Calculus of Variations

Volume minimising unit vector fields on three dimensional space forms of positive curvature

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Abstract

We prove that on a closed 3-manifold of constant positive curvature the unit vector fields of minimum volume are exactly the Hopf vector fields.

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1 Introduction

The seminal paper [7] where the authors show that on the unit S^3 the unit vector fields of minimum volume are Hopf vector fields and no others has served as inspiration of a wide research work.

Let's recall that any smooth vector field V on a Riemannian manifold (M, g) of dimension n defines an immersion $V : M \to TM$ and if we consider the natural metric g^S on the tangent bundle called the Sasaki metric, the n-dimensional volume of the submanifold $V(M) \subset (TM, g^S)$ is what is known as the volume of the vector field $\mathrm{Vol}(V)$; equivalently the volume of V is the volume of the Riemannian manifold (M, V^*g^S). It's immediate from the definition of the Sasaki metric that if the volume of the manifold $\mathrm{vol}(M, g)$ is finite then it is a lower bound of the volume functional; the equality holds if and only if V is a parallel vector field. For odd-dimensional spheres, probably the simplest manifolds admitting unit vector fields but not parallel ones, the question of finding the unit vector fields of minimum volume (or at least the infimum if the bound is not attained) is only solved in the case of $S^3(1)$ the unit sphere of dimension 3 ([7]); for greater dimensional spheres the problem is open although some advances have been obtained concerning the lower bounds in [3] and on the first and second variation of the volume functional in [5], [6] and [1].

In [7] the main ingredient of the proof was the use of a calibration in the unit tangent manifold $T^1 S^3(1)$ that calibrates exactly the images of Hopf vector fields. The calibration

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To A. M. Naveira, In memoriam.

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was obtained taking into account that $(T^1S^3(1), g^S)$ is isometric to the Stiefel manifold of orthonormal 2-frames of $\mathbf{R}^4$ with the homogeneous metric resulting from its diffeomorphism with $SO(4)/SO(2)$. It's worth recalling that unit tangent bundle $T^1S^3(c)$ with the Sasaki metric for $c \neq 1$ is not isometric to the standard Stiefel manifold and the proof in [7] is not valid.

The aim of this paper is to show that a unit vector field V on the sphere $S^3(c)$ of curvature $c > 0$ minimises the volume if and only if V is a Hopf vector field. The proof we provide is not an extension of the one in [7] but we rather use a completely different method. We obtain here that Hopf vector fields realise the minimum of the volume of unit vector fields from a result in [2] relating, on any 3-dimensional closed manifold, the volume of vector fields and the Ricci curvature tensor. Our proof of the more delicate part, namely the rigidity result, is based on the characterisation of the unit vector fields that are critical points of the volume functional given in [6] and the properties of the first two eigenvalues of the rough Laplacian acting on vector fields.

The rigidity of Hopf vector fields as volume minimisers among unit vector fields is extended to any complete three dimensional manifold of positive constant curvature; in fact, the proof is much simpler for constant curvature spaces different from the spheres themselves as can be seen in our main result, Theorem 9.

2 Preliminaries

The Sasaki metric of the tangent bundle of a Riemannian manifold (M, g) will be denoted by g^S and, for $(x, v) \in TM$, $\xi_1, \xi_2 \in T_{(x,v)}TM$, it is defined by

$$g^S(\xi_1, \xi_2) = g(\pi_*(\xi_1), \pi_*(\xi_2)) + g(\kappa(\xi_1), \kappa(\xi_2)),$$

where $\kappa : T(TM) \to TM$ is the connection map corresponding to the Levi Civita connection of g. An element $\xi \in T_{(x,v)}TM$ is said to be *vertical* if $\pi_*(\xi) = 0$ and it is said to be *horizontal* if $\kappa(\xi) = 0$.

To each smooth vector field X on M we can associate two vector fields on TM, a vertical one denoted by X^{ver} and the horizontal lift X^{hor}. See for example [10] pg. 53 for more details.

Any smooth vector field $V : M \to (TM, g^S)$ determines an embedded submanifold and by the definition of the vertical and horizontal lifts we obtain that for all vector field Y and $p \in M$

$$V_{*|p}(Y(p)) = Y^{hor}(V(p)) + (\nabla_Y V)^{ver}(V(p)) \tag{1}$$

where ∇ is the covariant derivative of the Levi Civita connection of g.

Definition 1 The volume of a vector field $V \in \Gamma^\infty(TM)$ is the n-dimensional volume of $V(M)$, or equivalently the volume of M endowed with the pullback metric V^*g^S and will be denoted Vol(V).

The volume is expressed in terms of the endomorphism field L_V relating the two metrics g and V^*g^S by $V^*g^S(X, Y) = g(L_V(X), Y)$. Using (1),

$$V^*g^S(X, Y) = g(X, Y) + g(\nabla_X V, \nabla_Y V) = g(X, Y) + g(\nabla V(X), \nabla V(Y))$$

and then $L_V = \text{Id} + (\nabla V)^t \circ \nabla V$. If we denote $f_V = \sqrt{\det L_V}$ then the volume functional $\text{Vol} : \Gamma^\infty(TM) \to \mathbf{R} \cup \{+\infty\}$ is given by

$$\text{Vol}(V) = \int_M f_V dv_g. \tag{2}$$

The covariant derivative acts on sections of a tensor bundle $\pi : P \to M$ over a Riemannian manifold as a differential operator $\nabla : \Gamma^\infty(P) \to \Gamma^\infty(P \otimes T^*M)$ given by $(\nabla\sigma)(X) = \nabla_X\sigma$ for $\sigma \in \Gamma^\infty(P)$ and $X \in \Gamma^\infty(TM)$. The divergence operator $\nabla^* : \Gamma^\infty(P \otimes T^*M) \to \Gamma^\infty(P)$ is a tensor contraction of the covariant derivative that for $K \in \Gamma^\infty(P \otimes T^*M)$ is expressed in an orthonormal local frame $\{E_i\}_{i=1}^n$ as

$$\nabla^* K = -\sum_{i=1}^n (\nabla_{E_i} K) E_i = -\sum_{i=1}^n \left\{ \nabla_{E_i} K(E_i) - K(\nabla_{E_i} E_i) \right\}.$$

In [6] we have computed the condition for a vector field to be a critical point of the volume and the more interesting condition of being critical of the volume functional restricted to unit vector fields, which is connected with the question raised in [7] of finding the minimisers. The next proposition summarises our results included in Propositions 4 and 6 of [6] and in Theorem 16 and Proposition 17 of [4].

Proposition 2 *Let V be a smooth unit vector field on a Riemannan manifold (M, g), let's denote by $\mathcal{D}_V$ the one dimensional subbundle of TM defined by V and by K_V the endomorphism field $K_V = f_V \nabla V \circ L_V^{-1}$.*

1. *V is a critical point of the volume functional restricted to unit vector fields if and only if $\nabla^* K_V \in \mathcal{D}_V$. This condition can be also written as $\nabla^* K_V = \langle K_V, \nabla V \rangle V = \text{tr}((\nabla V)^t \circ K_V)V$. Such a unit vector field will be called minimal.*
2. *If V is a critical point of the volume of vector fields then it is also a critical point of the volume for general immersions of M into T^1M and therefore $V \to (T^1M, g^S)$ is a minimal submanifold.*
3. *If V is a critical point of the volume the curvature tensor R verifies*

$$\sum_{i=1}^n \left\{ R((\nabla V)\tilde{E}_i, V, \tilde{E}_i, Y) + g(\nabla_{\tilde{E}_i}\tilde{E}_i - \tilde{\nabla}_{\tilde{E}_i}\tilde{E}_i, Y) \right\} = 0 \tag{3}$$

for all vector field Y, where $\tilde{\nabla}$ is the covariant derivative of the Levi-Civita connection of $\tilde{g} = V^*g^S$ and $\{\tilde{E}_i\}_{i=1}^n$ is a $\tilde{g}$-orthonormal local frame.

It is well known that Hopf fibration $\pi : S^{2m+1} \longrightarrow \mathbf{C}P^m$ determines a foliation of S^{2m+1} by great circles and that a unit vector field can be chosen as a generator of this distribution. It is given by $H = JN$ where N represents the unit normal to the sphere and J the usual complex structure on $\mathbf{R}^{2m+2}$. H is the standard Hopf vector field

Definition 3 A Hopf vector field will be any vector field in S^{2m+1} obtained as JN for J a complex structure on $\mathbf{R}^{2m+2}$, that is $J \in End(\mathbf{R}^{2m+2})$ such that $J^t \circ J = \text{Id}$, $J^2 = -\text{Id}$.

A useful characterisation is that Hopf vector fields of S^{2m+1} are exactly the unit Killing vector fields; more precisely

Proposition 4 *A unit vector field V on the sphere $S^{2m+1}(c)$ of radius $\frac{1}{\sqrt{c}}$ is a Killing vector field if and only if $V = JN$ for J a complex structure on $\mathbf{R}^{2m+2}$.*

Proof Let's recall that a vector field V is Killing if it verifies one of the following equivalente conditions:

- If $\{\varphi_t\}_{t\in\mathbf{R}}$ is the flow of V, all the φ_t are isometries.
- The Lie derivative verifies $\mathcal{L}_V g = 0$.
- The covariant derivative ∇V is skewsymmetric.

If $V = JN$ then $\nabla V = \sqrt{c}\, J$ on $\mathcal{D}_V^\perp$ and $\nabla V = 0$ on $\mathcal{D}_V$ and then ∇V is skewsymmetric. Conversely, if V is a unit Killing vector field we can define the map $J : \mathbf{R}^{2m+2} \to \mathbf{R}^{2m+2}$ by

$$J(p) = \|p\|\, V\left(\frac{1}{\|p\|\sqrt{c}}\, p\right)$$

for $p \neq 0$ and $J(0) = 0$ that clearly verifies $JN = V$. It's easy to check that for V Killing J is linear since the flow of V is given by isometries and then φ_t is the restriction to the sphere of some $\tilde{\varphi}_t \in O(2m+2)$; the linear map J is in fact an isometry because V is unit and J is skwesymmetric since V is tangent to the sphere and then $\langle J(p),\, p \rangle = 0$. Consequently, J is a complex structure of $\mathbf{R}^{2m+2}$ as we wanted to show.

Any complete manifold M of constant positive curvature c is isometric to the quotient of the sphere $S^n(c)$ by a subgroup $G \subset O(n+1)$ of isometries without fixed points; in particular, $G \subset U(m+1)$ if $n = 2m+1$. This is a classical result that can be found in [11] (Theorem 7.2.18). Vector fields of M are the projections of invariant vector fields on the sphere

$$\Gamma_G^\infty(TS^{2m+1}(c)) = \{X \in \Gamma^\infty(TS^{2m+1}(c)) \; ; \; \gamma \circ X = X \circ \gamma \;\; \forall \gamma \in G\}.$$

In view of Proposition 4 we can define Hopf vector fields on these manifolds as follows $\quad\square$

Definition 5 A vector field V on a complete manifold M of dimension $2m+1$ and constant positive curvature c will be said a Hopf vector field if one of the two equivalent conditions is verified:

- V is a unit Killing vector field.
- V is the projection of a Hopf vector field of $S^{2m+1}(c)$

In the case of the projective space $G = \mathbf{Z}_2$ all the Hopf vector fields project to the quotient but for general G we know that at least the set is not void since if J is the usual complex structure on $\mathbf{R}^{2m+2}$ the standard Hopf vector field $H = JN$ is G-invariant due to the fact that $G \subset U(m+1)$.

In [6] we have shown

Proposition 6 *Every unit Killing vector field V on a manifold M of constant curvature c is minimal and its volume is* $\mathrm{Vol}(V) = (1+c)^{(n-1)/2}\mathrm{vol}(M)$.

3 Characterisation of Hopf vector fields as the minimisers of the volume in the three dimensional case

First let's show that $(1+c)\mathrm{vol}(M)$ is a lower bound of the volume of unit vector fields not only for 3-spaces of constant curvature c but in general for Einstein 3-dimensional manifolds. Although the result is true independently of the sign of c, it's only relevant for $c > 0$ as we know that always $\mathrm{Vol}(V) \geq \mathrm{vol}(M)$. The main ingredient is Theorem 3 of [2] but we give the complete proof to establish some notation that will be used in the sequel.

Proposition 7 *Let M be a compact 3-manifold without boundary with an Einstein metric of Ricci curvature $2c > 0$, then the volume functional verifies $\mathrm{Vol}(V) \geq (1 + c)\mathrm{vol}(M)$ for every unit vector field V.*

Proof Let V be a unit vector field on M, we will denote by $\mathcal{D}_V$ the set of sections of the one-dimensional subbundle of TM determined by V. Since $g(\nabla_X V, V) = 0$ for all vector field X, we can define P as the restriction of ∇V to the subbundle $\mathcal{D}_V^\perp$. If for a matrix A we represent by $\sigma_i(A)$ the coefficients of the characteristic polynomial, we have that

$$\sigma_3((\nabla V)^t \circ \nabla V) = \det((\nabla V)^t \circ \nabla V) = \det(\nabla V)^2 = 0$$
$$\sigma_2((\nabla V)^t \circ \nabla V) = \sigma_2(P^t \circ P)$$
$$+ \sum_j \left(\|\nabla_V V\|^2 \|\nabla_{E_j} V\|^2 - g(\nabla_V V, \nabla_{E_j} V)^2 \right)$$
$$\sigma_1((\nabla V)^t \circ \nabla V) = \sigma_1(P^t \circ P) + \|\nabla_V V\|^2$$

where $\{E_1, E_2\}$ is a local orthonormal frame of $\mathcal{D}_V^\perp$. Then

$$\det(L_V) = \sigma_2((\nabla V)^t \circ \nabla V) + \sigma_1((\nabla V)^t \circ \nabla V) + 1$$
$$\geq \sigma_2(P^t \circ P) + \sigma_1(P^t \circ P) + 1 = \det(\mathrm{Id} + P^t \circ P),$$

where equality holds if and only if $\nabla_V V = 0$. Let μ_1^2, μ_2^2 with $\mu_1 \geq 0, \mu_2 \geq 0$ be the eigenvalues of the positive semidefinite symmetric endomorphism $P^t \circ P$. By definition $\det(\mathrm{Id} + P^t \circ P) = \mu_1^2 \mu_2^2 + \mu_1^2 + \mu_2^2 + 1$ and then $\det(\mathrm{Id} + P^t \circ P) \geq (1 + \mu_1 \mu_2)^2$ with equality if and only if $\mu_1 = \mu_2$. Finally

$$f_V = \sqrt{\det(L_V)} \geq (1 + \mu_1 \mu_2) \geq 1 + \sigma_2(P) = 1 + \sigma_2(\nabla V)$$

where the second inequality is an equality if and only if $\sigma_2(P) = \det P \geq 0$. Now the lower bound of the volume is a consequence of the integral formula, see for example [9] pg.170,

$$\int_M \rho(X, X) dv_g = 2 \int_M \sigma_2(\nabla X) dv_g$$

where X is any vector field and ρ is the Ricci curvature. □

Proposition 8 *For the sphere $S^{2m+1}(c)$ the first eigenvalue of the rough Laplacian acting on vector fields is $\lambda_1^* = c$ with $\mathcal{E}(\lambda_1^*) = \{\mathrm{grad}(\langle a, N \rangle) ;\ a \in \mathbf{R}^{2m+2}\}$ as eigenspace; the second eigenvalue is $\lambda_2^* = 2mc$ with eigenspace $\mathcal{E}(\lambda_2^*)$ being the space of Killing vector fields. For a constant positive curvature space $S^{2m+1}(c)/G$ different from the sphere the first eigenvalue is $2mc$ with the Killing vector fields as eigenvectors.*

Proof For the spheres, the spectrum of the Hodge Laplacian acting on p-forms is well known (see [8] for example) and in particular for 1-forms the first eigenvalue is $\lambda_1 = (2m + 1)c$ and the corresponding eigenspace is given by the differential of the eigenfunctions $f_a = \langle a, N \rangle$ corresponding to the first non zero eigenvalue of the Laplacian acting on functions

$$\mathcal{E}(\lambda_1) = \{df_a ;\ a \in \mathbf{R}^{2m+2}\ a \neq 0\}.$$

The second eigenvalue is $\lambda_2 = 4mc$ and the eigenspace $\mathcal{E}(\lambda_2)$ is the space of Killing 1-forms.

For $M = S^{2m+1}(c)/G, G \neq \{\mathrm{Id}\}$, the spectrum of the Hodge Laplacian is included in the spectrum of the sphere and consists on those λ_i such that the space of G-invariant eigenvectors

is different from $\{0\}$ and then the corresponding eigenspace consists in the projections of the elements of

$$\mathcal{E}^G(\lambda_i) = \{\omega \in \mathcal{E}(\lambda_i) ; \ \gamma^*\omega = \omega \ \forall \gamma \in G\}.$$

In particular, $\mathcal{E}^G(\lambda_1) = \{0\}$ since if $\omega = df_a$ would be G-invariant then $\forall \gamma \in G$ we will have $d(f_a \circ \gamma) = \gamma^* df_a = df_a$ and then

$$df_a = \frac{1}{\#(G)} d\left(\sum_{\gamma \in G} f_a \circ \gamma\right).$$

But for every $p \in S^{2m+1}(c)$,

$$\sum_{\gamma \in G} f_a(\gamma(p)) = \sqrt{c} \langle a, \sum_{\gamma \in G} \gamma(p)\rangle = 0$$

since $\sum_{\gamma \in G} \gamma(p)$ is a fixed point for the action of G which is fixed point free.

On the contrary, $\mathcal{E}^G(\lambda_2) \neq \{0\}$ since, as we have pointed out in the previous section, the standard Hopf vector field H is G-invariant and then the 1-form associated with H by the metric $\omega(X) = g(H, X)$ is an element of $\mathcal{E}^G(\lambda_2)$; so $4mc$ belongs to the spectrum and in that case it is the lowest eigenvalue.

To conclude we use the relation of the Hodge Laplacian Δ and the rough Laplacian $\nabla^*\nabla$ as can be seen for instance in [9] page 161

$$\Delta\alpha = \nabla^*\nabla\alpha + 2mc\alpha$$

and the correspondence between 1-forms and vector fields given by the metric g.   □

Remark When $M = \mathbf{R}P^n(c)$, it's easy to see that every Killing vector field of $S^n(c)$ is $\mathbf{Z}_2$-invariant and then $2(n-1)c$ has the same multiplicity in the spectra of both the sphere and the projective space.

Now we are going to show the main result of the paper

Theorem 9 *The only unit vector fields minimising the volume functional on a closed 3-manifold of constant positive curvature are the Hopf vector fields.*

Proof From Propositions 6 and 7 Hopf vector fields minimise the volume functional. Conversely, let V be a unit vector field that realises the lower bound $\mathrm{Vol}(V) = (1+c)\mathrm{vol}(M)$ that in turn implies, as we have seen in Proposition 7, that $\nabla_V V = 0$, $P^t \circ P = \mu^2\mathrm{Id}$, $\det P \geq 0$, then

$$\int_M (1+\mu^2) dv_g = \mathrm{Vol}(V) = (1+c)\mathrm{vol}(M)$$

and

$$\int_M \|\nabla V\|^2 dv_g = \int_M 2\mu^2 dv_g = 2c\,\mathrm{vol}(M) = 2c\int_M \|V\|^2 dv_g.$$

By Proposition 8, if M is not the sphere by Raigleigh formula V must be an eigenvector corresponding to the first eigenvalue of the rough Laplacian acting on vector fields and then a Killing vector field and since it's unit by Proposition 4, V is a Hopf vector field.

The case of the sphere is more complicated since to conclude that V must be an eigenvector corresponding to the second eigenvalue of the rough Laplacian we need to show that the vector field V must be L_2-orthogonal to the vector fields $\operatorname{grad} f_a$ with $f_a = \langle a, N \rangle$ for $a \in \mathbf{R}^4$, $a \neq 0$, that are the eigenvectors of the first eigenvalue of the rough Laplacian described in Proposition 8.

Being a minimum of the volume functional, V must be minimal and then to verify $\nabla^* K_V = \langle K_V, \nabla V \rangle V$ (Proposition 2, point 1). Under the conditions on ∇V above it is easy to see that $K_V = \nabla V$ and so the minimality condition can be written as

$$\nabla^* \nabla V = \|\nabla V\|^2 V = 2\mu^2 V. \tag{4}$$

Therefore we have

$$\begin{aligned}
\int_{S^3(c)} g(V, \operatorname{grad} f_a) dv_g &= \frac{1}{c} \int_{S^3(c)} g(V, \nabla^* \nabla \operatorname{grad} f_a) dv_g \\
&= \frac{1}{c} \int_{S^3(c)} g(\nabla^* \nabla V, \operatorname{grad} f_a) dv_g \\
&= \frac{2}{c} \int_{S^3(c)} \mu^2 g(V, \operatorname{grad} f_a) dv_g.
\end{aligned} \tag{5}$$

On the other hand, by point 3 of Proposition 2,

$$\sum_{i=1}^{3} \left\{ c\, g(\nabla_{\tilde{E}_i} V, \tilde{E}_i) V + \nabla_{\tilde{E}_i} \tilde{E}_i - \tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i \right\} = 0. \tag{6}$$

and since $\nabla_V V = 0$, $P^t \circ P = \mu^2 \mathrm{Id}$ we have that $\tilde{g} = g$ on $\mathcal{D}_V$ and $\tilde{g} = (1+\mu^2)g$ on $\mathcal{D}_V^\perp$. So if $\{E_1, E_2, V\}$ is an g-orthonormal local frame we can construct a local frame orthonormal with respect to the metric $\tilde{g}$ just by taking $\{\tilde{E}_1, \tilde{E}_2, V\}$ where $\tilde{E}_i = \frac{1}{\sqrt{1+\mu^2}} E_i$.

By Koszul formula, for $i = 1, 2$

$$\tilde{g}(\tilde{\nabla}_V V, \tilde{E}_i) = \tilde{g}([\tilde{E}_i, V], V) = g([\tilde{E}_i, V], V) = 0$$

and then $\tilde{\nabla}_V V = 0$ and expression (6) can be written

$$\frac{c}{1+\mu^2} \operatorname{div} V\, V + \frac{1}{1+\mu^2} \sum_{j=1}^{2} \left(\nabla_{E_j} E_j - \tilde{\nabla}_{E_j} E_j \right) = 0 \tag{7}$$

If $E = E_j$, $j = 1, 2$ and Z is any vector field, again by Koszul formula

$$\begin{aligned}
g(\nabla_E E - \tilde{\nabla}_E E, Z) &= g(\nabla_E E, Z) - \tilde{g}(\tilde{\nabla}_E E, L_V^{-1}(Z)) \\
&= Eg(E, Z) + g([Z, E], E) - E\tilde{g}(E, L_V^{-1}(Z)) \\
&\quad + \frac{1}{2} L_V^{-1}(Z)\tilde{g}(E, E) - \tilde{g}([L_V^{-1}(Z), E], E) \\
&= -g(\nabla_E Z, E) + g(\nabla_E L_V^{-1}(Z), L_V(E)) \\
&\quad + \frac{1}{2} L_V^{-1}(Z)(1+\mu^2).
\end{aligned}$$

In particular for $Z = V$ we obtain

$$g(\nabla_E E - \tilde{\nabla}_E E, V) = -g(\nabla_E V, E) + g(\nabla_E V, (1+\mu^2)E) + \frac{1}{2} V(1+\mu^2) \tag{8}$$

that used in (7) jointly with the equality $\operatorname{div}(fZ) = f \operatorname{div} Z + Z(f)$ gives

$$0 = c \operatorname{div} V + \sum_{j=1}^{2} \left(\mu^2 g(\nabla_{E_j} V, E_j) + \frac{1}{2} V(1 + \mu^2) \right)$$

$$= (c + \mu^2) \operatorname{div} V + V(1 + \mu^2) = \operatorname{div}((c + \mu^2)V)$$

and consequently

$$\int_{S^3(c)} (c + \mu^2) g(V, \operatorname{grad} f_a) dv_g = \int_{S^3(c)} (c + \mu^2) V(f_a) dv_g = 0$$

which is in contradiction with (5), unless $\int_{S^3(c)} g(V, \operatorname{grad} f_a) dv_g = 0$ which concludes the proof.   □

Remark In the Theorem above the condition of completeness of M is crucial since the unit vector fields $V = (1/\|\operatorname{grad} f_a\|)\operatorname{grad} f_a$ that are tangent to the radial geodesics issuing from the point $p = (1/\sqrt{c}\|a\|)a$ have the same volume as the Hopf vector fields for the 3 dimensional spheres but they are only defined in the sphere minus the two antipodal points $\{p, -p\}$

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