Calc. Var. (2021) 60:83
https://doi.org/10.1007/s00526-021-01965-z

Noncoercive quasilinear elliptic operators with singular lower order terms

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Received: 6 January 2020 / Accepted: 31 January 2021 / Published online: 17 April 2021

Abstract
We consider a family of quasilinear second order elliptic differential operators which are not coercive and are defined by functions in Marcinkiewicz spaces. We prove the existence of a solution to the corresponding Dirichlet problem. The associated obstacle problem is also solved. Finally, we show higher integrability of a solution to the Dirichlet problem when the datum is more regular.

Mathematics Subject Classification 35J25 · 35J87

1 Introduction

Given a bounded domain Ω of $\mathbb{R}^N$, $N \geq 2$, we consider

$$A\colon (x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \quad \mapsto \quad \mathbb{R}^N$$

Communicated by A. Malchiodi.

The authors are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of INdAM. The research of G.M. has been partially supported by the National Research Project PRIN "Gradient flows, Optimal Transport and Metric Measure Structures", code 2017TEXA3H.

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a Carathéodory vector field (i.e. measurable in $x \in \Omega$ and continuous in $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$) satisfying for a.e. $x \in \Omega$ and for every $u \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^N$ the following conditions:

coercivity condition

$$\langle A(x, u, \xi), \xi \rangle \geq \alpha |\xi|^p - \big(b(x)\,|u|\big)^p - \varphi(x)^p \tag{1}$$

growth condition

$$|A(x, u, \xi)| \leq \beta |\xi|^{p-1} + \big(\tilde{b}(x)\,|u|\big)^{p-1} + \varphi(x)^{p-1} \tag{2}$$

strict monotonicity

$$\langle A(x, u, \xi) - A(x, u, \eta), \xi - \eta \rangle > 0 \qquad \text{for } \xi \neq \eta \tag{3}$$

where $0 < \alpha < \beta$ are positive constants, $1 < p < N$, and $b, \tilde{b}$ and φ are positive functions verifying $b, \tilde{b} \in L^{N,\infty}(\Omega)$ and $\varphi \in L^p(\Omega)$. In view of Sobolev embedding theorem in Lorentz spaces [2,21,31], by (2) and the assumptions on b, $\tilde{b}$ and φ, for each $u \in W_0^{1,p}(\Omega)$ we have

$$A(x, u, \nabla u) \in L^{p'}(\Omega, \mathbb{R}^N)$$

Hence, we can define the quasilinear elliptic distributional operator

$$-\operatorname{div} A(x, u, \nabla u) \tag{4}$$

setting for any $w \in W_0^{1,p}(\Omega)$

$$\langle -\operatorname{div} A(x, u, \nabla u), w \rangle = \int_\Omega \langle A(x, u, \nabla u), \nabla w \rangle \,\mathrm{d}x. \tag{5}$$

Given $\Phi \in W^{-1,p'}(\Omega)$, we study the Dirichlet problem

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) = \Phi \\ \quad u \in W_0^{1,p}(\Omega) \end{cases} \tag{6}$$

By a solution to Problem (6) we mean a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega A(x, u, \nabla u) \nabla w \,\mathrm{d}x = \langle \Phi, w \rangle, \qquad \forall w \in C_0^\infty(\Omega), \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product of $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$. Clearly, (7) extends to all $w \in W_0^{1,p}(\Omega)$.

Our conditions allow us to consider operators with a singular coefficient in the lower-order term. As an example, we consider the following operator

$$A(x, u, \xi) := \langle \mathcal{H}(x)\xi, \xi \rangle^{\frac{p-2}{2}} \mathcal{H}(x)\xi + B(x)|u|^{p-2}u \tag{8}$$

with $1 < p < N$. Here $\mathcal{H} = \mathcal{H}(x) \colon \Omega \to \mathbb{R}^{N \times N}$ is a symmetric, bounded matrix field such that

$$\langle \mathcal{H}(x)\xi, \xi \rangle \geq \nu\,|\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^N,$$

for a given $\nu > 0$. The vector field $B \colon \Omega \to \mathbb{R}^N$ is a measurable function satisfying $|B(x)| \leq (\tilde{b}(x))^{p-1}$ for a.e. $x \in \Omega$ and for some $\tilde{b} \in L^{N,\infty}(\Omega)$.

The feature of Problem (6) is the lack of coercivity for the operator (4) and the singularity in the lower order term due to property of b and $\tilde{b}$. Indeed, as $\tilde{b} \in L^{N,\infty}(\Omega)$, by Sobolev embedding theorem the lower order term $\tilde{b}\,u \in L^p(\Omega)$ and it has a norm comparable with the norm of $|\nabla u|$. It is well known that, if the operator in (4)-(5) is coercive, then a solution to problem (6) exists. For instance it can be shown by monotone operator theory [4,7,8,28].

On the other hand, the existence of a bounded solution can be expected when Φ and $b, \tilde{b}$ are sufficiently smooth. For example, in the model case and even for the corresponding linear case, a solution to Problem (6) is bounded whenever Φ and $\tilde{b}$ are in $W^{-1,\frac{s}{p-1}}(\Omega)$ and $L^s(\Omega, \mathbb{R}^N)$, respectively, with $s > N$ (see [18,33]).

The space $L^{N,\infty}(\Omega)$ is slightly larger than $L^N(\Omega)$. Nevertheless, there are essential differences between the case in which the coefficients of the lower order term are in $L^N(\Omega)$ ([5,11]), or even $L^{N,q}(\Omega)$ ([30]), with $N \le q < \infty$, and the case in which $b, \tilde{b} \in L^{N,\infty}(\Omega)$. In $L^{N,\infty}(\Omega)$ bounded functions are not dense. Furthermore, in $L^{N,\infty}(\Omega)$ the norm is not absolutely continuous, namely a function can have large norm even if restricted to a set with small measure.

Our first result is the following

Theorem 1 *Let $\Phi \in W^{-1,p'}(\Omega)$. Under the assumptions (1), (2) and (3), if*

$$\operatorname{dist}_{L^{N,\infty}}(b, L^\infty(\Omega)) < \frac{\alpha^{\frac{1}{p}}}{S_{N,p}} \tag{9}$$

then Problem (6) admits a solution, where $S_{N,p} = \omega_N^{-1/N} p/(N-p)$ is the Sobolev constant.

Here ω_N denotes the measure of the unit ball in $\mathbb{R}^N$.

Note that the bound in (9) depends only on the coercivity condition (1), in particular, it is independent of $\tilde{b}$. Moreover, b^p belongs to $L^{\frac{N}{p},\infty}(\Omega)$ and assumption (9) can be rewritten as (see Sect. 2)

$$\operatorname{dist}_{L^{\frac{N}{p},\infty}}(b^p, L^\infty(\Omega)) < \frac{\alpha}{S_{N,p}^p} \tag{10}$$

Condition (9) does not imply smallness of the norm of b in $L^{N,\infty}(\Omega)$ (see [13] and [20]). It is more general than considering a condition on the norm and allows us to treat different settings of problem (6) in a unified way. Indeed, if we denote by $L_0^{N,\infty}(\Omega)$ the closure of $L^\infty(\Omega)$ in $L^{N,\infty}(\Omega)$, then the following immediate consequence of Theorem 1 holds.

Corollary 1 *Assume (1), (2) and (3), with $b \in L_0^{N,\infty}(\Omega)$. Then Problem (6) admits a solution, for every $\Phi \in W^{-1,p'}(\Omega)$.*

The closure $L_0^{N,\infty}(\Omega)$ contains for example all Lorentz spaces $L^{N,q}(\Omega)$, for $1 < q < \infty$, see Sect. 2.1.

We illustrate assumption (9) in the particular case of the operator

$$-\mu\,\Delta_p u - \operatorname{div}\left(D\frac{x}{|x|^p}\,|u|^{p-2}\,u\right), \tag{11}$$

in a ball Ω centered at the origin, where $\mu > 0$ and $D \in \mathbb{R}$. In the linear case $p = 2$ this has been considered in [6]. We denote

$$A(x, u, \xi) = \mu\,|\xi|^{p-2}\xi + D\frac{x}{|x|^p}\,|u|^{p-2}u$$

and by Young inequality,

$$A(x,u,\xi)\cdot\xi \ge \left(1-\frac{1}{p}\right)\left[\mu\,|\xi|^p - |D|\left(\frac{\mu}{|D|}\right)^{-\frac{1}{p-1}}\left(\frac{|u|}{|x|}\right)^p\right]$$

Therefore coercivity condition (1) holds with

$$\alpha=\left(1-\frac{1}{p}\right)\mu, \qquad b(x)^p=\left(1-\frac{1}{p}\right)\mu^{-\frac{1}{p-1}}|D|^{\frac{p}{p-1}}\frac{1}{|x|^p}$$

It is easy to verify (see also (20) below) that

$$\mathrm{dist}_{L^{N,\infty}(\Omega)}(b, L^\infty(\Omega)) = \left(1-\frac{1}{p}\right)^{\frac{1}{p}}\mu^{-\frac{1}{p(p-1)}}|D|^{\frac{1}{p-1}}\,\omega_N^{1/N}$$

and so condition (9) becomes

$$|D| < \mu\left(\frac{N-p}{p}\right)^{p-1}. \tag{12}$$

When $p=2$, Example 1 below shows that the bound (12) is sharp. This bound is comparable with the one given in Sect. 2 of [6] and it improves the bound found in [19].

In the case $p=2$ existence results in the same spirit of Theorem 1 have been proved in [13,20,34] and in [10,32,35] when the principal part has a coefficient bound in BMO (i.e. the space of functions of bounded mean oscillation). We explicitly remark that in this context the operator (4) has the same integrability properties of the principal part (see also [24]). The evolutionary counterpart of Problem (6) has been studied in [12]. Other related results can be found in [1,25,29].

An additional difficulty in proving Theorem 1 lies in the lack of compactness that the operator

$$u\in W_0^{1,p}(\Omega) \qquad \mapsto \qquad \left(\tilde b(x)\,|u|\right)^{p-1}\in L^{p'}(\Omega) \tag{13}$$

exhibits in the case $\tilde b\in L^{N,\infty}(\Omega)$, in contrast with the case $\tilde b\in L^N(\Omega)$ (see Example 3 in Sect. 2.3). In order to overcome this issue, first we consider the case in which $b,\tilde b\in L^\infty(\Omega)$. Under this assumption, we deduce the existence of a solution to Problem (6) by means of Leray–Schauder fixed point theorem. The a priori estimate required follows from a lemma that could be interesting in itself (see Lemma 2 below).

In order to reduce the general case $b,\tilde b\in L^{N,\infty}(\Omega)$ to the previous one, we consider a sequence of approximating problems, defined essentially by truncating the vector field $A=A(x,u,\xi)$ in the u–variable. A bound on the sequence of the solutions is achieved due to the assumption (9).

We emphasize that our result is new also when $b,\tilde b\in L^N(\Omega)$, in the sense that our approach allows us to treat the general nonlinear operator in (6).

Finally, by testing the problems with a suitable admissible test functions, we show that the sequence of solutions to the approximating problems is compact and its limit is a solution to the original problem (6).

In Sect. 5, we show that our approach is robust enough to handle also the corresponding obstacle problem. We prove an existence result in the same spirit of [19] (where the case $p=2$ is taken into account).

In Sect. 6 we present a regularity result. When $b,\tilde b\in L^N(\Omega)$, the study of the higher integrability of a solution to (6) has been developed in [14,15] by using the theory of quasi-minima. Local summability properties have been recently considered in [9,23] in the linear case. Here, we prove higher summability of a solution u to (6), in the spirit of [18] where the model case is treated.

Theorem 2 *Let $1 < p < r < N$ and $\Phi \in W^{-1,\frac{r}{p-1}}(\Omega)$. Assume that (1) and (2) hold with $\varphi \in L^r(\Omega)$. Under these hypotheses, if*

$$\mathrm{dist}_{L^{N,\infty}}(b, L^\infty(\Omega)) < \frac{\alpha^{\frac{1}{p}}}{S_{N,p}} \frac{p^*}{r^*}, \tag{14}$$

then any solution $u \in W_0^{1,p}(\Omega)$ of (6) satisfies

$$|u|^{r^*/p^*} \in W_0^{1,p}(\Omega) \tag{15}$$

In particular $u \in L^{r^*}(\Omega)$.

2 Preliminaries and examples

2.1 Notation and function spaces

Let Ω be a bounded domain in $\mathbb{R}^N$. Given $1 < p < \infty$ and $1 \leq q < \infty$, the Lorentz space $L^{p,q}(\Omega)$ consists of all measurable functions f defined on Ω for which the quantity

$$\|f\|_{p,q}^q = p \int_0^{+\infty} |\Omega_t|^{\frac{q}{p}} t^{q-1}\, \mathrm{d}t \tag{16}$$

is finite, where $\Omega_t = \{x \in \Omega : |f(x)| > t\}$ and $|\Omega_t|$ is the Lebesgue measure of Ω_t, that is, $\lambda_f(t) = |\Omega_t|$ is the distribution function of f. Note that $\|\cdot\|_{p,q}$ is equivalent to a norm and $L^{p,q}$ becomes a Banach space when endowed with it (see [3,17,31]). For $p = q$, the Lorentz space $L^{p,p}(\Omega)$ reduces to the Lebesgue space $L^p(\Omega)$. For $q = \infty$, the class $L^{p,\infty}(\Omega)$ consists of all measurable functions f defined on Ω such that

$$\|f\|_{p,\infty}^p = \sup_{t>0} t^p |\Omega_t| < +\infty$$

and it coincides with the Marcinkiewicz class, weak-$L^p(\Omega)$.

For Lorentz spaces the following inclusions hold

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega),$$

whenever $1 \leq q < p < r \leq \infty$. Moreover, for $1 < p < \infty$, $1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, if $f \in L^{p,q}(\Omega)$, $g \in L^{p',q'}(\Omega)$ we have the Hölder–type inequality

$$\int_\Omega |f(x)g(x)|\, \mathrm{d}x \leq \|f\|_{p,q} \|g\|_{p',q'}.$$

As it is well known, $L^\infty(\Omega)$ is not dense in $L^{p,\infty}(\Omega)$. For a function $f \in L^{p,\infty}(\Omega)$ we define

$$\mathrm{dist}_{L^{p,\infty}(\Omega)}(f, L^\infty(\Omega)) = \inf_{g \in L^\infty(\Omega)} \|f - g\|_{L^{p,\infty}(\Omega)}. \tag{17}$$

We can show that

$$\mathrm{dist}_{L^{p,\infty}(\Omega)}(f, L^\infty(\Omega)) = \lim_{k \to +\infty} \|f\, \chi_{\{|f|>k\}}\|_{p,\infty}. \tag{18}$$

To this end, it suffices to note the inequality

$$|f-g| \geq \left(1-\frac{\|g\|_\infty}{k}\right)|f|\,\chi_{\{|f|>k\}}$$

which holds for any $g \in L^\infty(\Omega)$ and $k > \|g\|_\infty$. For example, (18) implies that, if $\sigma p > 1$, then for a positive function f

$$\mathrm{dist}_{L^{p,\infty}(\Omega)}(f^\sigma, L^\infty(\Omega)) = \left[\mathrm{dist}_{L^{\sigma p,\infty}(\Omega)}(f, L^\infty(\Omega))\right]^\sigma.$$

We denote by $L_0^{p,\infty}(\Omega)$ the closure of $L^\infty(\Omega)$. We have (see [22, Lemma 2.3])

$$f \in L_0^{p,\infty}(\Omega) \iff \lim_{t\to+\infty} t\,[\lambda_f(t)]^{1/p} = 0. \tag{19}$$

Clearly, for $1 \leq q < \infty$ we have $L^{p,q}(\Omega) \subset L_0^{p,\infty}(\Omega)$, that is, any function in $L^{p,q}(\Omega)$ has vanishing distance zero to $L^\infty(\Omega)$. Indeed, $L^\infty(\Omega)$ is dense in $L^{p,q}(\Omega)$, the latter being continuously embedded into $L^{p,\infty}(\Omega)$. Actually, the inclusion also follows from (19), since $\lambda_f(t) = |\Omega_t|$ is decreasing and hence the convergence of the integral at (16) implies the condition on the right of (19).

Assuming the origin $0 \in \Omega$, a typical element of $L^{N,\infty}(\Omega)$ is $b(x) = B/|x|$, with B a positive constant. An elementary calculation shows that

$$\mathrm{dist}_{L^{N,\infty}(\Omega)}(b, L^\infty(\Omega)) = B\,\omega_N^{1/N} \tag{20}$$

where ω_N stands for the Lebesgue measure of the unit ball of $\mathbb{R}^N$.

The Sobolev embedding theorem in Lorentz spaces reads as

Theorem 3 ([2,21,31]) *Let us assume that $1 < p < N$, $1 \leq q \leq p$, then every function $g \in W_0^{1,1}(\Omega)$ verifying $|\nabla g| \in L^{p,q}(\Omega)$ actually belongs to $L^{p^*,q}(\Omega)$, with $p^* = \frac{Np}{N-p}$ and*

$$\|g\|_{p^*,q} \leq S_{N,p}\|\nabla g\|_{p,q}$$

where $S_{N,p} = \omega_N^{-1/N}\,p/(N-p)$ is the Sobolev constant.

2.2 A version of the Leray–Schauder fixed point theorem

We shall use the well known Leray–Schauder fixed point theorem in the following form (see [16, Theorem 11.3 pg. 280]). A continuous mapping between two Banach spaces is called compact if the images of bounded sets are precompact.

Theorem 4 *Let $\mathcal{F}$ be a compact mapping of a Banach space X into itself, and suppose there exists a constant M such that $\|x\|_X < M$ for all $x \in X$ and $t \in [0,1]$ satisfying $x = t\mathcal{F}(x)$. Then, $\mathcal{F}$ has a fixed point.*

2.3 Critical examples

Our first example shows that the only assumption that $b \in L^{N,\infty}(\Omega)$ does not guarantee the existence of a solution to Problem (6).

Example 1 Let Ω be the unit ball. For $\frac{N-2}{2} < D$, the problem

$$\begin{cases} -\Delta u - \mathrm{div}\left(D\,u\,\dfrac{x}{|x|^2}\right) = -\,\mathrm{div}\left(\dfrac{x}{|x|^{N-D}}\right) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{21}$$

does not admit a solution. Assume to the contrary that u is a solution of (21). In the right hand side of the equation we recognize that

$$\frac{x}{|x|^{N-D}} = \nabla v(x),$$

where $v \in W_0^{1,2}(\Omega)$ is given by

$$v(x) = \begin{cases} \dfrac{1}{2-N+D}\left(|x|^{2-N+D} - 1\right) & \text{for } D \neq N-2 \\ \log|x| & \text{for } D = N-2 \end{cases}$$

Moreover, v solves the adjoint problem

$$\begin{cases} -\Delta v + D\,\dfrac{x}{|x|^2} \cdot \nabla v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Testing the equation in (21) by v we have

$$\int_\Omega |\nabla v|^2 \,\mathrm{d}x = 0.$$

which readily implies $v \equiv 0$ in Ω, which is clearly not the case. □

Next example shows that for the complete operator

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u)$$

in general we do not have existence, even in the linear case.

Example 2 Let λ be an eigenvalue of Laplace operator and w a corresponding eigenfunction

$$\begin{cases} -\Delta w = \lambda\, w \\ w \in W_0^{1,2}(\Omega) \setminus \{0\} \end{cases}$$

Then the equation

$$-\Delta u - \lambda\, u = w$$

has no solution of class $W_0^{1,2}(\Omega)$.

Our final example shows that compactness of the operator (13) in the Introduction could fail.

Example 3 Assume $N \geq 2$ and $1 < p < N$. Let Ω be the ball of $\mathbb{R}^N$ centered at the origin of radius 3. Our aim is to construct a sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,p}(\Omega)$ and a function $\tilde{b} \in L^{N,\infty}(\Omega)$ such that $\{\nabla u_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\Omega, \mathbb{R}^N)$, however it is not possible to extract from $\{(\tilde{b}|u_n|)^{p-1}\}_{n\in\mathbb{N}}$ any subsequence strongly converging $L^{p'}(\Omega)$. To this aim, let

$$\tilde{b}(x) := \frac{1}{|x|}$$

and

$$\gamma := 1 - \frac{N}{p}.$$

We define a sequence $\{u_n\}_{n\in\mathbb{N}}$ setting for $x \in \Omega$

$$u_1(x) := \begin{cases} 1-2^\gamma & \text{if } |x|<1 \\ |x|^\gamma - 2^\gamma & \text{if } 1\le |x|<2 \\ 0 & \text{if } |x|\ge 2 \end{cases}$$
$$u_n(x) := n^{-\gamma} u_1(nx) \qquad \text{for } n \ge 2 \tag{22}$$

Observe that $u_n \in W_0^{1,p}(\Omega)$ since

$$|\nabla u_n(x)| = \begin{cases} |\gamma||x|^{\gamma-1} & \text{if } \frac{1}{n} \le |x| < \frac{2}{n} \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

and

$$\|\nabla u_n\|^p_{L^p(\Omega)} = |\gamma|^p N\omega_N \log 2, \tag{24}$$

where ω_N denotes the measure of the unit ball of $\mathbb{R}^N$. In particular, the norm $\|\nabla u_n\|^p_{L^p(\Omega)}$ is independent of n. On the other hand, a direct calculation shows that

$$\begin{aligned} \left\| \left(\tilde{b}|u_n|\right)^{p-1} \right\|^{p'}_{L^{p'}(\Omega)} &= \int_{|x|<\frac{3}{n}} (\tilde{b}|u_n|)^p \, \mathrm{d}x \\ &= \int_{|x|<\frac{1}{n}} (\tilde{b}|u_n|)^p \, dx + \int_{\frac{1}{n}\le|x|<\frac{2}{n}} (\tilde{b}|u_n|)^p \, \mathrm{d}x \\ &= N\omega_N \left[\frac{(1-2^\gamma)^p}{N-p} + \int_1^2 r^{N-p} \left(r^\gamma - 2^\gamma\right)^p \frac{\mathrm{d}r}{r} \right] \end{aligned} \tag{25}$$

Hence, we see that the norm of $(\tilde{b}|u_n|)^{p-1}$ in $L^{p'}(\Omega)$ is independent of n as well and strictly positive. On the other hand, $(\tilde{b}|u_n|)^{p-1} \to 0$ pointwise in Ω and this readily implies that there is no subsequence of $\{(\tilde{b}|u_n|)^{p-1}\}_{n\in\mathbb{N}}$ strongly converging in $L^{p'}(\Omega)$. □

2.4 An elementary lemma

Lemma 1 *Assume $f_n \to f$ a.e. Moreover, let g_n, $n \in \mathbb{N}$, and g in L^q, $1 \le q < +\infty$, verify $g_n \to g$ a.e., $|f_n| \le g_n$ a.e., $\forall n \in \mathbb{N}$, and*

$$\int_\Omega g_n^q \, \mathrm{d}x \to \int_\Omega g^q \, \mathrm{d}x.$$

Then $f_n, f \in L^q$ and

$$f_n \to f \text{ in } L^q.$$

It suffices to apply Fatou lemma to the sequence of nonnegative functions

$$F_n = 2^{q-1}(g_n^q + g^q) - |f_n - f|^q.$$

3 A weak compactness result

The aim of this section is to establish a weak compactness criterion in the space $W_0^{1,p}(\Omega)$ that has an interest by itself.

Lemma 2 *Let $\mathcal{B}$ be a nonempty subset of $W_0^{1,p}(\Omega)$. Assume that there exists a constant $C > 0$ such that*

$$\|\nabla u\|^p_{L^p(\Omega\setminus\Omega_\sigma)} \le C\left(1 + \|u\|^p_{L^p(\Omega\setminus\Omega_\sigma)}\right) \tag{26}$$

for any $\sigma > 0$ and $u \in \mathcal{B}$, where $\Omega_\sigma := \{x \in \Omega\colon\ |u(x)| \ge \sigma\}$. Then, there exists a constant $M > 0$ such that

$$\|u\|_{W^{1,p}(\Omega)} \le M \tag{27}$$

for any $u \in \mathcal{B}$.

Proof We argue by contradiction and assume $\mathcal{B}$ unbounded. Then we construct a sequence $\{u_k\}_k$ in $\mathcal{B}$ such that

$$\|u_k\| := \|\nabla u_k\|_p \to \infty$$

as $k \to \infty$. By (26) we get, for any $k \in \mathbb{N}$ and $\varepsilon > 0$

$$\int_\Omega |\nabla T_{\varepsilon\|u_k\|} u_k|^p \,\mathrm{d}x \le C\left(1 + \int_\Omega |u_k|^p \chi_{\{|u_k|<\varepsilon\|u_k\|\}} \,\mathrm{d}x\right) \tag{28}$$

We set

$$v_k = \frac{u_k}{\|u_k\|}.$$

Hence, there exists $v \in W_0^{1,p}(\Omega)$ such that (up to a subsequence) $v_k \rightharpoonup v$ weakly in $W_0^{1,p}$, $v_k \to v$ strongly in L^p and $v_k(x) \to v(x)$ for a.e. $x \in \Omega$. Notice that

$$\frac{T_{\varepsilon\|u_k\|} u_k}{\|u_k\|} = T_\varepsilon v_k,$$

thus $\nabla T_{\varepsilon\|u_k\|} u_k = 0$ on the set $\{x \in \Omega : |v_k(x)| \ge \varepsilon\}$. Dividing (28) by $\|u_k\|^p$ we have

$$\int_\Omega |\nabla T_\varepsilon v_k|^p \,\mathrm{d}x \le C\left(\|u_k\|^{-p} + \int_\Omega |v_k|^p \chi_{\{|v_k|<\varepsilon\}} \,\mathrm{d}x\right) \tag{29}$$

Now, we let $k \to +\infty$. To this end, we note that $T_\varepsilon v_k \rightharpoonup T_\varepsilon v$ weakly in $W_0^{1,p}(\Omega)$ and $T_\varepsilon v_k \to T_\varepsilon v$ strongly in $L^p(\Omega)$. In the left hand side of (29), we use semicontinuity of the norm with respect to weak convergence, while in the right hand side we observe that $\|u_k\|^{-1} \to 0$. Moreover, if

$$|\{x \in \Omega : |v(x)| = \varepsilon\}| = 0, \tag{30}$$

then we have $\chi_{\{|v_k|<\varepsilon\}} \to \chi_{\{|v|<\varepsilon\}}$ a.e. in Ω and hence

$$v_k\, \chi_{\{|v_k|<\varepsilon\}} \to v\, \chi_{\{|v|<\varepsilon\}}$$

strongly in L^p. Note that the set of values $\varepsilon > 0$ for which (30) fails is at most countable. Thus, we end up with the following estimate

$$\int_\Omega |\nabla T_\varepsilon v|^p \,\mathrm{d}x \le C \int_\Omega |v|^p \chi_{\{|v|<\varepsilon\}} \,\mathrm{d}x \tag{31}$$

Using Poincaré inequality in the left hand side, this yields

$$\varepsilon^p\, |\{x : |v| \ge \varepsilon\}| \le C\, \varepsilon^p\, |\{x : 0 < |v| < \varepsilon\}|.$$

Passing to the limit as $\varepsilon \downarrow 0$ (assuming (30)), we deduce

$$|\{x : |v| > 0\}| = 0,$$

that is, $v(x) = 0$ a.e. Once we know that $v_k \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$, the above argument (formally with $\varepsilon = +\infty$, i.e. without truncating v_k) actually shows that $v_k \to 0$ strongly in $W_0^{1,p}(\Omega)$, compare with (31), and this is not possible, as $\|v_k\| = 1$, for all k. □

4 Proof of Theorem 1

4.1 The case of bounded coefficient

In this subsection we assume $b, \tilde{b} \in L^\infty(\Omega)$. For a given function $v \in L^p(\Omega)$, we define the vector field on $\Omega \times \mathbb{R}^N$

$$A_v(x, \xi) := A(x, v(x), \xi) \tag{32}$$

which satisfies similar conditions as A, namely

$$\langle A_v(x, \xi), \xi \rangle \geq \alpha |\xi|^p - \big(b(x)\, |v|\big)^p - \varphi(x)^p \tag{33}$$

$$|A_v(x, \xi)| \leq \beta |\xi|^{p-1} + \big(\tilde{b}(x)\, |v|\big)^{p-1} + \varphi(x)^{p-1} \tag{34}$$

$$\langle A_v(x, \xi) - A_v(x, \eta), \xi - \eta \rangle > 0 \qquad \text{for } \xi \neq \eta \tag{35}$$

Hence, we can consider a quasilinear elliptic operator similar to (4)

$$u \in W_0^{1,p}(\Omega) \quad \mapsto \quad -\operatorname{div} A_v(x, \nabla u) \in W^{-1,p'}(\Omega) \tag{36}$$

defined by the rule

$$\langle -\operatorname{div} A_v(x, \nabla u), w \rangle = \int_\Omega \langle A(x, v, \nabla u), \nabla w \rangle \, \mathrm{d}x \tag{37}$$

for any $w \in W_0^{1,p}(\Omega)$. The operator at (36) is invertible. Indeed,

Proposition 1 *For every $\Phi \in W^{-1,p'}(\Omega)$, there exists a unique $u \in W_0^{1,p}(\Omega)$ such that*

$$-\operatorname{div} A_v(x, \nabla u) = \Phi \tag{38}$$

Moreover, the mapping

$$(v, \Phi) \in L^p(\Omega) \times W^{-1,p'}(\Omega) \quad \mapsto \quad u \in W_0^{1,p}(\Omega) \tag{39}$$

is continuous.

Proof Existence of a solution is classical, see e.g. [28], [8, pg. 27], or [27, Théorème 2.8, pg. 183]. Uniqueness trivially holds by monotonicity.

For the sake of completeness, we prove continuity of the map (39). Given $v_n \to v$ in $L^p(\Omega)$ and $\Phi_n \to \Phi$ in $W^{-1,p'}(\Omega)$, let $u_n \in W_0^{1,p}(\Omega)$ solve

$$-\operatorname{div} A(x, v_n, \nabla u_n) = \Phi_n. \tag{40}$$

The sequence $\{u_n\}_n$ is clearly bounded, hence we may assume $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. Moreover, testing equation (40) with $u_n - u$, we have

$$\lim_{n\to\infty} \int_\Omega A(x, v_n, \nabla u_n)(\nabla u_n - \nabla u) \, \mathrm{d}x = \lim_{n\to\infty} \langle \Phi_n, u_n - u \rangle = 0. \tag{41}$$

On the other hand, we easily see that $A(x, v_n, \nabla u) \to A(x, v, \nabla u)$ strongly in $L^{p'}(\Omega, \mathbb{R}^N)$ and thus (41) implies

$$\lim_{n\to\infty} \int_\Omega [A(x, v_n, \nabla u_n) - A(x, v_n, \nabla u)] \nabla(u_n - u)\, dx = 0. \tag{42}$$

The integrands in (42) are nonnegative by monotonicity. Hence, arguing as in the proof of [28, Lemma 3.3], we also get $\nabla u_n(x) \to \nabla u(x)$ a.e. in Ω, and

$$A(x, v_n, \nabla u_n) \rightharpoonup A(x, v, \nabla u)$$

weakly in $L^{p'}(\Omega, \mathbb{R}^N)$. Combining this with (41) yields

$$\lim_{n\to\infty} \int_\Omega A(x, v_n, \nabla u_n)\nabla u_n\, dx = \int_\Omega A(x, v, \nabla u)\nabla u\, dx. \tag{43}$$

By coercivity condition (1), we deduce

$$\alpha|\nabla u_n|^p \le A(x, v_n, \nabla u_n)\nabla u_n + (b|v_n|)^p + \varphi^p$$

Trivially $\int_\Omega (b|v_n|)^p\, dx$ converges to $\int_\Omega (b|v|)^p\, dx$. In view of (43), by Lemma 1 we get $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$, and u solves the equation

$$-\operatorname{div} A(x, v, \nabla u) = \Phi.$$

□

In view of Rellich Theorem, we have

Corollary 2 *For fixed $\Phi \in W^{-1,p'}(\Omega)$, the mapping*

$$\mathcal{F}\colon v \in W_0^{1,p}(\Omega) \quad \mapsto \quad u \in W_0^{1,p}(\Omega) \tag{44}$$

which takes v to the unique solution u of equation (38) is compact.

Now we state an existence result to Problem (6) when $b, \tilde{b} \in L^\infty(\Omega)$.

Proposition 2 *Let (1), (2) and (3) be in charge with $b, \tilde{b} \in L^\infty(\Omega)$. Then Problem (6) has a solution $u \in W_0^{1,p}(\Omega)$.*

Proof If $\mathcal{F}$ is the operator defined in Corollary 2, clearly a fixed point of $\mathcal{F}$ is a solution to Problem (6). To apply Leray-Schauder theorem, we need an a priori estimate on the solution $u \in W_0^{1,p}(\Omega)$ of the equation

$$u = t\mathcal{F}[u]$$

that is

$$-\operatorname{div} A\left(x, u, \frac{1}{t}\nabla u\right) = \Phi, \tag{45}$$

as $t \in\,]0, 1]$ varies. By using $T_\sigma u$ with $\sigma > 0$ as a test function in (45) we get

$$\int_\Omega \left\langle A\left(x, u, \frac{1}{t}\nabla u\right), \nabla T_\sigma u\right\rangle dx = \langle \Phi, T_\sigma u\rangle \tag{46}$$

Therefore, using the point-wise condition (33) we get

$$\alpha\, t^{1-p} \int_\Omega |\nabla T_\sigma u|^p\, dx \le \|\Phi\|\, \|\nabla T_\sigma u_k\|_p + \int_\Omega \left[b(x)^p |u|^p \chi_{\{|u|<\sigma\}} + \varphi(x)^p\right] dx \tag{47}$$

As $0 < t \le 1$, by Young inequality (47) yields

$$\frac{\alpha}{2}\int_\Omega |\nabla T_\sigma u|^p \,\mathrm{d}x \le \|\Phi\|^{p'} + \|b\|_\infty^p \int_\Omega |u|^p \chi_{\{|u|<\sigma\}} \,\mathrm{d}x + \|\varphi\|_p^p \tag{48}$$

The conclusion follows by Lemma 2. □

4.2 The approximating problems

For each $n \in \mathbb{N}$, we set

$$\vartheta_n(x) = \frac{T_n \max\{b(x), \tilde{b}(x)\}}{\max\{b(x), \tilde{b}(x)\}}, \qquad \text{a.e. } x \in \Omega, \tag{49}$$

and define the vector field

$$A_n\colon (x, u, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \quad \mapsto \quad \mathbb{R}^N \tag{50}$$

letting

$$A_n(x, u, \xi) = A(x, \vartheta_n u, \xi) \tag{51}$$

The vector field A_n has similar properties as A. More precisely,

$$\langle A_n(x, u, \xi), \xi\rangle \ge \alpha|\xi|^p - \big(T_n b(x)\,|u|\big)^p - \varphi(x)^p \tag{52}$$

$$|A_n(x, u, \xi)| \le \beta|\xi|^{p-1} + \big(T_n\tilde{b}(x)\,|u|\big)^{p-1} + \varphi(x)^{p-1} \tag{53}$$

$$\langle A_n(x, u, \xi) - A_n(x, u, \eta), \xi - \eta\rangle > 0 \qquad \text{for } \xi \ne \eta \tag{54}$$

Applying Proposition 2 with A_n in place of A, fixed $\Phi \in W^{-1,p'}(\Omega)$, we find $u_n \in W_0^{1,p}(\Omega)$ such that

$$-\operatorname{div} A_n(x, u_n, \nabla u_n) = \Phi. \tag{55}$$

Notice that we have, for $\sigma > 0$

$$\alpha \int_\Omega |\nabla T_\sigma u_n|^p \,\mathrm{d}x \le \|\Phi\|\,\|\nabla T_\sigma u_n\|_p + \int_\Omega \Big[b^p\,|u_n|^p \chi_{\{|u_n|<\sigma\}} + \varphi^p\Big]\,\mathrm{d}x \tag{56}$$

which implies

$$\alpha^{\frac{1}{p}}\|\nabla T_\sigma u_n\|_p \le (\|\Phi\|\,\|\nabla T_\sigma u_n\|_p)^{\frac{1}{p}} + \|b\,u_n \chi_{\{|u_n|<\sigma\}}\|_p + \|\varphi\|_p \tag{57}$$

Our next step consists in showing that the sequence $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$. Let m be a positive integer to be chosen later. We have

$$b = T_m b + (b - T_m b)$$

and hence

$$\|b\,u_n \chi_{\{|u_n|<\sigma\}}\|_p \le \|(T_m b)\,u_n \chi_{\{|u_n|<\sigma\}}\|_p + \|(b - T_m b)\,u_n \chi_{\{|u_n|<\sigma\}}\|_p \tag{58}$$

Using Hölder and Sobolev inequalities we get

$$\begin{aligned}\|(b - T_m b)\,u_n \chi_{\{|u_n|<\sigma\}}\|_p &\le \|b - T_m b\|_{N,\infty}\|T_\sigma u_n\|_{p^*,p}\\ &\le S_{N,p}\,\|b - T_m b\|_{N,\infty}\|\nabla T_\sigma u_n\|_p\end{aligned}$$

Then (57) and (58) give

$$\alpha^{\frac{1}{p}}\|\nabla T_\sigma u_n\|_p \le (\|\Phi\|\|\nabla T_\sigma u_n\|_p)^{\frac{1}{p}} + \|(T_m b)\, u_n \chi_{\{|u_n|<\sigma\}}\|_p + \|\varphi\|_p$$
$$+S_{N,p}\,\|b - T_m b\|_{L^{N,\infty}(\Omega)}\|\nabla T_\sigma u_n\|_{L^p(\Omega)} \tag{59}$$

By our assumption (9), the level m can be chosen large enough so that

$$S_{N,p}\|b - T_m b\|_{L^{N,\infty}(\Omega)} < \alpha^{\frac{1}{p}}$$

Then, by absorbing in (59) the latter term of the right hand side in the left hand side, we get

$$C\int_\Omega |\nabla T_\sigma u_n|^p\,\mathrm{d}x \le \|\Phi\|\,\|\nabla T_\sigma u_n\|_p + \int_\Omega \Big[(T_m b)^p\,|T_\sigma u_n|^p \chi_{\{|u_n|<\sigma\}} + \varphi^p\Big]\mathrm{d}x \tag{60}$$

for a positive constant C which is independent of n. Now, it is clear that (60), via Young inequality, allows us to apply Lemma 2, then

$$\|u_n\| \le M \tag{61}$$

for a constant M independent of n.

In the model case (8), it is easy to show that the operator $\mathcal{F}$ defined in (44) is compact, also for $b, \tilde{b} \in L^N(\Omega)$ (see Remark 1 below). In the general case, in which $b, \tilde{b} \in L^{N,\infty}(\Omega)$ we need more work.

4.3 Passing to the limit

Now, we are in a position to conclude the proof of Theorem 1. Taking into account estimate (61) we may assume

$$\begin{aligned} u_n &\rightharpoonup u \quad && \text{in } W_0^{1,p}(\Omega) \text{ weakly} \\ u_n &\to u \quad && \text{in } L^q(\Omega) \text{ strongly for any } q < p^*, \text{ and also a.e. in } \Omega \end{aligned} \tag{62}$$

for some $u \in W_0^{1,p}(\Omega)$. We shall conclude our proof showing that u solves Problem (6). In the rest of our argument, we let for simplicity $\gamma(t) := \arctan t$. Obviously, $\gamma \in C^1(\mathbb{R})$, $|\gamma(t)| \le |t|$ and $0 \le \gamma'(t) \le 1$ for all $t \in \mathbb{R}$. In particular, γ is Lipschitz continuous in the whole of $\mathbb{R}$ and therefore

$$u_n, u \in W_0^{1,p}(\Omega) \implies \gamma(u_n - u) \in W_0^{1,p}(\Omega).$$

Moreover, since $\gamma(0) = 0$ we have

$$\gamma(u_n - u) \rightharpoonup 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ weakly.} \tag{63}$$

Testing equation (55) with the function $\gamma(u_n - u)$ we get

$$\int_\Omega A_n(x, u_n, \nabla u_n)\nabla\gamma(u_n - u)\,\mathrm{d}x = \langle \Phi, \gamma(u_n - u)\rangle$$

where $\nabla\gamma(u_n - u) = \gamma'(u_n - u)(\nabla u_n - \nabla u)$. In view of (63) we necessarily have

$$\lim_{n\to\infty}\int_\Omega A_n(x, u_n, \nabla u_n)\nabla\gamma(u_n - u)\,\mathrm{d}x = 0. \tag{64}$$

We claim that

$$\lim_{n\to\infty}\int_\Omega A_n(x, u_n, \nabla u)\nabla\gamma(u_n - u)\,\mathrm{d}x = 0. \tag{65}$$

In order to prove (65), since $\nabla u_n - \nabla u \rightharpoonup 0$, it suffices to show that

$$\gamma'(u_n - u)\, A_n(x, u_n, \nabla u) = \frac{A_n(x, u_n, \nabla u)}{1 + |u_n - u|^2} \qquad \text{is compact in } L^{p'}. \tag{66}$$

Preliminarily, we observe that combining (62) with the property that $\vartheta_n \to 1$ as $n \to \infty$, we have

$$\frac{A_n(x, u_n, \nabla u)}{1 + |u_n - u|^2} \to A(x, u, \nabla u) \qquad \text{a.e. in } \Omega.$$

We are going to use Lemma 1. To this end, by (53) we deduce that

$$\left|\frac{A_n(x, u_n, \nabla u)}{1 + |u_n - u|^2}\right|^{p'} \le C\left[|\nabla u|^p + \varphi^p + (\tilde{b}|u|)^p + \frac{(\tilde{b}|u_n - u|)^p}{1 + |u_n - u|^2}\right]$$

for a positive constant $C = C(p, \beta)$. Hence, we can pass to the limit if $1 < p \le 2$. For $p > 2$ we choose s satisfying

$$\frac{p^*}{p} < s < \frac{p^*}{p - 2},$$

so that $ps' < N$, and we conclude also in this case, further estimating with the aid of Young inequality

$$\frac{(\tilde{b}|u_n - u|)^p}{1 + |u_n - u|^2} \le \tilde{b}^{ps'} + |u_n - u|^{(p-2)s}.$$

Now, from (64) and (65) we get

$$\lim_{n \to \infty} \int_\Omega [A_n(x, u_n, \nabla u_n) - A_n(x, u_n, \nabla u)]\, \nabla\gamma(u_n - u)\, \mathrm{d}x = 0. \tag{67}$$

As the integrand is nonnegative, we have (up to a subsequence)

$$[A_n(x, u_n, \nabla u_n) - A_n(x, u_n, \nabla u)]\, \nabla\gamma(u_n - u) \to 0$$

a.e. in Ω. Moreover, since $\gamma'(u_n - u) \to 1$ a.e. in Ω, the above in turn implies

$$[A_n(x, u_n, \nabla u_n) - A_n(x, u_n, \nabla u)]\ (\nabla u_n - \nabla u) \to 0 \tag{68}$$

Arguing as in the proof of [28, Lemma 3.3], we see that

$$\nabla u_n \to \nabla u \qquad \text{a.e. in } \Omega \tag{69}$$

and

$$A_n(x, u_n, \nabla u_n) \rightharpoonup A(x, u, \nabla u) \qquad \text{in } L^{p'}(\Omega, \mathbb{R}^N) \text{ weakly} \tag{70}$$

and we conclude that u is a solution to the original problem (6).

Remark 1 We discuss briefly the particular case in which the operator has the form

$$A(x, v, \xi) = A'(x, \xi) + A''(x, v),$$

with

$$A'(x, \xi) \cdot \xi \approx |\xi|^p, \quad |A''(x, v)| \le (\tilde{b}(x)\,|v|)^{p-1} + \varphi(x)^{p-1}$$

and $\tilde{b} \in L^N(\Omega)$ (see also [5]). We can easily show that the operator $\mathcal{F}$ defined in (44) is compact, also for $\tilde{b} \in L^N(\Omega)$. Indeed, equation (38) in this case becomes

$$- \operatorname{div} A'(x, \nabla u) = \Phi + \operatorname{div} A''(x, v) \tag{71}$$

and we can take $b = c\tilde{b}$ for a constant $c > 1$. Defined ϑ_n as in (49), each mapping

$$v \in W_0^{1,p}(\Omega) \quad \mapsto \quad A''(x, \vartheta_n\, v) \in L^{p'}(\Omega, \mathbb{R}^N)$$

is clearly compact. Moreover,

$$|A''(x, v) - A''(x, \vartheta_n\, v)| \leq 2[(\tilde{b}\,|v|)^{p-1} + \varphi^{p-1}]\, \chi_{E_n}, \tag{72}$$

where

$$E_n = \{x \in \Omega : b(x) > n\}.$$

Therefore, as $n \to +\infty$ we have

$$A''(x, \vartheta_n\, v) \;\to\; A''(x, v) \qquad \text{strongly in } L^{p'}(\Omega, \mathbb{R}^N),$$

the convergence being uniform when v varies in a bounded subset of $W_0^{1,p}(\Omega)$, and compactness is preserved for the limit mapping.

An a priori bound for solutions of equation

$$u = t\, \mathcal{F}[u]$$

can be easily obtained as above, splitting $b \in L^N(\Omega)$ as

$$b = T_m b + (b - T_m b)$$

for a sufficiently large m. Therefore, in this particular case the existence result of Theorem 1 follows simply applying Leray–Schauder fixed point theorem.

5 The obstacle problem

This section is devoted to the obstacle problem naturally related with problem (6). (See [26] for a comprehensive treatment of the topic.) We again assume that (1), (2) and (3) are in charge and we let $\Phi \in W^{-1,p}(\Omega)$. Given a measurable function $\psi\colon \Omega \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, \infty]$, we consider the convex subset of $\mathcal{K}_\psi(\Omega)$ of $W_0^{1,p}(\Omega)$ given by

$$\mathcal{K}_\psi(\Omega) := \left\{ w \in W_0^{1,p}(\Omega)\colon\; w \geq \psi \text{ a.e. in } \Omega \right\}. \tag{73}$$

We will assume that $\mathcal{K}_\psi(\Omega)$ is nonempty. An element $u \in \mathcal{K}_\psi(\Omega)$ is a solution to the obstacle problem associated with (6) if the following variational inequality holds

$$\int_\Omega \langle A(x, u, \nabla u), \nabla(w - u)\rangle \,\mathrm{d}x \geq \langle \Phi, w - u\rangle, \qquad \forall w \in \mathcal{K}_\psi(\Omega). \tag{74}$$

As $\mathcal{K}_\psi(\Omega) \neq \emptyset$, we may assume without loss of generality that

$$\psi \leq 0 \text{ a.e. in } \Omega. \tag{75}$$

In fact, if $g \in \mathcal{K}_\psi(\Omega)$, then one can consider the operator defined by the vector field

$$\tilde{A}(x, u, \xi) := A(x, u + g(x), \xi + \nabla g(x)),$$

satisfying conditions similar to (1), (2) and (3). Now it is clear that, if a function $\tilde{u} \in \mathcal{K}_{\psi-g}(\Omega)$ satisfies the following variational inequality

$$\int_\Omega \langle \tilde{A}(x, \tilde{u}, \nabla \tilde{u}), \nabla (w - \tilde{u}) \rangle \, dx \geq \langle \Phi, w - \tilde{u} \rangle \qquad \forall w \in \mathcal{K}_{\psi-g}(\Omega) \tag{76}$$

correspondingly $u = \tilde{u} + g$ is a solution to (74). Notice that the obstacle function for problem (76) is nonpositive, as we are assuming for the original problem.

Theorem 5 *Let $\Phi \in W^{-1,p'}(\Omega)$ and $\psi : \Omega \to [-\infty, 0]$ be a measurable function. Under the assumption* (1)*,* (2) *and* (3)*, if* (9) *holds, then the obstacle problem* (74) *admits a solution.*

Proof We follow closely the arguments of Sect. 4. For each $n \in \mathbb{N}$, we consider the function ϑ_n as in (49) and define the vector fields $A_n = A_n(x, u, \xi)$ as in (51). We consider a sequence of obstacle problems provided by

$$\int_\Omega \langle A_n(x, u_n, \nabla u_n), \nabla (w - u_n) \rangle \, dx \geq \langle \Phi, w - u_n \rangle, \qquad \forall w \in \mathcal{K}_\psi(\Omega). \tag{77}$$

The existence of a solution $u_n \in \mathcal{K}_\psi(\Omega)$ to (77) is proven applying [27, Théorème 8.2, pg. 247] to the operator

$$-\operatorname{div} A_n(x, v, \nabla u),$$

for a fixed $v \in W_0^{1,p}(\Omega)$, and then using Leray–Schauder Theorem, arguing as in Sect. 4.1. Due to (75), for every $k > 0$ the function

$$w := u_n - T_k u_n \in \mathcal{K}_\psi(\Omega)$$

is a test function for (77). Arguing as in Sect. 4.2 we obtain

$$\|u_n\| \leq M$$

with M independent of n (as in (61)). Therefore (62) holds for some $u \in W_0^{1,p}(\Omega)$. It is clear from (62) itself that

$$u \in \mathcal{K}_\psi(\Omega) \tag{78}$$

As for Theorem 1, we shall prove that u is a solution to the original problem (74). We proceed as follows. We use

$$w := u_n - \gamma(u_n - v) \tag{79}$$

in (77), where $\gamma(s) = \lambda \arctan(s/\lambda)$, for $\lambda > 0$, and $v \in \mathcal{K}_\psi(\Omega)$ is arbitrary. Note that this is a legitimate test function, that is $w \in \mathcal{K}_\psi(\Omega)$. Indeed, on the set where $u_n \geq v$ we have $\gamma(u_n - v) \leq u_n - v$ and so $w \geq v$; on the other hand, on the set where $u_n \leq v$ we have $\gamma(u_n - v) \leq 0$ and so $w \geq u_n$. Therefore, from (77) we get

$$\int_\Omega A_n(x, u_n, \nabla u_n) \nabla \gamma(u_n - v) \, dx \leq \langle \Phi, \gamma(u_n - v) \rangle \,. \tag{80}$$

Following the lines of the proof of Theorem 1 (where $\lambda = 1$), we get in turn (67), (69) and finally (70). To pass to the limit for fixed general $\lambda > 0$ in (80), we rewrite it as follows:

$$\begin{aligned} &\int_\Omega [A_n(x, u_n, \nabla u_n) - A_n(x, u_n, \nabla v)] \, \nabla \gamma(u_n - v) \, dx \\ &\quad \leq \langle \Phi, \gamma(u_n - v) \rangle - \int_\Omega A_n(x, u_n, \nabla v) \nabla \gamma(u_n - v) \, dx. \end{aligned} \tag{81}$$

In the left hand side we use Fatou lemma, as by condition (3) the integrand is nonnegative. In the right hand side, we note that $A_n(x, u_n, \nabla v)\,\gamma'(u_n - v)$ converges to $A(x, u, \nabla v)\,\gamma'(u - v)$ in $L^{p'}$, compare with (66) where we did not use that $u_n \to u$. Hence, we deduce from (81)

$$\int_\Omega [A(x, u, \nabla u) - A(x, u, \nabla v)]\,\nabla\gamma(u - v)\,\mathrm{d}x$$
$$\leq \langle \Phi, \gamma(u - v)\rangle - \int_\Omega A(x, u, \nabla v)\nabla\gamma(u - v)\,\mathrm{d}x,$$

that is

$$\int_\Omega A(x, u, \nabla u)\,\nabla\gamma(u - v)\,\mathrm{d}x \leq \langle \Phi, \gamma(u - v)\rangle\,. \tag{82}$$

Now we let $\lambda \to \infty$ in (82), noting that $\gamma(u - v) \to u - v$ strongly in $W_0^{1,p}(\Omega)$. Therefore, we get

$$\int_\Omega A(x, u, \nabla u)\,\nabla(u - v)\,\mathrm{d}x \leq \langle \Phi, u - v\rangle\,,$$

for all $v \in K_\psi(\Omega)$, which means exactly that u is a solution to our obstacle problem. □

Remark 2 Clearly, Theorem 5 is more general than Theorem 1 since we are allowed to choose $\psi \equiv -\infty$. Indeed, in this case, the obstacle problem (74) reduces to (6).

6 Regularity of the solution

In this Section, following [18] we study regularity of the problem (6).

Proof of Theorem 2 Let $u \in W_0^{1,p}(\Omega)$ be a solution of (6). We may write $\Phi \in W^{-1,\frac{r}{p-1}}(\Omega)$ as

$$\Phi = \mathrm{div}(|F|^{p-2}F)$$

for a suitable $F \in L^r(\Omega, \mathbb{R}^N)$.

For fixed $k > 0$, we use $v := u - T_k u$ as a test function in (7) to get

$$\alpha \int_{\Omega_k} |\nabla u|^p\,\mathrm{d}x \leq \int_{\Omega_k} |F|^{p-1}\,|\nabla u|\,\mathrm{d}x + \int_{\Omega_k} (b^p|u|^p + \varphi^p)\,\mathrm{d}x \tag{83}$$

where Ω_k denotes the superlevel set $\{|u| > k\}$. For $0 < \varepsilon < \alpha$, by Young inequality we get

$$(\alpha - \varepsilon)\int_{\Omega_k} |\nabla u|^p\,\mathrm{d}x \leq \int_{\Omega_k} (C\,|F|^p + b^p|u|^p + \varphi^p)\,\mathrm{d}x \tag{84}$$

with $C = C(p, \varepsilon) > 0$. We let

$$\lambda = \frac{r^*}{p^*} - 1 \tag{85}$$

and multiply both sides of (84) by $k^{p\lambda-1}$ and integrate w.r.t. k over the interval $[0, K]$, for $K > 0$ fixed. By Fubini theorem we have

$$(\alpha - \varepsilon)\int_\Omega |\nabla u|^p\,|T_K u|^{p\lambda}\,\mathrm{d}x \leq \int_\Omega (C\,|F|^p + b^p|u|^p + \varphi^p)\,|T_K u|^{p\lambda}\,\mathrm{d}x \tag{86}$$

which implies

$$(\alpha-\varepsilon)^{\frac{1}{p}}\,\|\nabla u\,|T_K u|^{\lambda}\|_p \le C\|F\,|T_K u|^{\lambda}\|_p + \|b\,u\,|T_K u|^{\lambda}\|_p + \|\varphi\,|T_K u|^{\lambda}\|_p \tag{87}$$

For $M>0$ we write

$$\|b\,u\,|T_K u|^{\lambda}\|_p \le \|(b-T_M b)\,u\,|T_K u|^{\lambda}\|_p + M\|u\,|T_K u|^{\lambda}\|_p \tag{88}$$

By Hölder inequality and Sobolev embedding Theorem 3

$$\begin{aligned}\|(b-T_M b)\,u\,|T_K u|^{\lambda}\|_p &\le \|b-T_M b\|_{N,\infty}\,\|u\,|T_K u|^{\lambda}\|_{p^*,p}\\ &\le \|b-T_M b\|_{N,\infty}\,S_{N,p}\,\|\nabla(u\,|T_K u|^{\lambda})\|_p\end{aligned} \tag{89}$$

Moreover,

$$|\nabla(u\,|T_K u|^{\lambda})| \le (1+\lambda)\,|\nabla u|\,|T_K u|^{\lambda} \tag{90}$$

Therefore

$$\|(b-T_M b)\,u\,|T_K u|^{\lambda}\|_{L^p(\Omega)} \le \|b-T_M b\|_{N,\infty}\,S_{N,p}\,(1+\lambda)\,\|\nabla u\,|T_K u|^{\lambda}\|_p \tag{91}$$

Under the assumption

$$\|b-T_M b\|_{N,\infty}\,S_{N,p}\,(1+\lambda) < \alpha^{\frac{1}{p}} \tag{92}$$

choosing ε small enough we get from (87)

$$\|\nabla u\,|T_K u|^{\lambda}\|_p \le C\,\|G\,|T_K u|^{\lambda}\|_p \tag{93}$$

with $C=C(p,r,M,\alpha)>0$, where we set

$$G^p = |F|^p + |u|^p + \varphi^p. \tag{94}$$

We first show the claim under the additional assumption $u\in L^r(\Omega)$, so that $G\in L^r(\Omega)$. By Hölder inequality we have

$$\|G\,|T_K u|^{\lambda}\|_p \le \|G\|_r\,\|T_K u\|^{\lambda}_{\lambda\frac{rp}{r-p}} \tag{95}$$

From (85) we get

$$\lambda\,\frac{rp}{r-p} = r^*. \tag{96}$$

Hence, by Sobolev embedding theorem we have

$$\begin{aligned}\|T_K u\|^{\lambda}_{\lambda\frac{rp}{r-p}} = \|T_K u\|^{\lambda}_{r^*} = \||T_K u|^{\frac{r^*}{p^*}}\|^{\lambda\frac{p^*}{r^*}}_{p^*} \le C\,\|\nabla |T_K u|^{\frac{r^*}{p^*}}\|^{\lambda\frac{p^*}{r^*}}_{p}\\ \le C\,\|\nabla u\,|T_K u|^{\lambda}\|_p^{\frac{\lambda}{\lambda+1}}\end{aligned} \tag{97}$$

Then, combining (93), (95) and (97), we get

$$\|\nabla u\,|T_K u|^{\lambda}\|_p^{\frac{p^*}{r^*}} \le C\,\|G\|_r \tag{98}$$

Passing to the limit as $K\to+\infty$ and recalling (94), we have

$$\|\nabla u\,|u|^{\lambda}\|_p^{\frac{p^*}{r^*}} \le C\,(\|F\|_r + \|\varphi\|_r + \|u\|_r) \tag{99}$$

that is

$$\|\nabla |u|^{\frac{r^*}{p^*}}\|_p \leq C\,(\|F\|_r + \|\varphi\|_r + \|u\|_r)^{\frac{r^*}{p^*}} \tag{100}$$

Hence, (15) holds as long as $u \in L^r(\Omega)$. At this point we observe that if $r \leq p^*$, using the Sobolev embedding theorem, $u \in L^{p^*}(\Omega)$ and the proof is concluded. In the complementary case $r > p^*$, we use a bootstrap approach. Precisely, we repeat the previous argument replacing r with p^* to get $u \in L^{p^{**}}(\Omega)$. Using this information, if $r \leq p^{**}$, there is nothing left to prove. Otherwise we repeat previous argument again. In a finite number of similar steps we can conclude our proof. □

Remark 3 In the case of the operator (11), the bound (14) becomes

$$|D| < \mu \left(\frac{N-r}{r}\right)^{p-1},$$

which for $p = 2$ reduces to

$$|D| < \mu \frac{N-r}{r},$$

compare with [6, Theorem 2.3].

Funding Open access funding provided by Universitá degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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