hooke tensor from the non-empty set $\mathcal{H}_1(\epsilon(\bar{x}))$, and $\delta_{\bar{x}}$ is the Dirac delta measure at $\bar{x}$. It is trivial to check that $\int c(\bar{\lambda}_{\epsilon}) = C_0$, whilst

$$\int j(\bar{\lambda}_{\epsilon}, \epsilon) = C_0 \int j(\bar{H}_{\epsilon(\bar{x})}, \epsilon(x)) \delta_{\bar{x}}(dx) = C_0 j(\bar{H}_{\epsilon(\bar{x})}, \epsilon(\bar{x})) = C_0 \bar{j}(\epsilon(\bar{x})),$$

which proves that indeed $\bar{J}(\epsilon) = C_0 \|\bar{j}(\epsilon)\|_{\infty}$ or that $\bar{J}(\epsilon) = \frac{C_0}{p} (\|\rho(\epsilon)\|_{\infty})^p$.

Next we use a technique that was e.g. applied in [24]: by substitution $u = t u_1$ we obtain

$$\begin{aligned} C_{\min} &= \sup_{u \in (\mathcal{D}(\mathbb{R}^d))^d} \left\{ \int \langle u, F \rangle - \frac{C_0}{p} (\|\rho(e(u))\|_{\infty})^p \right\} \\ &= \sup_{u_1 \in (\mathcal{D}(\mathbb{R}^d))^d, t \geq 0} \left\{ \left(\int \langle u_1, F \rangle \right) t - \frac{C_0}{p} t^p : \|\rho(e(u_1))\|_{\infty} \leq 1 \right\} \\ &= \sup_{u_1 \in (\mathcal{D}(\mathbb{R}^d))^d} \left\{ \frac{1}{p' C_0^{p'-1}} \left(\int \langle u_1, F \rangle \right)^{p'} : \|\rho(e(u_1))\|_{\infty} \leq 1 \right\}, \end{aligned}$$

where, under the assumption that $\int \langle u_1, F \rangle$ is non-negative, in the last step we have computed the maximum with respect to t which was attained for $\bar{t} = (\int \langle u_1, F \rangle / C_0)^{p'-1}$. Since the function $(\cdot)^{p'}$ is increasing for non-negative arguments, the thesis follows. $\square$

The remainder of this subsection focuses on the analysis of $(\mathcal{P})$ and its dual $(\mathcal{P}^*)$. This pair of problems has been already studied in [8, 10, 13]. In what follows we shall rely on the arguments known from those papers, and the details shall be skipped for brevity.

The closed gauge function $\rho^0 : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ will stand for the polar to ρ ; namely, for a stress tensor $\sigma \in \mathcal{S}^{d \times d}$

$$\rho^0(\sigma) = \sup_{\xi \in \mathcal{S}^{d \times d}} \left\{ \langle \xi, \sigma \rangle : \rho(\xi) \leq 1 \right\},$$

where we recall that $\langle \xi, \sigma \rangle := \sum_{i,j=1}^d \xi_{ij} \sigma_{ij}$. Then, to any measure $\tau \in \mathcal{M}(\bar{\Omega}; \mathcal{S}^{d \times d})$ we may assign a non-negative Borel measure $\rho^0(\tau)$ that for any Borel set $B \subset \bar{\Omega}$ can be defined via the integral formula $\rho^0(\tau)(B) := \int_B \rho^0\left(\frac{d\tau}{d\mu}(x)\right) \mu(dx)$ where $\mu \in \mathcal{M}_+(\bar{\Omega})$ is any measure with respect to which τ is absolutely continuous. Owing to [33, Theorem 6], we obtain the integral representation of the support function of a set in $C(\bar{\Omega}; \mathcal{S}^{d \times d})$:

$$\int \rho^0(\tau) = \sup \left\{ \int \langle \epsilon, \tau \rangle : \epsilon \in C(\bar{\Omega}; \mathcal{S}^{d \times d}), \rho(\epsilon) \leq 1 \text{ in } \bar{\Omega} \right\}. \quad (3.5)$$

The function ρ is a continuous gauge; hence, there exists a constant $C_2 > 0$ such that $\|\rho(\epsilon)\|_{\infty} \leq C_2 \|\epsilon\|_{\infty}$ for each $\epsilon \in C(\bar{\Omega}; \mathcal{S}^{d \times d})$. As a result, the set $\{\epsilon \in C(\bar{\Omega}; \mathcal{S}^{d \times d}) : \rho(\epsilon) \leq 1 \text{ in } \bar{\Omega}\}$

$\rho(\epsilon) \leq 1$ in $\overline{\Omega}$ has a non-empty interior in $C(\overline{\Omega}; \mathcal{S}^{d \times d})$ (with respect to the topology of uniform convergence). Then, by duality argument that employs a standard algorithm from [22, Chapter III], we arrive at the dual problem with the zero duality gap:

$$Z = \inf \left\{ \int \rho^0(\tau) : \tau \in \mathcal{M}(\overline{\Omega}; \mathcal{S}^{d \times d}), -\operatorname{div} \tau = F \right\} \quad (\mathcal{P}^*)$$

where the equilibrium equation is intended in the sense of distributions on the whole space $\mathbb{R}^d$, more precisely:

$$-\operatorname{div} \tau = F \quad \Leftrightarrow \quad \int \langle e(\varphi), \tau \rangle = \int \langle \varphi, F \rangle \quad \forall \varphi \in (\mathcal{D}(\mathbb{R}^d))^d.$$

The tensor valued measure $\tau \in \mathcal{M}(\overline{\Omega}; \mathcal{S}^{d \times d})$ models the *stress field*. With the trial functions φ treated as *virtual displacement functions*, the right hand side above is known as the *virtual work principle*. Note that φ above may not vanish on $\partial\Omega$, and, therefore, a Neumann boundary condition is accounted for in $-\operatorname{div} \tau = F$, possibly a non-homogeneous one if F charges $\partial\Omega$. Solution $\hat{\tau}$ of $(\mathcal{P}^*)$ exists as long as Z is finite, which is a part of the duality result. According to [13, Proposition 2.1], for existence of a stress field τ that equilibrates a load F it is necessary and sufficient that the latter is *balanced* in the sense of Definition 1.1 (we shall independently recover this result).

Let us define the space $\mathcal{U}_0$ of *rigid body displacement functions*:

$$\mathcal{U}_0 := \left\{ u \in (\mathcal{D}(\mathbb{R}^d))^d : e(u) = 0 \text{ in } \overline{\Omega} \right\}.$$

It is well established that $\mathcal{U}_0$ consists precisely of those functions $u \in (\mathcal{D}(\mathbb{R}^d))^d$ which for $x \in \overline{\Omega}$ satisfy $u(x) = Ax + b$, where $A \in \mathbb{R}^{d \times d}$ is a skew-symmetric matrix and $b \in \mathbb{R}^d$. Consequently, it is straightforward to show that a load $F \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^d)$ is balanced if and only if $\int \langle u, F \rangle = 0$ for any $u \in \mathcal{U}_0$.

At this point of the work the ellipticity assumption (H5) enters. It may seem weak as it allows degenerate tensors $H \in \mathcal{H}$ for which there exist non-zero strains ξ such that $j(H, \xi) = 0$. Nevertheless, (H5) guarantees that $\bar{j}(\xi) = 0$ if and only if $\xi = 0$, and, therefore, the same must hold for ρ . It is thus straightforward that:

Proposition 3.1 *Under assumptions (H1)–(H5) the mutually polar functions $\rho, \rho^0 : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ are finite, continuous, convex positively one-homogeneous, and they satisfy for some positive constants C_1, C_2 :*

$$\begin{aligned} C_1 |\xi| &\leq \rho(\xi) \leq C_2 |\xi| & \forall \xi \in \mathcal{S}^{d \times d}, \\ \frac{1}{C_2} |\sigma| &\leq \rho^0(\sigma) \leq \frac{1}{C_1} |\sigma| & \forall \sigma \in \mathcal{S}^{d \times d}. \end{aligned}$$

In other words, ρ, ρ^0 are mutually dual norms on $\mathcal{S}^{d \times d}$ if and only if they are symmetric.

Since in general the problem $(\mathcal{P})$ does not attain a solution, it must be relaxed, which shall consist in taking the closure of the set of admissible smooth displacement functions

$$\mathcal{U}_1 := \left\{ u \in (\mathcal{D}(\mathbb{R}^d))^d : \rho(e(u)) \leq 1 \text{ in } \overline{\Omega} \right\} \quad (3.6)$$

in the topology of uniform convergence on $\overline{\Omega}$ and showing suitable compactness. Clearly, as a subset of $C(\overline{\Omega}; \mathbb{R}^d)$, the set $\mathcal{U}_1$ is unbounded for it contains the linear subspace $\mathcal{U}_0$. The

"rigid body part" of a function u can be eliminated by means of a linear projection operator $P_{\mathcal{U}_0} : (\mathcal{D}(\mathbb{R}^d))^d \rightarrow \mathcal{U}_0$. It may be defined such that

$$(P_{\mathcal{U}_0}u)(x) = a_0(u) + A_0(u)x \quad \forall x \in \overline{\Omega}$$

where $A_0(u) := \int_{\Omega} \frac{1}{2}(\nabla u - (\nabla u)^{\top}) d\mathcal{L}^d$ and $a_0(u) := \int_{\Omega} (u(x) - A_0(u)x) d\mathcal{L}^d(dx)$ with $\int_{\Omega} (\cdot) d\mathcal{L}^d = \frac{1}{|\Omega|} \int_{\Omega} (\cdot) d\mathcal{L}^d$.

Thanks to coerciveness of ρ stated in Proposition 3.1, the compactness result can be obtained identically as in [8]: it requires using Korn's inequality in L^q twice for some $q > d$ and then exploiting the Morrey's embedding theorem.

Proposition 3.2 *Let Ω be a bounded domain with Lipschitz boundary. Then, under assumptions (H1)–(H5) the set $\{u - P_{\mathcal{U}_0}u : u \in \mathcal{U}_1\}$ is pre-compact as a subset of the space $C(\overline{\Omega}; \mathbb{R}^d)$ endowed with the topology of uniform convergence.*

Proof See [8, Proposition 3.1]. $\square$

With the compactness result at our disposal, we may readily propose the relaxation of the problem $(\mathcal{P})$:

$$\sup \left\{ \int \langle u, F \rangle : u \in \overline{\mathcal{U}}_1 \right\} \quad (\overline{\mathcal{P}})$$

where $\overline{\mathcal{U}}_1 \subset C(\overline{\Omega}; \mathbb{R}^d)$ is the closure of $\mathcal{U}_1$ in the topology of uniform convergence on $\overline{\Omega}$. Let us note that each function $u \in \overline{\mathcal{U}}_1$ is $\mathcal{L}^d$ -a.e. differentiable with $e(u) \in L^\infty(\Omega; \mathcal{S}^{d \times d})$, yet there are $u \notin \text{Lip}(\Omega; \mathbb{R}^d)$ belonging to $\overline{\mathcal{U}}_1$, which is reflected in the lack of Korn's inequality in L^∞ .

Recall that F is balanced if and only if $\int \langle u, F \rangle = 0 \quad \forall u \in \mathcal{U}_0$. Then, once F is balanced, it is straightforward to show that any maximizing sequence $u_n \in \mathcal{U}_1$ can be modified to another maximizing sequence $u_n - P_{\mathcal{U}_0}u_n$. As a result we obtain:

Corollary 3.1 *Under the prerequisites of Proposition 3.2 the number Z is finite if and only if F is balanced. In that case*

$$Z = \max \overline{\mathcal{P}} = \min \mathcal{P}^*. \quad (3.7)$$

In particular, problems $(\overline{\mathcal{P}})$ and $(\mathcal{P}^)$ attain their solutions.*

Thanks to Proposition 3.1 and equality $Z = \inf \mathcal{P}^*$, from Corollary 3.1 we recover the result that the equilibrium equation $-\text{div } \tau = F$ has a solution τ if and only if F is balanced. Finally, by combining Corollary 3.1 and Proposition 2.3, we establish the existence result for the (FMD) problem stated in Theorem 1.1 in the introduction.

In the sequel of this work we assume that $\partial\Omega$ is Lipschitz regular and that (H1)–(H5) hold for the function j (unless clearly stressed otherwise).

3.2 The displacement and stress solutions of the elasticity problem

Classically, we say that a vector displacement function u and a tensor stress function σ solve the elasticity problem for a load F whenever: (i) σ equilibrates the load F ; (ii) σ and the strain $e(u)$ point-wise satisfy the constitutive law of elasticity. With the Hooke tensor field given by a measure $\lambda = \mathcal{C}\mu \in \mathcal{M}(\overline{\Omega}; \mathcal{H})$ the two conditions may be written as (i) $-\text{div}(\sigma\mu) = F$ and (ii) $\sigma \in \partial j(\mathcal{C}, e(u))$ μ -a.e. The constitutive law written in this fashion is meaningful for any measure λ only if u is of C^1 class (see also Corollary 4.2).

In general, both solutions u and σ can be found by solving independent variational problems. In case of the displacement function, it is well established that one must solve the maximization problem that defines the compliance; we recall the formula:

$$\mathcal{C}(\lambda) = \sup \left\{ \int \langle u, F \rangle - \int j(\mathcal{C}, e(u)) d\mu : u \in (\mathcal{D}(\mathbb{R}^d))^d \right\}. \quad (3.8)$$

To establish existence of solution, the maximization problem must be relaxed. Once $j(\mathcal{C}, \cdot)$ satisfies a suitable ellipticity condition, the relaxed solution may be found in a Sobolev space with respect to measure μ denoted by $W_{\mu}^{1,p}$ which was proposed in [12] and then developed in e.g. [10]; see also [6] on the application of higher order weighted Sobolev space in elasticity of beams with degenerate width distribution. In the present paper it is crucial that the energy potential may be degenerate, e.g. in the sense that $j(\mathcal{C}, \epsilon)$ may vanish for some non-zero $\epsilon \in L_{\mu}^p(\bar{\Omega}; \mathcal{S}^{d \times d})$ on a set of non-zero measure μ . Therefore, the theory put forward in [12] cannot be directly applied to every pair of measures F and λ . One would have to design another functional space that depends on the structure of $j(\mathcal{C}, \cdot)$ and, in general, that may contain functions u of low regularity, including those of jump-type discontinuities.

The situation is better if λ is optimally chosen for F , i.e. it solves (FMD). Then, by a straightforward generalization of [10, Proposition 2], we could show that amongst the sequences $u_n \in (\mathcal{D}(\mathbb{R}^d))^d$ that are maximizing for (3.8) we can always choose one satisfying $\sup_n \|\rho(e(u_n))\|_{\infty} < \infty$. More precisely, it is justified to introduce the following notion:

Definition 3.1 For a balanced load F assume that λ solves the (FMD) problem. Then, by a *relaxed solution* of (3.8) we shall understand a function $u \in C(\bar{\Omega}; \mathbb{R}^d)$ such that there exists a sequence $u_n \in (\mathcal{D}(\mathbb{R}^d))^d$ that is maximizing for (3.8) and satisfies: $u_n \rightarrow u$ uniformly in $\bar{\Omega}$, and $\rho(e(u_n)) \leq (Z/C_0)^{p'/p}$ in $\bar{\Omega}$ for every n .

The problem whose solution occurs to be the stress σ is the one that is dual to problem (3.8). Similarly as in the previous section, the duality argument will be standard. Beforehand, however, for a fixed $\lambda = \mathcal{C}\mu$ we must examine the functional $\int j(\mathcal{C}, \cdot) d\mu : L_{\mu}^p(\bar{\Omega}; \mathcal{S}^{d \times d}) \rightarrow \bar{\mathbb{R}}$. As a first step we shall make sure that conjugation and integration operations commute, i.e. that $(\int j(\mathcal{C}, \cdot) d\mu)^* : L_{\mu}^{p'}(\bar{\Omega}; \mathcal{S}^{d \times d}) \rightarrow \bar{\mathbb{R}}$ is equal to $\int j^*(\mathcal{C}, \cdot) d\mu$, where $j^* : \mathcal{H} \times \mathcal{S}^{d \times d} \rightarrow \bar{\mathbb{R}}$ is the convex conjugate with respect to the second variable. A simple scenario when such commutation holds is when the integrand $j(\mathcal{C}(\cdot), \cdot) : \bar{\Omega} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ is a Carathéodory function, cf. [22]:

Proposition 3.3 For a given Radon measure $\mu \in \mathcal{M}_+(\bar{\Omega})$ let $\mathcal{C} : \bar{\Omega} \rightarrow \mathcal{H}$ be a μ -measurable function. Then $j(\mathcal{C}(\cdot), \cdot) : \bar{\Omega} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ is a Carathéodory function, i.e.

- (i) for μ -a.e. x the function $j(\mathcal{C}(x), \cdot)$ is continuous;
- (ii) for every $\xi \in \mathcal{S}^{d \times d}$ the function $j(\mathcal{C}(\cdot), \xi)$ is μ -measurable.

Proof The statement (i) follows easily from the assumption (H1) since every convex function that is finite on the whole finite dimensional space is automatically continuous. For any $\xi \in \mathcal{S}^{d \times d}$ the function $j(\mathcal{C}(\cdot), \xi)$ is a composition of an upper semi-continuous function $j(\cdot, \xi)$ (cf. assumption (H3)) and a μ -measurable function $\mathcal{C}$, hence the claim (ii). $\square$

Next, by employing assumptions (H1)–(H4), it is straightforward to infer that for $\mathcal{C} \in L_{\mu}^{\infty}(\bar{\Omega}; \mathcal{C})$ the functional $\int j(\mathcal{C}, \cdot) d\mu$ is convex and continuous on the Lebesgue space $L_{\mu}^p(\bar{\Omega}; \mathcal{S}^{d \times d})$. Combining this fact with Proposition 3.3 unlocks, once again, the standard

algorithm from [22, Chapter III], and we readily arrive at the problem dual to (3.8):

$$\mathcal{C}(\lambda) = \inf \left\{ \int j^*(\mathcal{C}, \sigma) d\mu : \sigma \in L_{\mu}^{p'}(\overline{\Omega}; \mathcal{S}^{d \times d}), -\operatorname{div}(\sigma \mu) = F \right\}, \quad (3.9)$$

where again $\lambda = \mathcal{C}\mu$. As a part of the duality result we infer that the minimizer of (3.9) exists whenever $\mathcal{C}(\lambda) < \infty$; therefore, contrarily to the displacement-based problem (3.8), the problem (3.9) does not require relaxation. The infimum in (3.9) yields an alternative, dual definition of compliance $\mathcal{C}(\lambda)$; it allows to give upper bounds for $\mathcal{C}(\lambda)$, which will be well utilized while proving Theorem 1.3.

Remark 3.1 Should it exist, solution σ of the problem (3.9) depends on the particular choice of μ and $\mathcal{C}$ that gives $\lambda = \mathcal{C}\mu$. For instance, if for given μ we put $\mu_1 = \alpha\mu$, $\mathcal{C}_1 = \frac{1}{\alpha}\mathcal{C}$ for some $\alpha > 0$, solution σ_1 for this pair would satisfy the scaling property: $\sigma_1 = \frac{1}{\alpha}\sigma$. To put it differently, it is the field $\tau = \sigma\mu$ that is invariant of the chosen representation $\lambda = \mathcal{C}\mu$. Accordingly, τ may be considered an *absolute stress field* whilst $\sigma = \frac{d\tau}{d\mu}$ (the Radon-Nikodym derivative) can be interpreted as the *relative stress field*, i.e. relative with respect to the elastic material's distribution μ . Since in this work we always enforce that $c(\mathcal{C}) = 1$ μ -a.e., speaking of solution σ of (3.9) (see (iii) in Definition 1.2) should not cause any confusion.

3.3 Designing the anisotropy at a point—the underlying finite dimensional program

The function $\bar{j}$ and, therefore, also the function ρ are expressed via finite dimensional program (3.3) where function j enters. In the present subsection it will appear that a "mirror" relation may be established between the polar ρ^0 and the conjugate function j^* , which will be fundamental for connecting two of minimization problems in this work: $(\mathcal{P}^*)$ and (3.9).

By definition of polar ρ^0 , for any pair $(\xi, \sigma) \in \mathcal{S}^{d \times d} \times \mathcal{S}^{d \times d}$ there always holds $\langle \xi, \sigma \rangle \leq \rho(\xi) \rho^0(\sigma)$. We shall say that a pair (ξ, σ) satisfies the *extremality condition* for ρ and its polar whenever we have an equality $\langle \xi, \sigma \rangle = \rho(\xi) \rho^0(\sigma)$. Another result of this subsection will link this extremality condition to satisfying the constitutive law $\sigma \in \partial j(\check{H}, \xi)$ for $\check{H} \in \mathcal{H}_1$ optimally chosen for σ , namely for $\check{H}$ belonging to the set

$$\mathcal{H}_1(\sigma) := \arg \min_{H \in \mathcal{H}_1} j^*(H, \sigma).$$

This relation will be utilized whilst formulating the optimality conditions for the (FMD) problem in Sect. 4.

We first investigate the properties of the convex conjugate j^* ; by its definition, for a fixed $H \in \mathcal{H}$ we get a function $j^*(H, \cdot) : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}$ expressed by the formula

$$j^*(H, \sigma) = \sup_{\zeta \in \mathcal{S}^{d \times d}} \left\{ \langle \zeta, \sigma \rangle - j(H, \zeta) \right\}. \quad (3.10)$$

Convexity and l.s.c. of $j^*(H, \cdot)$ follow by the well established properties of convex conjugates. We look at the subdifferential $\partial j(H, \cdot) : \mathcal{S}^{d \times d} \rightarrow 2^{\mathcal{S}^{d \times d}}$. Almost by definition for $\xi, \sigma \in \mathcal{S}^{d \times d}$

$$\sigma \in \partial j(H, \xi) \quad \Leftrightarrow \quad \langle \xi, \sigma \rangle \geq j(H, \xi) + j^*(H, \sigma), \quad (3.11)$$

while the opposite inequality on the right hand side, known as Fenchel's inequality, holds always. By recalling positive p -homogeneity of $j(H, \cdot)$, it is well known that the following

repartition of energy holds (see e.g. [32])

$$\sigma \in \partial j(H, \xi) \quad \Leftrightarrow \quad \begin{cases} \langle \xi, \sigma \rangle = p \, j(H, \xi), \\ \langle \xi, \sigma \rangle = p' \, j^*(H, \sigma). \end{cases} \quad (3.12)$$

Below we state the properties of j^* that are less straightforward:

Proposition 3.4 *The function j^* is convex and lower semi-continuous on $\mathcal{L}(\mathcal{S}^{d \times d}) \times \mathcal{S}^{d \times d}$, i.e. jointly in arguments H and σ . Moreover, for each $H \in \mathcal{H}$ the function $j^*(H, \cdot) : \mathcal{S}^{d \times d} \rightarrow \overline{\mathbb{R}}$ is proper, non-negative, and positively p' -homogeneous, while*

$$j^*(tH, \sigma) = \frac{1}{t^{1/(p-1)}} j^*(H, \sigma) \quad \forall t > 0. \quad (3.13)$$

Proof Since $j(\cdot, \zeta)$ is concave and upper semi-continuous for every $\zeta \in \mathcal{S}^{d \times d}$, the mapping $(H, \sigma) \mapsto \langle \zeta, \sigma \rangle - j(H, \zeta)$ is convex and lower semi-continuous jointly in H and σ . As a result, the function $j^* : \mathcal{L}(\mathcal{S}^{d \times d}) \times \mathcal{S}^{d \times d} \rightarrow \overline{\mathbb{R}}$ is also jointly convex and lower semi-continuous as a point-wise supremum with respect to ζ , cf. the definition (3.10). Next, once $H \in \mathcal{H}$, we have $j(H, 0) = 0$ guaranteeing that $j^*(H, \cdot)$ is non-negative, while non-negativity of $j(H, \cdot)$ implies that $j^*(H, 0) = 0$; hence, $j^*(H, \cdot)$ is proper. For the positive p' -homogeneity of $j^*(H, \cdot)$ we refer to [32, Corollary 15.3.1]. The formula (3.13) follows from the known equality $(tf)^*(\cdot) = tf^*(\cdot/t)$ (see e.g. [22, Chapter I, Eq. (4.7)]) and positive 1-homogeneity of $j(\cdot, \xi)$:

$$\begin{aligned} j^*(tH, \sigma) &= (j(tH, \cdot))^*(\sigma) = (t j(H, \cdot))^*(\sigma) = t (j(H, \cdot))^*(\sigma/t) \\ &= t j^*(H, \sigma/t) = \frac{t}{t^{p'}} j^*(H, \sigma). \end{aligned}$$

□

Remark 3.2 For a given $H \in \mathcal{H}$ the function $j^*(H, \cdot)$ may admit infinite values: take for instance $H \in \mathcal{H}$ that is a singular operator and $j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle$; then, $j^*(H, \sigma) = \infty$ for any σ lying outside the range of H . At this point it is not clear whether the function $j^*(\cdot, \sigma)$ is proper for arbitrary $\sigma \in \mathcal{S}^{d \times d}$, i.e. we question the strength of the assumption (H5). A positive answer to this question shall be a part of the theorem that we state below.

Theorem 3.1 *Let $\rho : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ be the real closed gauge function defined by (3.4), and by ρ^0 denote its polar. Then, the following statements hold:*

- (i) *For every stress $\sigma \in \mathcal{S}^{d \times d}$ the set $\mathcal{H}_1(\sigma)$ is a non-empty compact convex set, and*

$$\min_{H \in \mathcal{H}_1} j^*(H, \sigma) = \bar{j}^*(\sigma) = \frac{1}{p'} (\rho^0(\sigma))^{p'} \quad (3.14)$$

where the continuous function $\bar{j}^ : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ is the convex conjugate of $\bar{j}$. Moreover,*

$$\sigma \neq 0 \quad \Rightarrow \quad \mathcal{H}_1(\sigma) \subset \{H \in \mathcal{H} : c(H) = 1\}.$$

- (ii) *For a triple $(\xi, \sigma, \check{H}) \in \mathcal{S}^{d \times d} \times \mathcal{S}^{d \times d} \times \mathcal{H}$ satisfying: $\rho(\xi) \leq 1$, $\sigma \neq 0$, $\check{H} \in \mathcal{H}_1$ the following conditions are equivalent:*

- (1) *there hold the extremality conditions:*

$$\langle \xi, \sigma \rangle = \rho^0(\sigma) \quad \text{and} \quad \check{H} \in \mathcal{H}_1(\sigma);$$

(2) the constitutive law is satisfied:

$$\frac{1}{\rho^0(\sigma)} \sigma \in \partial j(\check{H}, \xi). \quad (3.15)$$

Moreover, for each of the conditions (1), (2) to be true, it is necessary that $\rho(\xi) = 1$.

(iii) The following implication is true for every non-zero $\xi, \sigma \in \mathcal{S}^{d \times d}$

$$\langle \xi, \sigma \rangle = \rho(\xi) \rho^0(\sigma) \quad \Rightarrow \quad \mathcal{H}_1(\sigma) \subset \bar{\mathcal{H}}_1(\xi),$$

while, in general, $\mathcal{H}_1(\sigma)$ may be a proper subset of $\bar{\mathcal{H}}_1(\xi)$.

Proof For the proof of the statement (i) we fix a matrix $\sigma \in \mathcal{S}^{d \times d}$; then, directly by definition of the convex conjugate (3.10) we obtain a min-max problem:

$$\begin{aligned} \inf_{H \in \mathcal{H}_1} j^*(H, \sigma) &= \inf_{H \in \mathcal{H}_1} \sup_{\zeta \in \mathcal{S}^{d \times d}} \left\{ \langle \zeta, \sigma \rangle - j(H, \zeta) \right\} = \sup_{\zeta \in \mathcal{S}^{d \times d}} \min_{H \in \mathcal{H}_1} \left\{ \langle \zeta, \sigma \rangle - j(H, \zeta) \right\} \\ &= \sup_{\zeta \in \mathcal{S}^{d \times d}} \left\{ \langle \zeta, \sigma \rangle - \max_{H \in \mathcal{H}_1} j(H, \zeta) \right\} = \sup_{\zeta \in \mathcal{S}^{d \times d}} \left\{ \langle \zeta, \sigma \rangle - \bar{j}(\zeta) \right\} = \bar{j}^*(\sigma). \end{aligned} \quad (3.16)$$

Above, to interchange the order of inf and sup we again used Ky Fan's theorem. The first equality in (3.14) is proved, while, considering (3.4), the second one is well established, see [33, Corollary 15.3.1]. Consequently, $\bar{j}^*(\sigma) < \infty$ by Proposition 3.1; therefore, owing to convexity and lower semi-continuity of $H \mapsto j^*(H, \sigma)$ (cf. Proposition 3.4) and compactness of $\mathcal{H}_1$, we infer non-emptiness, convexity, and compactness of $\mathcal{H}_1(\sigma)$. Since $\rho^0(\sigma) > 0$ for a non-zero σ (cf. Proposition 3.1), the moreover part of the claim (i) follows by (3.13).

We move on to the claim (ii). We fix $\xi \in \mathcal{S}^{d \times d}$ with $\rho(\xi) \leq 1$, non-zero $\sigma \in \mathcal{S}^{d \times d}$, and $\check{H} \in \mathcal{H}_1$. Since $\rho^0(\sigma) > 0$, it is not restrictive to assume that $\rho^0(\sigma) = 1$.

Let us first assume that (1) holds, i.e. that $\langle \xi, \sigma \rangle = \rho^0(\sigma) = 1$ and $\check{H}$ is an element of the non-empty set $\mathcal{H}_1(\sigma)$, so that $j^*(\check{H}, \sigma) = \min_{H \in \mathcal{H}_1} j^*(H, \sigma) = \bar{j}^*(\sigma)$. Since $\langle \xi, \sigma \rangle = \rho^0(\sigma) > 0$ and $\langle \xi, \sigma \rangle \leq \rho(\xi) \rho^0(\sigma)$, at the same time we must have $\rho(\xi) = 1$. As a result, $\langle \xi, \sigma \rangle = \rho(\xi) = \rho^0(\sigma) = 1$; therefore, $\sigma \in \partial \bar{j}(\xi)$ by the fact that $\langle \xi, \sigma \rangle = \frac{1}{p}(\rho(\xi))^p + \frac{1}{p'}(\rho^0(\sigma))^{p'}$, see (3.11) and (3.4), (3.14). Accordingly, we see that $\zeta = \xi$ solves the last maximization problem in (3.16), and, consequently, it solves all three last problems in ζ . On the other hand, condition $\check{H} \in \mathcal{H}_1(\sigma)$ means that $j^*(\check{H}, \sigma) = \min_{H \in \mathcal{H}_1} j^*(H, \sigma) = \bar{j}^*(\sigma)$, i.e. $\check{H}$ solves the first two minimization problems in (3.16). We thus infer that $(\check{H}, \xi)$ is a saddle point for the functional $(H, \zeta) \mapsto \langle \zeta, \sigma \rangle - j(H, \zeta)$, i.e.

$$\begin{aligned} \langle \xi, \sigma \rangle - j(\check{H}, \xi) &= \max_{\zeta \in \mathcal{S}^{d \times d}} \min_{H \in \mathcal{H}_1} \left\{ \langle \zeta, \sigma \rangle - j(H, \zeta) \right\} = \max_{\zeta \in \mathcal{S}^{d \times d}} \left\{ \langle \zeta, \sigma \rangle - j(\check{H}, \zeta) \right\} = j^*(\check{H}, \sigma) \\ &= \min_{H \in \mathcal{H}_1} \max_{\zeta \in \mathcal{S}^{d \times d}} \left\{ \langle \zeta, \sigma \rangle - j(H, \zeta) \right\} = \min_{H \in \mathcal{H}_1} \left\{ \langle \xi, \sigma \rangle - j(H, \xi) \right\} = \langle \xi, \sigma \rangle - \bar{j}(\xi) \end{aligned}$$

furnishing two equalities: $\langle \xi, \sigma \rangle - j(\check{H}, \xi) = j^*(\check{H}, \sigma)$ and $\langle \xi, \sigma \rangle - j(\check{H}, \xi) = \langle \xi, \sigma \rangle - \bar{j}(\xi)$, from which we infer that $\sigma \in \partial j(\check{H}, \xi)$ and, respectively, $\check{H} \in \mathcal{H}_1(\xi)$. The former conclusion gives the implication (1) $\Rightarrow$ (2), while the latter, since $\check{H}$ was an arbitrary element of $\mathcal{H}_1(\sigma)$ and $\langle \xi, \sigma \rangle = \rho^0(\sigma)$, establishes the implication in the claim (iii) for the case when $\rho(\xi) = 1$. For arbitrary non-zero ξ the implication follows by the fact that $\bar{\mathcal{H}}_1(\xi) = \bar{\mathcal{H}}_1(t\xi)$ for any $t > 0$. This establishes the statement (iii).

For the implication (2) $\Rightarrow$ (1) we assume that for a triple $\xi, \sigma, \check{H}$ with $\rho(\xi) \leq 1, \rho^0(\sigma) = 1, \check{H} \in \mathcal{H}_1$ the constitutive law (3.15) is satisfied. Then, by (3.12) we have the repartition of

energy: $\langle \xi, \sigma \rangle = p j(\check{H}, \xi) = p' j^*(\check{H}, \sigma)$. The following chain of estimates holds:

$$1 = \rho^0(\sigma) = p' \left(\frac{1}{p'} (\rho^0(\sigma))^{p'} \right) \leq p' j^*(\check{H}, \sigma) = \langle \xi, \sigma \rangle = p j(\check{H}, \xi) \leq p \left(\frac{1}{p} (\rho(\xi))^p \right) \leq 1,$$

and, therefore, all the inequalities above are in fact equalities; in particular we have

$$\langle \xi, \sigma \rangle = \rho^0(\sigma) = \rho(\xi) = 1, \quad \bar{j}^*(\sigma) = \frac{1}{p'} (\rho^0(\sigma))^{p'} = j^*(\check{H}, \sigma) \Rightarrow \check{H} \in \mathcal{H}_1(\sigma),$$

which proves implication (2) $\Rightarrow$ (1) and the "moreover part" of the assertion (ii). $\square$

Remark 3.3 Assume an energy function j that satisfies (H1)–(H4) but does not necessarily satisfy the ellipticity condition (H5). From the course of the foregoing proof we can deduce that the statements (i), (ii), (iii) in Theorem 3.1 hold true if the additional assumption is imposed in each one of them: $\rho^0(\sigma) < \infty$. Additionally, from (3.16) we infer that $\rho^0(\sigma) = \infty$ if and only if $j^*(H, \sigma) = \infty$ for every $H \in \mathcal{H}_1$.

We demonstrate some of the statements of Theorem 3.1 for the AMD setting of the Free Material Design problem:

Example 3.1 (The Anisotropic Material Design setting) We shall compute the functions ρ , ρ^0 together with the sets $\mathcal{H}_1(\xi)$, $\mathcal{H}_1(\sigma)$ in the setting of (FMD) problem mostly discussed in the literature: the *Anisotropic Material Design* (AMD) setting for the linearly elastic body, more precisely we choose

$$\mathcal{H} = \mathcal{L}_+(S^{d \times d}), \quad j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle \quad (p = 2), \quad c(H) = \text{Tr } H;$$

namely, $\mathcal{H}$ contains all possible Hooke tensors. Upon recalling that here $\mathcal{H}_1 = \{H \in \mathcal{L}_+(S^{d \times d}) : \text{Tr } H \leq 1\}$, for each $H \in \mathcal{H}_1$ we may write down the estimates

$$j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle \leq \frac{1}{2} \left(\max_{i \in \{1, \dots, N(d)\}} \lambda_i(H) \right) |\xi|^2 \leq \frac{1}{2} (\text{Tr } H) |\xi|^2 \leq \frac{1}{2} |\xi|^2$$

($|\xi| = (\langle \xi, \xi \rangle)^{1/2}$ is the Euclidean norm of ξ) and therefore $\frac{1}{2} (\rho(\xi))^2 = \bar{j}(\xi) \leq \frac{1}{2} |\xi|^2$. On the other hand, we may define for a fixed non-zero $\xi \in S^{d \times d}$

$$\bar{H}_\xi = \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} \quad (3.17)$$

that is an operator with only one non-zero eigenvalue being equal to 1 and the corresponding unit eigenvector $\xi/|\xi|$ (in fact a symmetric matrix); obviously we have $\text{Tr } \bar{H}_\xi = 1$ and $j(\bar{H}_\xi, \xi) = \frac{1}{2} \langle \bar{H}_\xi \xi, \xi \rangle = \frac{1}{2} |\xi|^2$, which shows that in fact $\bar{j}(\xi) = j(\bar{H}_\xi, \xi) = \frac{1}{2} |\xi|^2$, thus

$$\rho(\xi) = |\xi|, \quad \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} \in \mathcal{H}_1(\xi).$$

It is straightforward to check that for a non-zero ξ the tensor $\bar{H}_\xi$ is the unique element of the set $\mathcal{H}_1(\xi)$. It is well known that the polar ρ^0 to the Euclidean norm $\rho = |\cdot|$ is again this very norm. Furthermore, it is obvious that

$$\langle \xi, \sigma \rangle = \rho(\xi) \rho^0(\sigma) = |\xi| |\sigma| \Leftrightarrow t_1 \xi = t_2 \sigma \text{ for some } t_1, t_2 \geq 0.$$

Next, for a non-zero σ the point (iii) of Theorem 3.1 furnishes:

$$\xi_\sigma = t \sigma \text{ for } t > 0 \Rightarrow \mathcal{H}_1(\sigma) = \mathcal{H}_1(\xi_\sigma) = \left\{ \frac{\xi_\sigma}{|\xi_\sigma|} \otimes \frac{\xi_\sigma}{|\xi_\sigma|} \right\} = \left\{ \frac{\sigma}{|\sigma|} \otimes \frac{\sigma}{|\sigma|} \right\},$$

where we used the fact that $\tilde{\mathcal{H}}_1(\xi_\sigma)$ is a singleton. Therefore, we have obtained

$$\rho^0(\sigma) = |\sigma|, \quad \mathcal{H}_1(\sigma) = \left\{ \frac{\sigma}{|\sigma|} \otimes \frac{\sigma}{|\sigma|} \right\}. \quad (3.18)$$

The latter results were obtained in [18] by solving the problem $\min_{H \in \mathcal{H}_1} j^*(H, \sigma)$ directly.

Remark 3.4 It is worth noting that, in general, neither of the sets $\tilde{\mathcal{H}}_1(\xi)$ or even $\mathcal{H}_1(\sigma)$ is a singleton, see Examples 5.1 and 5.2 in Sect. 5.1.

3.4 Recasting a solution of the Free Material Design problem from a solution of the Linear Constrained Problem

We are finally moving to the result that is central for this work: we shall prove Theorem 1.3 that allows to move from a quadruple $(\hat{u}, \hat{\mu}, \hat{\sigma}, \hat{\mathcal{C}})$ that solves the Linear Constrained Problem in the sense of Definition 1.3 to a quadruple $(\hat{u}, \hat{\mu}, \hat{\sigma}, \hat{\mathcal{C}})$ solving the Free Material Design problem in the sense of Definition 1.2, and *vice versa*. As noted in the introduction, all the fields constituting the two solutions are already proved to exist except for the Hooke function $\hat{\mathcal{C}}$ being any $\hat{\mu}$ -measurable selection of the multivalued map $x \mapsto \mathcal{H}_1(\hat{\sigma}(x))$. The following result settles this issue:

Lemma 3.1 *For a given Radon measure $\mu \in \mathcal{M}_+(\bar{\Omega})$ let $\gamma : \bar{\Omega} \rightarrow \mathcal{S}^{d \times d}$ be a μ -measurable function. We consider the multifunction $\Gamma_\gamma : \bar{\Omega} \rightarrow 2^{\mathcal{H}_1 \setminus \emptyset}$ that is closed-valued and convex-valued:*

$$\Gamma_\gamma(x) := \mathcal{H}_1(\gamma(x)) = \arg \min_{H \in \mathcal{H}_1} j^*(H, \gamma(x)).$$

Then, there exists a μ -measurable selection $\mathcal{C}_\gamma : \bar{\Omega} \rightarrow \mathcal{H}_1$ of the multifunction Γ_γ , namely

$$\mathcal{C}_\gamma(x) \in \mathcal{H}_1(\gamma(x)) \quad \text{for } \mu\text{-a.e. } x \in \bar{\Omega}.$$

Proof It suffices to prove that the multifunction $\sigma \mapsto \mathcal{H}_1(\sigma)$ is upper semi-continuous on $\mathcal{S}^{d \times d}$. Then, it is also a measurable multifunction, and, thus, there exists a Borel measurable selection $\bar{H} : \mathcal{S}^{d \times d} \rightarrow \mathcal{H}_1$, i.e. $\bar{H}(\sigma) \in \mathcal{H}_1(\sigma)$ for every $\sigma \in \mathcal{S}^{d \times d}$, see [15, Corollary III.3 and Theorem III.6]. Then, $\mathcal{C}_\gamma := \bar{H} \circ \gamma : \bar{\Omega} \rightarrow \mathcal{H}_1$ is μ -measurable as a composition of Borel measurable and μ -measurable functions.

We now show upper semi-continuity of the multifunction $\sigma \mapsto \mathcal{H}_1(\sigma)$. Since $\mathcal{H}_1$ is compact, it is enough to show that its graph $\mathcal{G}$ is closed, see [27]. Thanks to equality (3.14) one can write

$$\mathcal{G} = \left\{ (\sigma, H) \in \mathcal{S}^{d \times d} \times \mathcal{H}_1 : j^*(H, \sigma) - \bar{j}^*(\sigma) \leq 0 \right\}.$$

The function $\bar{j}^* = \frac{1}{p'}(\rho^0(\cdot))^{p'}$ is real-valued and continuous, see Proposition 3.1. The function $j^* : \mathcal{H}_1 \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}$ is jointly lower semi-continuous by Proposition 3.4; therefore, so is the mapping $(H, \sigma) \mapsto j^*(H, \sigma) - \bar{j}^*(\sigma)$. As a result, the graph $\mathcal{G}$ is closed in $\mathcal{S}^{d \times d} \times \mathcal{H}_1$, which finishes the proof. $\square$

By combining Lemma 3.1 with Corollary 3.1 we obtain:

Proposition 3.5 *There exists a solution $(\hat{u}, \hat{\mu}, \hat{\mathcal{C}}, \hat{\sigma})$ of the Linear Constrained Problem in the understanding of Definition 1.3.*

This result renders the statement of Theorem 1.3 an effective method of solving the Free Material Design problem. We prove the theorem now:

Proof of Theorem 1.3 Let us first assume that the quadruple $(\hat{u}, \hat{\mu}, \hat{\mathcal{C}}, \hat{\sigma})$ is a solution of (LCP). By definition $\hat{\tau} = \hat{\sigma} \hat{\mu}$ is a solution of the problem $(\mathcal{P}^*)$ and $\rho^0(\hat{\tau}) = \rho^0(\hat{\sigma}) \hat{\mu} = \hat{\mu}$. Since $\rho^0(\hat{\sigma}) = 1$ $\hat{\mu}$ -a.e., owing to the "moreover part" of the claim (i) in Theorem 3.1, we deduce that $c(\hat{\mathcal{C}}) = 1$ $\hat{\mu}$ -a.e. as well, and the same concerns $\check{\mathcal{C}}$. We verify that $\check{\lambda} = \check{\mathcal{C}} \check{\mu}$ is a feasible Hooke tensor field by computing: $\int c(\check{\lambda}) = \int c(\check{\mathcal{C}}) d\check{\mu} = \int d\check{\mu} = \frac{C_0}{Z} \int d\hat{\mu} = \frac{C_0}{Z} \int \rho^0(\hat{\tau}) = C_0$ since $\hat{\tau}$ is a minimizer for $(\mathcal{P}^*)$. In order to prove that $\check{\lambda}$ is a solution for $\mathcal{C}_{\min}$, it suffices to show that $\mathcal{C}(\check{\lambda}) \leq \mathcal{C}_{\min}$ where $\mathcal{C}_{\min} = Z^{p'} / (p' C_0^{p'-1})$ by Theorem 1.2. We observe that $\hat{\mu}$ -a.e. $\rho^0(\check{\sigma}) = \frac{Z}{C_0} \rho^0(\hat{\sigma}) = \frac{Z}{C_0}$. Since there holds $\check{\sigma} \check{\mu} = \hat{\sigma} \hat{\mu} = \hat{\tau}$, the equilibrium equation $-\operatorname{div}(\check{\sigma} \check{\mu}) = F$ is satisfied. Due to the p' -homogeneity of $j^*(H, \cdot)$ (see Proposition 3.4), the field $\check{\mathcal{C}} = \check{\mathcal{C}}$ is both a measurable selection for $x \mapsto \mathcal{H}_1(\hat{\sigma}(x))$ and $x \mapsto \mathcal{H}_1(\check{\sigma}(x))$. Then, by the dual stress-based version of the elasticity problem (3.9)

$$\mathcal{C}(\check{\lambda}) \leq \int j^*(\check{\mathcal{C}}, \check{\sigma}) d\check{\mu} = \int \bar{j}^*(\check{\sigma}) d\check{\mu} = \int \frac{1}{p'} (\rho^0(\check{\sigma}))^{p'} d\check{\mu} = \int \frac{1}{p'} \left(\frac{Z}{C_0} \right)^{p'} d\check{\mu} = \mathcal{C}_{\min},$$

where in the first equality we have used the fact that $\check{\mathcal{C}}(x) \in \mathcal{H}_1(\check{\sigma}(x))$ for $\hat{\mu}$ -a.e. x , the second one is by the assertion (i) in Theorem 3.1, whilst the last equality is due to the fact that $\int d\check{\mu} = C_0$. This proves optimality of $\check{\lambda}$, and we have only equalities in the chain above. The first equality implies that $\check{\sigma}$ solves the dual elasticity problem (3.9) for $\lambda = \check{\lambda} = \check{\mathcal{C}} \check{\mu}$.

Next, we must show that $\check{u}$ is a relaxed solution for (1.1), see Definition 3.1. Since $\hat{u}$ solves $(\bar{\mathcal{P}})$, there exists a sequence $\hat{u}_n \in \mathcal{U}_1$ such that $\|\hat{u}_n - \hat{u}\|_{\infty} \rightarrow 0$ and $\int \langle \hat{u}_n, F \rangle \rightarrow Z$. By definition of $\mathcal{U}_1$, we have $\rho(e(\hat{u}_n)) \leq 1$, and maximizing sequence for (1.1) may be found as $\check{u}_n := (Z/C_0)^{p'/p} \hat{u}_n$. Indeed, by (3.3), (3.4) we see that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left\{ \int \langle \check{u}_n, F \rangle - \int j(\check{\mathcal{C}}, e(\check{u}_n)) d\check{\mu} \right\} &\geq \liminf_{n \rightarrow \infty} \left\{ \int \langle \check{u}_n, F \rangle - \int \frac{1}{p} (\rho(e(\check{u}_n)))^p d\check{\mu} \right\} \\ &\geq \liminf_{n \rightarrow \infty} \left\{ \int \langle \check{u}_n, F \rangle - \int \frac{1}{p} \left(\frac{Z}{C_0} \right)^{p'} d\check{\mu} \right\} = \lim_{n \rightarrow \infty} \left\{ \int \langle \check{u}_n, F \rangle \right\} - \frac{Z^{p'}}{p C_0^{p'-1}} = \mathcal{C}(\check{\lambda}), \end{aligned}$$

where we have used the fact that $\lim_{n \rightarrow \infty} \int \langle \check{u}_n, F \rangle = (Z/C_0)^{p'/p} \lim_{n \rightarrow \infty} \int \langle \hat{u}_n, F \rangle = (Z/C_0)^{p'-1} Z$ and that $Z^{p'} / (p' C_0^{p'-1}) = \mathcal{C}_{\min} = \mathcal{C}(\check{\lambda})$ by optimality of $\check{\lambda}$. Therefore, $\check{u}_n$ is a maximizing sequence for (1.1), thus concluding the proof of the first implication.

Conversely, we assume that the quadruple $(\check{u}, \check{\mu}, \check{\mathcal{C}}, \check{\sigma})$ is a solution of the (FMD) problem (by definition we have $c(\check{\mathcal{C}}) = 1$ $\check{\mu}$ -a.e.). The Hölder inequality furnishes

$$\int \rho^0(\check{\sigma}) d\check{\mu} \leq \left(\int d\check{\mu} \right)^{1/p} \left(\int (\rho^0(\check{\sigma}))^{p'} d\check{\mu} \right)^{1/p'} \leq C_0^{1/p} \left(\int (\rho^0(\check{\sigma}))^{p'} d\check{\mu} \right)^{1/p'} \quad (3.19)$$

and the equalities hold only if $\rho^0(\check{\sigma})$ is $\check{\mu}$ -a.e. constant and only if $\int d\check{\mu} = C_0$ (excluding the case when $\check{\sigma}$ is a zero function, which is justified by the prerequisite $\mathcal{C}_{\min} > 0$). The following chain of inequalities holds true:

$$\begin{aligned} \mathcal{C}_{\min} = \mathcal{C}(\check{\lambda}) &= \int j^*(\check{\mathcal{C}}, \check{\sigma}) d\check{\mu} \geq \int \bar{j}^*(\check{\sigma}) d\check{\mu} = \int \frac{1}{p'} (\rho^0(\check{\sigma}))^{p'} d\check{\mu} \\ &\geq \frac{1}{p' C_0^{p'/p}} \left(\int \rho^0(\check{\sigma}) d\check{\mu} \right)^{p'} \geq \frac{Z^{p'}}{p' C_0^{p'/p}} = \mathcal{C}_{\min}, \end{aligned}$$

where:

- the first and the second equality acknowledge that $\check{\lambda} = \check{\mathcal{C}}\check{\mu}$ solves (FMD) and, respectively, that $\check{\sigma}$ is a minimizer in (3.9);
- the remaining inequality and equality in the first line are by Theorem 3.1, claim (i);
- to pass to the second line we use (3.19);
- the last inequality is by admissibility of $\check{\sigma}\check{\mu}$ for $(\mathcal{P}^*)$, which is due to $-\operatorname{div}(\check{\sigma}\check{\mu}) = F$ coming from admissibility of $\check{\sigma}$ for (3.9);
- the last equality is the statement of Theorem 1.2.

We see that above we have equalities everywhere, which, considering that $C_{\min} > 0$, implies several facts. First, we have $Z = \int \rho^0(\check{\sigma}) d\check{\mu}$, which shows that $\hat{\tau} := \check{\sigma}\check{\mu}$ is a solution for $(\mathcal{P}^*)$. Then, by (3.19) and the comment below it, we obtain that $\int d\check{\mu} = C_0$ and $\rho^0(\check{\sigma}) = t = \text{const } \check{\mu}\text{-a.e.}$ Combining those three facts we get $\rho^0(\check{\sigma}) = \frac{Z}{C_0}$. Since $\hat{\sigma}\hat{\mu} = \hat{\tau}$, from this follows that $\hat{\sigma} = \frac{C_0}{Z}\check{\sigma}$ and $\hat{\mu} = \frac{Z}{C_0}\check{\mu}$ are solutions for (LCP). As the last information from the chain of equalities we take the point-wise equality $j^*(\check{\mathcal{C}}, \check{\sigma}) = \bar{j}^*(\check{\sigma})$ $\check{\mu}\text{-a.e.}$ implying through claim (i) in Theorem 3.1 that $\check{\mathcal{C}}(x) \in \mathcal{H}_1(\check{\sigma}(x)) = \mathcal{H}_1(\hat{\sigma}(x))$ for $\hat{\mu}\text{-a.e. } x$, and, thus, $\hat{\mathcal{C}} = \check{\mathcal{C}}$ together with the pair $\hat{\sigma}, \hat{\mu}$ are solutions for (LCP).

To finish the proof we have to show that $\hat{u} = \left(\frac{C_0}{Z}\right)^{p'/p}\check{u}$ is a solution for $(\overline{\mathcal{P}})$. It is straightforward to show that $\hat{u} \in \overline{\mathcal{U}}_1$ based on Definition 3.1 of the relaxed solution for (1.1), and, thus, we only have to verify whether $\int \langle \hat{u}, F \rangle = Z$. One can easily show that for $\check{u}$ being a relaxed solution for $\mathcal{C}(\check{\lambda})$ there holds the repartition of energy $\int \langle \check{u}, F \rangle = p' \mathcal{C}(\check{\lambda})$ (see [10, Proposition 3]). Since $\mathcal{C}(\check{\lambda}) = C_{\min} = Z^{p'}/(p' C_0^{p'/p})$, we indeed obtain $\int \langle \hat{u}, F \rangle = (C_0/Z)^{p'/p} \int \langle \check{u}, F \rangle = Z$, and the proof ends here. $\square$

From Theorem 1.3 we infer boundedness of the stress function $\check{\sigma}$ in the optimal body, a priori lying only in $L_{\mu}^{p'}(\overline{\Omega}; S^{d \times d})$ (cf. Proposition 3.1):

Corollary 3.2 *The relative stress $\check{\sigma}$, that due to the load F occurs in the structure of the optimal Hooke tensor distribution $\check{\lambda} = \check{\mathcal{C}}\check{\mu}$, is uniform in the sense that*

$$\check{\sigma} \in L_{\mu}^{\infty}(\overline{\Omega}; S^{d \times d}), \quad \rho^0(\check{\sigma}) = \frac{Z}{C_0} \quad \check{\mu}\text{-a.e.}$$

4 Optimality conditions for the Free Material Design problem

In order to efficiently verify whether a given quadruple $(u, \mu, \sigma, \mathcal{C})$ is optimal for (FMD) problem, we shall derive the system of *optimality conditions*. Due to the much simpler structure of the problem (LCP) and the link between the two problems in Theorem 1.3, it is more natural to pose the optimality conditions for (LCP). Since the form of the latter problem is similar to the one from the paper [10] (see also the earlier work [8]), we will build upon the results given therein: in addition we must involve the Hooke tensor function $\mathcal{C}$.

In [10] one of the optimality conditions is local for it relates the stress $\sigma(x)$ to the strain $\epsilon(x)$ at $\mu\text{-a.e. } x$. Since solutions $u \in \overline{\mathcal{U}}_1$ of the problem $(\overline{\mathcal{P}})$ may be non-differentiable, the notion $e(u)$ is not, in general, well defined $\mu\text{-a.e.}$ in the classical or the weak sense. To fix this issue the authors of [10] employ the apparatus of μ -tangential calculus, introduced for the first time in [12], see also [9]. It allows to define a μ -tangential gradient $\nabla_{\mu}u$ that (for a scalar function u) is an element of $L_{\mu}^1(\overline{\Omega}; \mathbb{R}^d)$.

Below we quickly review the main aspects of the μ -tangential calculus. Its theory was generalized in [10, Section 4], making it possible to work with a wide range of linear differential operators $A : (\mathcal{D}(\mathbb{R}^d))^m \rightarrow C(\bar{\Omega}; V)$ for any natural m and any finite dimensional space V . In what follows we specify the general framework from [10] by choosing $m = d$, $V = \mathcal{S}^{d \times d}$ and $A = e$ to arrive at the μ -tangential strain operator $e_\mu = e_\mu(u)$.

Let $\mu \in \mathcal{M}_+(\bar{\Omega})$ be any Radon measure. For the graph of the operator e with the space of smooth functions as its domain, i.e. for $\mathcal{G} := \{(u, e(u)) : u \in (\mathcal{D}(\mathbb{R}^d))^d\}$, we take the following closure:

$$\bar{\mathcal{G}} := \left\{ (u, \epsilon) \in C(\bar{\Omega}; \mathbb{R}^d) \times L_\mu^\infty(\bar{\Omega}; \mathcal{S}^{d \times d}) : \exists \{u_n\} \subset (\mathcal{D}(\mathbb{R}^d))^d, \right. \\ \left. \|u_n - u\|_\infty \rightarrow 0, \quad e(u_n) \xrightarrow{*} \epsilon \text{ in } L_\mu^\infty(\bar{\Omega}; \mathcal{S}^{d \times d}) \right\}$$

where $\|u\|_\infty := \sup_{x \in \bar{\Omega}} |u(x)|$, and $\xrightarrow{*}$ stands for the weak-* convergence. In general $\bar{\mathcal{G}}$ is no longer a graph of any operator, namely the weakly-* closed space

$$\mathcal{N} := \left\{ \zeta \in L_\mu^\infty(\bar{\Omega}; \mathcal{S}^{d \times d}) : (0, \zeta) \in \bar{\mathcal{G}} \right\} \quad (4.1)$$

is non-trivial. According to [10, Proposition 4], the space $\mathcal{N}$ is decomposable in the following sense: there exists a μ -measurable multifunction $x \mapsto N_\mu(x) \subset \mathcal{S}^{d \times d}$ such that

$$\zeta \in \mathcal{N} \quad \Leftrightarrow \quad \zeta(x) \in N_\mu(x) \quad \text{for } \mu\text{-a.e. } x. \quad (4.2)$$

It is clear that $N_\mu(x)$ for μ -a.e. x is a linear subspace of $\mathcal{S}^{d \times d}$, it is therefore closed. By construction (and the terminology borrowed from the case of $A = \nabla$, see [9, 12]) point-wise $N_\mu(x)$ receives an interpretation of the space of matrices normal to μ at x . The next step consists in defining the space of μ -tangential matrices by taking the orthogonal complement of $N_\mu(x)$. This line of reasoning leads to the definition:

Definition 4.1 [The μ -tangential strain operator e_μ] For μ -a.e. x let

$$\Pi_\mu(x) : \mathcal{S}^{d \times d} \rightarrow M_\mu(x) := (N_\mu(x))^\perp$$

be the orthogonal projection onto the subspace of μ -tangential matrices $M_\mu(x) \subset \mathcal{S}^{d \times d}$. We introduce the linear operator $e_\mu : \text{dom}(e_\mu) \subset C(\bar{\Omega}; \mathbb{R}^d) \rightarrow L_\mu^\infty(\bar{\Omega}; \mathcal{S}^{d \times d})$ with the domain

$$\text{dom}(e_\mu) := \left\{ u \in C(\bar{\Omega}; \mathbb{R}^d) : \exists \epsilon \in L_\mu^\infty(\bar{\Omega}; \mathcal{S}^{d \times d}), (u, \epsilon) \in \bar{\mathcal{G}} \right\}$$

that is defined via

$$e_\mu(u) := \Pi_\mu \epsilon$$

where ϵ is any element of $L_\mu^\infty(\bar{\Omega}; \mathcal{S}^{d \times d})$ such that $(u, \epsilon) \in \bar{\mathcal{G}}$.

Remark 4.1 The choice of ϵ in the definition above may be ambiguous: in general, for $u \in \text{dom}(e_\mu)$ we have different pairs $(u, \epsilon_1), (u, \epsilon_2) \in \bar{\mathcal{G}}$ where $\zeta := \epsilon_1 - \epsilon_2 \neq 0$ is an element of $\mathcal{N}$ by the very definition of the latter. Nonetheless, thanks to characterization (4.2), we have $\Pi_\mu(x) \zeta(x) = 0$ for μ -a.e. x ; therefore, the operator e_μ is well defined after all.

Remark 4.2 It is straightforward to show that for any smooth function $u \in (\mathcal{D}(\mathbb{R}^d))^d$ there holds $e_\mu(u) = \Pi_\mu e(u)$. This property can be easily extended to functions u of C^1 class.

Remark 4.3 As an orthogonal projection, the operator $\Pi_\mu(x)$ is symmetric, namely $\langle \Pi_\mu(x)\xi, \sigma \rangle = \langle \xi, \Pi_\mu(x)\sigma \rangle$ for any $\xi, \sigma \in \mathcal{S}^{d \times d}$.

Remark 4.4 (Characterization of M_μ for the multi-junction measures μ) In order to effectively characterize the space M_μ , a fairly wide class of geometric measures μ can be considered. By a *multi-junction measure* we shall understand a measure of the form $\mu = \sum_{i=1}^N \mu_i$ where for each i : $\mu_i = m_i \mathcal{H}^{k_i} \llcorner S_i$ with m_i being a positive constant, k_i being an integer in $\{1, \dots, d\}$, and S_i being a k_i -dimensional manifold of C^2 class. In addition, we assume that $\mu_i(S_{i'}) = 0$ whenever $i \neq i'$. If for each $x \in S_i$ by $T_{S_i}(x)$ one denotes the classical tangent space, then:

$$M_\mu(x) = T_{S_i}(x) \otimes T_{S_i}(x) := \left\{ \sum_{j=1}^{k_i} a_j \eta_j \otimes \eta_j : a_j \in \mathbb{R}, \eta_j \in T_{S_i}(x) \right\} \quad \text{for } \mu_i\text{-a.e. } x,$$

and, as a result, for any matrix $\xi \in \mathcal{S}^{d \times d}$

$$\Pi_\mu(x) \xi = P_{S_i}(x) \xi P_{S_i}(x) \quad \text{for } \mu_i\text{-a.e. } x \quad (4.3)$$

where $P_{S_i}(x)$ is the matrix of orthogonal projection onto $T_{S_i}(x)$. For details see [9,10,12].

The μ -tangential objects defined above can be analysed from the "dual perspective" that was, in fact, a point of departure in the pioneering work [12]. Let us consider a stress field $\tau \in \mathcal{M}(\Omega; \mathcal{S}^{d \times d})$ and its decomposition to $\tau = \sigma \mu$ for $\mu \in \mathcal{M}_+(\Omega)$ and $\sigma \in L^1_\mu(\Omega; \mathcal{S}^{d \times d})$. If F is any balanced load, the condition on the relative stress σ point-wisely lying in M_μ turns out to be necessary for the equilibrium. Indeed, by virtue of [10, Lemma 2] we have:

$$-\operatorname{div}(\sigma \mu) = F \quad \Rightarrow \quad \begin{cases} \sigma(x) \in M_\mu(x) & \text{for } \mu\text{-a.e. } x, \\ \int \langle e_\mu(u), \sigma \rangle d\mu = \int \langle u, F \rangle \quad \forall u \in \operatorname{dom}(e_\mu). \end{cases} \quad (4.4)$$

The main argument behind (4.4) lies in proving that if $\operatorname{div}(\sigma \mu)$ is a measure, then $\sigma \in \mathcal{N}^\perp$, namely $\int \langle \zeta, \sigma \rangle d\mu = 0$ for any $\zeta \in \mathcal{N}$. The integration by parts above can be called *the μ -tangential virtual work principle*.

A consequence for the stress based elasticity formulation (3.9) follows from (4.4): the condition $\sigma(x) \in M_\mu(x)$ μ -a.e. can be added in (3.9) without affecting the infimum $\mathcal{C}(\lambda)$. In other words, to the integrand $j^*(\mathcal{C}, \sigma)$ we can add $\chi_{M_\mu(x)}(\sigma)$ where, for a set $A \subset \mathcal{S}^{d \times d}$, χ_A is the indicator function, i.e. $\chi_A(x) = 0$ for $x \in A$ and $\chi_A(x) = \infty$ for $x \notin A$.

The foregoing reasoning motivates introducing the notion of *the μ -tangential energy function*: for $H \in \mathcal{H}$ and μ -a.e. x

$$j_\mu(x, H, \cdot) := (j^*(H, \cdot) + \chi_{M_\mu(x)}(\cdot))^*, \quad (4.5)$$

which precisely furnishes $j_\mu^*(x, H, \sigma) = j^*(H, \sigma) + \chi_{M_\mu(x)}(\sigma)$ where the convex conjugate is taken with respect to the last argument. Below we show that the function $j_\mu(x, \cdot, \cdot)$ enjoys the desirable properties:

Proposition 4.1 Let $j : \mathcal{H} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}$ be any function that satisfies conditions (H1)–(H5). Then, for μ -a.e. x the following statements hold true:

(i) The function $j_\mu(x, \cdot, \cdot) : \mathcal{H} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ satisfies (H1)–(H4), whilst

$$j_\mu(x, H, \xi) = \inf_{\zeta \in N_\mu(x)} j(H, \xi + \zeta). \quad (4.6)$$

(ii) As a consequence of (i) we may define the corresponding functions $\bar{j}_\mu(x, \cdot) : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$, $\rho_\mu(x, \cdot) : \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ and the sets $\mathcal{H}_{1,\mu}(x, \cdot)$, $\mathcal{H}_{1,\mu}(x, \cdot)$. We have the following formulas

$$\rho_\mu(x, \xi) = \inf_{\zeta \in N_\mu(x)} \rho(\xi + \zeta) \quad \forall \xi \in \mathcal{S}^{d \times d} \quad (4.7)$$

$$\rho_\mu^0(x, \sigma) = \rho^0(\sigma) + \chi_{M_\mu(x)}(\sigma) \quad \forall \sigma \in \mathcal{S}^{d \times d} \quad (4.8)$$

$$\mathcal{H}_{1,\mu}(x, \sigma) = \mathcal{H}_1(\sigma) \quad \forall \sigma \in M_\mu(x) \subset \mathcal{S}^{d \times d} \quad (4.9)$$

where $\rho, \rho^0, \mathcal{H}_1(\cdot)$ are computed for j .

Proof Let us fix $H \in \mathcal{H}$. Coerciveness $j^*(H, \sigma) \geq C|\sigma|^{p'}$ for $C = C(H) > 0$ can be easily checked by combining (3.13), (3.14) and Proposition 3.1. The following chain of equalities can be written down by introducing ζ as the Lagrange multiplier:

$$\begin{aligned} j_\mu(x, H, \xi) &= \max_{\sigma \in \mathcal{S}^{d \times d}} \left\{ \langle \xi, \sigma \rangle - j_\mu^*(x, H, \sigma) \right\} = \max_{\sigma \in M_\mu(x)} \left\{ \langle \xi, \sigma \rangle - j^*(H, \sigma) \right\} \\ &= \sup_{\sigma \in \mathcal{S}^{d \times d}} \inf_{\zeta \in N_\mu(x)} \left\{ \langle \xi + \zeta, \sigma \rangle - j^*(H, \sigma) \right\} = \inf_{\zeta \in N_\mu(x)} \sup_{\sigma \in \mathcal{S}^{d \times d}} \left\{ \langle \xi + \zeta, \sigma \rangle - j^*(H, \sigma) \right\} \end{aligned}$$

where, thanks to the aforementioned coerciveness, in each maximization problem the solution exists, while in order to swap inf and sup we may use [22, Chapter VI, Proposition 2.3]. Formula (4.6) is proved since the last supremum above is $j(H, \xi + \zeta)$. Conditions (H1), (H2) for $j_\mu(x, \cdot, \cdot)$ follow directly from definition (4.5) and properties of convex conjugates. Condition (H3) can be inferred from (4.6), where function $j_\mu(x, \cdot, \xi)$ is a point-wise infimum of concave upper semi-continuous functions $j(\cdot, \xi + \zeta)$; one similarly shows (H4).

The formula (4.8) is straightforward owing to Remark 3.3. Indeed, when $\sigma \notin M_\mu(x)$, then $j_\mu^*(x, H, \sigma) = \infty$ for all $H \in \mathcal{H}_1$ and, thus, $\rho_\mu^0(x, \sigma) = \infty$; when $\sigma \in M_\mu(x)$, then $\frac{1}{p'}(\rho_\mu^0(x, \sigma))^{p'} = \min_{H \in \mathcal{H}_1} j_\mu^*(x, H, \sigma)$, which in turn is equal to $\frac{1}{p'}(\rho^0(\sigma))^{p'}$ since $j_\mu^*(x, H, \sigma) = j^*(H, \sigma)$ for every $H \in \mathcal{H}_1$. The latter equality also furnishes (4.9). The proof of (4.7) is analogous to the proof of (4.6) considering that $\frac{1}{p}(\rho_\mu(x, \xi))^p = \bar{j}_\mu(x, \xi)$ and $\frac{1}{p'}(\rho_\mu^0(x, \sigma))^{p'} = \bar{j}_\mu^*(x, \sigma) = \bar{j}^*(\sigma) + \chi_{M_\mu(x)}(\sigma)$. $\square$

Remark 4.5 The formula in (4.6) was originally given as the definition of j_μ in the work [12] and then subsequently in [8–10]. It was naturally obtained in the process of lower semi-continuous regularization of the energy functional of the type $u \mapsto \int j(\mathcal{C}, e(u))d\mu$. The definition through (4.5) herein offers a more *ad hoc* perspective on j_μ .

Remark 4.6 At given x the function $j_\mu(x, \cdot, \cdot)$ satisfies the ellipticity condition (H5) if and only if $M_\mu(x) = \mathcal{S}^{d \times d}$.

Remark 4.7 Let $\mathcal{K} \subset \mathcal{S}^{d \times d}$ be any closed convex cone and by $\mathcal{K}^*$ denote its dual cone, i.e. $\mathcal{K}^* := \{\sigma \in \mathcal{S}^{d \times d} : \langle \xi, \sigma \rangle \geq 0 \quad \forall \xi \in \mathcal{K}\}$. From the proof of Proposition 4.1 we deduce a more general construction of a function $j_{\mathcal{K}}$ departing from j satisfying (H1)–(H5):

$$j_{\mathcal{K}}(H, \xi) = \inf_{\zeta \in \mathcal{K}} j(H, \xi + \zeta), \quad j_{\mathcal{K}}^*(H, \sigma) = j^*(H, \sigma) + \chi_{\mathcal{K}^*}(\sigma). \quad (4.10)$$

Note that, in the special case when the cone $\mathcal{K}$ is a subspace $N_\mu(x)$, its dual cone is the orthogonal complement, i.e. $\mathcal{K}^* = M_\mu(x)$. The function $j_{\mathcal{K}}$ satisfies conditions (H1)–(H4), and formulas similar to the ones in the assertion (ii) of Proposition 4.1 hold true. See an application of the construction $j_{\mathcal{K}}$ in Example 5.3.

Thanks to Remark 3.3, the properties of $j_\mu(x, \cdot, \cdot)$ stated above unlock the assertion (ii) of Theorem 3.1 provided that the considered stress σ lies in $M_\mu(x)$. In some places the lower index μ can be disposed of, namely:

Corollary 4.1 *Let us take any $\mu \in \mathcal{M}_+(\overline{\Omega})$ and any $j : \mathcal{H} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}$ satisfy conditions (H1)–(H5). For μ -a.e. x and for any triple $(\xi, \sigma, \check{H}) \in \mathcal{S}^{d \times d} \times \mathcal{S}^{d \times d} \times \mathcal{H}$ satisfying:*

$$\rho_\mu(x, \xi) \leq 1, \quad \sigma \neq 0, \quad \sigma \in M_\mu(x), \quad \check{H} \in \mathcal{H}_1$$

the following conditions are equivalent:

- (1) $\langle \xi, \sigma \rangle = \rho^0(\sigma)$ and $\check{H} \in \mathcal{H}_1(\sigma)$,
- (2) $\frac{1}{\rho^0(\sigma)} \sigma \in \partial j_\mu(x, \check{H}, \xi)$.

Moreover, for each of the conditions (1), (2) to be true it is necessary that $\rho_\mu(x, \xi) = 1$.

Proof The point of departure is the assertion (ii) in Theorem 3.1 written down for $j_\mu(x, \cdot, \cdot)$, $\rho_\mu(x, \cdot)$ and $\mathcal{H}_{1,\mu}(x, \cdot)$. It holds true whenever $\rho_\mu^0(x, \sigma) < \infty$ by virtue of Remark 3.3 and the claim (i) in Proposition 4.1. Since (H5) is met by j , we have $\rho^0(\sigma) < \infty$; therefore, by (4.8) the condition $\rho_\mu^0(x, \sigma) < \infty$ is equivalent to $\sigma \in M_\mu(x)$. Consequently, by exploiting (4.9) and again (4.8), we may write $\mathcal{H}_1(\sigma)$ and $\rho^0(\sigma)$ in place of $\mathcal{H}_{1,\mu}(x, \sigma)$ and $\rho_\mu^0(x, \sigma)$, respectively. $\square$

Below we prove that any admissible function $u \in \overline{\mathcal{U}}_1$ is an element of the domain of e_μ . On top of that, we show how the condition $\rho(e(u)) \leq 1$ in $\overline{\Omega}$ holding for smooth $u \in \mathcal{U}_1$ can be translated to $\overline{\mathcal{U}}_1$ for μ -a.e. x . The result below is almost identical to [10, Lemma 1], but we display the proof for the reader's convenience:

Proposition 4.2 *For any Radon measure $\mu \in \mathcal{M}_+(\overline{\Omega})$ there holds the inclusion*

$$\overline{\mathcal{U}}_1 \subset \text{dom}(e_\mu).$$

Moreover, for any $u \in \overline{\mathcal{U}}_1$

$$\rho_\mu(x, e_\mu(u)(x)) \leq 1 \quad \text{for } \mu\text{-a.e. } x.$$

Proof For a fixed $u \in \overline{\mathcal{U}}_1$ let $\{u_n\} \subset \mathcal{U}_1 \subset (\mathcal{D}(\mathbb{R}^d))^d$ be a sequence such that $u_n \rightarrow u$ uniformly on $\overline{\Omega}$. Due to the coerciveness of ρ guaranteed by Proposition 3.1, from the fact that $\rho(e(u_n)) \leq 1$ in $\overline{\Omega}$ we infer that $\sup_n \|e(u_n)\|_{L_\mu^\infty} < \infty$. Therefore, (up to choosing a subsequence) $e(u_n) \xrightarrow{*} \epsilon$ in $L_\mu^\infty(\overline{\Omega}; \mathcal{S}^{d \times d})$ for some function ϵ , and, thus, $u \in \text{dom}(e_\mu)$ by Definition 4.1 and, moreover, $\epsilon = e_\mu(u) + \zeta$ where $\zeta \in \mathcal{N}$. For any Borel set $B \subset \overline{\Omega}$ the convex functional $\epsilon \mapsto \int_B \rho(\epsilon) d\mu$ is lower semi-continuous for the weak-* topology on $L_\mu^\infty(\overline{\Omega}; \mathcal{S}^{d \times d})$, and, thus, starting by acknowledging (4.2) and formula (4.7):

$$\int_B \rho_\mu(x, e_\mu(u)) d\mu \leq \int_B \rho(e_\mu(u) + \zeta) d\mu \leq \liminf_n \int_B \rho(e(u_n)) d\mu \leq \int_B d\mu.$$

The "moreover part" readily follows due to arbitrariness of the Borel set B . $\square$

The optimality conditions for the Linear Constrained Problem may readily be given:

Theorem 4.1 *Let us take a quadruple $u \in C(\overline{\Omega}; \mathbb{R}^d)$, $\mu \in \mathcal{M}_+(\overline{\Omega})$, $\sigma \in L^\infty_\mu(\overline{\Omega}; \mathcal{S}^{d \times d})$, $\mathcal{C} \in L^\infty_\mu(\overline{\Omega}; \mathcal{H})$ with $\rho^0(\sigma) = 1$, and $\mathcal{C} \in \mathcal{H}_1$ μ -a.e.*

The quadruple solves (LCP) if and only if the following optimality conditions are met:

$$\begin{cases} (i) & -\operatorname{div}(\sigma\mu) = F, \\ (ii) & u \in \overline{U}_1, \\ (iii) & \langle e_\mu(u)(x), \sigma(x) \rangle = 1 \quad \text{for } \mu\text{-a.e. } x, \\ (iv) & \mathcal{C}(x) \in \mathcal{H}_1(\sigma(x)) \quad \text{for } \mu\text{-a.e. } x. \end{cases} \quad (4.11)$$

Moreover, the pair of conditions (iii), (iv) may be equivalently put as a constitutive law of elasticity:

$$(iii, iv)' \quad \sigma(x) \in \partial j_\mu(x, \mathcal{C}(x), e_\mu(u)(x)) \quad \text{for } \mu\text{-a.e. } x. \quad (4.12)$$

Proof Considering Definition 1.3 that explicitly stipulates the condition (iv), the first part of the claim will follow once we prove that: u and $\tau = \sigma\mu$ solve problems $(\overline{\mathcal{P}})$ and, respectively, $(\mathcal{P}^*)$ if and only if conditions (i), (ii), (iii) hold true. Since (i) and (ii) are the admissibility criteria for $(\mathcal{P}^*)$ and $(\overline{\mathcal{P}})$, they may be further assumed to be true. As a result the following chain may be written down

$$\int \langle e_\mu(u), \sigma \rangle d\mu = \int \langle u, F \rangle \leq \int \rho^0(\tau) = \int \rho^0(\sigma) d\mu = \int \rho_\mu^0(x, \sigma) d\mu \quad (4.13)$$

where:

- the first equality is the integration by parts formula in (4.4) that holds since $-\operatorname{div}(\sigma\mu) = F$ and $u \in \overline{U}_1 \subset \operatorname{dom}(e_\mu)$;
- the inequality acknowledges that $\max \overline{\mathcal{P}} = \min \mathcal{P}^*$, cf. Corollary 3.1;
- the last equality exploits the fact that $\sigma(x) \in M_\mu(x)$ for μ -a.e. x (see (4.4) again) and the formula (4.8): we have $\rho^0(\sigma) = \rho_\mu^0(x, \sigma)$ μ -a.e. as a result.

Using the zero-gap result $\max \overline{\mathcal{P}} = \min \mathcal{P}^*$ again, we deduce that u and $\tau = \sigma\mu$ are solutions if and only if in (4.13) we have equalities only. By the "moreover part" of Proposition 4.2 and by the very definition of polarity, we have $\langle e_\mu(u), \sigma \rangle \leq \rho_\mu^0(x, \sigma)$ μ -a.e. Consequently, equalities in (4.13) hold if and only if $\langle e_\mu(u), \sigma \rangle = \rho_\mu^0(x, \sigma)$ μ -a.e., which is exactly (iii) since $\rho_\mu^0(x, \sigma) = \rho^0(\sigma) = 1$. The first part of the claim is proved.

The "moreover part" of the assertion follows directly by Corollary 4.1 and the fact that $\rho^0(\sigma) = 1$ μ -a.e. Indeed, by acknowledging Proposition 4.2 again, for μ -a.e. x the triple $\xi = e_\mu(u)(x)$, $\sigma(x)$, $\check{H} = \mathcal{C}(x)$ satisfies the prerequisites of the corollary. $\square$

Owing to the "moreover part" of the theorem above, we can recover the equations of elasticity for the optimal body undergoing a non-smooth displacement $\check{u}$ (cf. (1.4) for their smooth variant):

Corollary 4.2 *For any quadruple $(\check{u}, \check{\mu}, \check{\mathcal{C}}, \check{\sigma})$ solving (FMD) in the understanding of Definition 1.2 there hold the equilibrium equation and the μ -tangential constitutive law of elasticity:*

$$-\operatorname{div}(\check{\sigma}\check{\mu}) = F, \quad \check{\sigma} \in \partial j_\mu(x, \check{\mathcal{C}}, e_\mu(\check{u})) \quad \check{\mu}\text{-a.e.} \quad (4.14)$$

Proof The stated pair of conditions holds true for any solution $(\hat{u}, \hat{\mu}, \hat{\mathcal{C}}, \hat{\sigma})$ of (LCP) by virtue of Theorem 4.1. Considering the characterization (3.12) written for j_μ , one easily checks that this pair of conditions is preserved under the transformation to $(\check{u}, \check{\mu}, \check{\mathcal{C}}, \check{\sigma})$ in accordance with Theorem 1.3. $\square$

5 Case study and examples of optimal structures

In Example 3.1 we have computed: ρ , ρ^0 together with the extremality conditions for ξ , σ and the sets of optimal Hooke tensors $\mathcal{H}_1(\xi)$, $\mathcal{H}_1(\sigma)$ in the Anisotropic Material Design setting (AMD) which assumed that $\mathcal{H} = \mathcal{L}_+(S^{d \times d})$ (all Hooke tensors are admissible) and $j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle$ (linearly elastic material). The computed functions and sets virtually define the problem (LCP) (in the AMD setting) which, via Theorem 1.3, paves the way to solution of the original (FMD) problem.

In Sect. 5.1 below we will compute ρ , ρ^0 and $\mathcal{H}_1(\xi)$, $\mathcal{H}_1(\sigma)$ for other settings of the (FMD) problem. They will vary in the choice of both $\mathcal{H}$ and j . Afterwards, in Sect. 5.2, a simple (FMD) problem will be solved analytically in all of the proposed settings.

5.1 Other examples of Free Material Design settings

The first two settings of the Free Material Design below will involve the linear constitutive law, namely $j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle$:

Example 5.1 (The Isotropic Material Design setting) The following setting of the (FMD) problem differs from the AMD setting in Example 3.1 only by the choice of the admissible set of Hooke tensors. It is known as the *Isotropic Material Design* setting (IMD), cf. [17]:

$$\mathcal{H} = \mathcal{H}_{iso}, \quad j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle, \quad c(H) = \text{Tr } H, \quad (5.1)$$

where $\mathcal{H}_{iso} = \{dK(\frac{1}{d} \text{I} \otimes \text{I}) + 2G(\text{Id} - \frac{1}{d} \text{I} \otimes \text{I}) : K, G \geq 0\}$ is a two-dimensional closed convex cone of isotropic Hooke tensors in a d -dimensional body, $d \in \{2, 3\}$. For any $H \in \mathcal{H}_{iso}$ and $\xi \in S^{d \times d}$ we have $j(H, \xi) = \frac{1}{2}(K|\text{Tr } \xi|^2 + 2G|\text{dev } \xi|^2)$ where $\text{dev } \xi = \xi - \frac{1}{d}(\text{Tr } \xi)\text{I} = (\text{Id} - \frac{1}{d} \text{I} \otimes \text{I})\xi$, and $|\text{dev } \xi|$ stands for the Euclidean norm. It is well established that H has a single eigenvalue dK and $N(d) - 1$ eigenvalues $2G$ (we recall that $N(d) = d(d+1)/2$), therefore $\text{Tr } H = dK + (N(d) - 1)2G$. Upon introducing auxiliary variables $A_1 = dK$ and $A_2 = (N(d) - 1)2G$, we obtain $\text{Tr } H = A_1 + A_2$ and

$$j(H, \xi) = \frac{1}{2} \left(A_1 \left(\frac{|\text{Tr } \xi|}{\sqrt{d}} \right)^2 + A_2 \left(\frac{|\text{dev } \xi|}{\sqrt{N(d) - 1}} \right)^2 \right).$$

Thus, we have

$$\bar{j}(\xi) = \max_{H \in \mathcal{H}_1} j(H, \xi) = \max_{A_1, A_2 \geq 0} \{j(H, \xi) : A_1 + A_2 \leq 1\} = \frac{1}{2}(\rho(\xi))^2$$

where

$$\rho(\xi) = \max \left\{ \frac{|\text{Tr } \xi|}{\sqrt{d}}, \frac{|\text{dev } \xi|}{\sqrt{N(d) - 1}} \right\}, \quad (5.2)$$

while for non-zero ξ

$$\bar{\mathcal{H}}_1(\xi) = \left\{ H \in \mathcal{H}_{iso} : dK + (N(d) - 1)2G = 1, \begin{pmatrix} \frac{|\text{Tr } \xi|}{\sqrt{d}} - \rho(\xi) \\ \frac{|\text{dev } \xi|}{\sqrt{N(d) - 1}} - \rho(\xi) \end{pmatrix} \begin{pmatrix} K \\ G \end{pmatrix} = 0 \right\}.$$

By acknowledging that $\langle \xi, \sigma \rangle = \frac{1}{d}(\text{Tr } \xi)(\text{Tr } \sigma) + \langle \text{dev } \xi, \text{dev } \sigma \rangle$ we arrive at the polar

$$\rho^0(\sigma) = \frac{1}{\sqrt{d}} |\text{Tr } \sigma| + \sqrt{N(d) - 1} |\text{dev } \sigma| \quad (5.3)$$

and the extremality conditions for non-zero ξ, σ follow:

$$\langle \xi, \sigma \rangle = \rho(\xi) \rho^0(\sigma) \quad \Leftrightarrow \quad \begin{cases} \text{Tr } \sigma = |\text{Tr } \sigma| \frac{\text{Tr } \xi}{\sqrt{d} \rho(\xi)}, \\ \text{dev } \sigma = |\text{dev } \sigma| \frac{\text{dev } \xi}{\sqrt{N(d)-1} \rho(\xi)}. \end{cases} \quad (5.4)$$

In order to characterize optimal Hooke tensors for non-zero σ , we use point (ii) of Theorem 3.1: H is an element of $\mathcal{H}_1(\sigma)$ if and only if, for any $\xi = \xi_\sigma$ satisfying $\rho(\xi_\sigma) = 1$ and the extremality conditions above, the constitutive law the constitutive law (3.15) holds. Since the function j was chosen as a quadratic form, the constitutive law reads $\sigma/\rho^0(\sigma) = H \xi_\sigma$ which, considering (5.4), may be rewritten as:

$$\frac{1}{\rho^0(\sigma)} \left(\frac{1}{d} |\text{Tr } \sigma| \frac{\text{Tr } \xi_\sigma}{\sqrt{d}} \text{I} + |\text{dev } \sigma| \frac{\text{dev } \xi_\sigma}{\sqrt{N(d)-1}} \right) = K (\text{Tr } \xi_\sigma) \text{I} + 2G \text{dev } \xi_\sigma.$$

It is easy to see that for any σ the tensor ξ_σ may be chosen so that both $\text{Tr } \xi_\sigma \neq 0$ and $\text{dev } \xi_\sigma \neq 0$, and, then, comparing the left and right hand side above yields

$$\mathcal{H}_1(\sigma) = \left\{ H \in \mathcal{H}_{iso} : K = \frac{1}{d\sqrt{d}} \frac{|\text{Tr } \sigma|}{\rho^0(\sigma)}, G = \frac{1}{2\sqrt{N(d)-1}} \frac{|\text{dev } \sigma|}{\rho^0(\sigma)} \right\}. \quad (5.5)$$

We notice that $\mathcal{H}_1(\sigma)$ is always a singleton for non-zero σ , while $\bar{\mathcal{H}}_1(\xi)$ may be a one dimensional affine subset of $\mathcal{H}_{iso}$, provided $|\text{Tr } \xi|/\sqrt{d} = |\text{dev } \xi|/\sqrt{N(d)-1} = \rho(\xi) \neq 0$.

Example 5.2 (The Fibrous Material Design setting)

We present the new setting of *Fibrous Material Design* (FibMD):

$$\mathcal{H} = \text{co}(\mathcal{H}_{axial}), \quad j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle, \quad c(H) = \text{Tr } H, \quad (5.6)$$

where $\mathcal{H}_{axial}$ was defined in Example 2.2 as a closed, yet non-convex cone $\mathcal{H}_{axial}$ of uni-axial Hooke tensors $a \eta \otimes \eta \otimes \eta \otimes \eta$. We first observe that, for each $H \in \mathcal{H}_{axial}$ with $c(H) \leq 1$, i.e. with $\text{Tr } H = a \leq 1$, there holds

$$j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle = \frac{a}{2} (\langle \xi, \eta \otimes \eta \rangle)^2 \leq \frac{1}{2} \left(\max_{i \in \{1, \dots, d\}} |\lambda_i(\xi)| \right)^2, \quad (5.7)$$

and, at the same time,

$$j(\bar{H}_\xi, \xi) = \frac{1}{2} \left(\max_{i \in \{1, \dots, d\}} |\lambda_i(\xi)| \right)^2 \quad \text{for } \bar{H}_\xi = \bar{v}(\xi) \otimes \bar{v}(\xi) \otimes \bar{v}(\xi) \otimes \bar{v}(\xi) \quad (5.8)$$

where $\bar{v}(\xi)$ is any unit eigenvector of ξ corresponding to eigenvalue of the maximal absolute value.

Let us now take any $\tilde{H} \in \text{co}(\mathcal{H}_{axial})$, namely $\tilde{H} = \sum_{i=1}^m \alpha_i H_i$ for some $\alpha_i \geq 0$ and $H_i \in \mathcal{H}_{axial}$ with $c(H_i) > 0$. Since both $c = \text{Tr}$ and $j(\cdot, \xi)$ are linear, there hold $c(\tilde{H}) = \sum_{i=1}^m \alpha_i c(H_i)$ and

$$\begin{aligned} j(\tilde{H}, \xi) &= \sum_{i=1}^m \alpha_i j(H_i, \xi) = \sum_{i=1}^m \alpha_i c(H_i) j\left(\frac{H_i}{c(H_i)}, \xi\right) \\ &\leq \left(\sup_{\substack{H \in \mathcal{H}_{axial} \\ c(H) \leq 1}} j(H, \xi) \right) \sum_{i=1}^m \alpha_i c(H_i) = \left(\sup_{\substack{H \in \mathcal{H}_{axial} \\ c(H) \leq 1}} j(H, \xi) \right) c(\tilde{H}). \end{aligned}$$

By recalling (5.7) and (5.8) we arrive at

$$\bar{j}(\xi) = \max_{\substack{H \in \text{co}(\mathcal{H}_{axial}) \\ c(H) \leq 1}} j(H, \xi) = \max_{\substack{H \in \mathcal{H}_{axial} \\ c(H) \leq 1}} j(H, \xi) = \frac{1}{2} \left(\max_{i \in \{1, \dots, d\}} |\lambda_i(\xi)| \right)^2, \quad (5.9)$$

where the first equality is by definition of $\bar{j}$; moreover

$$\bar{\mathcal{H}}_1(\xi) = \text{co} \left(\left\{ \bar{v} \otimes \bar{v} \otimes \bar{v} \otimes \bar{v} : \bar{v} \text{ is any eigenvector } v_i(\xi) \text{ with maximal } |\lambda_i(\xi)| \right\} \right). \quad (5.10)$$

As a consequence, ρ becomes the spectral norm on the space of symmetric matrices $\mathcal{S}^{d \times d}$; we display it next to the well-established formula for its polar:

$$\rho(\xi) = \max_{i \in \{1, \dots, d\}} |\lambda_i(\xi)|, \quad \rho^0(\sigma) = \sum_{i=1}^d |\lambda_i(\sigma)|. \quad (5.11)$$

For non-zero ξ, σ the extremality condition for the pair ρ, ρ^0 may be characterized as follows:

$$\langle \xi, \sigma \rangle = \rho(\xi) \rho^0(\sigma) \Leftrightarrow \begin{cases} \text{every eigenvector of } \sigma \text{ is an eigenvector of } \xi \text{ and} \\ \lambda_i(\sigma) = |\lambda_i(\sigma)| \frac{\lambda_i(\xi)}{\rho(\xi)} \quad \forall i. \end{cases} \quad (5.12)$$

It is thus only left to characterize the set $\mathcal{H}_1(\sigma)$; we see that this time around we are forced to search the set $\mathcal{H} = \text{co}(\mathcal{H}_{axial})$, instead of just $\mathcal{H}_{axial}$ being the case while maximizing $j(\cdot, \xi)$. Indeed, any σ of at least two non-zero eigenvalues yields $j^*(H, \sigma) = \infty$ for each $H \in \mathcal{H}_{axial}$. According to point (ii) of Theorem 3.1, for a given non-zero σ the Hooke tensor $H \in \mathcal{H}_1$ is an element of $\mathcal{H}_1(\sigma)$ if and only if the constitutive law, that here reads

$$\frac{\sigma}{\rho^0(\sigma)} = H \xi_\sigma, \quad (5.13)$$

holds true for any $\xi = \xi_\sigma$ that satisfies: $\rho(\xi_\sigma) = 1$ and the extremal relation (5.12) with σ . With $v_i(\sigma)$ being unit eigenvectors of σ , for a non-zero σ we propose the Hooke tensor

$$\bar{H}_\sigma = \sum_{i=1}^d \frac{|\lambda_i(\sigma)|}{\rho^0(\sigma)} v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma) \quad (5.14)$$

that is an element of $\mathcal{H}_1$, i.e. $\bar{H}_\sigma \in \text{co}(\mathcal{H}_{axial})$ and $\text{Tr } \bar{H}_\sigma = 1$. Since the pair ξ_σ, σ satisfies (5.12), each $v_i(\sigma)$ is an eigenvector for ξ_σ and, moreover, $\langle \xi_\sigma, v_i(\sigma) \otimes v_i(\sigma) \rangle = \lambda_i(\xi_\sigma) = \text{sign}(\lambda_i(\sigma))$ where we use the fact that $\rho(\xi_\sigma) = 1$, therefore

$$\bar{H}_\sigma \xi_\sigma = \sum_{i=1}^d \frac{|\lambda_i(\sigma)|}{\rho^0(\sigma)} \text{sign}(\lambda_i(\sigma)) v_i(\sigma) \otimes v_i(\sigma) = \frac{\sigma}{\rho^0(\sigma)},$$

which proves that $\bar{H}_\sigma \in \mathcal{H}_1(\sigma)$. The full characterization of the set $\mathcal{H}_1(\sigma)$ is difficult to write down for arbitrary d ; hence, further we shall proceed in dimension $d = 2$ where three cases must be examined:

Case a) the determinant of σ is negative

In this case σ has two non-zero eigenvalues of opposite sign, let us say: $\lambda_1(\sigma) < 0$ and $\lambda_2(\sigma) > 0$. Therefore, there exists a unique $\xi = \xi_\sigma$ that satisfies $\rho(\xi_\sigma) \leq 1$ and is in the extremal relation (5.12) with σ : there must hold $\xi_\sigma = -v_1(\sigma) \otimes v_1(\sigma) + v_2(\sigma) \otimes v_2(\sigma)$ where $v_1(\sigma), v_2(\sigma)$ are the respective eigenvectors of σ . According to point (iii) of Theorem 3.1,

there must hold $\mathcal{H}_1(\sigma) \subset \tilde{\mathcal{H}}_1(\xi_\sigma)$, and, thus, from (5.10) we deduce that each $H \in \tilde{\mathcal{H}}_1(\sigma)$ satisfies $H = \sum_{i=1}^2 \alpha_i v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma)$ for $\alpha_1 + \alpha_2 = 1$. Then, the constitutive law (5.13) enforces $\sigma/\rho^0(\sigma) = -\alpha_1 v_1(\sigma) \otimes v_1(\sigma) + \alpha_2 v_2(\sigma) \otimes v_2(\sigma)$, and we immediately obtain that $\alpha_i = |\lambda_i(\sigma)|/\rho^0(\sigma)$, and, therefore, H must coincide with $\tilde{H}_\sigma$ from (5.14). In summary, in the case when $d = 2$ and $\det \sigma < 0$, the set $\mathcal{H}_1(\sigma)$ is a singleton:

$$\mathcal{H}_1(\sigma) = \left\{ \sum_{i=1}^2 \frac{|\lambda_i(\sigma)|}{\rho^0(\sigma)} v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma) \right\}, \quad (5.15)$$

while $\tilde{\mathcal{H}}_1(\xi_\sigma)$ is the convex hull of $\{v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma) : i \in \{1, 2\}\}$.

Case b) the determinant of σ is positive

We shall focus on the case when $\lambda_1(\sigma), \lambda_2(\sigma) > 0$. Once again, there is unique ξ_σ with $\rho(\xi_\sigma) = 1$ and satisfying (5.12): necessarily $\xi_\sigma = \mathbf{I}$. Therefore, any unit vector η is an eigenvector of ξ_σ (but not necessarily of σ) with eigenvalue equal to one and, thus, $\tilde{\mathcal{H}}_1(\xi_\sigma) = \text{co}(\{\eta \otimes \eta \otimes \eta \otimes \eta : \eta \in S^{d-1}\})$. Therefore, the inclusion $\mathcal{H}_1(\sigma) \subset \tilde{\mathcal{H}}_1(\xi_\sigma)$ merely indicates that for $H \in \mathcal{H}_1(\sigma)$ there must hold $H = \sum_{i=1}^m \alpha_i \eta_i \otimes \eta_i \otimes \eta_i \otimes \eta_i$ where $m \in \mathbb{N}$, $\eta_i \in S^{d-1}$, $\alpha_i \geq 0$ and $\sum_{i=1}^m \alpha_i = 1$. By plugging this form of H into (5.13) we obtain the characterization for $\sigma \in S_+^{d \times d}$ (recall that $\langle \eta_i \otimes \eta_i, \xi_\sigma \rangle = 1$ for each i)

$$\mathcal{H}_1(\sigma) = \left\{ \sum_{i=1}^m \alpha_i \eta_i \otimes \eta_i \otimes \eta_i \otimes \eta_i : \eta_i \in S^1, \sum_{i=1}^m \alpha_i = 1, \frac{\sigma}{\rho^0(\sigma)} = \sum_{i=1}^m \alpha_i \eta_i \otimes \eta_i \right\},$$

which is not a singleton. The set $\mathcal{H}_1(\sigma)$ is in fact very rich: it may be showed that non-zero fibres may be laid out in any direction $\eta_i \in S^{d-1}$.

Case c) σ is of rank one

It is not restrictive to assume that $\lambda_1(\sigma) = 0, \lambda_2(\sigma) > 0$ and so $\sigma = \lambda_2(\sigma) v_2(\sigma) \otimes v_2(\sigma)$. In this case there are infinitely many ξ_σ such that $\rho(\xi_\sigma) = 1$ and (5.12) holds. We can, however, test (5.13) with only one: $\xi_\sigma := v_2(\sigma) \otimes v_2(\sigma)$ for which $\tilde{\mathcal{H}}_1(\xi_\sigma) = \{v_2(\sigma) \otimes v_2(\sigma) \otimes v_2(\sigma) \otimes v_2(\sigma)\}$, which is necessarily equal to $\mathcal{H}_1(\sigma)$ due to point (iii) of Theorem 3.1. Eventually, for a rank-one stress σ the set of optimal Hooke tensors may be written as a singleton

$$\mathcal{H}_1(\sigma) = \left\{ \frac{\sigma}{|\sigma|} \otimes \frac{\sigma}{|\sigma|} \right\}, \quad (5.16)$$

where we used the fact that $\rho^0(\sigma) = |\sigma|$ in this case. By comparing to Example 3.1, we learn that AMD and FibMD share the optimal Hooke tensor at points where σ is rank-one.

For the next step, we wish to consider an energy function different than $j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle$. For isotropic tensors $H \in \mathcal{H}_{iso}$ the quadratic function can be easily generalized to exponents $p \in (1, \infty)$ different than 2, thus arriving at the so called *power-laws*, see e.g. [16]. Instead, in order to make the fairly general assumptions (H1)–(H5) more worthwhile, we shall construct an important energy integrand that is non-linear with respect to H :

Example 5.3 (Constitutive law of elastic material that is dissymmetric in tension and compression) For a chosen closed convex cone $\mathcal{H}$ let $j : \mathcal{H} \times S^{d \times d} \rightarrow \mathbb{R}_+$ be any energy function that meets assumptions (H1)–(H5). We define two functions $j_+, j_- : \mathcal{H} \times S^{d \times d} \rightarrow \mathbb{R}_+$ by employing the construction of j_K in Remark 4.7 for $K = S_+^{d \times d}, K = S_-^{d \times d}$ respectively, where $S_+^{d \times d}$ and $S_-^{d \times d}$ are the closed convex cones of positive and negative

semi-definite symmetric matrices. Since both $\mathcal{S}_+^{d \times d}$, $\mathcal{S}_-^{d \times d}$ are self-dual cones, we arrive at

$$j_+(H, \xi) = \inf_{\zeta \in \mathcal{S}_+^{d \times d}} j(H, \xi + \zeta), \quad j_+^*(H, \sigma) = j^*(H, \sigma) + \chi_{\mathcal{S}_+^{d \times d}}(\sigma), \quad (5.17)$$

$$j_-(H, \xi) = \inf_{\zeta \in \mathcal{S}_-^{d \times d}} j(H, \xi + \zeta), \quad j_-^*(H, \sigma) = j^*(H, \sigma) + \chi_{\mathcal{S}_-^{d \times d}}(\sigma). \quad (5.18)$$

According to Remark 4.7, the functions j_+ and j_- satisfy conditions (H1)–(H4). They are proposals of elastic potentials of materials that are incapable of withstanding compressive and, respectively, tensile stresses. In fact, once the baseline energy function is chosen as $j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle$ for isotropic Hooke tensors $H = H(K, G)$ (see Example 2.1), then the potential $j_-(H, \cdot)$ can be employed to recast the 2D masonry material model that was put forth in [23] in a more elementary way. Indeed, if $\varepsilon_1 \leq \varepsilon_2$ stand for the two eigenvalues of $\xi \in \mathcal{S}^{2 \times 2}$, then one may compute that:

$$j_-(H, \xi) = \begin{cases} j(H, \xi) & \text{if } \varepsilon_1 + \varepsilon_2 \leq \frac{G}{K} (\varepsilon_1 - \varepsilon_2), \\ \frac{1}{2} \frac{4KG}{K+G} (\varepsilon_1)^2 & \text{if } \frac{G}{K} (\varepsilon_1 - \varepsilon_2) < \varepsilon_1 + \varepsilon_2 < \varepsilon_2 - \varepsilon_1, \\ 0 & \text{if } \xi \in \mathcal{S}_+^{2 \times 2} \end{cases} \quad (5.19)$$

for $K > 0$, while $j(H, \xi) = 0$ otherwise. The formula above coincides with [23, Eq. (3.19)]. Let us note that $j_-(H, \xi)$ varies in H (i.e. in K, G) non-linearly provided that we move within the second regime.

Although the starting function j is assumed to satisfy the ellipticity condition (H5), it is lost for either of the functions j_+ , j_- . More precisely, owing to first formulas in (5.17), (5.18), we infer that $j_+(H, \xi) = 0$ (resp. $j_-(H, \xi) = 0$) for every $H \in \mathcal{H}$ if and only if $\xi \in \mathcal{S}_-^{d \times d}$ (resp. $\xi \in \mathcal{S}_+^{d \times d}$). In order to restore the condition (H5), we define a function $j_{\pm} : \mathcal{H} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ that shall model a composite material that is dissymmetric for tension and compression:

$$j_{\pm}(H, \xi) := (\kappa_+)^p j_+(H, \xi) + (\kappa_-)^p j_-(H, \xi),$$

where κ_+, κ_- are positive reals and p is the homogeneity exponent of $j(H, \cdot)$. Effectively $j_{\pm}(H, \xi) = 0$ for each $H \in \mathcal{H}_1$ if and only if $\xi \in \mathcal{S}_+^{d \times d} \cap \mathcal{S}_-^{d \times d} = \{0\}$. In summary, the function $j_{\pm} : \mathcal{H} \times \mathcal{S}^{d \times d} \rightarrow \mathbb{R}_+$ satisfies all the conditions (H1)–(H5), and, thus, the Free Material Design problem is well posed for the material that $j_{\pm}$ models. At the same time, based on the analysis of the formula (5.19) we infer that we can easily obtain $j_{\pm}$ that is non-linear with respect to H , which justifies the need for a more general assumption (H3) kept throughout this work.

The newly proposed energy function $j_{\pm}$ can be now exploited to modify each of the foregoing settings of (FMD). We shall focus on the FibMD setting only:

Example 5.4 (The setting of Fibrous Material Design with dissymmetry in tension and compression) We revisit the setting of Fibrous Material Design with the linear constitutive law replaced by the constitutive law for material that responds differently in tension and compression (the setting will be further abbreviated by FibMD $\pm$), i.e. we take

$$\mathcal{H} = \text{co}(\mathcal{H}_{axial}), \quad j_{\pm}(H, \xi) = (\kappa_+)^2 j_+(H, \xi) + (\kappa_-)^2 j_-(H, \xi), \quad c(H) = \text{Tr } H$$

where j_+, j_- are computed for $j(H, \xi) = \frac{1}{2} \langle H \xi, \xi \rangle$ as in Example 5.3. In contrast to Example 5.2, we have no linearity of $j_{\pm}$ with respect to H , and, therefore, for given $\xi \in \mathcal{S}^{d \times d}$ we must test $j_{\pm}(H, \xi)$ with tensors H in the whole $\text{co}(\mathcal{H}_{axial})$ instead of just $\mathcal{H}_{axial}$.

Nevertheless, computations similar to those from Example 5.2 show that $\bar{j}_{\pm}(\xi) = \frac{1}{2}(\rho_{\pm}(\xi))^2$ with the $\rho_{\pm}$ being a non-symmetric generalization of the spectral norm:

$$\rho_{\pm}(\xi) = \max_{i \in \{1, \dots, d\}} \left\{ \max \{ \kappa_+ \lambda_i(\xi), -\kappa_- \lambda_i(\xi) \} \right\}.$$

For $\sigma \in \mathcal{S}^{d \times d}$ the polar $\rho_{\pm}^0$ reads

$$\rho_{\pm}^0(\sigma) = \sum_{i=1}^d \max \left\{ \frac{\lambda_i(\sigma)}{\kappa_+}, -\frac{\lambda_i(\sigma)}{\kappa_-} \right\} = \frac{1}{2} \left(\frac{1}{\kappa_+} - \frac{1}{\kappa_-} \right) \text{Tr } \sigma + \frac{1}{2} \left(\frac{1}{\kappa_+} + \frac{1}{\kappa_-} \right) \rho^0(\sigma)$$

where ρ^0 is the polar to the spectral norm, see (5.11); it is worth to note that $\text{Tr } \sigma$ enters the formula with a sign. The formula for $\rho_{\pm}^0$ was already reported in [28, Section 3.5]. The extremality conditions between ξ and σ for $\rho_{\pm}$ and $\rho_{\pm}^0$ are very similar to those displayed for the FibMD setting (see (5.12)); thus, we shall neglect to write them down. The same goes for characterizations of the sets $\mathcal{H}_1(\xi)$ and $\mathcal{H}_1(\sigma)$; we merely show a formula for

$$\bar{H}_{\sigma} = \sum_{i=1}^d \frac{\max \left\{ \frac{1}{\kappa_+} \lambda_i(\sigma), -\frac{1}{\kappa_-} \lambda_i(\sigma) \right\}}{\rho_{\pm}^0(\sigma)} v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma) \otimes v_i(\sigma)$$

being a universal (but in general non-unique) element of the set $\mathcal{H}_1(\sigma)$.

Remark 5.1 The pair of variational problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ with ρ and ρ^0 specified in Example 5.2 are well known to constitute the *Michell problem* which is the one of finding the least-weight truss-resembling structure for the permissible stresses equal in tension and compression: $\bar{s}_+ = \bar{s}_- \in \mathbb{R}_+$, cf. [34] and [13]. An extensive coverage of the Michell structures may be found in [28]. Herein, the Michell problem is recovered as the FibMD setting of the Free Material Design problem. Another works, where a link between the Michell problem and optimal design of elastic body was made, are [1, 3, 7, 14, 30], where the Michell problem was recast through an asymptotic analysis of optimal shape obtained by homogenization method in the high-porosity regime. To the knowledge of the present authors, however, until now no elastic design problem discussed in the literature was showed to be equivalent to the Michell problem for uneven permissible stresses in tension and compression $\bar{s}_+ \neq \bar{s}_-$, whereas here it naturally emerges through the pair of problems $(\mathcal{P})$ and $(\mathcal{P}^*)$ in the FibMD $\pm$ setting discussed in Example 5.4 provided one chooses $\kappa_+/\kappa_- = \bar{s}_+/\bar{s}_-$.

5.2 Examples of solutions of the Free Material Design problem in settings: AMD, IMD, FibMD and FibMD $\pm$

For a load F that simulates the uni-axial tension test, we are to solve the Free Material Design problem in the several settings listed in this paper. Thanks to Theorem 1.3, we may solve the corresponding (LCP) problem instead, for which we have at our disposal the optimality conditions from Theorem 4.1. Our strategy will be to first propose a competitor $(u, \mu, \sigma, \mathcal{C})$ for which we shall validate the optimality conditions. The displacement solutions u are non-unique as they can be modified by any rigid body displacement function $u_0 \in \mathcal{U}_0$.

Example 5.5 (Optimal material design of a plate under uni-axial tension test) For a rectangle being a closed set $R = \text{co}(\{A_1, A_2, B_1, B_2\}) \subset \mathbb{R}^2$ (we set $d = 2$ in this example) with $A_1 = (-a/2, -b/2)$, $A_2 = (-a/2, b/2)$, $B_1 = (a/2, -b/2)$, $B_2 = (a/2, b/2)$ we consider

Fig. 1 Graphical representation of the load $F = F_q + F_Q$ and of an optimal elastic material distribution μ

a load $F = F_q + F_Q$ where, for e_1, e_2 being the Cartesian frame,

$$F_q = -q e_1 \mathcal{H}^1 \llcorner [A_1, A_2] + q e_1 \mathcal{H}^1 \llcorner [B_1, B_2], \quad F_Q = -Q e_1 \delta_{A_0} + Q e_1 \delta_{B_0}$$

where $A_0 = (-a/2, 0)$, $B_0 = (a/2, 0)$, and q and Q are non-negative constants that represent, respectively, loads diffused along segments and point loads, see Fig. 1. It is straightforward to check that F is balanced. For Ω we can take any domain such that $R \subset \overline{\Omega}$.

Case a) the Anisotropic Material Design setting

In the AMD setting where $\rho = \rho^0 = |\cdot|$, cf. Example 3.1, we propose the quadruple

$$u(x) = x_1 e_1, \quad \mu = q \mathcal{L}^2 \llcorner R + Q \mathcal{H}^1 \llcorner [A_0, B_0], \quad (5.20)$$

$$\sigma = e_1 \otimes e_1, \quad \mathcal{C} = e_1 \otimes e_1 \otimes e_1 \otimes e_1. \quad (5.21)$$

We see that $\rho^0(\sigma) = |\sigma| = 1$ and $\text{Tr } \mathcal{C} = 1$, which are the initial prerequisites in Theorem 4.1. An elementary computation shows that $-\text{div}(\sigma\mu) = F$, which gives the optimality condition (i) in Theorem 4.1. The function u is smooth, and, thus, checking the condition $u \in \overline{\mathcal{U}}_1$ boils down to verifying whether $\rho(e(u)) = |e(u)| \leq 1$ in $\overline{\Omega}$. We have $e(u) = e_1 \otimes e_1$, and, clearly, the optimality condition (ii) follows. Next, we can choose which of the conditions (iii,iv) or (iii,iv)' in Theorem 4.1 we shall check. First we list essential elements of μ -tangential calculus. Measure μ is a multi-junction measure in the understanding of Remark 4.4; therefore, for μ -a.e. x :

$$M_\mu(x) = \begin{cases} \mathcal{S}^{2 \times 2} & \text{for } \mathcal{L}^2\text{-a.e. } x \in R, \\ \text{span}\{e_1 \otimes e_1\} & \text{for } \mathcal{H}^1\text{-a.e. } x \in [A_0, B_0], \end{cases}$$

$$\Pi_\mu(x) \xi = \begin{cases} \xi & \text{for } \mathcal{L}^2\text{-a.e. } x \in R, \\ (e_1 \otimes e_1) \xi (e_1 \otimes e_1) & \text{for } \mathcal{H}^1\text{-a.e. } x \in [A_0, B_0] \end{cases} \quad \forall \xi \in \mathcal{S}^{2 \times 2},$$

while $N_\mu(x)$ is the orthogonal complement of $M_\mu(x)$. Since u is smooth, we simply compute $e_\mu(u)(x) = \Pi_\mu(x) e(u)(x)$; having $e(u) = e_1 \otimes e_1$ we clearly obtain that $e_\mu(u) = e_1 \otimes e_1$ μ -a.e. as well. Accordingly, we check that $\langle \sigma, e_\mu(u) \rangle = 1$ μ -a.e. furnishing the condition (iii). In addition, since $\mathcal{C} = \sigma \otimes \sigma$, we have $\mathcal{C} \in \mathcal{H}_1(\sigma)$ (see (3.18)), and the last optimality condition (iv) follows. We have thus already proved that the quadruple $(u, \mu, \sigma, \mathcal{C})$ is an optimal solution for (LCP), and Theorem 1.3 furnishes a solution of the Free Material Design problem in the AMD setting.

For the sake of demonstration, we will in addition check the condition (iii,iv)' as well: to this purpose we must compute the formula for $j_\mu(x, \mathcal{C}(x), \cdot)$. For $\mathcal{L}^2$ -a.e. $x \in R$ clearly $j_\mu(x, \mathcal{C}(x), \xi) = j(\mathcal{C}(x), \xi)$ since for such x we have $N_\mu(x) = \{0\}$. For $\mathcal{H}^1$ -a.e. $x \in [A_0, B_0]$ we have $\langle e_1 \otimes e_1, \xi \rangle = 0$ whenever $\xi \in N_\mu(x)$; hence, for any $\xi \in \mathcal{S}^{2 \times 2}$, we use

(4.6) to find

$$j_\mu(x, \mathcal{C}(x), \xi) = \inf_{\zeta \in N_\mu(x)} j(\mathcal{C}(x), \xi + \zeta) = \inf_{\zeta \in N_\mu(x)} \frac{1}{2} (\langle e_1 \otimes e_1, \xi + \zeta \rangle)^2 = \frac{1}{2} (\langle e_1 \otimes e_1, \xi \rangle)^2,$$

and ultimately $j_\mu(x, \mathcal{C}(x), \cdot) = j(\mathcal{C}(x), \cdot)$ for μ -a.e. x . Therefore, verifying the condition (iii,iv)' boils down to checking if $\sigma = \mathcal{C} e_\mu(u)$ μ -a.e., and this is straightforward.

Case b) the Isotropic Material Design setting

For the IMD setting, the norms ρ and ρ^0 are given in (5.2) and (5.3), respectively. We put forward a quadruple that shall be checked for optimality:

$$\begin{aligned} u(x) &= \frac{2 + \sqrt{2}}{2} x_1 e_1 - \frac{2 - \sqrt{2}}{2} x_2 e_2, \quad \mu = \frac{2 + \sqrt{2}}{2} \left(q \mathcal{L}^2 \llcorner R + \mathcal{Q} \mathcal{H}^1 \llcorner [A_0, B_0] \right), \\ \sigma &= \frac{2}{2 + \sqrt{2}} e_1 \otimes e_1, \quad \mathcal{C} = 2K \left(\frac{1}{2} \mathbf{I} \otimes \mathbf{I} \right) + 2G \left(\text{Id} - \frac{1}{2} \mathbf{I} \otimes \mathbf{I} \right) \end{aligned}$$

with

$$K = \frac{1}{2 + 2\sqrt{2}}, \quad G = \frac{1}{4 + 2\sqrt{2}}. \quad (5.22)$$

First we check that $\text{Tr } \mathcal{C} = 2K + 2 \cdot 2G = 1$, thus $\mathcal{C} \in \mathcal{H}_1$ as assumed in Theorem 4.1. Since the stress field $\tau = \sigma \mu$ is identical to the one from Case a), the optimality condition (i) in Theorem 4.1 clearly holds. The function u is again smooth, so we compute $e(u) = \frac{2 + \sqrt{2}}{2} e_1 \otimes e_1 - \frac{2 - \sqrt{2}}{2} e_2 \otimes e_2$ and

$$\text{Tr}(e(u)) = \sqrt{2}, \quad |\text{dev}(e(u))| = \sqrt{2}, \quad \text{Tr } \sigma = \frac{2}{2 + \sqrt{2}}, \quad |\text{dev } \sigma| = \frac{1}{1 + \sqrt{2}}$$

yielding

$$\rho(e(u)) = \max \left\{ \frac{|\text{Tr}(e(u))|}{\sqrt{2}}, \frac{|\text{dev}(e(u))|}{\sqrt{2}} \right\} = 1, \quad \rho^0(\sigma) = \frac{|\text{Tr } \sigma|}{\sqrt{2}} + \sqrt{2} |\text{dev } \sigma| = 1$$

and therefore $u \in \overline{U}_1$, which validates the optimality condition (ii); moreover $\rho^0(\sigma) = 1$ as required in Theorem 4.1. We move on to check the remaining optimality conditions in the version (iii,iv). Since μ above coincides with the one from Case a) (up to a multiplicative constant), the formulas for M_μ and Π_μ given therein are also correct here. By acknowledging Remarks 4.2, 4.3, we have μ -a.e. $\langle \sigma, e_\mu(u) \rangle = \langle \sigma, \Pi_\mu e(u) \rangle = \langle \sigma, e(u) \rangle$, where we used the fact that $\sigma \in M_\mu$ μ -a.e. We easily check that $\langle \sigma, e(u) \rangle = 1$, and condition (iii) follows: $\langle \sigma, e_\mu(u) \rangle = 1$ μ -a.e. Then, one may find that the moduli K, G agree with the characterization of the set $\mathcal{H}_1(\sigma)$ in (5.5), hence the condition (iv), namely $\mathcal{C} \in \mathcal{H}_1(\sigma)$ μ -a.e., which proves that quadruple $(u, \mu, \sigma, \mathcal{C})$ is optimal for (LCP) in the IMD setting.

In order to be complete, we will show that the optimality condition (iii,iv)' holds as well. It is clear that for $\mathcal{L}^2$ -a.e. $x \in R$, where $M_\mu(x) = \mathcal{S}^{2 \times 2}$, we have $j_\mu(x, \mathcal{C}(x), \cdot) = j(\mathcal{C}(x), \cdot)$. Meanwhile, for $\mathcal{H}^1$ -a.e. $x \in [A_0, B_0]$ the tensors $\zeta \in N_\mu(x)$ are exactly those of the form $\zeta = e_2 \overset{s}{\otimes} \eta$, where $\eta \in \mathbb{R}^2$ and $\overset{s}{\otimes}$ is the symmetrized tensor product. Hence, for $\mathcal{H}^1$ -a.e. $x \in [A_0, B_0]$, after performing the minimization (being non-trivial here) we obtain

$$j_\mu(x, \mathcal{C}(x), \xi) = \inf_{\eta \in \mathbb{R}^2} j(\mathcal{C}(x), \xi + e_2 \overset{s}{\otimes} \eta) = \frac{1}{2} \frac{4KG}{K + G} \langle e_1 \otimes e_1 \otimes e_1 \otimes e_1, \xi \otimes \xi \rangle,$$

where constant $\frac{4KG}{K+G}$ can be readily recognized as Young modulus E , cf. (2.3). For chosen ξ the minimizer $\eta = \eta_\xi$ above is exactly the one for which $\mathcal{C}(x)$ ($\xi + e_2 \otimes \eta_\xi = s e_1 \otimes e_1$). The potential j_μ induces the well-known uni-axial constitutive law in the bar $[A_0, B_0]$. Upon computing: $e_\mu(u)(x) = e(u)(x)$ for $\mathcal{L}^2$ -a.e. $x \in R$ and $e_\mu(u)(x) = \frac{2+\sqrt{2}}{2} e_1 \otimes e_1$ for $\mathcal{H}^1$ -a.e. $x \in [A_0, B_0]$, we see that verifying condition (iii,iv)' boils down to checking if

$$\sigma(x) = \begin{cases} \mathcal{C}(x) e(u)(x) & \text{for } \mathcal{L}^2\text{-a.e. } x \in R, \\ \left(\frac{4KG}{K+G} e_1 \otimes e_1 \otimes e_1 \otimes e_1 \right) \left(\frac{2+\sqrt{2}}{2} e_1 \otimes e_1 \right) & \text{for } \mathcal{H}^1\text{-a.e. } x \in [A_0, B_0]. \end{cases}$$

The equations above are verified after elementary computations; in particular, using formulas (5.22) for optimal K, G gives the Young modulus and the Poisson ratio:

$$E = \frac{4KG}{K+G} = \left(\frac{2}{2+\sqrt{2}} \right)^2 = 6 - 4\sqrt{2}, \quad \nu = \frac{K-G}{K+G} = 3 - 2\sqrt{2}.$$

The computed value of Young modulus E turns out to be maximal among all pairs $K, G \geq 0$ satisfying $\text{Tr } \mathcal{C} = 2K + 2 \cdot 2G \leq 1$.

Case c) the Fibrous Material Design setting

In the case of the Fibrous Material Design setting it is enough to shortly note that the quadruple $(u, \mu, \sigma, \mathcal{C})$ proposed in Case a) is also optimal in the FibMD setting: indeed, both $e(u)$ and σ are of rank one; thus, spectral norm $\rho(e(u))$ and its polar $\rho^0(\sigma)$ (see (5.11)) coincide with $|e(u)|$ and $|\sigma|$, respectively. Moreover, again for a rank-one field σ , the sets $\mathcal{H}_1(\sigma)$ are identical for AMD and FibMD, see (5.16) and the comment below.

Further, the same solution (5.20), (5.21) of (LCP) will be shared by the FibMD $\pm$ setting provided that one assumes $\kappa_+ = 1$. This is a consequence of σ being positive definite μ -a.e.

Acknowledgements The authors would like to thank the National Science Centre (Poland) for the financial support: the first author would like to acknowledge the Research Grant no 2015/19/N/ST8/00474 (Poland), entitled "Topology optimization of thin elastic shells - a method synthesizing shape and free material design"; the second author would like to acknowledge the Research Grant no 2019/33/B/ST8/00325 entitled "Merging the optimum design problems of structural topology and of the optimal choice of material characteristics. The theoretical foundations and numerical methods".

Moreover, both the authors thank Professor Piotr Rybka for fruitful discussions that influenced this work.

Data Availability Statement data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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