Two-term expansion of the ground state one-body density matrix of a mean-field Bose gas

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Received: 9 October 2020 / Accepted: 17 February 2021 / Published online: 27 April 2021
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Abstract

We consider the homogeneous Bose gas on a unit torus in the mean-field regime when the interaction strength is proportional to the inverse of the particle number. In the limit when the number of particles becomes large, we derive a two-term expansion of the one-body density matrix of the ground state. The proof is based on a cubic correction to Bogoliubov's approximation of the ground state energy and the ground state.

MSC code 81Q05 · 47A75

1 Introduction

We consider a homogeneous system of N bosons on the unit torus $\mathbb{T}^d$, for any dimension $d \geq 1$. The system is governed by the mean-field Hamiltonian

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \frac{1}{N-1} \sum_{1 \leq j < k \leq N} w(x_j - x_k) \quad (1)$$

which acts on the bosonic Hilbert space

$$\mathcal{H}^N = L^2_{\text{sym}}((\mathbb{T}^d)^N).$$

Here the kinetic operator $-\Delta$ is the usual Laplacian (with periodic boundary conditions). The interaction potential w is a real-valued, even function. We assume that its Fourier transform

Communicated by A. Malchiodi.

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is non-negative and integrable, namely

$$w(x) = \sum_{p \in 2\pi\mathbb{Z}^d} \widehat{w}(p) e^{ip \cdot x} \quad \text{with } 0 \leq \widehat{w} \in \ell^1(2\pi\mathbb{Z}^d).$$

In particular, w is bounded. Since w is even, $\widehat{w}$ is also even.

Under the above conditions, H_N is well defined on the core domain of smooth functions. Moreover, it is well-known that H_N is bounded from below and can be extended to be a self-adjoint operator by Friedrichs' method. The self-adjoint extension, still denoted by H_N , has a unique ground state Ψ_N (up to a complex phase) which solves the variational problem

$$E_N = \inf_{\|\Psi\|_{\mathcal{H}^N}=1} \langle \Psi, H_N \Psi \rangle.$$

Here $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}^N$.¹

In the present paper, we are interested in the asymptotic behavior of the ground state $\Psi_N \in \mathcal{H}^N$ of H_N in the limit when $N \rightarrow \infty$. More precisely, we will focus on the one-body density matrix $\gamma_{\Psi_N}^{(1)}$ which is a trace class operator on $L^2(\mathbb{T}^d)$ with kernel

$$\gamma_{\Psi_N}^{(1)}(x, y) = N \int_{\mathbb{T}^{d(N-1)}} \Psi_N(x, x_1, \dots, x_N) \overline{\Psi_N(y, x_2, \dots, x_N)} dx_2 \dots dx_N.$$

Note that $\gamma_{\Psi_N}^{(1)} \geq 0$ and $\text{Tr} \gamma_{\Psi_N}^{(1)} = N$.

1.1 Main result

Our main theorem is

Theorem 1 (Ground state density matrix) *Assume that $0 \leq \widehat{w} \in \ell^1((2\pi\mathbb{Z})^d)$. Then the ground state Ψ_N of the Hamiltonian H_N in (1) satisfies*

$$\lim_{N \rightarrow \infty} \text{Tr} \left| \gamma_{\Psi_N}^{(1)} - \left(N - \sum_{p \neq 0} \gamma_p^2 \right) |u_0\rangle \langle u_0| - \sum_{p \neq 0} \gamma_p^2 |u_p\rangle \langle u_p| \right| = 0$$

where

$$u_p(x) = e^{ip \cdot x}, \quad \gamma_p = \frac{\alpha_p}{\sqrt{1 - \alpha_p^2}}, \quad \alpha_p = \frac{\widehat{w}(p)}{p^2 + \widehat{w}(p) + \sqrt{p^4 + 2p^2 \widehat{w}(p)}}.$$

Here $|u\rangle \langle u|$ is the orthogonal projection on u . We use the bra-ket notation, where $|u\rangle = u$ is a vector in the Hilbert space $\mathcal{H}$ and $\langle u|$ is an element in the dual space of $\mathcal{H}$ which maps any vector $v \in \mathcal{H}$ to the inner product $\langle u, v \rangle_{\mathcal{H}}$.

To the leading order, our result implies Bose–Einstein condensation, namely

$$\lim_{N \rightarrow \infty} \frac{1}{N} \gamma_{\Psi_N}^{(1)} = |u_0\rangle \langle u_0|$$

in the trace norm. This result is well-known and it follows easily from Onsager's inequality

$$\frac{1}{N-1} \sum_{1 \leq i < j \leq N} w(x_i - x_j) \geq \frac{N}{2} \widehat{w}(0) - \frac{N}{N-1} w(0) \quad (2)$$

(see [18]). The significance of Theorem 1 is that it gives the next order correction to $\gamma_{\Psi_N}^{(1)}$, thus justifying Bogoliubov's approximation in a rather strong sense as we will explain.

¹ We use the convention that $\langle \cdot, \cdot \rangle$ is linear in the second argument and anti-linear in the first.

1.2 Bogoliubov's approximation

It is convenient to turn to the grand canonical setting. Let us introduce the Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^n = \mathbb{C} \oplus \mathcal{H} \oplus \mathcal{H}^2 \oplus \dots$$

For any Fock space vector $\Psi = (\Psi_n)_{n=0}^{\infty} \in \mathcal{F}$ with $\Psi_n \in \mathcal{H}^n$, we define its norm by

$$\|\Psi\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} \|\Psi_n\|_{\mathcal{H}^n}^2.$$

and define the particle number expectation by

$$\langle \Psi, \mathcal{N} \Psi \rangle = \sum_{n=0}^{\infty} n \|\Psi_n\|_{\mathcal{H}^n}^2.$$

In particular, the vacuum state $|0\rangle = (1, 0, 0, \dots)$ is a normalized vector on Fock space which has the particle number expectation $\langle 0 | \mathcal{N} | 0 \rangle = 0$.

For any $f \in \mathcal{H}$, the creation operator $a^*(f)$ on Fock space maps from $\mathcal{H}^n$ to $\mathcal{H}^{n+1}$ for every $n \geq 0$ and satisfies

$$\begin{aligned} (a^*(f)\Psi_n)(x_1, \dots, x_{n+1}) \\ = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} f(x_j) \Psi(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}), \quad \forall \Psi_n \in \mathcal{H}^n. \end{aligned}$$

Its adjoint is the annihilation operator $a(f)$, which maps from $\mathcal{H}^n$ to $\mathcal{H}^{n-1}$ for every $n \geq 0$ (with convention $\mathcal{H}^{-1} = \{0\}$) and satisfies

$$(a(f)\Psi_n)(x_1, \dots, x_{n-1}) = \sqrt{n} \int_{\mathbb{R}^d} \overline{f(x_n)} \Psi(x_1, \dots, x_n) dx_n, \quad \forall \Psi_n \in \mathcal{H}^n.$$

We will denote by a_p^* and a_p the creation and annihilation operators with momentum $p \in 2\pi\mathbb{Z}^d$, namely

$$a_p^* = a^*(u_p), \quad a_p = a(u_p), \quad u_p(x) = e^{ip \cdot x}.$$

They satisfy the canonical commutation relation (CCR)

$$[a_p, a_q] = 0 = [a_p^*, a_q^*], \quad [a_p, a_q^*] = \delta_{p,q} \quad (3)$$

where $[X, Y] = XY - YX$.

The creation and annihilation operators can be used to express several operators on Fock space. For example, the number operator can be written as

$$\mathcal{N} = \sum_{n=0}^{\infty} n \mathbb{1}_{\mathcal{H}^n} = \sum_{p \in 2\pi\mathbb{Z}^d} a_p^* a_p.$$

Similarly, the Hamiltonian H_N in (1) can be rewritten as

$$H_N = \sum_{p \in 2\pi\mathbb{Z}^d} p^2 a_p^* a_p + \frac{1}{2(N-1)} \sum_{p, q, k \in \mathbb{Z}^d} \widehat{w}(k) a_{p-k}^* a_{q+k}^* a_p a_q. \quad (4)$$

The right side of (4) is an operator on Fock space, which coincides with (1) when being restricted to $\mathcal{H}^N$. In the following we will only use the grand-canonical formula (4).

In 1947, Bogoliubov [4] suggested a heuristic argument to compute the low-lying spectrum of the operator H_N by using a perturbation around the condensation. Roughly speaking, he proposed to first substitute all operators a_0 and a_0^* in (4) by the scalar number $\sqrt{N}$ (c-number substitution²), and then ignore all interaction terms which are coupled with coefficients of order $o(1)_{N \rightarrow \infty}$. All this leads to the formal expression

$$H_N \approx \frac{N}{2} \widehat{w}(0) + \mathbb{H}_{\text{Bog}} \quad (5)$$

where

$$\mathbb{H}_{\text{Bog}} = \sum_{p \neq 0} \left((p^2 + \widehat{w}(p)) a_p^* a_p + \frac{1}{2} \widehat{w}(p) (a_p^* a_{-p}^* + a_p a_{-p}) \right). \quad (6)$$

Note that the expression (5) is formal since H_N acts on the N -body Hilbert space $\mathcal{H}^N$ while the Bogoliubov Hamiltonian $\mathbb{H}_{\text{Bog}}$ acts on the excited Fock space

$$\mathcal{F}_+ = \bigoplus_{n=0}^{\infty} \mathcal{H}_+^n = \mathbb{C} \oplus \mathcal{H}_+ \oplus \mathcal{H}_+^2 \oplus \cdots, \quad \mathcal{H}_+ = \mathcal{QH}$$

where we have introduced the projections

$$Q = \sum_{p \neq 0} |u_p\rangle \langle u_p| = 1 - P, \quad P = |u_0\rangle \langle u_0|.$$

In particular, unlike H_N , the quadratic Hamiltonian $\mathbb{H}_{\text{Bog}}$ does not preserve the number of particles. Nevertheless, $\mathbb{H}_{\text{Bog}}$ can be *diagonalized* by the following unitary transformation on $\mathcal{F}_+$

$$U_{\text{B}} = \exp \left(\sum_{p \neq 0} \beta_p (a_p^* a_{-p}^* - a_p a_{-p}) \right) \quad (7)$$

where the coefficients $\beta_p > 0$ are determined by

$$\tanh(2\beta_p) = \alpha_p = \frac{\widehat{w}(p)}{p^2 + \widehat{w}(p) + \sqrt{p^4 + 2p^2 \widehat{w}(p)}}.$$

In fact, by using the CCR (3) it is straightforward to check that

$$U_{\text{B}} a_p U_{\text{B}}^* = \frac{a_p + \alpha_p a_{-p}^*}{\sqrt{1 - \alpha_p^2}} =: \sigma_p a_p + \gamma_p a_{-p}^*, \quad \forall p \neq 0 \quad (8)$$

where

$$\sigma_p := \frac{1}{\sqrt{1 - \alpha_p^2}} = \cosh(\beta_p), \quad \gamma_p := \frac{\alpha_p}{\sqrt{1 - \alpha_p^2}} = \sinh(\beta_p).$$

Consequently,

$$U_{\text{B}} \mathbb{H}_{\text{Bog}} U_{\text{B}}^* = E_{\text{Bog}} + \sum_{p \neq 0} e(p) a_p^* a_p, \quad (9)$$

² Strictly speaking, for $a_0^* a_0^* a_0 a_0$ we should rewrite it as $(a_0^* a_0)^2 - a_0^* a_0$ before doing the substitution

where

$$E_{\text{Bog}} = -\frac{1}{2} \sum_{p \neq 0} (|p|^2 + \widehat{w}(p) - e(p)), \quad e_p = \sqrt{|p|^4 + 2|p|^2 \widehat{w}(p)}.$$

Note that the assumption $0 \leq \widehat{w} \in \ell^1(2\pi\mathbb{Z}^d)$ ensures that E_{Bog} is finite. Moreover we have the uniform bounds

$$\sum_{p \neq 0} \gamma_p \leq C, \quad \sup_{p \neq 0} \sigma_p \leq C. \quad (10)$$

Thus Bogoliubov's approximation predicts that the ground state energy of H_N is

$$E_N = \frac{N}{2} \widehat{w}(0) + E_{\text{Bog}} + o(1)_{N \rightarrow \infty}. \quad (11)$$

In 2011, Seiringer [18] gave the first rigorous proof of (11). He also proved that the low-lying spectrum of H_N is given approximately by the *elementary excitation* e_p . These results have been extended to inhomogeneous trapped systems in [11], to more general interaction potentials in [12], to a large volume limit in [9], and to situations of multiple-condensation in [14, 17].

Let us recall the approach in [12] which also provides the convergence of the ground state of the mean-field Hamiltonian H_N in (1). Mathematically, the formal expression (5) can be made rigorous using the unitary operator introduced in [12]

$$U_N : \mathcal{H}^N \rightarrow \mathcal{F}_+^{\leq N} = \mathbb{1}^{\leq N} \mathcal{F}_+, \quad \mathbb{1}^{\leq N} = \mathbb{1}(\mathcal{N}_+ \leq N)$$

which is defined by

$$U_N = \sum_{j=0}^N Q^{\otimes j} \left(\frac{a_0^{N-j}}{\sqrt{(N-j)!}} \right), \quad U_N^* = \bigoplus_{j=0}^N \left(\frac{(a_0^*)^{N-j}}{\sqrt{(N-j)!}} \right). \quad (12)$$

Recall from [12, Proposition 4.2] that

$$U_N a_p^* a_q U_N^* = a_p^* a_q, \quad U_N a_p^* a_0 U_N^* = a_p^* \sqrt{N - \mathcal{N}_+}, \quad \forall p, q \neq 0 \quad (13)$$

where $\mathcal{N}_+$ is the number operator on the excited Fock space $\mathcal{F}_+$,

$$\mathcal{N}_+ = \sum_{p \neq 0} a_p^* a_p.$$

Thus U_N implements the c-number substitution in Bogoliubov's argument because it replaces a_0 by $\sqrt{N - \mathcal{N}_+} \approx \sqrt{N}$ (we have $\mathcal{N}_+ \ll N$ due to the condensation). Then the formal expression (5) can be reformulated as

$$U_N H_N U_N^* \approx \frac{N}{2} \widehat{w}(0) + \mathbb{H}_{\text{Bog}} \quad (14)$$

which is rigorous since the operators on both sides act on the same excited Fock space. By justifying (14), the authors of [12] recovered the convergence of eigenvalues of H_N first obtained in [18], and also obtained the convergence of eigenfunctions of H_N to those of $\mathbb{H}_{\text{Bog}}$. In particular, for the ground state, we have from [12, Theorem 2.2] that

$$\lim_{N \rightarrow \infty} U_N \Psi_N = U_B |0\rangle \quad (15)$$

where $|0\rangle$ is the vacuum in Fock space. The convergence (15) holds strongly in norm of $\mathcal{F}_+$, and also strongly in the norm induced by the quadratic form of $\mathbb{H}_{\text{Bog}}$ in $\mathcal{F}_+$. In particular, this implies the convergence of one-body density matrix

$$\lim_{N \rightarrow \infty} Q \gamma_{\Psi_N}^{(1)} Q = \sum_{p \neq 0} \gamma_p^2 |u_p\rangle \langle u_p| \quad (16)$$

in trace class (see (68) for a detailed explanation). Since $\text{Tr} \gamma_{\Psi_N}^{(1)} = N$, (16) is equivalent to

$$\lim_{N \rightarrow \infty} \text{Tr} \left| P \gamma_{\Psi_N}^{(1)} P + Q \gamma_{\Psi_N}^{(1)} Q - \left(N - \sum_{p \neq 0} \gamma_p^2 \right) |u_0\rangle \langle u_0| - \sum_{p \neq 0} \gamma_p^2 |u_p\rangle \langle u_p| \right| = 0. \quad (17)$$

Recall that $P = |u_0\rangle \langle u_0| = 1 - Q$. The formula (17) looks similar to the result in Theorem 1, except that the cross term $P \gamma_{\Psi_N}^{(1)} Q + Q \gamma_{\Psi_N}^{(1)} P$ is missing. Putting differently, to get the result in Theorem 1 we have to show that

$$\lim_{N \rightarrow \infty} \text{Tr} \left| P \gamma_{\Psi_N}^{(1)} Q + Q \gamma_{\Psi_N}^{(1)} P \right| = 0. \quad (18)$$

As explained in [12, Eq. (2.19)], from (16) and the Cauchy–Schwarz inequality one only obtains that the left side of (18) is of order $O(\sqrt{N})$. Moreover, (18) implies that

$$\lim_{N \rightarrow \infty} \sqrt{N} \langle U_N \Psi_N, a_p U_N \Psi_N \rangle = 0, \quad \forall p \neq 0, \quad (19)$$

thus answering an open question in [13]. As explained in [13, Section 5], (19) would follow if we could replace $U_N \Psi_N$ by $U_B |0\rangle$ (which is a quasi-free state, and thus satisfies Wick's Theorem [19, Chapter 10]). However, the norm convergence (15) is not strong enough to justify (19).

1.3 Outline of the proof

To prove Theorem 1 we have to extract some information going beyond Bogoliubov's approximation. Roughly speaking, we will refine (14) by computing exactly the term of order $O(N^{-1})$. Our proof consists of three main steps.

Step 1 (Excitation Hamiltonian). After implementing the c-number substitution, instead of ignoring all terms with coefficients of order $o(1)_{N \rightarrow \infty}$, we will keep all terms of order $O(N^{-1})$. More precisely, in Lemma 7 below we show that

$$U_N H_N U_N^* = \frac{N}{2} \widehat{w}(0) + \mathcal{G}_N + O(N^{-3/2}) \quad (20)$$

in an appropriate sense, where

$$\begin{aligned} \mathcal{G}_N = & \mathbb{H}_{\text{Bog}} + \frac{\mathcal{N}_+(1 - \mathcal{N}_+)}{2(N-1)} \widehat{w}(0) + \sum_{p \neq 0} \frac{1 - \mathcal{N}_+}{N-1} \widehat{w}(p) a_p^* a_p \\ & + \left(\frac{1}{2} \sum_{p \neq 0} \widehat{w}(p) a_p^* a_{-p}^* \frac{1 - 2\mathcal{N}_+}{2N} + \text{h.c.} \right) \\ & + \left(\frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell + p \neq 0} \widehat{w}(\ell) a_{p+\ell}^* a_{-\ell}^* a_p + \text{h.c.} \right) \end{aligned}$$

$$+ \frac{1}{2(N-1)} \sum_{\substack{k, p \neq 0 \\ \ell \neq -p, k}} \widehat{w}(\ell) a_{p+\ell}^* a_{k-\ell}^* a_p a_k.$$

The formula (20) is obtained by a direct computation using the actions of U_N as in [12], plus an expansion of $\sqrt{N - \mathcal{N}_+}$ and $\sqrt{(N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1)}$ in the regime $\mathcal{N}_+ \ll N$. The advantage of using $\mathcal{G}_N$ is that it is well-defined on the full Fock space $\mathcal{F}_+$. This idea has been used to study the norm approximation for the many-body quantum dynamics in [8].

Step 2 (Quadratic transformation). Then we conjugate the operator on the right side of (20) by the Bogoliubov transformation U_B in (7). In Lemma 8 we prove that

$$U_B \mathcal{G}_N U_B^* = \langle 0 | U_B \mathcal{G}_N U_B^* | 0 \rangle + \sum_{p \neq 0} e(p) a_p^* a_p + \mathcal{C}_N + R_2 \quad (21)$$

where

$$\mathcal{C}_N = \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} \widehat{w}(p) \left[(\sigma_{p+q} \sigma_{-p} \gamma_q + \gamma_{p+q} \gamma_p \sigma_q) a_{p+q}^* a_{-p}^* a_{-q}^* + \text{h.c.} \right]$$

and R_2 is an error term whose expectation against the ground state is of order $O(N^{-3/2})$.

Note that in $\mathcal{C}_N$ we keep only cubic terms with three creation operators or three annihilation operators. These are the most problematic terms. All other cubic terms, as well as all quartic terms, are of lower order and can be estimated by the Cauchy–Schwarz inequality (the quartic terms always come with a factor N^{-1} instead of $N^{-1/2}$ and this helps).

As we will see, the energy contribution of the cubic term $\mathcal{C}_N$ is of order $O(N^{-1})$. Thus (21) implies that

$$E_N = \frac{N}{2} \widehat{w}(0) + \langle 0 | U_B \mathcal{G}_N U_B^* | 0 \rangle + O(N^{-1}) \quad (22)$$

which improves (11). Moreover, for the ground state we have

$$\langle U_B U_N \Psi_N, \mathcal{N}_+ U_B U_N \Psi_N \rangle \leq C N^{-1} \quad (23)$$

which in turn implies the norm approximation (up to an appropriate choice of the phase factor for Ψ_N)

$$\|U_N \Psi_N - U_B^* |0\rangle\|_{\mathcal{F}_+}^2 \leq C N^{-1}. \quad (24)$$

and the following bound on the one-body density matrix

$$\lim_{N \rightarrow \infty} \text{Tr} \left| P \gamma_{\Psi_N}^{(1)} Q + Q \gamma_{\Psi_N}^{(1)} P \right| \leq C.$$

Unfortunately the latter bound is still weaker than (18). Thus the desired result (18) cannot be obtained within Bogoliubov's theory.

Step 3 (Cubic transformation). To factor out the energy contribution of the cubic term $\mathcal{C}_N$ in (21), we will use a cubic transformation. It is given by

$$U_S = e^S, \quad S = \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} \eta_{p,q} \left(a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}^{\leq N} - \mathbb{1}^{\leq N} a_{p+q} a_{-p} a_{-q} \right) \quad (25)$$

where

$$\eta_{p,q} = \frac{\widehat{w}(p)(\sigma_{p+q}\sigma_p\gamma_q + \gamma_{p+q}\gamma_p\sigma_q)}{e_{p+q} + e_p + e_q}. \quad (26)$$

From the assumption $\widehat{w} \in \ell^1(2\pi\mathbb{Z}^d)$ and the bounds (10) we have the summability

$$\sum_{p,q \neq 0} |\eta_{p,q}| \leq C. \quad (27)$$

Here we insert the cut-off $\mathbb{1}^{\leq N}$ in the definition of U_S to make sure that it does not change the particle number operator $\mathcal{N}_+$ too much; see Lemma 5 for details.

The choice of the cubic transformation above can be deduced on an abstract level. Consider an operator of the form

$$A = A_0 + X$$

where X stands for some perturbation. Then, in principle, we can remove X by conjugating A with e^S provided that

$$X + [S, A_0] = 0$$

and that $[S, [S, A_0]] = -[S, X]$ is small in an appropriate sense. This can be seen by the simple expansions

$$e^S X e^{-S} = X + \int_0^1 e^{sS} [X, S] e^{-sS} ds$$

and

$$e^S A_0 e^{-S} = A_0 + [S, A_0] + \int_0^1 \int_0^t e^{sS} [S, [S, A_0]] e^{-sS} ds dt.$$

In our situation, $A_0 = \sum_{p \neq 0} e(p) a_p^* a_p$ and $X = \mathcal{C}_N$, allowing to find S explicitly in (25).

In Lemma 9 we prove that

$$U_S U_B \mathcal{G}_N U_B^* U_S^* = \left\langle 0 \left| U_S U_B \mathcal{G}_N U_B^* U_S^* \right| 0 \right\rangle + \sum_{p \neq 0} e(p) a_p^* a_p + R_3$$

with an error term R_3 whose expectation against the ground state is of order $O(N^{-3/2})$. This allows us to obtain the following improvements of (22), (23) and (24).

Theorem 2 (Refined ground state estimates) *Assume that $0 \leq \widehat{w} \in \ell^1((2\pi\mathbb{Z})^d)$. Then the ground state energy of the Hamiltonian H_N in (1) satisfies*

$$E_N = \frac{N}{2} \widehat{w}(0) + \left\langle 0 \left| U_S U_B \mathcal{G}_N U_B^* U_S^* \right| 0 \right\rangle + O(N^{-3/2}).$$

Moreover, if Ψ_N is the ground state of H_N , then $\Phi = U_S U_B U_N \Psi_N$ satisfies

$$\langle \Phi, \mathcal{N}_+ \Phi \rangle \leq C N^{-3/2}.$$

Consequently, we have the norm approximation (up to an appropriate choice of the phase factor for Ψ_N)

$$\|U_N \Psi_N - U_B^* U_S^* |0\rangle\|_{\mathcal{F}_+}^2 \leq C N^{-3/2}.$$

As we will explain, Theorem 2 implies (18) and thus justifies Theorem 1.

The idea of using cubic transformations has been developed to handle dilute Bose gases in [1–3, 16, 20], where the interaction potential has a much shorter range but the interaction strength is much larger in its range. In this case, the contribution of the cubic terms is much bigger, and Bogoliubov's approximation has to be modified appropriately to capture the short-range scattering effect. Results similar to (11) have been proved recently for the Gross-Pitaevskii limit [2] and for the thermodynamic limit [10, 20]. It is unclear to us how to extend Theorem 1 to the dilute regime.

Our work shows that in the mean-field regime, in contrast to the dilute regime, the cubic terms are smaller, and they actually contribute only to the next order correction to Bogoliubov's approximation (there are also some quadratic and quartic terms which contribute to the same order of the cubic term). On the other hand, it is interesting that the contribution of the cubic terms is not visible in the expansion of the one-body density matrix in Theorem 1; putting differently the approximation in Theorem 1 can be guessed using only Bogoliubov's theory (although its proof requires more information).

There have been also remarkable works concerning higher order expansions in powers of N^{-1} in the mean-field regime; see [15] for a study of the ground state, [7] for the low-energy spectrum, and [5, 6] for the quantum dynamics. These works are based on perturbative approaches which are very different from ours. Note that the method of Bossmann, Petrat and Seiringer in [7] also gives access to the higher order expansion of the reduced density matrices (see [7, Eq. (3.15)] for a comparison). We hope that our rather explicit strategy complements the previous analysis in [5–7, 15] concerning the correction to Bogoliubov's theory in the mean-field regime.

Organization of the paper

In Sect. 2 we will derive some useful estimates for the particle number operator $\mathcal{N}_+$. Then we analyze the actions of the transformations U_N , U_B , U_S in Sects. 3, 4 and 5, respectively. Finally, we prove Theorem 2 in Sect. 7 and conclude Theorem 1 in Sect. 7.

2 Moment estimates for the particle number operator

In this section we justify the Bose–Einstein condensation by showing that the ground state has a bounded number of excited particles. As explained in [18], the uniform bound on the expectation of $\mathcal{N}_+$ follows easily from Onsager's inequality (2). For our purpose, we will need uniform bounds for higher moments of $\mathcal{N}_+$. The following lemma is an extension of [13, Lemma 5].

Lemma 3 (Number of excited particles) *If Ψ_N is the ground state of H_N , then*

$$\langle \Psi_N, \mathcal{N}_+^s \Psi_N \rangle \leq C_s, \quad \forall s \in \mathbb{N}.$$

Proof As in [13, Lemma 5], from the operator inequality

$$H_N \geq (2\pi)^2 \mathcal{N}_+ + \frac{N^2}{2(N-1)} \hat{w}(0) - \frac{N}{2(N-1)} w(0) \quad (28)$$

we obtain

$$|E_N| \leq \frac{N}{2} \hat{w}(0) \quad \text{and} \quad \langle \Psi_N, \mathcal{N}_+^s \Psi_N \rangle \leq C \quad (29)$$

for $s = 1, 2, 3$. Let us assume that $s \in \mathbb{N}$ is even. We will show that $\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \rangle \leq C$.

Since Ψ_N is a ground state of H_N , it solves the Schrödinger equation

$$H_N \Psi_N = E_N \Psi_N.$$

Consequently, we get the identity

$$\left\langle \Psi_N, \mathcal{N}_+^{\frac{s}{2}} \left(H_N - E_N \right) \mathcal{N}_+^{\frac{s}{2}} \Psi_N \right\rangle = \left\langle \Psi_N, \mathcal{N}_+^{\frac{s}{2}} [H_N, \mathcal{N}_+^{\frac{s}{2}}] \Psi_N \right\rangle. \quad (30)$$

The left side of (30) can be estimated using (28) and (29) as

$$\left\langle \Psi_N, \mathcal{N}_+^{\frac{s}{2}} \left(H_N - E_N \right) \mathcal{N}_+^{\frac{s}{2}} \Psi_N \right\rangle \geq \left\langle \Psi_N, \left((2\pi)^2 \mathcal{N}_+^{s+1} - C \mathcal{N}_+^s \right) \Psi_N \right\rangle. \quad (31)$$

For the right side of (30), since

$$[A, B^k] = \sum_{j=0}^{k-1} B^j [A, B] B^{k-j-1},$$

using (4) and the CCR (3) we write

$$\begin{aligned} \mathcal{N}_+^{\frac{s}{2}} [H_N, \mathcal{N}_+^{\frac{s}{2}}] &= \frac{1}{2(N-1)} \sum_{j=0}^{\frac{s}{2}-1} \sum_{\ell \neq 0} \sum_{p,q} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} [a_{p-\ell}^* a_{q+\ell}^* a_p a_q, \mathcal{N}_+] \mathcal{N}_+^{\frac{s}{2}-j-1} \\ &= \frac{1}{2(N-1)} \sum_{j=0}^{\frac{s}{2}-1} \sum_{\ell \neq 0} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} \left(2a_0^* a_0^* a_{\ell} a_{-\ell} - 2a_{-\ell}^* a_{\ell}^* a_0 a_0 \right) \mathcal{N}_+^{\frac{s}{2}-j-1} \\ &\quad + \frac{1}{2(N-1)} \sum_{j=0}^{\frac{s}{2}-1} \sum_{\ell \neq 0 \neq p \neq \ell} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} \left(a_{p-\ell}^* a_0^* a_p a_{-\ell} - a_{p-\ell}^* a_{\ell}^* a_p a_0 \right) \mathcal{N}_+^{\frac{s}{2}-j-1} \\ &\quad + \frac{1}{2(N-1)} \sum_{j=0}^{\frac{s}{2}-1} \sum_{\ell \neq 0 \neq q \neq -\ell} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} \left(a_0^* a_{q+\ell}^* a_{\ell} a_q - a_{-\ell}^* a_{q+\ell}^* a_0 a_q \right) \mathcal{N}_+^{\frac{s}{2}-j-1}. \quad (32) \end{aligned}$$

Now we take the expectation against Ψ_N and estimate. For the first term on the right side of (32), by the Cauchy–Schwarz inequality, we get for a given j

$$\begin{aligned} &\left| \left\langle \Psi_N, \sum_{\ell \neq 0} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} a_0^* a_0^* a_{\ell} a_{-\ell} \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\rangle \right| \\ &= \left| \left\langle \Psi_N, \sum_{\ell \neq 0} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} a_0^* a_0^* (\mathcal{N}_+ + 1)^{-j} (\mathcal{N}_+ + 1)^j a_{\ell} a_{-\ell} \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\rangle \right| \\ &\leq \sum_{\ell \neq 0} \left\| (\mathcal{N}_+ + 1)^{-j} a_0 a_0 \mathcal{N}_+^{\frac{s}{2}+j} \Psi_N \right\| |\widehat{w}(\ell)| \left\| (\mathcal{N}_+ + 1)^j a_{\ell} a_{-\ell} \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\| \\ &= \sum_{\ell \neq 0} \left\| a_0 a_0 \mathcal{N}_+^{\frac{s}{2}} \Psi_N \right\| |\widehat{w}(\ell)| \left\| a_{\ell} a_{-\ell} (\mathcal{N}_+ - 1)^j \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\| \\ &\leq \left\| a_0 a_0 \mathcal{N}_+^{\frac{s}{2}} \Psi_N \right\| \left(\sum_{\ell \neq 0} |\widehat{w}(\ell)|^2 \right)^{1/2} \left(\sum_{\ell \neq 0} \left\| a_{\ell} a_{-\ell} (\mathcal{N}_+ - 1)^j \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\|^2 \right)^{1/2} \end{aligned}$$

$$\leq CN \langle \Psi_N, \mathcal{N}_+^s \Psi_N \rangle.$$

Here we have used that $a_0 a_0$ commutes with $\mathcal{N}_+$, that $a_0^* a_0 \leq N$ on $\mathcal{H}^N$ and that $\sum |\widehat{w}(\ell)|^2 = \|w\|_{L^2}^2 < \infty$. Similarly, for the second term, we have

$$\begin{aligned} & \left| \left\langle \Psi_N, \sum_{\ell \neq 0} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} a_{-\ell}^* a_{\ell}^* a_0 a_0 \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\rangle \right| \\ &= \left| \left\langle \Psi_N, \sum_{\ell \neq 0} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} a_{-\ell}^* a_{\ell}^* (\mathcal{N}_+ + 1)^{-j-1} (\mathcal{N}_+ + 1)^{j+1} a_0 a_0 \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\rangle \right| \\ &\leq \sum_{\ell \neq 0} \left\| (\mathcal{N}_+ + 1)^{-j-1} a_{-\ell} a_{\ell} \mathcal{N}_+^{\frac{s}{2}+j} \Psi_N \right\| |\widehat{w}(\ell)| \left\| (\mathcal{N}_+ + 1)^{j+1} a_0 a_0 \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\| \\ &\leq CN \langle \Psi_N, (\mathcal{N}_+ + 1)^s \Psi_N \rangle \end{aligned}$$

as before. For the third term, we can bound

$$\begin{aligned} & \left| \left\langle \Psi_N, \sum_{\ell \neq 0 \neq p \neq \ell} \widehat{w}(\ell) \mathcal{N}_+^{\frac{s}{2}+j} a_{p-\ell}^* a_0^* a_p a_{-\ell} \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\rangle \right| \\ &\leq \sum_{\ell \neq 0 \neq p \neq \ell} |\widehat{w}(\ell)| \left\| (\mathcal{N}_+ + 1)^{-j} a_0 a_{p-\ell} \mathcal{N}_+^{\frac{s}{2}+j} \Psi_N \right\| \left\| (\mathcal{N}_+ + 1)^j a_p a_{-\ell} \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\| \\ &\leq \left(\sum_{\ell \neq 0 \neq p \neq \ell} |\widehat{w}(\ell)|^2 \left\| a_0 a_{p-\ell} \mathcal{N}_+^{\frac{s}{2}} \Psi_N \right\|^2 \right)^{1/2} \\ &\quad \left(\sum_{\ell \neq 0 \neq p \neq \ell} \left\| a_p a_{-\ell} (\mathcal{N}_+ - 1)^{-j} \mathcal{N}_+^{\frac{s}{2}-j-1} \Psi_N \right\|^2 \right)^{1/2} \\ &\leq CN^{1/2} \langle \Psi_N, \mathcal{N}_+^s \Psi_N \rangle^{1/2} \langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \rangle^{1/2} \end{aligned}$$

and proceed similarly for other terms. Thus in summary, from (32) we get

$$\begin{aligned} \left| \left\langle \Psi_N, \mathcal{N}_+^{\frac{s}{2}} [H_N, \mathcal{N}_+^{\frac{s}{2}}] \Psi_N \right\rangle \right| &\leq C \langle \Psi_N, (\mathcal{N}_+ + 1)^s \Psi_N \rangle \\ &\quad + CN^{-1/2} \langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \rangle^{1/2} \langle \Psi_N, \mathcal{N}_+^s \Psi_N \rangle^{1/2}. \end{aligned} \quad (33)$$

Inserting (31) and (33) into (30), we obtain

$$\begin{aligned} \left\langle \Psi_N, \left((2\pi)^2 \mathcal{N}_+^{s+1} - C \mathcal{N}_+^s \right) \Psi_N \right\rangle &\leq C \langle \Psi_N, (\mathcal{N}_+ + 1)^s \Psi_N \rangle \\ &\quad + CN^{-1/2} \langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \rangle^{1/2} \langle \Psi_N, \mathcal{N}_+^s \Psi_N \rangle^{1/2}. \end{aligned}$$

By the Cauchy–Schwarz inequality

$$\langle \Psi_N, \mathcal{N}_+^s \Psi_N \rangle \leq \left\langle \Psi_N, \mathcal{N}_+^{s-1} \Psi_N \right\rangle^{1/2} \left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle^{1/2} \quad (34)$$

we get

$$\left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle \leq C \left\langle \Psi_N, \mathcal{N}_+^{s-1} \Psi_N \right\rangle^{1/2} \left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle^{1/2}$$

$$+ CN^{-1/2} \langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \rangle^{3/4} \langle \Psi_N, \mathcal{N}_+^{s-1} \Psi_N \rangle^{1/4}$$

which implies

$$\begin{aligned} \left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle^{1/2} &\leq C \left\langle \Psi_N, \mathcal{N}_+^{s-1} \Psi_N \right\rangle^{1/2} \\ &\quad + CN^{-1/2} \langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \rangle^{1/4} \langle \Psi_N, \mathcal{N}_+^{s-1} \Psi_N \rangle^{1/4}. \end{aligned}$$

We can now use

$$\left\langle \Psi_N, \mathcal{N}_+^{s-1} \Psi_N \right\rangle \leq \left\langle \Psi_N, \mathcal{N}_+^{s-3} \Psi_N \right\rangle^{1/2} \left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle^{1/2}$$

and obtain

$$\begin{aligned} \left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle^{1/4} &\leq C \left\langle \Psi_N, \mathcal{N}_+^{s-3} \Psi_N \right\rangle^{1/4} \\ &\quad + CN^{-1/2} \langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \rangle^{1/8} \left\langle \Psi_N, \mathcal{N}_+^{s-3} \Psi_N \right\rangle^{1/8}. \end{aligned}$$

Telescoping this inequality and using [13, Lemma 5] we arrive at a bound on $\left\langle \Psi_N, \mathcal{N}_+^{s+1} \Psi_N \right\rangle$ that is uniform in N . This gives the desired result for odd powers of $\mathcal{N}_+$. Finally, using (34), we obtain the bound for any $s \in \mathbb{N}$ and this ends the proof. $\square$

In order to put Lemma 3 in a good use, we will also need the fact that the moments of $\mathcal{N}_+$ are essentially stable under the actions of the Bogoliubov transformation and the cubic transformation.

Lemma 4 *Let U_B be given in (7). Then*

$$U_B \mathcal{N}_+^k U_B^* \leq C_k (\mathcal{N}_+ + 1)^k, \quad \forall k \in \mathbb{N}. \quad (35)$$

Lemma 5 *Let $U_S = e^S$ be given in (25). Then for all $t \in [-1, 1]$ and $k \in \mathbb{N}$,*

$$e^{tS} (\mathcal{N}_+ + 1)^k e^{-tS} \leq C_k (\mathcal{N}_+ + 1)^k. \quad (36)$$

The results in Lemma 4 and Lemma 5 are well-known. For the completeness, let us quickly explain the proof of Lemma 5, following the strategy in [2, Proposition 4.2] (the proof of Lemma 4 is similar and simpler).

Proof of Lemma 5 Take a normalized vector $\Phi \in \mathcal{F}_+$ and define

$$f(t) = \langle \Phi, e^{tS} (\mathcal{N}_+ + 1)^k e^{-tS} \Phi \rangle, \quad \forall t \in [-1, 1].$$

Then

$$\begin{aligned} \partial_t f(t) &= \langle \Phi, e^{tS} [S, (\mathcal{N}_+ + 1)^k] e^{-tS} \Phi \rangle \\ &= \frac{2}{\sqrt{N}} \operatorname{Re} \left\langle \Phi, e^{tS} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} \eta_{p, q} a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}^{\leq N} \Theta_k(\mathcal{N}_+) e^{-tS} \Phi \right\rangle \end{aligned}$$

with

$$\Theta_k(\mathcal{N}_+) = (\mathcal{N}_+ + 1)^k - (\mathcal{N}_+ + 4)^k.$$

Here we have used

$$[a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}^{\leq N}, (\mathcal{N}_+ + 1)^k] = a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}^{\leq N} \Theta_k(\mathcal{N}_+).$$

It is obvious that $|\Theta_k(\mathcal{N}_+)| \leq C_k(\mathcal{N}_+ + 1)^{k-1}$. Combining with the summability (27) and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} |\partial_t f(t)| &\leq \frac{2}{\sqrt{N}} \left(\sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} \|(\mathcal{N}_+ + 1)^{(k-3)/2} a_{p+q} a_{-p} a_{-q} e^{-tS} \Phi\|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} |\eta_{p,q}|^2 \|\mathbb{1}^{\leq N} (\mathcal{N}_+ + 1)^{(3-k)/2} \Theta_k(\mathcal{N}_+) e^{-tS} \Phi\|^2 \right)^{1/2} \\ &\leq \frac{C_k}{\sqrt{N}} \|\mathcal{N}_+^{k/2} e^{-tS} \Phi\| \|\mathbb{1}^{\leq N} (\mathcal{N}_+ + 1)^{(k+1)/2} e^{-tS} \Phi\|. \end{aligned} \quad (37)$$

Thanks to the cut-off, we can bound

$$\mathbb{1}^{\leq N} (\mathcal{N}_+ + 1)^{(k+1)/2} \leq \sqrt{N+1} \mathcal{N}_+^{k/2}.$$

Thus (37) implies that

$$|\partial_t f(t)| \leq C_k \|(\mathcal{N}_+ + 1)^{(k+1)/2} e^{-tS} \Phi\|^2 = C_k f(t)$$

From Grönwall's lemma, it follows that

$$f(t) \leq C_k f(0), \quad \forall t \in [-1, 1]. \quad (38)$$

Since the latter bound is uniform in Φ , we get the desired operator inequality. $\square$

We will also need the following refinement of Lemma 5.

Lemma 6 *Let $U_S = e^S$ be given in (25). Then for all $t \in [-1, 1]$ and $k \in \mathbb{N}$,*

$$e^{tS} \mathcal{N}_+^k e^{-tS} \leq C_k \left(\mathcal{N}_+^k + \frac{(\mathcal{N}_+ + 1)^{k+1}}{N} \right). \quad (39)$$

Proof Take a normalized vector $\Phi \in \mathcal{F}_+$ and define

$$g(t) = \langle \Phi, e^{tS} \mathcal{N}_+^k e^{-tS} \Phi \rangle, \quad \forall t \in [-1, 1].$$

Then proceeding similarly to (37), we have

$$|\partial_t g(t)| \leq \frac{C_k}{\sqrt{N}} \|\mathcal{N}_+^{k/2} e^{-tS} \Phi\| \|\mathbb{1}^{\leq N} \mathcal{N}_+^{(k+1)/2} e^{-tS} \Phi\| \leq \frac{C_k}{\sqrt{N}} \sqrt{g(t)f(t)}$$

with $f(t)$ being defined in the proof of Lemma 5. Using (38) and the Cauchy–Schwarz inequality we obtain

$$|\partial_t g(t)| \leq C_k \left(g(t) + \frac{\langle \Phi, (\mathcal{N}_+ + 1)^{k+1} \Phi \rangle}{N} \right), \quad \forall t \in [-1, 1].$$

From Grönwall's lemma, it follows that

$$g(t) \leq C_k \left(g(0) + \frac{1}{N} \langle \Phi, (\mathcal{N}_+ + 1)^{k+1} \Phi \rangle \right), \quad \forall t \in [-1, 1].$$

The latter bound is uniform in Φ and it implies the desired conclusion. $\square$

3 Excitation Hamiltonian

In this section, we study the action of the transformation U_N in (12). By conjugating H_N with U_N , we can factor out the contribution of the condensation. More precisely, we have

Lemma 7 *We have the operator identity on $\mathcal{F}_+^{\leq N}$*

$$U_N H_N U_N^* = \frac{N}{2} \widehat{w}(0) + \mathbb{1}^{\leq N} (\mathcal{G}_N + R_1) \mathbb{1}^{\leq N}$$

where

$$\begin{aligned} \mathcal{G}_N = & \mathbb{H}_{\text{Bog}} + \frac{\mathcal{N}_+(1 - \mathcal{N}_+)}{2(N-1)} \widehat{w}(0) + \sum_{p \neq 0} \frac{1 - \mathcal{N}_+}{N-1} \widehat{w}(p) a_p^* a_p \\ & + \left(\frac{1}{2} \sum_{p \neq 0} \widehat{w}(p) a_p^* a_{-p}^* \frac{1 - 2\mathcal{N}_+}{2N} + \text{h.c.} \right) \\ & + \left(\frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \widehat{w}(\ell) a_{p+\ell}^* a_{-\ell}^* a_p + \text{h.c.} \right) \\ & + \frac{1}{2(N-1)} \sum_{\substack{k, p \neq 0 \\ \ell \neq -p, k}} \widehat{w}(\ell) a_{p+\ell}^* a_{k-\ell}^* a_p a_k \end{aligned}$$

and the error term R_1 satisfies the quadratic form estimate

$$\pm R_1 \leq \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}}.$$

Moreover, we have the operator inequality on $\mathcal{F}_+$

$$\mathbb{1}^{\leq N} U_N H_N U_N^* \mathbb{1}^{\leq N} \leq \frac{N}{2} \widehat{w}(0) + \mathcal{G}_N + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}}.$$

Proof A straightforward computation using the relations (12) shows that

$$\begin{aligned} U_N H_N U_N^* = & \frac{N}{2} \widehat{w}(0) + \frac{\mathcal{N}_+(1 - \mathcal{N}_+)}{2(N-1)} \widehat{w}(0) + \sum_{p \neq 0} \left(p^2 + \frac{N - \mathcal{N}_+}{N-1} \widehat{w}(p) \right) a_p^* a_p \\ & + \frac{1}{2} \left(\sum_{p \neq 0} \widehat{w}(p) a_p^* a_{-p}^* \frac{\sqrt{(N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1)}}{N-1} + \text{h.c.} \right) \\ & + \left(\sum_{\ell, p \neq 0, \ell+p \neq 0} \widehat{w}(\ell) a_{p+\ell}^* a_{-\ell}^* a_p \frac{\sqrt{N - \mathcal{N}_+}}{N-1} + \text{h.c.} \right) \\ & + \frac{1}{2(N-1)} \sum_{\substack{k, p \neq 0 \\ \ell \neq -p, k}} \widehat{w}(\ell) a_{p+\ell}^* a_{k-\ell}^* a_p a_k. \end{aligned}$$

This operator identity holds on $\mathcal{F}_+^{\leq N}$. For further analysis, we will expand $\sqrt{N - \mathcal{N}_+}$ and $\sqrt{(N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1)}$, making the effective expressions well-defined on the whole

Fock space $\mathcal{F}_+$. This idea has been used before in [8]. Here it suffices to use

$$\left| \frac{\sqrt{N - \mathcal{N}_+}}{N - 1} - \frac{1}{N^{1/2}} \right| \leq \frac{C(\mathcal{N}_+ + 1)}{N^{3/2}} \quad (40)$$

and

$$\left| \frac{\sqrt{(N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1)}}{N - 1} - 1 - \frac{1 - 2\mathcal{N}_+}{2N} \right| \leq \frac{C(\mathcal{N}_+ + 1)^2}{N^2}. \quad (41)$$

The operator inequalities (40) and (41) hold on $\mathcal{F}_+^{\leq N}$. Thus we can write

$$U_N H_N U_N^* = \frac{N}{2} \widehat{w}(0) + \mathbb{H}_{\text{Bog}} + \mathcal{G}_N + R_1$$

with $\mathcal{G}_N$ given in the statement of Lemma 7 and with the error term $R_1 = R_{1a} + R_{1b}$ where

$$R_{1a} = \frac{1}{2} \sum_{p \neq 0} \widehat{w}(p) a_p^* a_{-p}^* \left(\frac{\sqrt{(N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1)}}{N - 1} - 1 - \frac{1 - 2\mathcal{N}_+}{2N} \right) + \text{h.c.},$$

$$R_{1b} = \sum_{\ell, p \neq 0, \ell + p \neq 0} \widehat{w}(\ell) a_{p+\ell}^* a_{-\ell}^* a_p \left(\frac{\sqrt{N - \mathcal{N}_+}}{N - 1} - \frac{1}{\sqrt{N}} \right) + \text{h.c.}$$

By the Cauchy–Schwarz inequality, we have the quadratic form estimates

$$\begin{aligned} \pm R_{1a} &\leq N^{-2} \sum_{p \neq 0} a_p^* a_{-p}^* (\mathcal{N}_+ + 1) a_{-p} a_p + N^2 \sum_{p \neq 0} |\widehat{w}(p)|^2 (\mathcal{N}_+ + 1)^{-1/2} \times \\ &\quad \times \left(\frac{\sqrt{(N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1)}}{N - 1} - 1 - \frac{1 - 2\mathcal{N}_+}{2N} \right)^2 (\mathcal{N}_+ + 1)^{-1/2} \\ &\leq \frac{C(\mathcal{N}_+ + 1)^3}{N^2} \end{aligned}$$

and

$$\begin{aligned} \pm R_{1b} &\leq N^{-3/2} \sum_{\ell, p \neq 0, \ell + p \neq 0} a_{p+\ell}^* a_{-\ell}^* a_{-\ell} a_{p+\ell} \\ &\quad + N^{3/2} \sum_{\ell, p \neq 0, \ell + p \neq 0} |\widehat{w}(\ell)|^2 \left(\frac{\sqrt{N - \mathcal{N}_+}}{N - 1} - \frac{1}{\sqrt{N}} \right) a_p^* a_p \left(\frac{\sqrt{N - \mathcal{N}_+}}{N - 1} - \frac{1}{\sqrt{N}} \right) \\ &\leq \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}}. \end{aligned}$$

This completes the first part of Lemma 7.

Now let us turn to the operator inequality on the Fock space $\mathcal{F}_+$. We have proved that

$$\mathbb{1}^{\leq N} U_N H_N U_N^* \mathbb{1}^{\leq N} \leq \mathbb{1}^{\leq N} \left(\frac{N}{2} \widehat{w}(0) + \mathcal{G}_N + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}} \right) \mathbb{1}^{\leq N}. \quad (42)$$

Let us compare the right side of (42) with the corresponding version without the cut-off $\mathbb{1}^{\leq N}$. First, consider the terms commuting with $\mathcal{N}_+$. Since

$$\frac{N}{2} \widehat{w}(0) + \sum_{p \neq 0} |p|^2 a_p^* a_p + \frac{1}{2(N - 1)} \sum_{\substack{k, p \neq 0 \\ \ell \neq -p, k}} \widehat{w}(\ell) a_{p+\ell}^* a_{k-\ell}^* a_p a_k + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}} \geq 0,$$

this operator is not smaller than its product with the cut-off $\mathbb{1}^{\leq N}$. Moreover, using

$$\mathbb{1}^{>N} = \mathbb{1} - \mathbb{1}^{\leq N} = \mathbb{1}(\mathcal{N}_+ > N) \leq \frac{\mathcal{N}_+}{N}$$

we have

$$\begin{aligned} & \pm \mathbb{1}^{>N} \left(\frac{\mathcal{N}_+(1 - \mathcal{N}_+)}{2(N-1)} \widehat{w}(0) + \sum_{p \neq 0} \frac{1 - \mathcal{N}_+}{N-1} \widehat{w}(p) a_p^* a_p \right) \\ & \leq \mathbb{1}^{>N} \frac{C(\mathcal{N}_+ + 1)^2}{N} \leq \frac{C(\mathcal{N}_+ + 1)^3}{N^2}. \end{aligned}$$

Finally, consider

$$\begin{aligned} X := & \left(\frac{1}{2} \sum_{p \neq 0} \widehat{w}(p) a_p^* a_{-p}^* \left(1 + \frac{1 - \mathcal{N}_+}{N-1} \right) + \text{h.c.} \right) \\ & + \left(\frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell + p \neq 0} \widehat{w}(\ell) a_{p+\ell}^* a_{-\ell}^* a_p + \text{h.c.} \right). \end{aligned}$$

By the Cauchy–Schwarz inequality $\pm(Y^*Z + Z^*Y) \leq Y^*Y + Z^*Z$ we can bound

$$\begin{aligned} \pm X & \leq \sum_{p \neq 0} a_p^* a_{-p}^* (\mathcal{N}_+ + 1)^{-1} a_{-p} a_p + \sum_{p \neq 0} |\widehat{w}(p)|^2 (\mathcal{N}_+ + 1) \left(1 + \frac{1 - \mathcal{N}_+}{N-1} \right)^2 \\ & + \frac{1}{N} \sum_{\ell, p \neq 0, \ell + p \neq 0} a_{p+\ell}^* a_{-\ell}^* a_{-\ell} a_{p+\ell} + \sum_{\ell, p \neq 0, \ell + p \neq 0} |\widehat{w}(\ell)|^2 a_p^* a_p \\ & \leq C \left[(\mathcal{N}_+ + 1) + \frac{(\mathcal{N}_+ + 1)^2}{N} + \frac{(\mathcal{N}_+ + 1)^3}{N^2} \right]. \end{aligned}$$

Moreover, since X changes the number of particles by at most 2, we have

$$X + \mathbb{1}^{>N} X \mathbb{1}^{>N} - \mathbb{1}^{\leq N} X \mathbb{1}^{\leq N} = \mathbb{1}^{>N} X + X \mathbb{1}^{>N} = \mathbb{1}^{>N} X \mathbb{1}^{>N-2} + \mathbb{1}^{>N-2} X \mathbb{1}^{>N}.$$

Hence, combining with the above bound on $\pm X$ we find that

$$\begin{aligned} \pm(X - \mathbb{1}^{\leq N} X \mathbb{1}^{\leq N}) & = \pm \left(\mathbb{1}^{>N} X \mathbb{1}^{>N-2} + \mathbb{1}^{>N-2} X \mathbb{1}^{>N} - \mathbb{1}^{>N} X \mathbb{1}^{>N} \right) \\ & \leq C \left[(\mathcal{N}_+ + 1) + \frac{(\mathcal{N}_+ + 1)^2}{N} + \frac{(\mathcal{N}_+ + 1)^3}{N^2} \right] \mathbb{1}^{>N-2} \\ & \leq \frac{C(\mathcal{N}_+ + 1)^3}{N^2}. \end{aligned}$$

This completes the proof of the operator inequality on $\mathcal{F}_+$ in Lemma 7. $\square$

4 Quadratic transformation

Recall that the Bogoliubov transformation U_B in (7) diagonalizes $\mathbb{H}_{\text{Bog}}$ as in (9). In this section, we will study the action of U_N on the operator $\mathcal{G}_N$. We have

Lemma 8 Let $\mathcal{G}_N$ be given in Lemma 7. Then we have the operator identity on $\mathcal{F}_+$

$$U_B \mathcal{G}_N U_B^* = \langle 0 | U_B \mathcal{G}_N U_B^* | 0 \rangle + \sum_{p \neq 0} e(p) a_p^* a_p + C_N + R_2$$

where

$$C_N = \frac{1}{\sqrt{N}} \sum_{\substack{p, q \neq 0 \\ p+q \neq 0}} \widehat{w}(p) \left[(\sigma_{p+q} \sigma_{-p} \gamma_q + \gamma_{p+q} \gamma_p \sigma_q) a_{p+q}^* a_{-p}^* a_{-q}^* + \text{h.c.} \right]$$

and the error term R_2 satisfies

$$\pm R_2 \leq \frac{C}{\sqrt{N}} \mathcal{N}_+^2 + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}}.$$

Proof Let us decompose

$$\mathcal{G}_N - \mathbb{H}_{\text{Bog}} = \widetilde{\mathcal{D}}_N + \widetilde{C}_N$$

where

$$\widetilde{C}_N = \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \widehat{w}(\ell) a_{p+\ell}^* a_{-\ell}^* a_p + \text{h.c.}$$

Non-cubic terms. Let us prove that $U_B \widetilde{\mathcal{D}}_N U_B^* - \langle 0 | U_B \widetilde{\mathcal{D}}_N U_B^* | 0 \rangle$ contains only the terms of the form

$$\sum_{m_1, \dots, m_s, n_1, \dots, n_t \neq 0} A_{m_1, \dots, m_s, n_1, \dots, n_t} a_{m_1}^* \dots a_{m_s}^* a_{n_1} \dots a_{n_t} \quad (43)$$

with $1 \leq s + t \leq 4$ and the coefficients $A_{m_1, \dots, m_s, n_1, \dots, n_t}$ satisfy

$$\sup_{m_1, \dots, m_s \neq 0} \sum_{n_1, \dots, n_t \neq 0} |A_{m_1, \dots, m_s, n_1, \dots, n_t}| \leq \frac{C}{N}, \quad \sup_{n_1, \dots, n_t \neq 0} \sum_{m_1, \dots, m_s \neq 0} |A_{m_1, \dots, m_s, n_1, \dots, n_t}| \leq \frac{C}{N}. \quad (44)$$

Let us start with the quadratic terms involving $a_p^* a_{-p}^*$. Using (8) and the CCR (3) we have

$$\begin{aligned} U_B \left(\frac{1}{4N} \sum_{p \neq 0} \widehat{w}(p) a_p^* a_{-p}^* \right) U_B^* &= \frac{1}{4N} \sum_{p \neq 0} \widehat{w}(p) (\sigma_p a_p^* + \gamma_p a_{-p}) (\sigma_p a_{-p}^* + \gamma_p a_p) \\ &= \frac{1}{4N} \sum_{p \neq 0} \widehat{w}(p) \left[\sigma_p^2 a_p^* a_{-p}^* + 2\sigma_p \gamma_p a_p^* a_p + \gamma_p^2 a_{-p} a_p + \sigma_p \gamma_p \right]. \end{aligned} \quad (45)$$

Obviously the constant in (45) satisfies

$$\frac{1}{4N} \sum_{p \neq 0} \widehat{w}(p) \sigma_p \gamma_p = \left\langle 0 \left| U_B \left(\frac{1}{4N} \sum_{p \neq 0} \widehat{w}(p) a_p^* a_{-p}^* \right) U_B^* \right| 0 \right\rangle.$$

Moreover, the other terms in (45) can be rewritten as

$$\frac{1}{4N} \sum_{p \neq 0} \widehat{w}(p) \sigma_p^2 a_p^* a_{-p}^* = \frac{1}{4N} \sum_{p, q} \widehat{w}(p) \sigma_p^2 \delta_{p=-q} a_p^* a_q^*, \quad (46)$$

$$\frac{1}{2N} \sum_{p \neq 0} \widehat{w}(p) \sigma_p \gamma_p a_p^* a_p = \frac{1}{2N} \sum_{p, q} \widehat{w}(p) \sigma_p \gamma_p \delta_{q=p} a_p^* a_q, \quad (47)$$

$$\frac{1}{4N} \sum_{p \neq 0} \widehat{w}(p) \gamma_p^2 a_{-p} a_p = \frac{1}{4N} \sum_{p,q} \widehat{w}(p) \gamma_p^2 \delta_{p=-q} a_p a_q. \quad (48)$$

All of the sums in (46), (47), (48) are of the general form (43)–(44), thanks to the uniform bounds (10). The quadratic terms involving $a_p^* a_p$ can be treated similarly.

Next, consider

$$\begin{aligned} U_B \left(\frac{1}{2N} \sum_{p \neq 0} \widehat{w}(p) a_p^* a_{-p} \mathcal{N}_+ \right) U_B^* &= U_B \left(\frac{1}{2N} \sum_{p,q \neq 0} \widehat{w}(p) a_p^* a_{-p} a_q^* a_q \right) U_B^* \\ &= \frac{1}{2N} \sum_{p,q \neq 0} \widehat{w}(p) (\sigma_p a_p^* + \gamma_p a_{-p}) (\sigma_p a_{-p}^* + \gamma_p a_p) (\sigma_q a_q^* + \gamma_q a_{-q}) (\sigma_q a_q + \gamma_q a_{-q}^*) \\ &= \frac{1}{2N} \sum_{p,q \neq 0} \widehat{w}(p) \left[\sigma_p^2 a_p^* a_{-p}^* + 2\sigma_p \gamma_p a_p^* a_p + \gamma_p^2 a_{-p} a_p + \sigma_p \gamma_p \right] \times \\ &\quad \times \left[(\sigma_q^2 + \gamma_q^2) a_q^* a_q + \sigma_q \gamma_q (a_q^* a_{-q}^* + a_{-q} a_q) + \gamma_q^2 \right] \\ &= \frac{1}{2N} \sum_{p,q \neq 0} \widehat{w}(p) \left[\sigma_p^2 a_p^* a_{-p}^* + \sigma_p \gamma_p \right] \left[(\sigma_q^2 + \gamma_q^2) a_q^* a_q + \sigma_q \gamma_q (a_q^* a_{-q}^* + a_{-q} a_q) + \gamma_q^2 \right] \\ &\quad + \frac{1}{N} \sum_{p,q} \widehat{w}(p) \sigma_p \gamma_p a_p^* \left[(\sigma_q^2 + \gamma_q^2) a_q^* a_q + \sigma_q \gamma_q (a_q^* a_{-q}^* + a_{-q} a_q) + \gamma_q^2 \right] a_p \\ &\quad + \frac{1}{N} \sum_{p,q \neq 0} \widehat{w}(p) \sigma_p \gamma_p \left[(\sigma_q^2 + \gamma_q^2) a_p^* a_p \delta_{p,q} + \sigma_q \gamma_q a_p^* a_{-p}^* (\delta_{p,q} + \delta_{p,-q}) \right] \\ &\quad + \frac{1}{2N} \sum_{p,q} \widehat{w}(p) \gamma_p^2 \left[(\sigma_q^2 + \gamma_q^2) a_q^* a_q + \sigma_q \gamma_q (a_q^* a_{-q}^* + a_{-q} a_q) + \gamma_q^2 \right] a_p a_{-p} \\ &\quad + \frac{1}{2N} \sum_{p,q \neq 0} \widehat{w}(p) \gamma_p^2 \left[(\sigma_q^2 + \gamma_q^2) a_p a_{-p} (\delta_{p,q} + \delta_{p,-q}) \right. \\ &\quad \left. + \sigma_q \gamma_q (a_p^* a_p + a_{-p}^* a_{-p} + 1) (\delta_{p,q} + \delta_{p,-q}) \right]. \end{aligned} \quad (49)$$

It is straightforward to see that, except the constant

$$\begin{aligned} &\frac{1}{2N} \sum_{p,q \neq 0} \widehat{w}(p) \sigma_p \gamma_p \gamma_q^2 + \frac{1}{2N} \sum_{p,q \neq 0} \widehat{w}(p) \gamma_p^2 \sigma_q \gamma_q (\delta_{p,q} + \delta_{p,-q}) \\ &= \left\langle 0 \left| U_B \left(\frac{1}{2N} \sum_p \widehat{w}(p) a_p^* a_{-p} \mathcal{N}_+ \right) U_B^* \right| 0 \right\rangle, \end{aligned}$$

all other terms in (49) can be written as in (43), with the corresponding bound (44) following from (10). By the same argument, we can show that the terms involving $a_p^* a_p \mathcal{N}_+$, $a_{p+\ell}^* a_{q-\ell}^* a_p a_q$ and $\mathcal{N}_+ (\mathcal{N}_+ - 1)$ are of the general form (43) and (44).

Next, let us bound the terms of the general form (43) and (44). We consider the case $s \geq t$ (the other case is treated similarly). By the Cauchy–Schwarz inequality

$$Y^* Z + Z^* Y \leq Y^* Y + Z^* Z$$

we have

$$\pm \left(\sum_{m_1, \dots, m_s, n_1, \dots, n_t} A_{m_1, \dots, m_s, n_1, \dots, n_t} a_{m_1}^* \dots a_{m_s}^* a_{n_1} \dots a_{n_t} + \text{h.c.} \right)$$

$$\begin{aligned}
&\leq \varepsilon^{-1} \sum_{m_1, \dots, m_s, n_1, \dots, n_t} |A_{m_1, \dots, m_s, n_1, \dots, n_t}| a_{m_1}^* \dots a_{m_s}^* (\mathcal{N}_+ + 5)^{1-s} a_{m_s} \dots a_{m_1} \\
&\quad + \varepsilon \sum_{m_1, \dots, m_s, n_1, \dots, n_t} |A_{m_1, \dots, m_s, n_1, \dots, n_t}| a_{n_t}^* \dots a_{n_1}^* (\mathcal{N}_+ + 5)^{s-1} a_{n_1} \dots a_{n_t} \\
&\leq \varepsilon^{-1} \left(\sup_{m_1, \dots, m_s} \sum_{n_1, \dots, n_t} |A_{m_1, \dots, m_s, n_1, \dots, n_t}| \right) \sum_{m_1, \dots, m_s} a_{m_1}^* \dots a_{m_s}^* (\mathcal{N}_+ + 5)^{1-s} a_{m_s} \dots a_{m_1} \\
&\quad + \varepsilon \left(\sup_{n_1, \dots, n_t} \sum_{m_1, \dots, m_s} |A_{m_1, \dots, m_s, n_1, \dots, n_t}| \right) \sum_{n_1, \dots, n_t} a_{n_t}^* \dots a_{n_1}^* (\mathcal{N}_+ + 5)^{s-1} a_{n_1} \dots a_{n_t} \\
&\leq \varepsilon^{-1} \frac{C}{N} \mathcal{N}_+ + \varepsilon \frac{C}{N} (\mathcal{N}_+ + 1)^{t+s-1} \tag{50}
\end{aligned}$$

for all $\varepsilon > 0$. Note that if $\min(t, s) \geq 1$, then on the right side of (50) we can replace $(\mathcal{N}_+ + 1)^{t+s-1}$ by $\mathcal{N}_+^{t+s-1}$.

In particular, for the non-cubic term $\tilde{\mathcal{D}}_N$, using (50) with $\varepsilon = N^{-1/2}$ and $t + s \leq 4$ we get

$$\pm \left(U_B \tilde{\mathcal{D}}_N U_B^* - \langle 0 | U_B \tilde{\mathcal{D}}_N U_B^* | 0 \rangle \right) \leq \frac{C}{\sqrt{N}} \mathcal{N}_+ + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}}. \tag{51}$$

Cubic terms. By using (8) we have

$$\begin{aligned}
&U_B \left(\frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \hat{w}(\ell) a_{p+\ell}^* a_{-\ell}^* a_p \right) U_B^* \\
&= \frac{1}{\sqrt{N}} \sum_{\ell, p \neq 0, \ell+p \neq 0} \hat{w}(\ell) (\sigma_{p+\ell} a_{p+\ell}^* + \gamma_{p+\ell} a_{-p-\ell}) (\sigma_\ell a_{-\ell}^* + \gamma_\ell a_\ell) (\sigma_p a_p + \gamma_p a_{-p}^*) \\
&= \frac{1}{\sqrt{N}} \sum_{\substack{\ell, p \neq 0 \\ \ell+p \neq 0}} \hat{w}(\ell) \left(\sigma_{p+\ell} \sigma_\ell \gamma_p a_{p+\ell}^* a_{-\ell}^* a_{-p}^* + \gamma_{p+\ell} \gamma_\ell \sigma_p a_{-p-\ell} a_\ell a_p \right) \\
&\quad + \frac{1}{\sqrt{N}} \sum_{\substack{\ell, p \neq 0 \\ \ell+p \neq 0}} \left(\hat{w}(\ell) \sigma_{p+\ell} \sigma_\ell \sigma_p + \hat{w}(p+\ell) \sigma_{p+\ell} \gamma_\ell \gamma_{-p} + \hat{w}(p) \sigma_{p+\ell} \gamma_p \gamma_\ell \right) a_{p+\ell}^* a_{-\ell}^* a_p \\
&\quad + \frac{1}{\sqrt{N}} \sum_{\substack{\ell, p \neq 0 \\ \ell+p \neq 0}} \left(\hat{w}(\ell) \sigma_{p+\ell} \gamma_\ell \sigma_p + \hat{w}(p+\ell) \sigma_{p+\ell} \sigma_\ell \gamma_p + \hat{w}(p) \gamma_{-p-\ell} \gamma_p \gamma_\ell \right) a_{p+\ell}^* a_p a_\ell. \tag{52}
\end{aligned}$$

By using (10), we can write the last sum of (52) as

$$\sum_{p, q, r} \tilde{A}_{p, q, r} a_p^* a_q a_r$$

with

$$\sup_p \sum_{q, r} |\tilde{A}_{p, q, r}| \leq \frac{C}{\sqrt{N}}, \quad \sup_{q, r} \sum_p |\tilde{A}_{p, q, r}| \leq \frac{C}{\sqrt{N}}.$$

Using (50) with $\varepsilon = 1$, $t = 1$, $s = 2$, we get

$$\pm \left(\sum_{p, q, r} \tilde{A}_{p, q, r} a_p^* a_q a_r + \text{h.c.} \right) \leq \frac{C}{N} (\mathcal{N}_+ + \mathcal{N}_+^2) \leq \frac{C \mathcal{N}_+^2}{N}. \tag{53}$$

Here $\mathcal{N}_+ \leq \mathcal{N}_+^2$ since the spectrum of $\mathcal{N}_+$ is $\{0, 1, 2, \dots\}$. The second sum on the right side of (52) can be treated by the same way. Thus from (52) and its adjoint, we have

$$\begin{aligned} & \pm \left(U_B \tilde{\mathcal{C}}_N U_B^* - \frac{1}{\sqrt{N}} \sum_{\substack{\ell, p \neq 0 \\ \ell+p \neq 0}} \widehat{w}(\ell) \left[\left(\sigma_{p+\ell} \sigma_\ell \gamma_p a_{p+\ell}^* a_{-\ell}^* a_{-p}^* \right. \right. \right. \\ & \quad \left. \left. \left. + \gamma_{p+\ell} \gamma_\ell \sigma_p a_{-p-\ell} a_\ell a_p \right) + \text{h.c.} \right] \right) \\ & \leq \frac{C \mathcal{N}_+^2}{N} \end{aligned}$$

which is equivalent to

$$\pm \left(U_B \tilde{\mathcal{C}}_N U_B^* - \mathcal{C}_N \right) \leq \frac{C \mathcal{N}_+^2}{N}. \quad (54)$$

In particular, (54) implies that

$$\langle 0 | U_B \tilde{\mathcal{C}}_N U_B^* | 0 \rangle = 0.$$

Therefore, from (9), (51) and (54) we obtain the desired conclusion of Lemma 8. $\square$

5 Cubic transformation

To factor out the cubic term $\mathcal{C}_N$ in Lemma 8, we will use a cubic renormalization. We will prove

Lemma 9 *Let $\mathcal{C}_N$ be the cubic term in Lemma 8 and let U_S be given in (25). Then we have the operator identity on Fock space $\mathcal{F}_+$*

$$U_S U_B \mathcal{G}_N U_B^* U_S^* = \left\langle 0 \left| U_S U_B \mathcal{G}_N U_B^* U_S^* \right| 0 \right\rangle + \sum_{p \neq 0} e(p) a_p^* a_p + R_3$$

where

$$\pm R_3 \leq C \frac{\mathcal{N}_+^2}{\sqrt{N}} + \frac{C(\mathcal{N}_+ + 1)^4}{N^{3/2}}.$$

Proof Recall that from Lemma 8 we have

$$U_S U_B \mathcal{G}_N U_B^* U_S^* = \left\langle 0 \left| U_B \mathcal{G}_N U_B^* \right| 0 \right\rangle + U_S \left(d\Gamma(\xi) + \mathcal{C}_N \right) U_S^* + U_S R_2 U_S^* \quad (55)$$

with

$$d\Gamma(\xi) = \sum_{p \neq 0} e(p) a_p^* a_p, \quad \pm R_2 \leq \frac{C}{\sqrt{N}} \mathcal{N}_+^2 + C \frac{(\mathcal{N}_+ + 1)^3}{N^{3/2}}.$$

Thanks to Lemmas 5 and 6, we find that

$$\pm U_S R_2 U_S^* \leq \frac{C}{\sqrt{N}} \mathcal{N}_+^2 + C \frac{(\mathcal{N}_+ + 1)^3}{N^{3/2}}.$$

Thus this error term is part of R_3 .

For the main term, we use $U_S = e^S$ and the Duhamel formula

$$e^X Y e^{-X} = Y + \int_0^1 e^{tX} [X, Y] e^{-tX} dt \quad (56)$$

we can write

$$\begin{aligned} e^S (d\Gamma(\xi) + C_N) e^{-S} &= d\Gamma(\xi) + C_N + \int_0^1 e^{tS} ([S, d\Gamma(\xi)] + [S, C_N]) e^{-tS} dt \\ &= d\Gamma(\xi) + \int_0^1 e^{tS} (C_N + [S, d\Gamma(\xi)] + [S, C_N]) e^{-tS} dt - \int_0^1 \int_0^t e^{sS} [S, C_N] e^{-sS} ds dt. \end{aligned} \quad (57)$$

Controlling $C_N + [S, d\Gamma(\xi)]$. Since $d\Gamma(\xi)$ commutes with $\mathcal{N}_+$ and

$$[a_k^* a_k, a_{p+q}^* a_{-p}^* a_{-q}^*] = (\delta_{k,p+q} + \delta_{k,-p} + \delta_{k,-q}) a_{p+q}^* a_{-p}^* a_{-q}^*$$

we find that

$$\begin{aligned} [d\Gamma(\xi), S] &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} \sum_{k \neq 0} e(k) \eta_{p,q} [a_k^* a_k, a_{p+q}^* a_{-p}^* a_{-q}^*] \mathbb{1}^{\leq N} + \text{h.c.} \\ &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} (e(p+q) + e(p) + e(q)) \eta_{p,q} a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}^{\leq N} + \text{h.c.} \\ &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} \widehat{w}(p) (\sigma_{p+q} \sigma_p \gamma_q + \gamma_{p+q} \gamma_p \sigma_q) a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}^{\leq N} + \text{h.c.} \end{aligned}$$

which is equivalent to

$$C_N + [S, d\Gamma(\xi)] = \frac{1}{\sqrt{N}} \sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} \widehat{w}(p) (\sigma_{p+q} \sigma_p \gamma_q + \gamma_{p+q} \gamma_p \sigma_q) a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}^{>N} + \text{h.c.}$$

where $\mathbb{1}^{>N} = \mathbb{1} - \mathbb{1}^{\leq N} = \mathbb{1}(\mathcal{N}_+ > N)$. Thanks to the summability (10), we can use the Cauchy–Schwarz inequality similarly to (50) (with $\varepsilon = 1$) to get

$$\pm (C_N + [S, d\Gamma(\xi)]) \leq \frac{C}{\sqrt{N}} \mathcal{N}_+ + \frac{C(\mathcal{N}_+ + 1)^2}{\sqrt{N}} \mathbb{1}^{>N} \leq \frac{C}{\sqrt{N}} \mathcal{N}_+ + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}}$$

Combining with Lemmas 5 and 6 we obtain

$$e^{tS} (C_N + [S, d\Gamma(\xi)]) e^{-tS} \leq \frac{C}{\sqrt{N}} \mathcal{N}_+ + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}}, \quad \forall t \in [-1, 1]. \quad (58)$$

Controlling $[S, C_N]$. Let us decompose $S = \widetilde{S} - S^>$ where

$$\begin{aligned} \widetilde{S} &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} \eta_{p,q} a_{p+q}^* a_{-p}^* a_{-q}^* - \text{h.c.}, \\ S^> &= \frac{1}{\sqrt{N}} \sum_{\substack{p,q \neq 0 \\ p+q \neq 0}} \eta_{p,q} a_{p+q}^* a_{-p}^* a_{-q}^* \mathbb{1}^{>N} - \text{h.c.} \end{aligned}$$

The main contribution comes from

$$\begin{aligned}
 [C_N, \tilde{S}] &= \frac{1}{N} \sum_{p,q,r \neq 0} \sum_{p',q',r' \neq 0} \delta_{p+q+r=0} \delta_{p'+q'+r'=0} \eta_{p',q'} \widehat{w}(p) (\sigma_r \sigma_p \gamma_q + \gamma_r \gamma_p \sigma_q) \times \\
 &\quad \times \left[a_r a_p a_q, a_{r'}^* a_{p'}^* a_{q'}^* \right] + \text{h.c.} \\
 &= \frac{1}{N} \sum_{p,q,r \neq 0} \sum_{p',q',r' \neq 0} \delta_{p+q+r=0} \delta_{p'+q'+r'=0} \eta_{p',q'} \widehat{w}(p) (\sigma_r \sigma_p \gamma_q + \gamma_r \gamma_p \sigma_q) \times \\
 &\quad \times \left(\delta_{r=r'} a_p a_q a_{p'}^* a_{q'}^* + \delta_{r=p'} a_p a_q a_{r'}^* a_{q'}^* + \delta_{r=q'} a_p a_q a_{r'}^* a_{p'}^* \right. \\
 &\quad + \delta_{p=r'} a_q a_{p'}^* a_{q'}^* a_r + \delta_{p=p'} a_q a_{r'}^* a_{q'}^* a_r + \delta_{p=q'} a_q a_{r'}^* a_{p'}^* a_r \\
 &\quad \left. + \delta_{q=r'} a_p^* a_{q'}^* a_r a_p + \delta_{q=p'} a_r^* a_{q'}^* a_r a_p + \delta_{q=q'} a_r^* a_{p'}^* a_r a_p \right) + \text{h.c.}
 \end{aligned}$$

By using the CCR (3) as in (49), we can write

$$[\tilde{S}, C_N] = \langle 0 | [\tilde{S}, C_N] | 0 \rangle + \sum_{p,q \neq 0} A_{pq} a_p^* a_q + \sum_{p,q,r,k \neq 0} B_{pqrk} a_p^* a_q^* a_r a_k$$

where

$$\begin{aligned}
 \sup_{q \neq 0} \sum_{p \neq 0} |A_{pq}| &\leq \frac{C}{N}, \quad \sup_{p \neq 0} \sum_{q \neq 0} |A_{pq}| \leq \frac{C}{N}, \\
 \sup_{p,q \neq 0} \sum_{r,k \neq 0} |B_{pqrk}| &\leq \frac{C}{N}, \quad \sup_{r,k \neq 0} \sum_{p,q \neq 0} |B_{pqrk}| \leq \frac{C}{N}.
 \end{aligned}$$

By the Cauchy–Schwarz inequality as in (50), we get

$$\pm \left([\tilde{S}, C_N] - \langle 0 | [\tilde{S}, C_N] | 0 \rangle \right) \leq C \frac{\mathcal{N}_+}{\sqrt{N}} + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}}. \quad (59)$$

It remains to bound $[S^>, C_N]$. From the explicit form of $S^>$ and C_N , it is straightforward to check that

$$\pm [S^>, C_N] = \pm \left((S^>) C_N + C_N (S^>)^* \right) \leq (S^>) (S^>)^* + C_N^2 \leq \frac{C}{N} (\mathcal{N}_+ + 1)^3.$$

On the other hand, we observe that

$$S^> = \mathbb{1}^{>N-4} S^> \mathbb{1}^{>N-4}$$

and that C_N does not change the number of particles more than 3. Therefore,

$$\pm [S^>, C_N] = \pm \mathbb{1}^{>N-7} [S^>, C_N] \mathbb{1}^{>N-7} \leq \frac{C}{N} (\mathcal{N}_+ + 1)^3 \mathbb{1}^{>N-7} \leq \frac{C}{N^2} (\mathcal{N}_+ + 1)^4. \quad (60)$$

Moreover, it is obvious that

$$\langle 0 | [S^>, C_N] | 0 \rangle = 0$$

for $N \geq 10$. Thus from (59) and (60) we obtain

$$\pm \left([S, C_N] - \langle 0 | [S, C_N] | 0 \rangle \right) \leq C \frac{\mathcal{N}_+}{\sqrt{N}} + \frac{C(\mathcal{N}_+ + 1)^4}{N^{3/2}}.$$

Combining with Lemma 5 we conclude that

$$\pm e^{tS}([S, C_N] - \langle 0|[S, C_N]|0 \rangle) e^{-tS} \leq C \frac{\mathcal{N}_+}{\sqrt{N}} + \frac{C(\mathcal{N}_+ + 1)^4}{N^{3/2}}, \quad \forall t \in [-1, 1]. \quad (61)$$

Conclusion. Inserting (58) and (61) in (57) we find that

$$\pm \left(e^S (d\Gamma(\xi) + C_N) e^{-S} - d\Gamma(\xi) - \frac{1}{2} \langle 0|[S, C_N]|0 \rangle \right) \leq C \frac{\mathcal{N}_+}{\sqrt{N}} + \frac{C(\mathcal{N}_+ + 1)^4}{N^{3/2}}.$$

Combining with (55) we deduce that

$$\pm \left(U_S U_B \mathcal{G}_N U_B^* U_S^* - d\Gamma(\xi) - \langle 0|U_B \mathcal{G}_N U_B^*|0 \rangle - \frac{1}{2} \langle 0|[S, C_N]|0 \rangle \right) \leq C \frac{\mathcal{N}_+^2}{\sqrt{N}} + \frac{C(\mathcal{N}_+ + 1)^4}{N^{3/2}}.$$

Taking the expectation of the latter bound again the vacuum, we find that

$$\pm \left(\langle 0|U_S U_B \mathcal{G}_N U_B^* U_S^*|0 \rangle - \langle 0|U_B \mathcal{G}_N U_B^*|0 \rangle - \frac{1}{2} \langle 0|[S, C_N]|0 \rangle \right) \leq \frac{C}{N^{3/2}}.$$

Thus we obtain the desired conclusion

$$\pm \left(U_S U_B \mathcal{G}_N U_B^* U_S^* - d\Gamma(\xi) - \langle 0|U_S U_B \mathcal{G}_N U_B^* U_S^*|0 \rangle \right) \leq C \frac{\mathcal{N}_+^2}{\sqrt{N}} + \frac{C(\mathcal{N}_+ + 1)^4}{N^{3/2}}.$$

This completes the proof of Lemma 9. $\square$

6 Proof of Theorem 2

Proof We will prove the ground state energy estimate

$$E_N = \frac{N}{2} \widehat{w}(0) + \langle 0|U_S U_B \mathcal{G}_N U_B^* U_S^*|0 \rangle + O(N^{-3/2}).$$

Upper bound. We use the following N -body trial state

$$\widetilde{\Psi}_N = \frac{1}{\|U_N^* \mathbb{1}^{\leq N} U_B^* U_S^*|0 \rangle\|} U_N^* \mathbb{1}^{\leq N} U_B^* U_S^*|0 \rangle.$$

Then by the variational principle and the operator inequality on $\mathcal{F}_+$ in Lemma 7 we have

$$\begin{aligned} E_N &\leq \langle \widetilde{\Psi}_N, H_N \widetilde{\Psi}_N \rangle = \frac{1}{\|U_N^* \mathbb{1}^{\leq N} U_B^* U_S^*|0 \rangle\|^2} \langle 0|U_S U_B \mathbb{1}^{\leq N} U_N H_N U_N^* \mathbb{1}^{\leq N} U_B^* U_S^*|0 \rangle \\ &\leq \frac{1}{\|U_N^* \mathbb{1}^{\leq N} U_B^* U_S^*|0 \rangle\|^2} \langle 0|U_S U_B \left(\frac{N \widehat{w}(0)}{2} + \mathcal{G}_N + \frac{C(\mathcal{N}_+ + 1)^3}{N^{3/2}} \right) U_B^* U_S^*|0 \rangle. \end{aligned}$$

By Lemmas 4 and 5 we know that

$$\langle 0|U_S U_B (\mathcal{N}_+ + 1)^3 U_B^* U_S^*|0 \rangle \leq C.$$

Consequently,

$$\begin{aligned} \|U_N^* \mathbb{1}^{\leq N} U_B^* U_S^*|0 \rangle\|^2 &= 1 - \langle 0|U_S U_B \mathbb{1}^{>N} U_B^* U_S^*|0 \rangle \\ &\geq 1 - \langle 0|U_S U_B (\mathcal{N}_+^3/N^3) U_B^* U_S^*|0 \rangle \geq 1 - C N^{-3}. \end{aligned}$$

Combining with Lemma 9 we find that

$$\begin{aligned} E_N &\leq \frac{1}{\|U_N^* \mathbb{1}^{\leq N} U_B^* U_S^* |0\rangle\|^2} \left(\frac{N\widehat{w}(0)}{2} + \langle 0 | U_S U_B \mathcal{G}_N U_B^* U_S^* | 0 \rangle + \frac{C}{N^{3/2}} \right) \\ &\leq \frac{N\widehat{w}(0)}{2} + \langle 0 | U_S U_B \mathcal{G}_N U_B^* U_S^* | 0 \rangle + \frac{C}{N^{3/2}}. \end{aligned}$$

In the last estimate, we have also used the simple upper bound

$$\langle 0 | U_S U_B \mathcal{G}_N U_B^* U_S^* | 0 \rangle \leq C \quad (62)$$

which will be justified below.

Lower bound. Let Ψ_N be the ground state of H_N and denote $\Phi := U_S U_B U_N \Psi_N \in \mathcal{F}_+$. By Lemmas 3, 35 and 36, we have

$$\langle \Phi, (\mathcal{N}_+ + 1)^4 \Phi \rangle \leq C.$$

Then from the operator identity on $\mathcal{F}_+^{\leq N}$ in Lemma 7 it follows that

$$\begin{aligned} E_N &= \langle \Psi_N, H_N \Psi_N \rangle = \langle U_N \Psi_N, U_N H_N U_N^* U_N \Psi_N \rangle \\ &\geq \langle U_N \Psi_N, \left(\frac{N\widehat{w}(0)}{2} + \mathcal{G}_N - C \frac{(\mathcal{N}_+ + 1)^3}{N^{3/2}} \right) U_N \Psi_N \rangle \\ &\geq \frac{N\widehat{w}(0)}{2} + \langle \Phi, U_S U_B \mathcal{G}_N U_B^* U_S^* \Phi \rangle - C N^{-3/2}. \end{aligned}$$

Next, using Lemma 9 together with two simple estimates:

$$\sum_{p \neq 0} e(p) a_p^* a_p \geq \left(\inf_{q \neq 0} e(q) \right) \sum_{p \neq 0} a_p^* a_p \geq (2\pi)^2 \mathcal{N}_+$$

and

$$\frac{\mathcal{N}_+^2}{\sqrt{N}} \leq \varepsilon \mathcal{N}_+ + \frac{\mathcal{N}_+^2}{\sqrt{N}} \mathbb{1}(\mathcal{N}_+ > \varepsilon \sqrt{N}) \leq \varepsilon \mathcal{N}_+ + C_\varepsilon \frac{\mathcal{N}_+^4}{N^{3/2}}$$

for $\varepsilon > 0$ small (but independent of N), we obtain

$$U_S U_B \mathcal{G}_N U_B^* U_S^* \geq \langle 0 | U_S U_B \mathcal{G}_N U_B^* U_S^* | 0 \rangle + \mathcal{N}_+ - C \frac{(\mathcal{N}_+ + 1)^4}{N^{3/2}}.$$

Therefore,

$$\langle \Phi, U_S U_B \mathcal{G}_N U_B^* U_S^* \Phi \rangle \geq \langle 0 | U_S U_B \mathcal{G}_N U_B^* U_S^* | 0 \rangle + \langle \Phi, \mathcal{N}_+ \Phi \rangle - C N^{-3/2}.$$

Thus

$$E_N \geq \frac{N\widehat{w}(0)}{2} + \langle 0 | U_S U_B \mathcal{G}_N U_B^* U_S^* | 0 \rangle + \langle \Phi, \mathcal{N}_+ \Phi \rangle - C N^{-3/2}. \quad (63)$$

From (63), since $\langle \Phi, \mathcal{N}_+ \Phi \rangle \geq 0$ we obtain the desired energy lower bound

$$E_N \geq \frac{N\widehat{w}(0)}{2} + \langle 0 | U_S U_B \mathcal{G}_N U_B^* U_S^* | 0 \rangle + O(N^{-3/2}).$$

This and the obvious upper bound $E_N \leq \widehat{w}(0)(N/2)$ imply the simple estimate (62). Thus the matching energy upper bound is valid, and hence we conclude that

$$E_N = \frac{N\widehat{w}(0)}{2} + \langle 0 | U_S U_B \mathcal{G}_N U_B^* U_S^* | 0 \rangle + O(N^{-3/2}). \quad (64)$$

Ground state estimates. By comparing the ground state energy expansion (64) with the lower bound (63) we deduce that

$$\langle \Phi, \mathcal{N}_+ \Phi \rangle \leq C N^{-3/2}. \quad (65)$$

Let us write $\Phi = (\Phi_j)_{j=0}^\infty$ with $\Phi_j \in \mathcal{H}_+^j$. We can choose a phase factor for Ψ_N such that $\Phi_0 \geq 0$. Then

$$\|\Phi - |0\rangle\|^2 = |\Phi_0 - 1|^2 \leq 1 - |\Phi_0|^2 = \sum_{j \geq 1} |\Phi_j|^2 \leq \langle \Phi, \mathcal{N}_+ \Phi \rangle \leq C N^{-3/2}.$$

Putting back the definition $\Phi = U_S U_B U_N \Psi_N$ we obtain the norm approximation

$$\|U_N \Psi_N - U_B^* U_S^* |0\rangle\| = \|\Phi - |0\rangle\| \leq C N^{-3/2}.$$

This completes the proof of Theorem 2. $\square$

7 Proof of Theorem 1

Proof Let Ψ_N be the ground state for H_N . As explained in the introduction, we will decompose

$$\gamma_{\Psi_N}^{(1)} = P \gamma_{\Psi_N}^{(1)} P + Q \gamma_{\Psi_N}^{(1)} Q + P \gamma_{\Psi_N}^{(1)} Q + Q \gamma_{\Psi_N}^{(1)} P.$$

Diagonal terms. For $Q \gamma_{\Psi_N}^{(1)} Q$, recall from [12, Theorem 2.2 (iii)] that

$$U_N \Psi_N \rightarrow U_B^* |0\rangle$$

strongly in the quadratic form of $\mathbb{H}_{\text{Bog}}$ on $\mathcal{F}_+$. Moreover, it is easy to see that

$$\mathbb{H}_{\text{Bog}} \geq \frac{1}{2} \sum_{p \neq 0} |p|^2 a_p^* a_p - C \geq \mathcal{N}_+ - C$$

(see e.g. [13, Proof of Theorem 1]). Therefore, in the limit $N \rightarrow \infty$,

$$\begin{aligned} \text{Tr} Q \gamma_{\Psi_N}^{(1)} Q &= \langle U_N \Psi_N, \mathcal{N}_+ U_N \Psi_N \rangle \rightarrow \langle 0 | U_B \mathcal{N}_+ U_B^* | 0 \rangle \\ &= \left\langle 0 \left| \sum_{p \neq 0} (\sigma_p a_p^* + \gamma_p a_{-p}) (\sigma_p a_p + \gamma_p a_{-p}^*) \right| 0 \right\rangle = \sum_{p \neq 0} \gamma_p^2. \end{aligned} \quad (66)$$

Here we have used Bogoliubov's transformation (8). Similarly, for any $p, q \neq 0$ we have

$$\begin{aligned} \langle u_p, Q \gamma_{\Psi_N}^{(1)} Q u_q \rangle &= \langle U_N \Psi_N, a_p^* a_q U_N \Psi_N \rangle \rightarrow \langle 0 | U_B (a_p^* a_q) U_B^* | 0 \rangle \\ &= \langle 0 | (\sigma_p a_p^* + \gamma_p a_{-p}) (\sigma_q a_q + \gamma_q a_{-q}^*) | 0 \rangle = \gamma_p^2 \delta_{p,q}. \end{aligned} \quad (67)$$

From (66) and (67), we conclude that

$$Q \gamma_{\Psi_N}^{(1)} Q \rightarrow \sum_{p \neq 0} \gamma_p^2 |u_p\rangle \langle u_p| \quad (68)$$

strongly in trace class. Consequently,

$$\text{Tr}(P \gamma_{\Psi_N}^{(1)} P) = N - \text{Tr} Q \gamma_{\Psi_N}^{(1)} Q = N - \sum_{p \neq 0} \gamma_p^2$$

and hence

$$\mathrm{Tr} \left| P \gamma_{\Psi_N}^{(1)} P + Q \gamma_{\Psi_N}^{(1)} Q - \left(N - \sum_{p \neq 0} \gamma_p^2 \right) |u_0\rangle \langle u_0| - \sum_{p \neq 0} \gamma_p^2 |u_p\rangle \langle u_p| \right| \rightarrow 0.$$

Off-diagonal terms. Let us prove that

$$\mathrm{Tr} \left| P \gamma_{\Psi_N}^{(1)} Q + Q \gamma_{\Psi_N}^{(1)} P \right| \leq C N^{-1/4}. \quad (69)$$

By using $P = |u_0\rangle \langle u_0|$ and the Cauchy–Schwarz inequality, it suffices to show that

$$\|Q \gamma_{\Psi_N}^{(1)} u_0\|^2 \leq C N^{-1/2}.$$

Since $\{u_p\}_{p \neq 0}$ is an orthonormal basis for $\mathcal{H}_+$, we have

$$\|Q \gamma_{\Psi_N}^{(1)} u_0\|^2 = \sum_{p \neq 0} |\langle u_p, \gamma_{\Psi_N}^{(1)} u_0 \rangle|^2 = \sum_{p \neq 0} |\langle \Psi_N, a_0^* a_p \Psi_N \rangle|^2.$$

Using the excitation map U_N and the relations (12) we can decompose

$$\begin{aligned} \langle \Psi_N, a_0^* a_p \Psi_N \rangle &= \langle U_N \Psi_N, \sqrt{N - \mathcal{N}_+} a_p U_N \Psi_N \rangle \\ &= \sqrt{N} \langle U_N \Psi_N, a_p U_N \Psi_N \rangle + \langle U_N \Psi_N, (\sqrt{N - \mathcal{N}_+} - \sqrt{N}) a_p U_N \Psi_N \rangle. \end{aligned}$$

Therefore, by the Cauchy–Schwarz inequality

$$\begin{aligned} \|Q \gamma_{\Psi_N}^{(1)} u_0\|^2 &\leq 2N \sum_{p \neq 0} |\langle U_N \Psi_N, a_p U_N \Psi_N \rangle|^2 \\ &\quad + 2 \sum_{p \neq 0} |\langle U_N \Psi_N, (\sqrt{N - \mathcal{N}_+} - \sqrt{N}) a_p U_N \Psi_N \rangle|^2. \end{aligned} \quad (70)$$

For the second sum in (70), using the Cauchy–Schwarz inequality, the simple bound

$$\left(\sqrt{N - \mathcal{N}_+} - \sqrt{N} \right)^2 = \left(\frac{\mathcal{N}_+}{\sqrt{N - \mathcal{N}_+} + \sqrt{N}} \right)^2 \leq \frac{\mathcal{N}_+^2}{N}$$

and Lemma 3, we find that

$$\begin{aligned} &\sum_{p \neq 0} |\langle U_N \Psi_N, (\sqrt{N - \mathcal{N}_+} - \sqrt{N}) a_p U_N \Psi_N \rangle|^2 \\ &\leq \sum_{p \neq 0} \|(\sqrt{N - \mathcal{N}_+} - \sqrt{N}) U_N \Psi_N\|^2 \|a_p U_N \Psi_N\|^2 \\ &\leq N^{-1} \langle U_N \Psi_N, \mathcal{N}_+^2 U_N \Psi_N \rangle \langle U_N \Psi_N, \mathcal{N}_+ U_N \Psi_N \rangle \\ &= N^{-1} \langle \Psi_N, \mathcal{N}_+^2 \Psi_N \rangle \langle \Psi_N, \mathcal{N}_+ \Psi_N \rangle \leq C N^{-1}. \end{aligned} \quad (71)$$

To control the first sum in (70), we will use the bound from Theorem 2:

$$\langle \Phi, \mathcal{N}_+ \Phi \rangle \leq C N^{-3/2}, \quad \text{with } \Phi = U_S U_B U_N \Psi_N. \quad (72)$$

Also, from Lemmas 3, 4 and 5 it follows that

$$\langle \Phi, (\mathcal{N}_+ + 1)^4 \Phi \rangle \leq C. \quad (73)$$

Using the action of Bogoliubov's transformation in (8) and the uniform bounds (10) we obtain

$$\begin{aligned} \sum_{p \neq 0} |\langle U_N \Psi_N, a_p U_N \Psi_N \rangle|^2 &= \sum_{p \neq 0} |\langle \Phi, U_S U_B a_p U_B^* U_S^* \Phi \rangle|^2 \\ &= \sum_{p \neq 0} |\langle \Phi, U_S (\sigma_p a_p + \gamma_p a_{-p}^*) U_S^* \Phi \rangle|^2 \leq C \sum_{p \neq 0} |\langle \Phi, U_S a_p U_S^* \Phi \rangle|^2. \end{aligned} \quad (74)$$

To estimate further the right side of (74), we use the Duhamel formula

$$U_S a_p U_S^* = a_p + \int_0^1 e^{tS} [S, a_p] e^{-tS} dt$$

and the Cauchy–Schwarz inequality to get

$$\sum_{p \neq 0} |\langle \Phi, U_S a_p U_S^* \Phi \rangle|^2 \leq 2 \sum_{p \neq 0} |\langle \Phi, a_p \Phi \rangle|^2 + 2 \sum_{p \neq 0} \int_0^1 |\langle \Phi, e^{tS} [S, a_p] e^{-tS} \Phi \rangle|^2 dt. \quad (75)$$

Thanks to (72) we can bound

$$\sum_{p \neq 0} |\langle \Phi, a_p \Phi \rangle|^2 \leq \sum_{p \neq 0} \|a_p \Phi\|^2 = \langle \Phi, \mathcal{N}_+ \Phi \rangle \leq C N^{-3/2}. \quad (76)$$

It remains to handle the term involving the commutator $[S, a_p]$ in (75). Using the CCR (3) and the identity $[a_p, \mathbb{1}^{\leq N}] = -\mathbb{1}(\mathcal{N}_+ = N) a_p$ we can decompose

$$\begin{aligned} [a_p, S] &= \frac{1}{\sqrt{N}} \sum_{\substack{m, n \neq 0 \\ m+n \neq 0}} \eta_{m,n} \left[a_p, a_{m+n}^* a_{-m}^* a_{-n}^* \mathbb{1}^{\leq N} - \mathbb{1}^{\leq N} a_{m+n} a_{-m} a_{-n} \right] \\ &= \frac{1}{\sqrt{N}} \sum_{\substack{m, n \neq 0 \\ m+n \neq 0}} \eta_{m,n} (\delta_{p, m+n} a_{-m}^* a_{-n}^* + \delta_{p, -m} a_{m+n}^* a_{-n}^* + \delta_{p, -n} a_{m+n}^* a_n^*) \mathbb{1}^{\leq N} \\ &\quad - \frac{1}{\sqrt{N}} \sum_{\substack{m, n \neq 0 \\ m+n \neq 0}} \eta_{m,n} a_{m+n}^* a_{-m}^* a_{-n}^* \mathbb{1}(\mathcal{N}_+ = N) a_p \\ &\quad + \frac{1}{\sqrt{N}} \sum_{\substack{m, n \neq 0 \\ m+n \neq 0}} \eta_{m,n} \mathbb{1}(\mathcal{N}_+ = N) a_p a_{m+n} a_{-m} a_{-n} \\ &=: I_1(p) + I_2(p) + I_3(p). \end{aligned}$$

Hence, by the Cauchy–Schwarz inequality we have for all $t \in [0, 1]$,

$$\sum_{p \neq 0} |\langle \Phi, e^{tS} [a_p, S] e^{-tS} \Phi \rangle|^2 \leq 3 \sum_{k=1}^3 \sum_{p \neq 0} |\langle \Phi, e^{tS} I_k(p) e^{-tS} \Phi \rangle|^2. \quad (77)$$

The right side of (77) can be bounded using the Cauchy–Schwarz inequality, the summability (27), Lemmas 5, 6, (72) and (73). For the terms involving $I_1(p)$, we have

$$\sum_{p \neq 0} |\langle \Phi, e^{tS} I_1(p) e^{-tS} \Phi \rangle|^2$$

$$\begin{aligned}
&\leq \frac{C}{N} \left(\sum_{\substack{m,n,p \neq 0 \\ m+n \neq 0}} |\eta_{m,n}|^2 \delta_{p,m+n} \right) \left(\sum_{\substack{m,n \neq 0 \\ m+n \neq 0}} \|(\mathcal{N}_+ + 3)^{-1/2} a_{-m} a_{-n} e^{-tS} \Phi\|^2 \right) \\
&\quad \|(\mathcal{N}_+ + 3)^{1/2} \mathbb{1}^{\leq N} e^{-tS} \Phi\|^2 \\
&+ \frac{C}{N} \left(\sum_{\substack{m,n,p \neq 0 \\ m+n \neq 0}} |\eta_{m,n}|^2 \delta_{p,-m} \right) \left(\sum_{\substack{m,n \neq 0 \\ m+n \neq 0}} \|(\mathcal{N}_+ + 3)^{-1/2} a_{m+n} a_{-n} e^{-tS} \Phi\|^2 \right) \\
&\quad \|(\mathcal{N}_+ + 3)^{1/2} \mathbb{1}^{\leq N} e^{-tS} \Phi\|^2 \\
&+ \frac{C}{N} \left(\sum_{\substack{m,n,p \neq 0 \\ m+n \neq 0}} |\eta_{m,n}|^2 \delta_{p,-n} \right) \left(\sum_{\substack{m,n \neq 0 \\ m+n \neq 0}} \|(\mathcal{N}_+ + 3)^{-1/2} a_{m+n} a_{-m} e^{-tS} \Phi\|^2 \right) \\
&\quad \|(\mathcal{N}_+ + 3)^{1/2} \mathbb{1}^{\leq N} e^{-tS} \Phi\|^2 \\
&\leq \frac{C}{N} \left\langle \Phi, e^{tS} \mathcal{N}_+ e^{-tS} \Phi \right\rangle \left\langle \Phi, e^{tS} (\mathcal{N}_+ + 3) e^{-tS} \Phi \right\rangle \\
&\leq \frac{C}{N} \left(\left\langle \Phi, \mathcal{N}_+ \Phi \right\rangle + \frac{1}{N} \left\langle \Phi, (\mathcal{N}_+ + 1)^2 \Phi \right\rangle \right) \left\langle \Phi, (\mathcal{N}_+ + 3) \Phi \right\rangle \leq \frac{C}{N^2}.
\end{aligned}$$

Similarly, the terms involving $I_2(p)$ are bounded by

$$\begin{aligned}
&\sum_{p \neq 0} |\langle \Phi, e^{tS} I_2(p) e^{-tS} \Phi \rangle|^2 \\
&\leq \frac{C}{N} \left(\sum_{\substack{m,n \neq 0 \\ m+n \neq 0}} |\eta_{m,n}|^2 \right) \left(\sum_{\substack{m,n \neq 0 \\ m+n \neq 0}} \|\mathbb{1}(\mathcal{N}_+ = N) a_{m+n} a_{-m} a_{-n} e^{-tS} \Phi\|^2 \right) \left(\sum_{p \neq 0} \|a_p e^{-tS} \Phi\|^2 \right) \\
&\leq \frac{C}{N} \left\langle \Phi, e^{tS} \mathcal{N}_+^3 e^{-tS} \Phi \right\rangle \left\langle \Phi, e^{tS} \mathcal{N}_+ e^{-tS} \Phi \right\rangle \\
&\leq \frac{C}{N} \left\langle \Phi, (\mathcal{N}_+ + 1)^3 \Phi \right\rangle \left(\left\langle \Phi, \mathcal{N}_+ \Phi \right\rangle + \frac{1}{N} \left\langle \Phi, (\mathcal{N}_+ + 1)^2 \Phi \right\rangle \right) \leq \frac{C}{N^2}.
\end{aligned}$$

Finally for the terms involving $I_3(p)$, using

$$\mathbb{1}(\mathcal{N}_+ = N) \leq (\mathcal{N}_+ / N)^4$$

we have

$$\begin{aligned}
&\sum_{p \neq 0} |\langle \Phi, e^{tS} I_3(p) e^{-tS} \Phi \rangle|^2 \\
&\leq \frac{C}{N} \left(\sum_{\substack{m,n \neq 0 \\ m+n \neq 0}} |\eta_{m,n}|^2 \right) \|\mathbb{1}(\mathcal{N}_+ = N) e^{-tS} \Phi\|^2 \left(\sum_{\substack{m,n,p \neq 0 \\ m+n \neq 0}} \|a_{m+n} a_{-m} a_{-n} a_p e^{-tS} \Phi\|^2 \right) \\
&\leq \frac{C}{N} \left\langle \Phi, e^{tS} (\mathcal{N}_+ / N)^4 e^{-tS} \Phi \right\rangle \left\langle \Phi, e^{tS} \mathcal{N}_+^4 e^{-tS} \Phi \right\rangle \\
&\leq \frac{C}{N^5} \left\langle \Phi, (\mathcal{N}_+ + 1)^4 \Phi \right\rangle^2 \leq \frac{C}{N^5}.
\end{aligned}$$

Thus we conclude from (77) that

$$\sum_{p \neq 0} |\langle \Phi, e^{tS} [S, a_p] e^{-tS} \Phi \rangle|^2 \leq \frac{C}{N^2}, \quad \forall t \in [0, 1]. \quad (78)$$

Consequently,

$$\sum_{p \neq 0} \int_0^1 |\langle \Phi, e^{tS} [S, a_p] e^{-tS} \Phi \rangle|^2 dt \leq \frac{C}{N^2}. \quad (79)$$

Inserting (79) and (76) in (75) and (74) we obtain

$$\sum_{p \neq 0} |\langle U_N \Psi_N, a_p U_N \Psi_N \rangle|^2 \leq C \sum_{p \neq 0} |\langle \Phi, U_S a_p U_S^* \Phi \rangle|^2 \leq C N^{-3/2}.$$

Using the latter bound and (71), we deduce from (70) that

$$\|Q\gamma_{\Psi_N}^{(1)} u_0\|^2 \leq C N^{-1/2}.$$

This implies (69) and completes the proof of Theorem 1. $\square$

Acknowledgements We thank Robert Seiringer and Nicolas Rougerie for helpful discussions. The research is funded by the Polish-German Beethoven Classic 3 project “Mathematics of many-body quantum systems”. PTN acknowledges the support from the Deutsche Forschungsgemeinschaft (DFG Project Nr. 426365943). MN acknowledges the support from the National Science Centre (NCN Project Nr. 2018/31/G/ST1/01166).

Funding Open Access funding enabled and organized by Projekt DEAL.

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