Isolated singularities for the n -Liouville equation

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Abstract

In dimension n isolated singularities—at a finite point or at infinity—for solutions of finite total mass to the n -Liouville equation are of logarithmic type. As a consequence, we simplify the classification argument in Esposito (Anal Non Linéaire 35(3):781–801, 2018) and establish a quantization result for entire solutions of the singular n -Liouville equation.

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1 Introduction

The behavior near an isolated singularity has been discussed by Serrin in [19,20] for a very general class of second-order quasi-linear equations. The simplest example is given by the prototypical equation $-\Delta_n u = f$, where $\Delta_n(\cdot) = \operatorname{div}(|\nabla(\cdot)|^{n-2}\nabla(\cdot))$, $n \geq 2$, is the n -Laplace operator. In dimension n , the case $f \in L^1$ is very delicate as it represents a limiting situation where Serrin's results do not apply. We will be interested in the n -Liouville equation, where f is taken as an exponential function of u according to Liouville's seminal paper [16], and the singularity might be at a finite point or at infinity.

To this aim, it is enough to consider the generalized n -Liouville equation

$$-\Delta_n u = |x|^{n\alpha} e^u \text{ in } \Omega \setminus \{0\}, \quad \int_{\Omega} |x|^{n\alpha} e^u < +\infty \quad (1.1)$$

on an open set $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$, and we will be concerned with describing the behavior of u at 0. A solution u of (1.1) stands for a function $u \in C_{loc}^{1,\eta}(\Omega \setminus \{0\})$ which satisfies

$$\int_{\Omega} |\nabla u|^{n-2} \langle \nabla u, \nabla \varphi \rangle = \int_{\Omega} |x|^{n\alpha} e^u \varphi \quad \forall \varphi \in C_0^1(\Omega \setminus \{0\}).$$

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The regularity assumption on u is not restrictive since a solution in $W_{\text{loc}}^{1,n}(\Omega \setminus \{0\})$ is automatically in $C_{\text{loc}}^{1,\eta}(\Omega \setminus \{0\})$, for some $\eta \in (0, 1)$, thanks to [9, 19, 22], see Theorem 2.3 in [11].

Concerning the behavior near an isolated singularity, our main result is

Theorem 1.1 *Let u be a solution of (1.1). Then there exists $\gamma > -n^n|\alpha + 1|^{n-2}(\alpha + 1)\omega_n$, $\omega_n = |B_1(0)|$, so that*

$$-\Delta_n u = |x|^{n\alpha} e^u - \gamma \delta_0 \text{ in } \Omega \quad (1.2)$$

with

$$u - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log |x| \in L_{\text{loc}}^\infty(\Omega) \quad (1.3)$$

and

$$\lim_{x \rightarrow 0} \left[|x| \nabla u(x) - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \frac{x}{|x|} \right] = 0. \quad (1.4)$$

The case $\alpha = -2$ is relevant for the asymptotic behavior at infinity for solutions u of

$$-\Delta_n u = e^u \text{ in } \Omega, \quad \int_{\Omega} e^u < +\infty, \quad (1.5)$$

where Ω is an unbounded open set so that $B_R(0)^c \subset \Omega$ for some $R > 0$. Indeed, let us recall that Δ_n is invariant under Kelvin transform: if u solves (1.1), then $\hat{u}(x) = u(\frac{x}{|x|^2})$ does satisfy

$$-\Delta_n \hat{u} = |x|^{-2n} (-\Delta_n u) \left(\frac{x}{|x|^2} \right) = |x|^{-n(\alpha+2)} e^{\hat{u}} \text{ in } \hat{\Omega} = \left\{ x \neq 0 : \frac{x}{|x|^2} \in \Omega \right\}. \quad (1.6)$$

By Theorem 1.1 applied with $\alpha = -2$ to $\hat{u}$ at 0 we find:

Corollary 1.2 *Let u be a solution of (1.5) on an unbounded open set Ω with $B_R(0)^c \subset \Omega$ for some $R > 0$. Then there holds*

$$u = - \left(\frac{\gamma_\infty}{n\omega_n} \right)^{\frac{1}{n-1}} \log |x| + O(1) \quad (1.7)$$

as $|x| \rightarrow \infty$ for some $\gamma_\infty > n^n \omega_n$. In particular, when $\Omega = \mathbb{R}^n$ there holds

$$u = - \left(\frac{1}{n\omega_n} \int_{\mathbb{R}^n} e^u \right)^{\frac{1}{n-1}} \log |x| + O(1) \quad (1.8)$$

as $|x| \rightarrow \infty$.

When $n = 2$ the asymptotic expansion (1.8) is a well known property established in [6] by means of the Green representation formula—unfortunately not available in the quasi-linear case—and of the growth properties of entire harmonic functions. Notice that

$$\gamma_\infty = \int_{\mathbb{R}^n} |x|^{-2n} e^{\hat{u}} = \int_{\mathbb{R}^n} e^u$$

follows by integrating (1.2) written for $\hat{u}$ on $\mathbb{R}^n$. Property (1.8) has been already proved in [11] under the assumption $\gamma_\infty > n^n \omega_n$ and the present full generality allows to simplify the classification argument in [11]: a Pohozaev identity leads for γ_∞ to the quantization property

$$\int_{\mathbb{R}^n} e^u = n \left(\frac{n^2}{n-1} \right)^{n-1} \omega_n \quad (1.9)$$

and an isoperimetric argument concludes the classification result thanks to (1.9).

In the punctured plane $\Omega = \mathbb{R}^n \setminus \{0\}$ the isoperimetric argument fails and in general the classification result is no longer available. The two-dimensional case $n = 2$ has been treated via complex analysis in [7, 17]: solutions u to

$$-\Delta u = e^u - \gamma \delta_0 \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < +\infty, \quad (1.10)$$

have been classified for $\gamma > -4\pi$ of the form

$$u(x) = \log \frac{8(\alpha + 1)^2 \lambda^2 |x|^{2\alpha}}{(1 + \lambda^2 |x|^{\alpha+1} + c|^2)^2}, \quad \alpha = \frac{\gamma}{4\pi},$$

with $\lambda > 0$ and a complex number $c = 0$ if $\alpha \notin \mathbb{N}$, and in particular satisfy

$$\int_{\mathbb{R}^n} e^u = 8\pi(\alpha + 1). \quad (1.11)$$

The structure of entire solutions u to (1.10) changes drastically passing from radial solutions when $\alpha \notin \mathbb{N}$ to non-radial solutions when $\alpha \in \mathbb{N}$ (and $c \neq 0$). Unfortunately a PDE approach is not available for $n = 2$ and a classification result is completely out of reach when $n \geq 3$. However, quantization properties are still in order as it follows by Theorem 1.1 and the Pohozaev identities:

Theorem 1.3 *Let u be a solution of*

$$-\Delta_n u = e^u - \gamma \delta_0 \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^u < +\infty. \quad (1.12)$$

Then $\gamma > -n^n \omega_n$ and

$$\int_{\mathbb{R}^n} e^u = \gamma + \gamma_\infty \quad (1.13)$$

with γ_∞ the unique solution in $(n^n \omega_n, +\infty)$ of

$$\frac{n-1}{n} (n\omega_n)^{-\frac{1}{n-1}} \gamma_\infty^{\frac{n}{n-1}} - n\gamma_\infty = n\gamma + \frac{n-1}{n} (n\omega_n)^{-\frac{1}{n-1}} |\gamma|^{\frac{n}{n-1}}. \quad (1.14)$$

When $n = 2$ notice that for $\gamma > -4\pi$ the unique solution $\gamma_\infty > 4\pi$ of (1.14) is given explicitly as $\gamma_\infty = \gamma + 8\pi$ and then $\int_{\mathbb{R}^n} e^u = 2\gamma + 8\pi = 8\pi(\alpha + 1)$ in accordance with (1.11). To have Theorem 1.3 meaningful, in Sect. 4 we will show the existence of a family of radial solutions u to (1.12) but we don't know whether other solutions might exist or not, depending on the value of γ . Notice that (1.10) is equivalent to

$$-\Delta v = |x|^{2\alpha} e^v \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2\alpha} e^v < +\infty,$$

in terms of $v = u - 2\alpha \log |x|$. For $n \geq 3$ such equivalence breaks down and the problem

$$-\Delta_n v = |x|^{n\alpha} e^v \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} |x|^{n\alpha} e^v < +\infty, \quad (1.15)$$

has its own interest, independently of (1.12). As in Theorem 1.3, for (1.15) we have the following quantization result:

Theorem 1.4 *Let v be a solution of (1.15). Then $\alpha > -1$ and*

$$\int_{\mathbb{R}^n} |x|^{n\alpha} e^v = n \left(\frac{n^2}{n-1} \right)^{n-1} (\alpha+1)^{n-1} \omega_n.$$

Radial solutions v of (1.15) can be easily obtained as $v = n \log(\alpha+1) + u(|x|^{\alpha+1})$ in terms of a radial entire solution u to (1.5). Thanks to the classification result in [11], for (1.15) we can therefore exhibit the following family of radial solutions:

$$v_\lambda = \log \frac{c_n(\alpha+1)^n \lambda^n}{\left(1 + \lambda^{\frac{n}{n-1}} |x|^{\frac{n(\alpha+1)}{n-1}}\right)^n}, \quad c_n = n \left(\frac{n^2}{n-1} \right)^{n-1}.$$

Problems with exponential nonlinearities on a bounded domain with given singular sources can exhibit non-compact solution-sequences, whose shape near a blow-up point is asymptotically described by the limiting problem (1.12). In the regular case (i.e. in absence of singular sources) a concentration-compactness principle has been established [5] for $n = 2$ and [1] for $n \geq 2$. In the non-compact situation the exponential nonlinearity concentrates at the blow-up points as a sum of Dirac measures. Theorem 1.3 gives information on the concentration mass of such Dirac measures at a singular blow-up point, which is expected to be a super-position of several masses $c_n \omega_n$ carried by multiple sharp collapsing peaks governed by (1.12) $_{\gamma=0}$ and possibly the mass (1.13) of a sharp peak described by (1.12). In the regular case such quantization property on the concentration masses has been proved [14] for $n = 2$ and extended [12] to $n \geq 2$ by requiring an additional boundary assumption, while the singular case has been addressed in [2,21] for $n = 2$. For Theorem 1.4 a similar comment is in order.

Let us briefly explain the main ideas behind Theorem 1.1. We can re-adapt the argument in [11] to show that $u \in \bigcap_{1 \leq q < n} W_{\text{loc}}^{1,q}(\Omega)$ and then u satisfies (1.2) for some $\gamma \in \mathbb{R}$. On a radial ball $B \subset \subset \Omega$ decompose u as $u = u_0 + h$, where h is given by

$$\Delta_n h = \gamma \delta_0 \text{ in } B, \quad h = u \text{ on } \partial B$$

and satisfies (1.3) thanks to [13,19,20]. The key property stems from a simple observation: $|x|^{n\alpha} e^h \in L^1$ near 0 implies $|x|^{n\alpha} e^h \in L^p$ near 0 for some $p > 1$ whenever h has a logarithmic singularity at 0. Back to [19,20] thanks to such improved integrability, one aims to show that $u_0 \in L^\infty(B)$ and then u has the same logarithmic behavior (1.3) as h . In order to develop a regularity theory for the solution u_0 of

$$-\Delta_n(u_0 + h) + \Delta_n h = |x|^{n\alpha} e^{u_0+h} \text{ in } B, \quad u_0 = 0 \text{ on } \partial B, \quad (1.16)$$

the crucial point is to establish several integral inequalities involving u_0 paralleling the estimates available for entropy solutions in [1,3] and for $W^{1,n}$ -solutions in [19]. To this aim, we make use of the deep uniqueness result [10] to show that u can be regarded as a Solution Obtained as Limit of Approximations (the so-called SOLA, see for example [4]).

The paper is organized as follows. In Sect. 2 we develop the above argument to prove Theorem 1.1. Section 3 is devoted to establish Theorems 1.3–1.4 via Pohozaev identities: going back to an idea of Y.Y. Li and N. Wolanski for $n = 2$, the Pohozaev identities have revealed to be a fundamental tool to derive information on the mass of a singularity (see for example [2,12,18]). In Sect. 4 a family of radial solutions u to (1.12) is constructed.

2 Proof of Theorem 1.1

Assume $B_1(0) \subset \subset \Omega$. Let us first establish the following property on u .

Proposition 2.1 *Let u be a solution of (1.1). There exists $C > 0$ so that*

$$u(x) \leq C - n(\alpha + 1) \log |x| \quad \text{in } B_1(0) \setminus \{0\}. \quad (2.1)$$

Proof Letting $U_r(y) = \hat{u}(\frac{y}{r}) + n(\alpha + 1) \log r = u(\frac{ry}{|y|^2}) + n(\alpha + 1) \log r$ for $0 < r \leq \frac{1}{2}$, we have that U_r solves

$$\begin{aligned} -\Delta_n U_r &= |y|^{-n(\alpha+2)} e^{U_r} \quad \text{in } \mathbb{R}^n \setminus B_{\frac{1}{2}}(0), \\ \int_{\mathbb{R}^n \setminus B_{\frac{1}{2}}(0)} |y|^{-n(\alpha+2)} e^{U_r} &= \int_{B_{2r}(0)} |x|^{n\alpha} e^u \end{aligned} \quad (2.2)$$

in view of (1.6). Given a ball $B_{\frac{1}{2}}(x_0)$ for $x_0 \in \mathbb{S}^{n-1}$, let us consider the n -harmonic function H_r in $B_{\frac{1}{2}}(x_0)$ so that $H_r = U_r$ on $\partial B_{\frac{1}{2}}(x_0)$. By the weak maximum principle we deduce that $H_r \leq U_r$ in $B_{\frac{1}{2}}(x_0)$ and then

$$\begin{aligned} \int_{B_{\frac{1}{2}}(x_0)} (H_r)_+^n &\leq \int_{B_{\frac{1}{2}}(x_0)} (U_r)_+^n \leq n! \int_{B_{\frac{1}{2}}(x_0)} e^{U_r} \\ &\leq C \int_{\mathbb{R}^n \setminus B_{\frac{1}{2}}(0)} |y|^{-n(\alpha+2)} e^{U_r} \leq C \int_{\Omega} |x|^{n\alpha} e^u < +\infty \end{aligned} \quad (2.3)$$

for all $0 < r \leq \frac{1}{2}$. By the estimates in [19] we have that there exists $C > 0$ so that

$$\|H_r\|_{\infty, B_{\frac{1}{4}}(x_0)} \leq C \quad (2.4)$$

for all $0 < r \leq \frac{1}{2}$. At the same time, by the exponential estimate in [1] we have that there exist $0 < r_0 \leq \frac{1}{2}$ and $C > 0$ so that

$$\int_{B_{\frac{1}{2}}(x_0)} e^{2|U_r - H_r|} \leq C \quad (2.5)$$

for all $0 < r \leq r_0$ in view of $\lim_{r \rightarrow 0^+} \int_{B_{\frac{1}{2}}(x_0)} |y|^{-n(\alpha+2)} e^{U_r} = 0$ thanks to (1.1) and (2.2). Since $|y|^{-n(\alpha+2)} e^{U_r} \leq C e^{|U_r - H_r|}$ on $B_{\frac{1}{4}}(x_0)$ for all $0 < r \leq \frac{1}{2}$ in view of (2.4), we deduce that $|y|^{-n(\alpha+2)} e^{U_r}$ and $(U_r)_+^{\frac{n}{2}}$ are uniformly bounded in $L^2(B_{\frac{1}{4}}(x_0))$ for all $0 < r \leq r_0$ in view of (2.3) and (2.5). Thanks again to the estimates in [19], we finally deduce that

$$\|U_r^+\|_{\infty, B_{\frac{1}{8}}(x_0)} \leq C \quad (2.6)$$

for all $0 < r \leq r_0$. Since $\mathbb{S}^{n-1}$ can be covered by a finite number of balls $B_{\frac{1}{8}}(x_0)$, $x_0 \in \mathbb{S}^n$, going back to u from (2.6) one deduces that

$$u(x) \leq C - n(\alpha + 1) \log |x|$$

for all $|x| = r \leq r_0$. Since this estimate does hold in $B_1(0) \setminus B_{r_0}(0)$ too, we have established the validity of (2.1). $\square$

From now on, set $B = B_r(0)$ for $0 < r \leq 1$. We are now ready to establish the starting point for the argument we will develop in the sequel. There holds

Proposition 2.2 *Let u be a solution of (1.1). Then*

$$u \in \bigcap_{1 \leq q < n} W^{1,q}(\Omega). \quad (2.7)$$

Proof Let us go through the argument in [11] to obtain $W^{1,q}$ -estimates on u . For $0 < \epsilon < r < 1$ let us introduce $h_{\epsilon,r} \in W^{1,n}(A_{\epsilon,r})$, $A_{\epsilon,r} := B \setminus \overline{B_\epsilon(0)}$, as the solution of

$$\Delta_n h_{\epsilon,r} = 0 \text{ in } A_{\epsilon,r}, \quad h_{\epsilon,r} = u \text{ on } \partial A_{\epsilon,r}.$$

Regularity issues for quasi-linear PDEs involving Δ_n are well established since the works of DiBenedetto, Evans, Lewis, Serrin, Tolksdorf, Uhlenbeck, Uraltseva. For example, by [9, 15, 19, 22] we deduce that $h_{\epsilon,r}$, $u_{\epsilon,r} = u - h_{\epsilon,r} \in C^{1,\eta}(\overline{A_{\epsilon,r}})$ and $u_{\epsilon,r}$ satisfies

$$-\Delta_n(u_{\epsilon,r} + h_{\epsilon,r}) + \Delta_n h_{\epsilon,r} = |x|^{n\alpha} e^u \text{ in } A_{\epsilon,r}, \quad u_{\epsilon,r} = 0 \text{ on } \partial A_{\epsilon,r}. \quad (2.8)$$

By the techniques in [1, 3, 4] we have the following estimates, see Proposition 2.1 in [11]: for all $1 \leq q < n$ and all $p \geq 1$ there exist $0 < r_0 < 1$ and $C > 0$ so that

$$\int_{A_{\epsilon,r}} |\nabla u_{\epsilon,r}|^q + \int_{A_{\epsilon,r}} e^{pu_{\epsilon,r}} \leq C \quad (2.9)$$

for all $0 < \epsilon < r \leq r_0$ thanks to (2.8) and $\lim_{r \rightarrow 0^+} \int_B |x|^{n\alpha} e^u = 0$. Since by the Sobolev embedding $W_0^{1,\frac{n}{2}}(B_1(0)) \hookrightarrow L^n(B_1(0))$ there holds $\int_{A_{\epsilon,r}} |u_{\epsilon,r}|^n \leq C$ for all $0 < \epsilon < r \leq r_0$ in view of (2.9) with $q = \frac{n}{2}$ and $A_{\epsilon,r} \subset B_1(0)$, we have that

$$\|h_{\epsilon,r}\|_{L^n(A)} \leq C(A) \quad \forall A \subset \subset \overline{B} \setminus \{0\}, \quad \forall 0 < \epsilon < r \leq r_0$$

in view of $u \in C_{loc}^{1,\eta}(B_1(0) \setminus \{0\})$ and then

$$\|h_{\epsilon,r}\|_{C^{1,\eta}(A)} \leq C(A) \quad \forall A \subset \subset \overline{B} \setminus \{0\}, \quad \forall 0 < \epsilon < r \leq r_0$$

thanks to [9, 15, 19, 22]. By the Ascoli–Arzelà's Theorem and a diagonal process we can find a sequence $\epsilon \rightarrow 0$ so that $h_{\epsilon,r} \rightarrow h_r$ and $u_{\epsilon,r} \rightarrow u_r := u - h_r$ in $C_{loc}^1(\overline{B} \setminus \{0\})$ as $\epsilon \rightarrow 0$, where $h_r \leq u$ is a n -harmonic function in $B \setminus \{0\}$ and u_r satisfies

$$u_r \in W_0^{1,q}(B), \quad e^{u_r} \in L^p(B) \quad (2.10)$$

for all $1 \leq q < n$ and all $p \geq 1$ if r is sufficiently small in view of (2.9). Since

$$h_r(x) \leq C - n(\alpha + 1) \log |x| \quad \text{in } B$$

in view of $h_r \leq u$ and (2.1), we have that $H^\lambda(y) = -\frac{h_r(\lambda y)}{\log \lambda}$ is a n -harmonic function in $B_{\frac{r}{\lambda}}(0)$ so that $H^\lambda \leq n(\alpha + 1) + 1$ in $B_2(0) \setminus B_{\frac{1}{2}}(0)$ for all $0 < \lambda \leq \lambda_0$, where $\lambda_0 \in (0, \frac{r}{2}]$ is a suitable small number. By the Harnack inequality in Theorem 7- [19] applied to $n(\alpha + 1) + 1 - H^\lambda \geq 0$ we deduce that

$$\max_{|y|=1} H^\lambda \leq C \left[\frac{n|\alpha + 1| + 1}{C} + \min_{|y|=1} H^\lambda \right] \quad (2.11)$$

for all $0 < \lambda \leq \lambda_0$, for a suitable $C \in (0, 1)$. There are two possibilities:

- either $\min_{|y|=1} H^\lambda \geq -\frac{n|\alpha+1|+1}{C}$ for all $0 < \lambda \leq \lambda_1$ and some $\lambda_1 \in (0, \lambda_0]$, which implies $\max_{|y|=1} |H^\lambda| \leq \frac{n|\alpha+1|+1}{C}$ for all $0 < \lambda \leq \lambda_1$ and in particular

$$|h_r| \leq -C_0 \log |x| \quad \text{in } B_{\lambda_1}(0) \quad (2.12)$$

for some $C_0 > 0$;

- or $\min_{|y|=1} H^{\lambda_n} \leq -\frac{n|\alpha+1|+1}{C}$ for a sequence $\lambda_n \downarrow 0$, which implies $\max_{|y|=1} H^{\lambda_n} \leq 0$ in view of (2.11) and in turn $h_r \leq 0$ on $|x| = \lambda_n$ for all $n \in \mathbb{N}$, leading to

$$h_r \leq 0 \quad \text{in } B_{\lambda_1}(0) \quad (2.13)$$

by the weak maximum principle.

Notice that (2.13) implies the validity of (2.12) for some $C_0 > 0$ in view of Theorem 12- [19]. Thanks to (2.12) one can apply Theorem 1.1- [13] to show that

$$h_r \in W^{1,q}(B) \quad (2.14)$$

for all $1 \leq q < n$ and there exists $\gamma_r \in \mathbb{R}$ so that

$$h_r - \gamma_r(n\omega_n|\gamma_r|^{n-2})^{-\frac{1}{n-1}} \log |x| \in L^\infty(B), \quad \Delta_n h_r = \gamma_r \delta_0 \text{ in } B. \quad (2.15)$$

In particular, $u \in \bigcap_{1 \leq q < n} W^{1,q}(\Omega)$ in view of (2.10) and (2.14), and (2.7) is established. $\square$

Even if $\gamma_r > -n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$, at this stage we cannot exclude that $\lim_{r \rightarrow 0} \gamma_r = -n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$. Therefore, we are not able to use (2.10) and (2.15) for improving the exponential integrability on u to reach $|x|^{n\alpha}e^u = |x|^{n\alpha}e^{h_r}e^{u_r} \in L^p$ near 0 for some $p > 1$ and r sufficiently small, as it would be necessary to prove L^∞ -bounds on u_r via (2.8) on $u_{\epsilon,r}$.

We need to argue in a different way. Since $u \in W^{1,n-1}(\Omega)$ in view of (2.7), we can extend (1.1) at 0 as

$$-\Delta_n u = |x|^{n\alpha}e^u - \gamma\delta_0 \quad \text{in } \Omega. \quad (2.16)$$

To see it, let $\varphi \in C_0^1(\Omega)$ and consider a function $\chi_\epsilon \in C^\infty(\Omega)$ with $0 \leq \chi_\epsilon \leq 1$, $\chi_\epsilon = 0$ in $B_{\frac{\epsilon}{2}}(0)$, $\chi_\epsilon = 1$ in $\Omega \setminus B_\epsilon(0)$ and $\epsilon|\nabla \chi_\epsilon| \leq C$. Taking $\chi_\epsilon \varphi \in C_0^1(\Omega \setminus \{0\})$ as a test function in (1.1) we have that

$$\int_\Omega |\nabla u|^{n-2} \langle \nabla u, \varphi \nabla \chi_\epsilon + \chi_\epsilon \nabla \varphi \rangle = \int_\Omega \chi_\epsilon |x|^{n\alpha} e^u \varphi. \quad (2.17)$$

Since $u \in W^{1,n-1}(\Omega)$ and $|x|^{n\alpha}e^u \in L^1(\Omega)$ it is easily seen that

$$\begin{aligned} \int_\Omega \chi_\epsilon |\nabla u|^{n-2} \langle \nabla u, \nabla \varphi \rangle &\rightarrow \int_\Omega |\nabla u|^{n-2} \langle \nabla u, \nabla \varphi \rangle, \\ \int_\Omega \chi_\epsilon |x|^{n\alpha} e^u \varphi &\rightarrow \int_\Omega |x|^{n\alpha} e^u \varphi \end{aligned} \quad (2.18)$$

as $\epsilon \rightarrow 0$. Since

$$\int_\Omega |\nabla u|^{n-1} |\varphi - \varphi(0)| |\nabla \chi_\epsilon| \leq C \int_{B_\epsilon(0) \setminus B_{\frac{\epsilon}{2}}(0)} |\nabla u|^{n-1} \rightarrow 0$$

as $\epsilon \rightarrow 0$ in view of $|\varphi - \varphi(0)| \leq C\epsilon$ in $B_\epsilon(0) \setminus B_{\frac{\epsilon}{2}}(0)$ and $u \in W^{1,n-1}(\Omega)$, the remaining term in (2.17) can be re-written as follows:

$$\int_{\Omega} |\nabla u|^{n-2} \varphi \langle \nabla u, \nabla \chi_\epsilon \rangle = \varphi(0) \int_{\Omega} |\nabla u|^{n-2} \langle \nabla u, \nabla \chi_\epsilon \rangle + o(1) \quad (2.19)$$

as $\epsilon \rightarrow 0$. By inserting (2.18)–(2.19) into (2.17) we deduce the existence of

$$\gamma = \lim_{\epsilon \rightarrow 0} \int_{\Omega} |\nabla u|^{n-2} \langle \nabla u, \nabla \chi_\epsilon \rangle$$

and the validity of (2.16) for u . Moreover, if we assume $u \in C^1(\overline{\Omega} \setminus \{0\})$, we can interpret γ as

$$\gamma = \lim_{\epsilon \rightarrow 0} \left[\int_{\Omega} |x|^{n\alpha} e^u \chi_\epsilon + \int_{\partial\Omega} |\nabla u|^{n-2} \partial_n u \right] = \int_{\Omega} |x|^{n\alpha} e^u + \int_{\partial\Omega} |\nabla u|^{n-2} \partial_n u.$$

Since $\gamma_r \geq \gamma + o(1)$ as $r \rightarrow 0$ according to (4.16)–[11], we find that h_r is possibly much lower than u and then needs to be compensated by an unbounded function $u_r \geq 0$ in order to keep the validity of $u = u_r + h_r$. Instead, thanks to Theorem 2.1–[13] introduce $h \in \bigcap_{1 \leq q < n} W^{1,q}(B)$ as the solution of

$$\Delta_n h = \gamma \delta_0 \text{ in } B, \quad h = u \text{ on } \partial B$$

so that

$$h - \gamma(n\omega_n |\gamma|^{n-2})^{-\frac{1}{n-1}} \log |x| \in L^\infty(B). \quad (2.20)$$

Decomposing u as $u = u_0 + h$, the solution u_0 of (1.16) on B is very likely a bounded function, as we will prove below.

In order to establish some crucial integral inequalities involving u_0 , let us introduce the following approximation scheme. By convolution with mollifiers consider sequences $f_j, g_j \in C_0^\infty(B)$ so that $f_j \rightarrow |x|^{n\alpha} e^u - \gamma \delta_0$ weakly in the sense of measures and $0 \leq f_j - g_j \rightarrow |x|^{n\alpha} e^u$ in $L^1(B)$ as $j \rightarrow +\infty$. Since $u \in C^{1,\eta}(\partial B)$, let $\varphi \in C^{1,\eta}(B)$ be the n -harmonic extension of $u|_{\partial B}$ in B . Let $v_j, w_j \in W_0^{1,n}(B)$ be the weak solutions of $-\operatorname{div} \mathbf{a}(x, \nabla v_j) = f_j$ and $-\operatorname{div} \mathbf{a}(x, \nabla w_j) = g_j$ in B , where $\mathbf{a}(x, p) = |p + \nabla \varphi|^{n-2}(p + \nabla \varphi) - |\nabla \varphi|^{n-2} \nabla \varphi$. In this way, $u_j = v_j + \varphi$ and $h_j = w_j + \varphi$ do solve

$$-\Delta_n u_j = f_j \text{ and } -\Delta_n h_j = g_j \text{ in } B, \quad u_j = h_j = u \text{ on } \partial B.$$

Since f_j, g_j are uniformly bounded in $L^1(B)$, by (21) in [4] we can assume that $v_j \rightarrow v$ and $w_j \rightarrow w$ in $W_0^{1,q}(B)$ for all $1 \leq q < n$ as $j \rightarrow +\infty$, where v and w do satisfy

$$-\operatorname{div} \mathbf{a}(x, \nabla v) = |x|^{n\alpha} e^u - \gamma \delta_0 \text{ and } -\operatorname{div} \mathbf{a}(x, \nabla w) = -\gamma \delta_0 \text{ in } B \quad (2.21)$$

in view of $g_j \rightarrow -\gamma \delta_0$ weakly in the sense of measures as $j \rightarrow +\infty$. Since $u - \varphi, h - \varphi \in \bigcap_{1 \leq q < n} W_0^{1,q}(B)$ do solve the first and the second equation in (2.21), respectively, by the uniqueness result in [10] (see Theorems 1.2 and 4.2 in [10]) we have that $v = u - \varphi$ and $w = h - \varphi$, i.e.

$$u_j \rightarrow u \text{ and } h_j \rightarrow h \text{ in } W^{1,q}(B) \text{ for all } 1 \leq q < n \text{ as } j \rightarrow +\infty.$$

Thanks to the approximation given by the u_j 's and h_j 's, we can now derive some crucial integral inequalities on u_0 .

Proposition 2.3 *Let u_0 be a solution of (1.16). Then $u_0 \geq 0$ and we have:*

$$\int_{\{k < |u_0| < k+a\}} |\nabla u_0|^n \leq \frac{a}{d} \int_B |x|^{n\alpha} e^u \quad \forall k, a > 0 \quad (2.22)$$

and, if $|x|^{n\alpha} e^u \in L^p(B)$ for some $p > 1$,

$$\left(\int_B u_0^{2mnq} \right)^{\frac{1}{2m}} \leq \frac{Cq^{n-1}}{d} |B|^{\frac{n-1}{mnq}} \left(\int_B |x|^{np\alpha} e^{pu} \right)^{\frac{1}{p}} \left(\int_B u_0^{mnq} \right)^{\frac{n(q-1)+1}{mnq}}, \quad (2.23)$$

where $m = \frac{p}{p-1}$ and

$$d = \inf_{X \neq Y} \frac{\langle |X|^{n-2} X - |Y|^{n-2} Y, X - Y \rangle}{|X - Y|^n} > 0. \quad (2.24)$$

Proof First use $-(v_j - w_j)_- \in W_0^{1,n}(B)$ as a test function for $-\operatorname{div} \mathbf{a}(x, \nabla v_j) + \operatorname{div} \mathbf{a}(x, \nabla w_j)$ to get

$$\begin{aligned} d \int_{\{v_j - w_j < 0\}} |\nabla(v_j - w_j)|^n &\leq - \int_B \langle \mathbf{a}(x, \nabla v_j) - \mathbf{a}(x, \nabla w_j), \nabla(v_j - w_j)_- \rangle \\ &= - \int_B (f_j - g_j)(v_j - w_j)_- \leq 0 \end{aligned}$$

in view of (2.24) and $f_j - g_j \geq 0$. Hence, $v_j - w_j \geq 0$ and then $u_0 \geq 0$ in view of $v_j - w_j \rightarrow u - h = u_0$ in $W_0^{1,q}(B)$ for all $1 \leq q < n$ as $j \rightarrow +\infty$. Now, introduce the truncature operator $T_{k,a}$, for $k, a > 0$, as

$$T_{k,a}(s) = \begin{cases} s - k \operatorname{sign}(s) & \text{if } k < |s| < k + a, \\ a \operatorname{sign}(s) & \text{if } |s| \geq k + a, \\ 0 & \text{if } |s| \leq k, \end{cases}$$

and use $T_{k,a}(v_j - w_j) \in W_0^{1,n}(B)$ as a test function for $-\operatorname{div} \mathbf{a}(x, \nabla v_j) + \operatorname{div} \mathbf{a}(x, \nabla w_j)$ to get

$$\begin{aligned} d \int_{\{k < |v_j - w_j| < k+a\}} |\nabla(v_j - w_j)|^n &\leq \int_B \langle \mathbf{a}(x, \nabla v_j) - \mathbf{a}(x, \nabla w_j), \nabla T_{k,a}(v_j - w_j) \rangle \\ &= \int_B (f_j - g_j) T_{k,a}(v_j - w_j) \end{aligned} \quad (2.25)$$

in view of (2.24). Since $v_j - w_j \rightarrow u_0$ in $W_0^{1,q}(B)$ for all $1 \leq q < n$ and $f_j - g_j \rightarrow |x|^{n\alpha} e^u$ in $L^1(B)$ as $j \rightarrow +\infty$, we can let $j \rightarrow +\infty$ in (2.25) and get by Fatou's Lemma that

$$d \int_{\{k < |u_0| < k+a\}} |\nabla u_0|^n \leq \int_B |x|^{n\alpha} e^u T_{k,a}(u_0) \leq a \int_B |x|^{n\alpha} e^u$$

yielding the validity of (2.22). Finally, if $|x|^{n\alpha} e^u \in L^p(B)$ for some $p > 1$, we can assume that $f_j - g_j \rightarrow |x|^{n\alpha} e^u$ in $L^p(B)$ as $j \rightarrow +\infty$ and use $T_a[v_j - w_j]^{n(q-1)}(v_j - w_j) \in W_0^{1,n}(B)$, where $T_a = T_{0,a}$ and $a > 0, q \geq 1$, as a test function for $-\operatorname{div} \mathbf{a}(x, \nabla v_j) +$

$\operatorname{div} \mathbf{a}(x, \nabla w_j)$ to get by Hölder's inequality

$$\begin{aligned} & d \frac{n(q-1)+1}{q^n} \int_{\{|v_j-w_j|^{n(q-1)+1} < a\}} |\nabla |v_j - w_j|^q|^n \\ & \leq \int_B |f_j - g_j| |v_j - w_j|^{n(q-1)+1} \\ & \leq |B|^{\frac{n-1}{mnq}} \|f_j - g_j\|_p \left(\int_B |v_j - w_j|^{mnq} \right)^{\frac{n(q-1)+1}{mnq}} \end{aligned}$$

in view of $|T_a(s)| \leq |s|$ and (2.24). We have used that $v_j - w_j \in W_0^{1,n}(B) \subset \bigcap_{q \geq 1} L^q(B)$ by the Sobolev embedding Theorem. Letting $a \rightarrow +\infty$, by Fatou's Lemma we get that

$$\int_B |\nabla |v_j - w_j|^q|^n \leq \frac{q^n}{d[n(q-1)+1]} |B|^{\frac{n-1}{mnq}} \|f_j - g_j\|_p \left(\int_B |v_j - w_j|^{mnq} \right)^{\frac{n(q-1)+1}{mnq}}.$$

In particular, $|v_j - w_j|^q \in W_0^{1,n}(B)$ and by the Sobolev embedding $W_0^{1,n}(B) \subset L^{2mn}(B)$ we have that

$$\left(\int_B |v_j - w_j|^{2mnq} \right)^{\frac{1}{2m}} \leq \frac{Cq^n}{d[n(q-1)+1]} |B|^{\frac{n-1}{mnq}} \|f_j - g_j\|_p \left(\int_B |v_j - w_j|^{mnq} \right)^{\frac{n(q-1)+1}{mnq}}.$$

Letting $j \rightarrow +\infty$, we finally deduce the validity of (2.23):

$$\left(\int_B u_0^{2mnq} \right)^{\frac{1}{2m}} \leq \frac{Cq^{n-1}}{d} |B|^{\frac{n-1}{mnq}} \left(\int_B |x|^{np\alpha} e^{pu} \right)^{\frac{1}{p}} \left(\int_B u_0^{mnq} \right)^{\frac{n(q-1)+1}{mnq}}$$

in view of $v_j - w_j \rightarrow u_0$ in $W_0^{1,q}(B)$ for all $1 \leq q < n$ and $f_j - g_j \rightarrow |x|^{n\alpha} e^u$ in $L^p(B)$ as $j \rightarrow +\infty$. $\square$

We are now ready to complete the proof of Theorem 1.1.

Proof (of Theorem 1.1) Since $u_0 \geq 0$ by Proposition 2.3, we have that $h \leq u$. By (1.1) and (2.20) we have that

$$\int_B |x|^{n\alpha + \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}}} \leq C \int_B |x|^{n\alpha} e^h \leq C \int_\Omega |x|^{n\alpha} e^u < +\infty,$$

which implies

$$n\alpha + \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} > -n \quad (2.26)$$

or equivalently

$$\gamma > -n^n |\alpha + 1|^{n-2} (\alpha + 1) \omega_n. \quad (2.27)$$

Since $|x|^{n\alpha} e^h \in L^1$ near 0 and h has a logarithmic singularity at 0, then, as already observed in the Introduction, a stronger integrability follows:

$$|x|^{n\alpha} e^h \in L^p(B) \quad (2.28)$$

for some $p > 1$. Inequality (2.22) is used in [1] to deduce exponential estimates on u_0 like

$$\int_B e^{\frac{\delta u_0}{\|f\|_1}} \leq C_r \quad (2.29)$$

for some $\delta > 0$ where $f = |x|^{n\alpha} e^u$. Since $\lim_{r \rightarrow 0} \int_B |x|^{n\alpha} e^u = 0$, by (2.29) we deduce that $e^{u_0} \in L^p(B)$ for all $p \geq 1$ if r is sufficiently small and then by (2.28)

$$|x|^{n\alpha} e^u = |x|^{n\alpha} e^h e^{u_0} \in L^p(B)$$

for some $p > 1$. Inequality (2.23) is used in Proposition 4.1-[11] (compare with (4.4) in [11]) to get $u_0 \in L^\infty(B)$ and then (2.20) does hold for u too, yielding the validity of (1.3). In order to prove (1.4), set $H = u - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log |x|$ and introduce the function

$$U_r(y) = u(ry) - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log r = \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log |y| + H(ry)$$

for a given sequence $r \rightarrow 0$. Since

$$-\Delta_n U_r = r^{n(1+\alpha)} |y|^{n\alpha} e^{u(ry)} = r^{n(1+\alpha) + \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}}} |y|^{n\alpha + \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}}} e^{H(ry)},$$

by (1.3) and (2.26)–(2.27) we have that U_r and $\Delta_n U_r$ are bounded in $L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, uniformly in r . By [9, 19, 22] we deduce that U_r is bounded in $C^{1,\eta}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, uniformly in r . By the Ascoli–Arzelà’s Theorem and a diagonal process, up to a sub-sequence we have that $U_r \rightarrow U_0$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where U_0 is a n -harmonic function in $\mathbb{R}^n \setminus \{0\}$. Setting $H_r(y) = H(ry)$, we deduce that $H_r \rightarrow H_0$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where $H_0 \in L^\infty(\mathbb{R}^n)$ in view of (1.3). Since $U_0 = \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log |y| + H_0$ with $H_0 \in L^\infty(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$, it is well known that H_0 is a constant function, as shown in Corollary 2.2-[13] (see also [11] for a direct proof). In particular we get that

$$\sup_{|x|=r} |x| \left| \nabla [u - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log |x|] \right| = \sup_{|y|=1} |\nabla H_r| \rightarrow \sup_{|y|=1} |\nabla H_0| = 0.$$

Since this is true for any sequence $r \rightarrow 0$ up to extracting a sub-sequence, we have established the validity of (1.4). The proof of Theorem 1.1 is concluded. $\square$

3 Quantization results

In this section we will make crucial use of the following integral identity: for any solution u of

$$-\Delta_n u = |x|^{n\alpha} e^u \text{ in } \mathbb{R}^n \setminus \{0\} \quad (3.1)$$

there holds

$$n(\alpha + 1) \int_A |x|^{n\alpha} e^u = \int_{\partial A} \left[|x|^{n\alpha} e^u \langle x, \nu \rangle + |\nabla u|^{n-2} \partial_\nu u \langle \nabla u, x \rangle - \frac{|\nabla u|^n}{n} \langle x, \nu \rangle \right], \quad (3.2)$$

where A is the annulus $A = B_R(0) \setminus B_\epsilon(0)$, $0 < \epsilon < R < +\infty$, and ν is the unit outward normal vector at ∂A . Notice that (3.2) is simply a special case of the well-known Pohozaev identities associated to (3.1). Even though the classical Pohozaev identities require more regularity than simply $u \in C^{1,\eta}(\mathbb{R}^n \setminus \{0\})$, (3.2) is still valid in the quasilinear case and we refer to [8] for a justification. Thanks to (3.2) we are able to show the following general result.

Proposition 3.1 *Let u be a solution of (3.1) so that (1.3)–(1.4) do hold at 0 and ∞ with γ and $-\gamma_\infty$, respectively, so that $\gamma > -n^n|\alpha + 1|^{n-2}(\alpha + 1)\omega_n$ and $\gamma_\infty > n^n|\alpha + 1|^{n-2}(\alpha + 1)\omega_n$. Then $\int_{\mathbb{R}^n} |x|^{n\alpha} e^u = \gamma + \gamma_\infty$ satisfies*

$$n(\alpha + 1)(\gamma + \gamma_\infty) = \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}} \left[|\gamma_\infty|^{\frac{n}{n-1}} - |\gamma|^{\frac{n}{n-1}} \right]. \quad (3.3)$$

Proof By (1.3)–(1.4) at 0 with $\gamma > -n^n|\alpha + 1|^{n-2}(\alpha + 1)\omega_n$ we deduce that

$$\begin{aligned} |\nabla u| &= \frac{1}{|x|} \left[\left(\frac{|\gamma|}{n\omega_n} \right)^{\frac{1}{n-1}} + o(1) \right], \quad \langle \nabla u, x \rangle = \gamma (n\omega_n |\gamma|^{n-2})^{-\frac{1}{n-1}} + o(1), \\ |x|^{n\alpha} e^u &= o\left(\frac{1}{|x|^n}\right) \end{aligned} \quad (3.4)$$

as $x \rightarrow 0$ thanks to the equivalence between (2.26) and (2.27). By (3.4) we have that

$$\int_{\partial B_\epsilon(0)} |x| \left[|x|^{n\alpha} e^u + |\nabla u|^{n-2} \langle \nabla u, \frac{x}{|x|} \rangle^2 - \frac{|\nabla u|^n}{n} \right] \rightarrow \frac{n-1}{n} (n\omega_n)^{-\frac{1}{n-1}} |\gamma|^{\frac{n}{n-1}} \quad (3.5)$$

as $\epsilon \rightarrow 0^+$ in view of $\text{Area}(\mathbb{S}^{n-1}) = n\omega_n$. Similarly, by (1.3)–(1.4) at ∞ with $-\gamma_\infty$ so that $\gamma_\infty > n^n|\alpha + 1|^{n-2}(\alpha + 1)\omega_n$ we deduce that

$$\begin{aligned} |\nabla u| &= \frac{1}{|x|} \left[\left(\frac{|\gamma_\infty|}{n\omega_n} \right)^{\frac{1}{n-1}} + o(1) \right], \quad \langle \nabla u, x \rangle = -\gamma_\infty (n\omega_n |\gamma_\infty|^{n-2})^{-\frac{1}{n-1}} + o(1), \\ |x|^{n\alpha} e^u &= o\left(\frac{1}{|x|^n}\right) \end{aligned}$$

as $|x| \rightarrow \infty$ and then

$$\int_{\partial B_R(0)} |x| \left[|x|^{n\alpha} e^u + |\nabla u|^{n-2} \langle \nabla u, \frac{x}{|x|} \rangle^2 - \frac{|\nabla u|^n}{n} \right] \rightarrow \frac{n-1}{n} (n\omega_n)^{-\frac{1}{n-1}} |\gamma_\infty|^{\frac{n}{n-1}} \quad (3.6)$$

as $R \rightarrow +\infty$. In view of (1.4) at 0 and ∞ we easily get that

$$-\Delta_n u = |x|^{n\alpha} e^u - \gamma \delta_0 - \gamma_\infty \delta_\infty \text{ in } \mathbb{R}^n$$

in the sense

$$\int_{\mathbb{R}^n} |\nabla u|^{n-2} \langle \nabla u, \nabla \varphi \rangle = \int_{\mathbb{R}^n} |x|^{n\alpha} e^u \varphi - \gamma \varphi(0) - \gamma_\infty \varphi(\infty)$$

for all $\varphi \in C^1(\mathbb{R}^n)$ so that $\varphi(\infty) := \lim_{|x| \rightarrow \infty} \varphi(x)$ does exist. Choosing $\varphi = 1$ we deduce that

$$\int_{\mathbb{R}^n} |x|^{n\alpha} e^u = \gamma + \gamma_\infty. \quad (3.7)$$

By inserting (3.5)–(3.7) into (3.2) and letting $\epsilon \rightarrow 0^+$, $R \rightarrow +\infty$ we deduce the validity of (3.3). $\square$

Let us now apply Proposition 3.1 to problems (1.12) and (1.15).

Proof (of Theorem 1.3) Let u be a solution of (1.12). By Theorem 1.1 (1.3)–(1.4) do hold for u at 0 with $\gamma > -n^n \omega_n$. By (1.6) the Kelvin transform $\hat{u}$ satisfies

$$-\Delta_n \hat{u} = |x|^{-2n} e^{\hat{u}} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

Let us apply Theorem 1.1 to deduce the validity of (1.3)–(1.4) for $\hat{u}$ at 0 with $\gamma_\infty > n^n \omega_n$. Back to u , (1.3)–(1.4) do hold for u at ∞ with $-\gamma_\infty$ so that $\gamma_\infty > n^n \omega_n$. Let us apply Proposition 3.1 with $\alpha = 0$ to get $\int_{\mathbb{R}^n} e^u = \gamma + \gamma_\infty$ with γ_∞ satisfying (1.14). Notice that the function $f(s) = ns + \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}}|s|^{\frac{n}{n-1}}$ is increasing in $(-n^n \omega_n, +\infty)$ and then $f(s) > f(-n^n \omega_n) = -n^n \omega_n$ for all $s \in (-n^n \omega_n, +\infty)$. At the same time the function $g(s) = \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}}s^{\frac{n}{n-1}} - ns$ is increasing in $(n^n \omega_n, +\infty)$ and then $g(s) > g(n^n \omega_n) = -n^n \omega_n$ for all $s \in (n^n \omega_n, +\infty)$. Therefore, for any $\gamma > -n^n \omega_n$ Equation (1.14) has a unique solution $\gamma_\infty > n^n \omega_n$. The proof of Theorem 1.3 is concluded. $\square$

Remark 3.2 Concerning Corollary 1.2, observe that in the argument above we have established (1.7) for problem (1.5) on $\Omega = \mathbb{R}^n$ and a similar proof is in order for a general unbounded open set Ω . Since $\gamma = 0$, we deduce the validity of (1.8) in view of (1.13).

Proof (of Theorem 1.4) Let v be a solution of (1.15). Applying Theorem 1.1 to the Kelvin transform $\hat{v}$, solution of

$$-\Delta_n \hat{v} = |x|^{-n(\alpha+2)} e^{\hat{v}} \text{ in } \mathbb{R}^n \setminus \{0\},$$

we deduce the validity of (1.3)–(1.4) for v at ∞ with $-\gamma_\infty$ so that $\gamma_\infty > n^n |\alpha + 1|^{n-2} (\alpha + 1) \omega_n$. By Proposition 3.1 with $\gamma = 0$ we deduce that $\gamma_\infty = \int_{\mathbb{R}^n} |x|^{n\alpha} e^v$ satisfies

$$n(\alpha + 1)\gamma_\infty = \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}} \gamma_\infty^{\frac{n}{n-1}}.$$

Therefore, $\alpha > -1$ and

$$\int_{\mathbb{R}^n} |x|^{n\alpha} e^v = n \left(\frac{n^2}{n-1} \right)^{n-1} (\alpha + 1)^{n-1} \omega_n,$$

concluding the proof of Theorem 1.4. $\square$

4 Radial solutions for (1.12)

Fix $M > 1$ and assume that

$$\frac{1}{M} \leq r_0 \leq M, \quad \alpha_0 \leq M, \quad \frac{1}{M} \leq |\alpha_1| \leq M. \quad (4.1)$$

Let us first discuss the local existence theory for the following Cauchy problem:

$$\begin{cases} -\frac{1}{r^{n-1}}(r^{n-1}|U'|^{n-2}U')' = e^U \\ U(r_0) = \alpha_0, \quad U'(r_0) = \alpha_1. \end{cases} \quad (4.2)$$

Given $0 < \delta < \frac{1}{2M}$, define $I = [r_0 - \delta, r_0 + \delta]$ and $E = \{U \in C(I, [\alpha_0 - 1, \alpha_0 + 1]) : U(r_0) = \alpha_0\}$, which is a Banach space endowed with $\|\cdot\|_\infty$ as a norm. We can re-formulate

(4.2) as $U = TU$, where

$$TU(r) = \alpha_0 + \int_{r_0}^r \frac{ds}{s} \left| r_0^{n-1} |\alpha_1|^{n-2} \alpha_1 - \int_{r_0}^s t^{n-1} e^{U(t)} dt \right|^{-\frac{n-2}{n-1}} \left(r_0^{n-1} |\alpha_1|^{n-2} \alpha_1 - \int_{r_0}^s t^{n-1} e^{U(t)} dt \right).$$

In view of

$$|s^n - r_0^n| \leq n(M+1)^{n-1} \delta \quad \forall s \in I \quad (4.3)$$

we have that $\max_I U \leq M+1$ and $\max_I \left| \int_{r_0}^s t^{n-1} e^{U(t)} dt \right| \leq e^{M+1} (M+1)^{n-1} \delta$ for all $U \in E$, and then for $0 < \delta < \frac{e^{-M-1}}{2(M+1)^{3n-3}}$ we have that

$$r_0^{n-1} |\alpha_1|^{n-2} \alpha_1 - \int_{r_0}^s t^{n-1} e^{U(t)} dt \text{ has the same sign as } \alpha_1 \quad \forall s \in I \quad (4.4)$$

and

$$\begin{aligned} \frac{1}{2M^{2n-2}} &\leq \frac{1}{2} r_0^{n-1} |\alpha_1|^{n-1} \leq |r_0^{n-1} |\alpha_1|^{n-2} \alpha_1| \\ &\quad - \int_{r_0}^s t^{n-1} e^{U(t)} dt \leq \frac{3}{2} r_0^{n-1} |\alpha_1|^{n-1} \leq \frac{3}{2} M^{2n-2} \end{aligned} \quad (4.5)$$

for all $U \in E$. Since $\log \frac{r_0+\delta}{r_0} \leq \log \frac{r_0}{r_0-\delta} \leq \frac{\delta}{r_0-\delta} \leq 2M\delta$ in view of $\delta < \frac{r_0}{2}$ and

$$||x|^{-\frac{n-2}{n-1}} x - |y|^{-\frac{n-2}{n-1}} y| \leq C_M |x - y| \quad \forall x, y \in \mathbb{R} : xy \geq 0, \min\{|x|, |y|\} \geq \frac{1}{2M^{2n-2}}$$

(for example, take $C_M = (1 + \frac{n-2}{n-1})(4M^{2n-2})^{\frac{n-2}{n-1}}$), by (4.4)–(4.5) we have that

$$\begin{aligned} \|TU - \alpha_0\|_{\infty, I} &\leq \sup_{r \in I} \left| \int_{r_0}^r \frac{ds}{s} \left| r_0^{n-1} |\alpha_1|^{n-2} \alpha_1 - \int_{r_0}^s t^{n-1} e^{U(t)} dt \right|^{\frac{1}{n-1}} \right| \\ &\leq 2 \left(\frac{3}{2} \right)^{\frac{1}{n-1}} M^3 \delta \leq 3M^3 \delta \end{aligned}$$

and

$$\begin{aligned} \|TU - TV\|_{\infty, I} &\leq C_M \sup_{r \in I} \left| \int_{r_0}^r \frac{ds}{s} \left| \int_{r_0}^s t^{n-1} [e^{U(t)} - e^{V(t)}] dt \right| \right| \\ &\leq 2C_M (M+1)^n e^{M+1} \delta \|U - V\|_{\infty, I} \end{aligned}$$

for all $U, V \in C^1(I)$ in view of $\delta < 1$ and (4.3). In conclusion, if

$$0 < \delta < \min \left\{ \frac{1}{3M^3}, \frac{e^{-M-1}}{2(M+1)^{3n-3}}, \frac{e^{-M-1}}{2C_M(M+1)^n} \right\}, \quad (4.6)$$

then T is a contraction map from E into itself and a unique fixed point $U \in E$ of T is found by the Contraction Mapping Theorem, providing a solution U of (4.2) in $I = [r_0 - \delta, r_0 + \delta]$.

Once a local existence result has been established for (4.2), we can turn the attention to global issues. Given $r_0 > 0$, α_0 and $\alpha_1 \neq 0$, let $I = (r_1, r_2)$, $0 \leq r_1 < r_0 < r_2 \leq +\infty$, be the maximal interval of existence for the solution U of (4.2). We claim that $r_1 = 0$ when $\alpha_1 > 0$ and $r_2 = +\infty$ when $\alpha_1 < 0$.

Consider first the case $\alpha_1 > 0$ and assume by contradiction $r_1 > 0$. Since

$$U'(r) = \frac{1}{r} \left(r_0^{n-1} \alpha_1^{n-1} + \int_r^{r_0} t^{n-1} e^{U(t)} dt \right)^{\frac{1}{n-1}} \geq \frac{r_0 \alpha_1}{r} > 0 \quad (4.7)$$

for all $r \in (r_1, r_0]$, one would have that

$$U(r) \leq \alpha_0, \quad \alpha_1 \leq U'(r) \leq \frac{1}{r_1} [r_0^{n-1} \alpha_1^{n-1} + \frac{r_0^n}{n} e^{\alpha_0}]^{\frac{1}{n-1}}$$

for all $r \in (r_1, r_0]$ and then (4.1) would hold for initial conditions $\alpha'_0 = U(r'_0)$, $\alpha'_1 = U'(r'_0)$ in (4.2) at r'_0 approaching r_1 from the right. Since this would allow to continue the solution U on the left of r_1 in view of the estimate (4.6) on the time for local existence, we would reach a contradiction and then the property $r_1 = 0$ has been established.

In the case $\alpha_1 < 0$ assume by contradiction $r_2 < +\infty$. Since

$$U'(r) = -\frac{1}{r} \left(r_0^{n-1} |\alpha_1|^{n-1} + \int_{r_0}^r t^{n-1} e^{U(t)} dt \right)^{\frac{1}{n-1}} \leq -\frac{r_0 |\alpha_1|}{r} < 0 \quad (4.8)$$

for all $r \in [r_0, r_2)$, one would have that

$$U(r) \leq \alpha_0, \quad -\frac{1}{r_0} [r_0^{n-1} |\alpha_1|^{n-1} + \frac{r_0^n}{n} e^{\alpha_0}]^{\frac{1}{n-1}} \leq U'(r) \leq -\frac{r_0 |\alpha_1|}{r_2}$$

for all $r \in [r_0, r_2)$ and then (4.1) would hold for initial conditions $\alpha'_0 = U(r'_0)$, $\alpha'_1 = U'(r'_0)$ in (4.2) at r'_0 approaching r_2 from the left. Since one could continue the solution U past r_2 thanks to (4.6), a contradiction would arise. Then, we have shown that $r_2 = +\infty$.

Given $\epsilon > 0$, let now $U_\epsilon^\pm$ be the maximal solution of

$$\begin{cases} -\frac{1}{r^{n-1}} (r^{n-1} |U'|^{n-2} U')' = e^U \\ U(1) = \alpha_0, \quad U'(1) = \pm \epsilon. \end{cases}$$

By the discussion above we have that U_ϵ^+ and U_ϵ^- are well defined in $(0, 1]$ and $[1, +\infty)$, respectively. According to (4.7)–(4.8) one has

$$\begin{aligned} (U_\epsilon^+)' &= \frac{1}{r} \left(\epsilon^{n-1} + \int_r^1 t^{n-1} e^{U_\epsilon^+(t)} dt \right)^{\frac{1}{n-1}} \quad \text{in } (0, 1], \\ (U_\epsilon^-)' &= -\frac{1}{r} \left(\epsilon^{n-1} + \int_1^r t^{n-1} e^{U_\epsilon^-(t)} dt \right)^{\frac{1}{n-1}} \quad \text{in } [1, +\infty) \end{aligned} \quad (4.9)$$

and then $U_\epsilon^+, U_\epsilon^-$ are uniformly bounded in $C_{loc}^{1,\gamma}(0, 1]$, $C_{loc}^{1,\gamma}[1, +\infty)$, respectively, in view of $U_\epsilon^+, U_\epsilon^- \leq \alpha_0$. Up to a subsequence and a diagonal argument, we can assume that $U_\epsilon^+ \rightarrow U^+$ in $C_{loc}^1(0, 1]$ and $U_\epsilon^- \rightarrow U^-$ in $C_{loc}^1[1, +\infty)$ as $\epsilon \rightarrow 0^+$, where

$$\begin{aligned} (U^+)' &= \frac{1}{r} \left(\int_r^1 t^{n-1} e^{U^+(t)} dt \right)^{\frac{1}{n-1}} \quad \text{in } (0, 1], \\ (U^-)' &= -\frac{1}{r} \left(\int_1^r t^{n-1} e^{U^-(t)} dt \right)^{\frac{1}{n-1}} \quad \text{in } [1, +\infty) \end{aligned} \quad (4.10)$$

thanks to (4.9). Since $U^+(1) = U^-(1) = \alpha_0$ and $(U^+)'(1) = (U^-)'(1) = 0$ in view of (4.10), we have that

$$U = \begin{cases} U^+ & \text{in } (0, 1] \\ U^- & \text{in } [1, +\infty) \end{cases}$$

is in $C^1(0, +\infty)$ with $U \leq U(1) = \alpha_0$, $U'(1) = 0$ and

$$U'(r) = \frac{1}{r} \left| \int_r^1 t^{n-1} e^{U(t)} dt \right|^{-\frac{n-2}{n-1}} \int_r^1 t^{n-1} e^{U(t)} dt \text{ in } (0, +\infty). \quad (4.11)$$

It is not difficult to check that U satisfies $-\Delta_n U = e^U$ in $\mathbb{R}^n \setminus \{0\}$ and

$$\lim_{r \rightarrow 0} \frac{U(r)}{\log r} = \lim_{r \rightarrow 0} r U'(r) = \left(\int_0^1 t^{n-1} e^{U(t)} dt \right)^{\frac{1}{n-1}} = \left(\frac{1}{n\omega_n} \int_{B_1(0)} e^U \right)^{\frac{1}{n-1}} \quad (4.12)$$

in view of (4.11). By Theorem 1.1 and (4.12) we deduce that U is a radial solution of

$$-\Delta_n U = e^U - \gamma \delta_0 \text{ in } \mathbb{R}^n, \quad U \leq U(1) = \alpha_0,$$

with $\gamma = \int_{B_1(0)} e^U$ depending on the choice of α_0 . By the Pohozaev identity (3.2) on $A = B_1(0) \setminus B_\epsilon(0)$, $\epsilon \in (0, 1)$, we have that

$$\omega_n [e^{\alpha_0} - \epsilon^n e^{U(\epsilon)}] = \int_{B_1(0) \setminus B_\epsilon(0)} e^U + \frac{n-1}{n} \omega_n [\epsilon U'(\epsilon)]^n$$

in view of $U(1) = \alpha_0$ and $U'(1) = 0$, and letting $\epsilon \rightarrow 0^+$ one deduces that

$$\omega_n e^{\alpha_0} = \gamma + \frac{n-1}{n} \omega_n \left(\frac{\gamma}{n\omega_n} \right)^{\frac{n}{n-1}}$$

in view of (4.12). Since $\gamma \in (0, +\infty) \rightarrow \gamma + \frac{n-1}{n} \omega_n \left(\frac{\gamma}{n\omega_n} \right)^{\frac{n}{n-1}} \in (0, +\infty)$ is a bijection, for any given $\gamma > 0$ let $\alpha_0 = \log \left[\frac{\gamma}{\omega_n} + \frac{n-1}{n} \left(\frac{\gamma}{n\omega_n} \right)^{\frac{n}{n-1}} \right]$ and the corresponding U is the solution of (1.12) we were searching for. Notice that $\int_{\mathbb{R}^n} e^U < +\infty$ in view of $\int_1^\infty t^{n-1} e^{U(t)} dt < +\infty$, as it can be deduced by

$$\lim_{r \rightarrow +\infty} \frac{U(r)}{\log r} = \lim_{r \rightarrow +\infty} r U'(r) = - \left(\int_1^\infty t^{n-1} e^{U(t)} dt \right)^{\frac{1}{n-1}}$$

due to (4.11). We have established the following result:

Theorem 4.1 *For any $\gamma > 0$ there exists a 1-parameter family of distinct solutions U_λ , $\lambda > 0$, to (1.12) given by $U_\lambda(x) = U(\lambda x) + n \log \lambda$ such that U_λ takes its unique absolute maximum point at $\frac{1}{\lambda}$.*

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References

1. Aguilar Crespo, J.A., Peral Alonso, I.: Blow-up behavior for solutions of $-\Delta_N u = V(x)e^u$ in bounded domains in $\mathbb{R}^N$. *Nonlinear Anal.* **29**(4), 365–384 (1997)
2. Bartolucci, D., Tarantello, G.: Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory. *Commun. Math. Phys.* **229**(1), 3–47 (2002)
3. Bénilan, P., Boccardo, L., Gallouët, T., Gariepy, R., Pierre, M., Vázquez, J.L.: An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **22**(2), 241–273 (1995)
4. Boccardo, L., Gallouët, T.: Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.* **87**(1), 149–169 (1989)
5. Brézis, H., Merle, F.: Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. *Commun. Partial Differ. Equ.* **16**(8–9), 1223–1253 (1991)
6. Chen, W., Li, C.: Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.* **63**(3), 615–623 (1991)
7. Chou, K.S., Wan, T.Y.H.: Asymptotic radial symmetry for solutions of $\Delta u + e^u = 0$ in a punctured disc. *Pac. J. Math.* **163**, 269–276 (1994)
8. Damascelli, L., Farina, A., Sciunzi, B., Valdinoci, E.: Liouville results for m -Laplace equations of Lane–Emden–Fowler type. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**(4), 1099–1119 (2009)
9. DiBenedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.* **7**(8), 827–850 (1983)
10. Dolzmann, G., Hungerbühler, N., Müller, S.: Uniqueness and maximal regularity for nonlinear elliptic systems of n -Laplace type with measure valued right hand side. *J. Reine Angew. Math.* **520**, 1–35 (2000)
11. Esposito, P.: A classification result for the quasi-linear Liouville equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **35**(3), 781–801 (2018)
12. Esposito, P., Morlando, F.: On a quasilinear mean field equation with an exponential nonlinearity. *J. Math. Pures Appl.* (9) **104**(2), 354–382 (2015)
13. Kichenassamy, S., Veron, L.: Singular solutions of the p -Laplace equation. *Math. Ann.* **275**(4), 599–615 (1986)
14. Li, Y.Y., Shafrir, I.: Blow-up analysis for solutions of $-\Delta u = V e^u$ in dimension two. *Indiana Univ. Math. J.* **43**(4), 1255–1270 (1994)
15. Lieberman, G.M.: Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.* **12**(11), 1203–1219 (1988)
16. Liouville, J.: Sur l'équation aux dérivées partielles $\partial^2 \log \lambda / \partial u \partial v \pm 2\lambda a^2 = 0$. *J. de Math.* **18**, 71–72 (1853)
17. Prajapat, J., Tarantello, G.: On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results. *Proc. R. Soc. Edinb.* **131**(4), 967–985 (2001)
18. Robert, F., Wei, J.: Asymptotic behavior of a fourth order mean field equation with Dirichlet boundary condition. *Indiana Univ. Math. J.* **57**(5), 2039–2060 (2008)
19. Serrin, J.: Local behavior of solutions of quasilinear equations. *Acta Math.* **111**, 247–302 (1964)
20. Serrin, J.: Isolated singularities of solutions of quasi-linear equations. *Acta Math.* **113**, 219–240 (1965)
21. Tarantello, G.: A quantization property for blow-up solutions of singular Liouville-type equations. *J. Funct. Anal.* **219**, 368–399 (2005)
22. Tolksdorf, P.: Regularity for more general class of quasilinear elliptic equations. *J. Differ. Equ.* **51**(1), 126–150 (1984)