A diffused interface with the advection term in a Sobolev space

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Abstract

We study the asymptotic limit of diffused surface energy in the van der Waals–Cahn–Hillard theory when an advection term is added and the energy is uniformly bounded. We prove that the limit interface is an integral varifold and the generalized mean curvature vector is determined by the advection term. As the application, a prescribed mean curvature problem is solved using the min–max method.

Mathematics Subject Classification 49Q20

1 Introduction

The object of study in this paper is the energy functional appearing in the van der Waals–Cahn–Hillard theory [4, 7],

$$E_\varepsilon(u) = \int_\Omega \frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon}, \quad (1.1)$$

where $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ($n \geq 2$) is the normalized density distribution of two phases of a material, $|\nabla u|^2 = \sum_{k=1}^n (\partial u / \partial x_k)^2$ and $W : \mathbb{R} \rightarrow [0, \infty)$ is a double-well potential with two global minima at ± 1 . In the thermodynamic context, W corresponds to the Helmholtz free energy density and the typical example is $W(u) = (1 - u^2)^2$. When the positive parameter ε is small relative to the size of the domain Ω and $E_\varepsilon(u)$ is bounded, it is expected that u is close to $+1$ or -1 on most of Ω while a spatial change between ± 1 occurs within a hypersurface-like region of $O(\varepsilon)$ thickness which we may call the *diffused interface* of u . In this case, the quantity $E_\varepsilon(u)$ is expected to be proportional to the surface area of the diffused

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interface. Due to the importance of the surface area in calculus of variations, it is interesting to investigate the validity of such expectation and other salient properties of E_ε .

In this direction, there have been a number of works studying the asymptotic behavior of E_ε as $\varepsilon \rightarrow 0+$ under various assumptions. For the energy minimizers with appropriate side conditions, it is well-known that it Γ -converges to the area functional of the limit interface [10, 12–14, 20]. On the other hand, due in part to the non-convex nature of the functional, there may exist multiple and even infinite number of critical points of E_ε different from the energy minimizers. For general critical points, Hutchinson and the first author [8] proved that the limit is an integral stationary varifold [1]. For general *stable* critical points, the first author and Wickramasekera [26] proved that the limit is an embedded real-analytic minimal hypersurface except for a closed singular set of codimension seven. More recently, Guaraco [6] showed that a uniform Morse index bound is sufficient to conclude the same regularity for $n \geq 3$ and gave a new proof of Almgren–Pitts theorem [16] as the application. The new proof significantly simplifies the existence part of the proof even though one needs to use Wickramasekera’s hard regularity theorem [27].

While the investigations on the critical points of E_ε have direct links to the minimal surface theory as above, more generally, it turned out that suitable controls of the first variation of E_ε guarantee the analogous good asymptotic behaviors. For example, under the assumption that

$$\liminf_{\varepsilon \rightarrow 0+} (E_\varepsilon(u_\varepsilon) + \|f_\varepsilon\|_{W^{1,p}(\Omega)}) < \infty$$

with $f_\varepsilon := -\varepsilon \Delta u_\varepsilon + W'(u_\varepsilon)/\varepsilon$ and $p > n/2$, the first author [22, 25] proved that the limit interface is an integral varifold whose generalized mean curvature belongs to L^q ($q = p(n-1)/(n-p) > n-1$) with respect to the surface measure. Here $W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega)\}$. The mean curvature of the limit interface is characterized by the weak $W^{1,p}$ limit of f_ε [18]. Another example concerns one of De Giorgi’s conjectures. Under the assumption that (with f_ε as above)

$$\liminf_{\varepsilon \rightarrow 0+} (E_\varepsilon(u_\varepsilon) + \varepsilon^{-1} \|f_\varepsilon\|_{L^2(\Omega)}^2) < \infty$$

and $n = 2, 3$, Röger–Schätzle [17] (independently [15] for the case of $n = 2$) proved the similar result. In this case, the limit interface has an L^2 generalized mean curvature.

In this paper, along the line of research described above, we investigate the asymptotic behavior of u_ε satisfying

$$-\varepsilon \Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon} = \varepsilon v_\varepsilon \cdot \nabla u_\varepsilon, \quad (1.2)$$

where v_ε is considered here as a given vector field and we assume that

$$\liminf_{\varepsilon \rightarrow 0+} (E_\varepsilon(u_\varepsilon) + \|v_\varepsilon\|_{W^{1,p}(\Omega)}) < \infty$$

and $p > n/2$. The problem is related to (parabolic) Allen–Cahn-type equations studied in [11, 21], for example. It is also natural to investigate the effect of advection term as $\varepsilon \rightarrow 0+$. We prove the analogous result Theorem 2.1 to [22, 25], namely, the limit is an integral varifold with L^q (the same as above) generalized mean curvature which is characterized by the weak $W^{1,p}$ limit of v_ε . Using this result, we give some existence theorem for a *vectorial prescribed mean curvature problem*, as described in Theorem 2.2. Despite the simplicity of the problem, this is the first existence result in the setting of the min–max method, with minimal regularity assumptions on the prescribed vector field. As for the existence problem for *scalar* constant

or prescribed mean curvature using a min–max approach along the lines of Almgren–Pitts [16], we mention papers by Zhou and Zhu [28,29].

As for the proof, just as in the case of [8,22,25], the key point is to prove a certain monotonicity-type formula which is the essential tool in the setting of Geometric Measure Theory. We wish to treat $\varepsilon v_\varepsilon \cdot \nabla u_\varepsilon$ as a perturbative term, and to do so, we need to control a certain “trace” norm of v_ε on diffused interface. If an ε -independent upper density ratio estimate of diffused surface measure is available, then we can control $\varepsilon v_\varepsilon \cdot \nabla u_\varepsilon$ by the $W^{1,p}$ -norm of v_ε . For this purpose, we establish the key estimate, Theorem 3.8, which gives a local uniform upper density ratio estimate. Once this part is done, the rest proceeds just like [25] with minor modifications.

The paper is organized as follows. In Sect. 2 we state our assumptions and explain the main results. Section 3 contains the main estimates which ultimately give a monotonicity-type formula, Theorem 3.9. In Sect. 4, we prove the main theorem by modifying the proof in [22,25], and in Sect. 5, we give some concluding remarks.

2 Assumptions and main results

We use the notation that $U_r(a) := \{x \in \mathbb{R}^n : |x-a| < r\}$, $B_r(a) := \{x \in \mathbb{R}^n : |x-a| \leq r\}$, $U_r := U_r(0)$ and $B_r := B_r(0)$.

2.1 Assumptions

Throughout the paper, we assume that:

- (a) The function $W : \mathbb{R} \rightarrow [0, \infty)$ is C^3 and has two strict minima $W(\pm 1) = W'(\pm 1) = 0$.
- (b) For some $\gamma \in (-1, 1)$, $W' > 0$ on $(-1, \gamma)$ and $W' < 0$ on $(\gamma, 1)$.
- (c) For some $\alpha \in (0, 1)$ and $\kappa > 0$, $W''(x) \geq \kappa$ for all $|x| \geq \alpha$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We assume that we are given $W^{1,2}(\Omega)$ functions $\{u_i\}_{i=1}^\infty$, $W^{1,p}(\Omega; \mathbb{R}^n)$ vector fields $\{v_i\}_{i=1}^\infty$ and positive constants $\{\varepsilon_i\}_{i=1}^\infty$ satisfying

$$-\varepsilon_i \Delta u_i + \frac{W'(u_i)}{\varepsilon_i} = \varepsilon_i v_i \cdot \nabla u_i \quad (2.1)$$

weakly on Ω for each $i \in \mathbb{N}$. In addition, assume that

$$\lim_{i \rightarrow \infty} \varepsilon_i = 0, \quad \frac{n}{2} < p < n \quad (2.2)$$

and that there exist constants c_0 , E_0 and λ_0 such that, for all $i \in \mathbb{N}$, we have:

$$\|u_i\|_{L^\infty(\Omega)} \leq c_0, \quad (2.3)$$

$$\int_{\Omega} \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} \right) \leq E_0, \quad (2.4)$$

$$\|v_i\|_{L^{\frac{np}{n-p}}(\Omega)} + \|\nabla v_i\|_{L^p(\Omega)} \leq \lambda_0. \quad (2.5)$$

The condition (2.3) is not essential and can be often derived from the PDE or the proof of existence. Here we assume (2.3) for simplicity. Next, define

$$\Phi(s) := \int_{-1}^s \sqrt{W(t)/2} \, dt, \quad w_i(x) := \Phi(u_i(x)).$$

By the Cauchy–Schwarz inequality and (2.4), we obtain

$$\int_{\Omega} |\nabla w_i| \leq \frac{1}{2} \int_{\Omega} \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} \right) \leq \frac{1}{2} E_0.$$

Hence, by the compactness theorem for BV functions [30, Corollary 5.3.4], there exist a converging subsequence (which we denote by the same notation) $\{w_i\}$ in the L^1 norm and the limit BV function w . Define

$$u(x) := \Phi^{-1}(w(x)).$$

where Φ^{-1} is the inverse function of Φ . It follows that u_i converges to u a.e. on Ω . By Fatou's Lemma and (2.4), we have

$$\int_{\Omega} W(u) = \int_{\Omega} \lim_{i \rightarrow \infty} W(u_i) \leq \liminf_{i \rightarrow \infty} \int_{\Omega} W(u_i) = 0.$$

This shows that $u = \pm 1$ a.e. on Ω and u is a BV function. For simplicity we write $\partial^* \{u = 1\}$ as the reduced boundary [30] of $\{u = 1\}$ and $\|\partial^* \{u = 1\}\|$ as the boundary measure.

2.2 The associated varifolds

We associate to each solution of (1.2) a varifold in a natural way in the following. We refer to [1, 19] for a comprehensive treatment of varifolds.

Let $\mathbf{G}(n, n-1)$ be the Grassmannian, i.e. the space of unoriented $(n-1)$ -dimensional subspaces in $\mathbb{R}^n$. We also regard $S \in \mathbf{G}(n, n-1)$ as the $n \times n$ matrix representing the orthogonal projection of $\mathbb{R}^n$ onto S . For two given square-matrices S_1 and S_2 , we write $S_1 \cdot S_2 := \text{trace}(S_1^t \circ S_2)$, where the upper-script t indicates the transpose of the matrix and $\circ$ is the matrix multiplication. We say that V is an $(n-1)$ -dimensional varifold in $\Omega \subset \mathbb{R}^n$ if V is a Radon measure on $\mathbf{G}_{n-1}(\Omega) := \Omega \times \mathbf{G}(n, n-1)$. Let $\mathbf{V}_{n-1}(\Omega)$ be the set of all $(n-1)$ -dimensional varifolds in Ω . Convergence in the varifold sense means convergence in the usual sense of measures. For $V \in \mathbf{V}_{n-1}(\Omega)$, we let $\|V\|$ be the weight measure of V . For $V \in \mathbf{V}_{n-1}(\Omega)$, we define the first variation of V by

$$\delta V(g) := \int_{\mathbf{G}_{n-1}(\Omega)} \nabla g(x) \cdot S \, dV(x, S) \quad (2.6)$$

for any vector field $g \in C_c^1(\Omega; \mathbb{R}^n)$. We let $\|\delta V\|$ be the total variation of δV . If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, then the Radon–Nikodym derivative $\delta V / \|V\|$ exists as a vector-valued $\|V\|$ measurable function. In this case, we define the generalized mean curvature vector of V by $-\delta V / \|V\|$ and we use the notation H_V .

We associate to each function u_i a varifold V_i as follows. First, we define a Radon measure μ_i on Ω by

$$d\mu_i := \frac{1}{\sigma} \left(\frac{\varepsilon_i |\nabla u_i|^2}{2} + \frac{W(u_i)}{\varepsilon_i} \right) d\mathcal{L}^n, \quad (2.7)$$

where $\mathcal{L}^n$ is the n -dimensional Lebesgue measure and $\sigma := \int_{-1}^1 \sqrt{2W(s)} \, ds$. Define $V_i \in \mathbf{V}_{n-1}(\Omega)$ by

$$V_i(\phi) := \int_{\{|\nabla u_i| \neq 0\}} \phi \left(x, I - \frac{\nabla u_i(x)}{|\nabla u_i(x)|} \otimes \frac{\nabla u_i(x)}{|\nabla u_i(x)|} \right) d\mu_i(x) \quad (2.8)$$

for $\phi \in C_c(\mathbf{G}_{n-1}(\Omega))$, where I is the $n \times n$ identity matrix and $\otimes$ is the tensor product of the two vectors. Note that $I - \frac{\nabla u_i(x)}{|\nabla u_i(x)|} \otimes \frac{\nabla u_i(x)}{|\nabla u_i(x)|}$ represents the orthogonal projection to the $(n-1)$ -dimensional subspace $\{a \in \mathbb{R}^n : a \cdot \nabla u_i(x) = 0\}$. By definition, we have

$$\|V_i\| = \mu_i \llcorner_{\{|\nabla u_i| \neq 0\}}$$

and by (2.6), we have

$$\delta V_i(g) = \int_{\{|\nabla u_i| \neq 0\}} \nabla g \cdot \left(I - \frac{\nabla u_i}{|\nabla u_i|} \otimes \frac{\nabla u_i}{|\nabla u_i|} \right) d\mu_i \quad (2.9)$$

for each $g \in C_c^1(\Omega, \mathbb{R}^n)$.

2.3 Main Theorems

With the above assumptions and notation, we show:

Theorem 2.1 *Suppose that u_i, v_i, ε_i satisfy (2.1)–(2.5) and let V_i be the varifold associated with u_i as in (2.8). On passing to a subsequence we can assume that*

$$v_i \rightarrow v \text{ weakly in } W^{1,p}, \quad u_i \rightarrow u \text{ a.e.}, \quad V_i \rightarrow V \text{ in the varifold sense.}$$

Then we have the following properties.

(1) *For each $\phi \in C_c(\Omega)$,*

$$\begin{aligned} \frac{1}{2} \|V\|(\phi) &= \lim_{i \rightarrow \infty} \frac{1}{\sigma} \int_{\Omega} \frac{\varepsilon_i}{2} |\nabla u_i|^2 \phi = \lim_{i \rightarrow \infty} \frac{1}{\sigma} \int_{\Omega} \frac{W(u_i)}{\varepsilon_i} \phi \\ &= \lim_{i \rightarrow \infty} \frac{1}{\sigma} \int_{\Omega} |\nabla w_i| \phi. \end{aligned}$$

(2) *$\text{spt } \|\partial^* \{u = 1\}\| \subset \text{spt } \|V\|$ and $\{u_i\}$ converges locally uniformly to ± 1 on $\Omega \setminus \text{spt } \|V\|$.*

(3) *For each $0 < b < 1$, $\{|u_i| \leq 1 - b\}$ locally converges to $\text{spt } \|V\|$ in the Hausdorff distance sense in Ω .*

(4) *V is an integral varifold and the density $\theta(x)$ of V satisfies*

$$\theta(x) = \begin{cases} \text{odd} & \mathcal{H}^{n-1} \text{ a.e. } x \in \partial^* \{u = 1\}, \\ \text{even} & \mathcal{H}^{n-1} \text{ a.e. } x \in \text{spt } \|V\| \setminus \partial^* \{u = 1\}, \end{cases}$$

(5) *the generalized mean curvature vector H_V of V is given by*

$$H_V(x) = (T_x \text{spt } \|V\|)^\perp(v(x)),$$

for $\|V\|$ a.e. $x \in \Omega$.

(6) *For $\tilde{\Omega} \subset \subset \Omega$, there exists a constant λ_1 depending only on $c_0, \lambda_0, n, p, W, E_0$ and $\text{dist}(\tilde{\Omega}, \partial\Omega)$ such that*

$$\int_{\tilde{\Omega}} |H_V(x)|^{\frac{p(n-1)}{n-p}} d\|V\|(x) \leq \int_{\tilde{\Omega}} |v(x)|^{\frac{p(n-1)}{n-p}} d\|V\|(x) \leq \lambda_1.$$

Note that $\frac{p(n-1)}{n-p} > n-1$ due to (2.2).

Since V is integral and the generalized mean curvature vector is in the stated class, V satisfies various good properties described in [19, Section 17]. In particular, $\text{spt } \|V\|$ is a closed countably $(n-1)$ -rectifiable set (see [19, 17.9(1)]), and writing $\Gamma := \text{spt } \|V\|$, for any $\phi \in C_c(\mathbf{G}_{n-1}(\Omega))$,

$$\int_{\mathbf{G}_{n-1}(\Omega)} \phi(x, S) dV(x, S) = \int_{\Gamma} \phi(x, T_x \Gamma) \theta(x) d\mathcal{H}^{n-1}(x).$$

Here, $T_x \Gamma \in \mathbf{G}(n, n-1)$ is the approximate tangent space of Γ at x which exists $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma$. With this notation, (5) implies that $H_V(x) = (T_x \Gamma)^\perp(v(x))$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma$, i.e., the generalized mean curvature vector of V coincides with the projection of v to the orthogonal subspace $(T_x \Gamma)^\perp$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma$. We emphasize the difference of characterization of the mean curvature vector from [25], where the similar equality holds only on the reduced boundary of $\{u = 1\}$, while the equality here holds on the whole support of $\|V\|$ including on the “hidden boundary” $\Gamma \setminus \partial^* \{u = 1\}$. If we additionally assume that $\theta = 1$ for $\mathcal{H}^{n-1}$ a.e. $x \in \Gamma$, then because of the integrability of H_V and the Allard regularity theorem [1], except for a closed $\mathcal{H}^{n-1}$ -null set, Γ is locally a $C^{1,2-\frac{n}{p}}$ hypersurface. Without the assumption $\theta = 1$, we can still conclude that $\text{spt } \|V\|$ is $C^{1,2-\frac{n}{p}}$ hypersurface on a dense open set of $\text{spt } \|V\|$, even though we do not know if the complement is $\mathcal{H}^{n-1}$ -null or not.

2.4 A vectorial prescribed mean curvature problem

As an application¹ of Theorem 2.1 with suitable modifications, we prove the following:

Theorem 2.2 *Let (M, g) be a smooth compact n -dimensional Riemannian manifold and let $\rho \in W^{2,p}(M)$ be a given function, where $p > \frac{n}{2}$. Then, there exists a non-zero integral varifold V in M such that*

$$H_V(x) = (T_x \text{spt } \|V\|)^\perp(\nabla \rho(x))$$

for $\|V\|$ a.e. $x \in M$.

Proof We may assume $p < n$. Consider the following functional for $\varepsilon > 0$ and $u \in W^{1,2}(M)$:

$$F_\varepsilon(u) := \int_M \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right) \exp(\rho) d\omega_g.$$

By the Sobolev embedding, $\rho \in C^{0,2-\frac{n}{p}}(M)$ and thus $0 < \exp(\min \rho) \leq \exp(\rho) \leq \exp(\max \rho) < \infty$. By considering the path space in $W^{1,2}(M)$ connecting $u \equiv 1$ and $u \equiv -1$, the standard min–max method gives a non-trivial critical point u_ε for each $\varepsilon > 0$, with uniform strictly positive lower and upper bounds of $F_\varepsilon(u_\varepsilon)$. The critical point satisfies (2.1) with $v = \nabla \rho$ and $|u_\varepsilon| \leq 1$, with an appropriate modification of the equation. Take a sequence $\varepsilon_i \rightarrow 0+$ and a corresponding min–max critical points u_i . Then the sequence $u_i, \nabla \rho, \varepsilon_i$ satisfy all the assumptions of Theorem 2.1 with a small error terms coming from the metric of M (see Guaraco’s work [6] for an adaptation of the results to Riemannian manifolds in the case $v_\varepsilon = 0$). The limit varifold V thus has the desired property. $\square$

For more remarks on the main results, see Sect. 5.

¹ The authors thank Nick Edelen for a discussion which inspired this application.

3 The estimate for the upper density ratio

In this section, we prove Theorem 3.8–3.10, which give ε -independent estimates of the upper and lower density ratios of the energy. Throughout this section, we drop the index i and set $\Omega = U_1 = \{|x| < 1\}$ since the result is local. Assume $u \in W^{1,2}(U_1)$ and $v \in W^{1,p}(U_1; \mathbb{R}^n)$ satisfy (2.1) with a positive ε and (2.3)–(2.5) are satisfied for a given set of c_0, E_0, λ_0 . The exponent p satisfies (2.2). We first derive two preliminary properties for u , Lemmas 3.1 and 3.2.

Lemma 3.1 *There exists $c_1 > 0$ depending only on c_0, λ_0, n, p and W such that*

$$\sup_{x \in U_{1-\varepsilon}} \varepsilon |\nabla u(x)| \leq c_1 \quad (3.1)$$

and

$$\sup_{x, x' \in U_{1-\varepsilon}} \varepsilon^{3-\frac{n}{p}} \frac{|\nabla u(x) - \nabla u(x')|}{|x - x'|^{2-\frac{n}{p}}} \leq c_1 \quad (3.2)$$

for $0 < \varepsilon < 1/2$. If $\varepsilon \geq 1/2$, then we have for any $0 < s < 1$

$$\sup_{x \in U_s} |\nabla u(x)| \leq c_1 \quad (3.3)$$

where c_1 depends additionally on s . In both cases, we have $u \in W_{loc}^{3,p}(U_1)$.

Proof Consider the case $0 < \varepsilon < 1/2$. Define $\tilde{u}(x) := u(\varepsilon x)$ and $\tilde{v}(x) := \varepsilon v(\varepsilon x)$ for $x \in U_{\varepsilon^{-1}}$. After this change of variables, we obtain from (2.1) that

$$-\Delta \tilde{u} + W'(\tilde{u}) = \tilde{v} \cdot \nabla \tilde{u} \quad \text{weakly on } U_{\varepsilon^{-1}}. \quad (3.4)$$

Under the change of variables, we obtain from (2.5)

$$\|\tilde{v}\|_{L^{\frac{np}{n-p}}(U_{\varepsilon^{-1}})} + \|\nabla \tilde{v}\|_{L^p(U_{\varepsilon^{-1}})} \leq \lambda_0 \varepsilon^{2-\frac{n}{p}}. \quad (3.5)$$

For any $U_2(x) \subset U_{\varepsilon^{-1}}$, let $\phi \in C_c^1(U_2(x))$ be a function such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B_1(x)$ and $|\nabla \phi| \leq 4$ on $U_2(x)$. Use (3.4) with the test function $\tilde{u}\phi^2$. Using also (2.3), we obtain

$$\begin{aligned} \int |\nabla \tilde{u}|^2 \phi^2 &\leq c_0 \int (2\phi |\nabla \phi| |\nabla \tilde{u}| + |W'| \phi^2 + |\tilde{v}| |\nabla \tilde{u}| \phi^2) \\ &\leq \frac{1}{2} \int |\nabla \tilde{u}|^2 \phi^2 + \int (4c_0^2 |\nabla \phi|^2 + c_0 |W'| \phi^2 + c_0^2 |\tilde{v}|^2 \phi^2). \end{aligned} \quad (3.6)$$

Since $\frac{np}{n-p} > 2$, (3.5) and (3.6) give

$$\sup_{B_2(x) \subset U_{\varepsilon^{-1}}} \int_{B_1(x)} |\nabla \tilde{u}|^2 \leq c(c_0, \lambda_0, n, p, W). \quad (3.7)$$

We next note that the function $\tilde{u}\phi$ weakly satisfies the following equation:

$$-\Delta(\tilde{u}\phi) = -\tilde{u}\Delta\phi - 2\nabla\phi \cdot \nabla\tilde{u} + (\tilde{v} \cdot \nabla\tilde{u} - W'(\tilde{u}))\phi. \quad (3.8)$$

Using the standard L^p theory [5, Theorem 9.11] to (3.8), we may start a bootstrapping argument as follows. Starting with $q = 2$, we have

$$\begin{aligned}\nabla \tilde{u} \in L_{loc}^q &\implies \tilde{v} \cdot \nabla \tilde{u} \in L_{loc}^{\frac{npq}{np+q(n-p)}} \implies \tilde{u} \in W_{loc}^{2, \frac{npq}{np+q(n-p)}} \\ &\implies \nabla \tilde{u} \in L_{loc}^{\frac{npq}{np-q(2p-n)}}\end{aligned}$$

with the corresponding estimates relating these norms. Note that the exponent of integrability of $\nabla \tilde{u}$ is raised from q to $q \cdot \frac{np}{np-q(2p-n)}$, with the factor strictly larger than one. Thus, in a finite number of bootstrapping, we obtain the $W_{loc}^{2,s}$ (with $s > n$) estimate for $\tilde{u}$, and by the Sobolev inequality, the L_{loc}^∞ estimate for $\nabla \tilde{u}$. Again by the L^p theory, we obtain the $W_{loc}^{2, \frac{np}{n-p}}$ estimate of $\tilde{u}$. In particular, by the Sobolev inequality, we obtain (3.1) and (3.2). Since the right-hand side of (3.8) is in $W_{loc}^{1,p}$ (note that $\tilde{v} \cdot \nabla^2 \tilde{u} \in L_{loc}^{\frac{np}{2(n-p)}}$ and $\frac{np}{2(n-p)} > p$ by (2.2)), we have $\tilde{u} \in W_{loc}^{3,p}$ and the weak third-derivatives of $\tilde{u}$ exist. The case of $\varepsilon \geq 1/2$ does not require the change of variables as above and the proof is omitted. $\square$

Lemma 3.2 *Given $0 < s < 1$, there exist constants $0 < \varepsilon_1, \eta < 1$ depending only on c_0, λ_0, W, n, p and s such that*

$$\sup_{x \in B_s} |u(x)| \leq 1 + \varepsilon^\eta \quad (3.9)$$

for $\varepsilon \leq \varepsilon_1$.

Proof Let $q = \frac{np}{n-p} - 1$ and $\phi \in C_c^\infty(B_{\frac{s+1}{2}})$ with $\phi \geq 0$. Multiplying (2.1) by $[(u-1)_+]^q \phi^2$, we have

$$\begin{aligned}-\varepsilon \int q[(u-1)_+]^{q-1} |\nabla u|^2 \phi^2 + 2[(u-1)_+]^q \phi \nabla \phi \cdot \nabla u \\ = \int \frac{W'}{\varepsilon} [(u-1)_+]^q \phi^2 - \int \varepsilon \nabla u \cdot v [(u-1)_+]^q \phi^2.\end{aligned} \quad (3.10)$$

By $W'(u) \geq \kappa(u-1)$ for $u \geq 1$ and (3.1), we obtain

$$\begin{aligned}\frac{\kappa}{\varepsilon} \int [(u-1)_+]^{q+1} \phi^2 + \int \varepsilon q [(u-1)_+]^{q-1} |\nabla u|^2 \phi^2 \\ \leq 2\varepsilon \int [(u-1)_+]^q \phi |\nabla \phi| |\nabla u| + c_1 \int |v| [(u-1)_+]^q \phi^2 \\ \leq \frac{q\varepsilon}{2} \int [(u-1)_+]^{q-1} |\nabla u|^2 \phi^2 + \frac{8\varepsilon}{q} \int [(u-1)_+]^{q+1} |\nabla \phi|^2 \\ + \frac{\kappa}{2\varepsilon} \int [(u-1)_+]^{q+1} \phi^2 + \frac{\varepsilon^q c(q, c_1)}{\kappa^q} \int |v|^{q+1} \phi^2,\end{aligned} \quad (3.11)$$

which shows

$$\frac{\kappa}{2\varepsilon} \int [(u-1)_+]^{q+1} \phi^2 \leq \frac{8\varepsilon}{q} \int [(u-1)_+]^{q+1} |\nabla \phi|^2 + \frac{\varepsilon^q c(q, c_1)}{\kappa^q} \int |v|^{q+1} \phi^2. \quad (3.12)$$

By (2.3), (2.5) and iterating the computation above with suitable ϕ , we obtain

$$\int_{B_s} [(u-1)_+]^{q+1} \leq c_2(s, q, \lambda_0, n, p, W, c_0, c_1) \varepsilon^{q+1}. \quad (3.13)$$

To derive a contradiction, assume that $u(x_0) - 1 \geq \varepsilon^\eta$ for some $x_0 \in B_s$. By (3.1), for $y \in B_{\frac{\varepsilon^{1+\eta}}{2c_1}}(x_0)$,

$$u(y) - 1 \geq u(x_0) - 1 - \sup |\nabla u| \frac{\varepsilon^{1+\eta}}{2c_1} \geq \frac{\varepsilon^\eta}{2}. \quad (3.14)$$

Then we have

$$c_2 \varepsilon^{q+1} \geq \int_{B_{\frac{\varepsilon^{1+\eta}}{2c_1}}(x_0)} [(u-1)_+]^{q+1} \geq \left(\frac{\varepsilon^\eta}{2}\right)^{q+1} \omega_n \left(\frac{\varepsilon^{1+\eta}}{2c_1}\right)^n, \quad (3.15)$$

which show by $q = \frac{np}{n-p} - 1$

$$\varepsilon^{\eta \frac{np}{n-p} - \frac{np}{n-p} + n + n\eta} \leq c_3(s, q, \lambda_0, n, p, W, c_0, c_1). \quad (3.16)$$

This is a contradiction if η and ε are sufficiently small. $u \geq -1 - \varepsilon^\eta$ is proved similarly. $\square$

The next Lemma 3.3 is the starting point of the ultimate establishment of the monotonicity formula.

Lemma 3.3 For $B_r(x) \subset U_1$, we have

$$\begin{aligned} \frac{d}{dr} \left\{ \frac{1}{r^{n-1}} \int_{B_r(x)} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right) \right\} &= \frac{1}{r^n} \int_{B_r(x)} \left(\frac{W(u)}{\varepsilon} - \frac{\varepsilon |\nabla u|^2}{2} \right) \\ &+ \frac{\varepsilon}{r^{n+1}} \int_{\partial B_r(x)} ((y-x) \cdot \nabla u)^2 + \frac{\varepsilon}{r^n} \int_{B_r(x)} (v \cdot \nabla u)((y-x) \cdot \nabla u). \end{aligned} \quad (3.17)$$

Proof Multiply both sides of (2.1) by $\nabla u \cdot g$, where $g = (g^1, \dots, g^n) \in C_c^1(U_1; \mathbb{R}^n)$. By integration by parts, we obtain

$$\int \left(\left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W}{\varepsilon} \right) \operatorname{div} g - \varepsilon \sum_{i,j} u_{y_i} u_{y_j} g_{y_j}^i + \varepsilon (v \cdot \nabla u)(\nabla u \cdot g) \right) = 0. \quad (3.18)$$

We assume that $x = 0$ after a suitable translation and let $g^j(y) = y_j \rho(|y|)$. Writing $r = |y|$, (3.18) becomes

$$\begin{aligned} \int \left(\left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W}{\varepsilon} \right) (r\rho' + n\rho) - \varepsilon \frac{\rho'}{r} (y \cdot \nabla u)^2 \right. \\ \left. - \varepsilon |\nabla u|^2 \rho + \varepsilon (\nabla u \cdot v)(\nabla u \cdot y) \rho \right) = 0. \end{aligned}$$

We choose ρ which is a smooth approximation of χ_{B_r} , the characteristic function of B_r , and then we take a limit $\rho \rightarrow \chi_{B_r}$. Then we have

$$\begin{aligned} - (n-1) \int_{B_r} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W}{\varepsilon} \right) + r \int_{\partial B_r} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W}{\varepsilon} \right) \\ = \int_{B_r} \left(\frac{W}{\varepsilon} - \frac{\varepsilon |\nabla u|^2}{2} \right) + \frac{\varepsilon}{r} \int_{\partial B_r} (y \cdot \nabla u)^2 + \varepsilon \int_{B_r} (\nabla u \cdot v)(\nabla u \cdot y). \end{aligned}$$

By dividing the above equation by r^n , the lemma follows. $\square$

We need the following lemma to control the negative contribution of the right-hand side of (3.17).

Lemma 3.4 *Given $0 < s < 1$, there exist constants $0 < \beta_1 < 1$ and $0 < \varepsilon_2 < 1$ which depend only on c_0, λ_0, W, n, p and s such that, if $\varepsilon \leq \varepsilon_2$,*

$$\sup_{B_s} \left(\frac{\varepsilon}{2} |\nabla u|^2 - \frac{W(u)}{\varepsilon} \right) \leq \varepsilon^{-\beta_1}. \quad (3.19)$$

The proof of Lemma 3.4 is deferred to the end of this section. Next, for $B_r(x) \subset U_1$ and $0 < r < \text{dist}(x, \partial U_1)$, define

$$E(r, x) := \frac{1}{r^{n-1}} \int_{B_r(x)} \left(\frac{\varepsilon |\nabla u|^2}{2} + \frac{W(u)}{\varepsilon} \right).$$

Using Lemmas 3.3 and 3.4, we prove:

Lemma 3.5 *Given $0 < s < 1$, there exist constants $0 < \varepsilon_3, c_4, c_5 < 1$ which depend only on c_0, λ_0, W, n, p and s such that, if $B_{\varepsilon\beta_1}(x) \subset U_s$, $|u(x)| \leq \alpha$ and $\varepsilon \leq \varepsilon_3$, then*

$$E(r, x) \geq c_4 \quad \text{for all } \varepsilon \leq r \leq c_5 \varepsilon^{\beta_1}. \quad (3.20)$$

Proof By integrating (3.17) over $[\varepsilon, r]$, we have

$$\begin{aligned} E(r, x) - E(\varepsilon, x) &\geq - \int_{\varepsilon}^r \frac{d\tau}{\tau^n} \int_{B_{\tau}(x)} \left(\frac{\varepsilon}{2} |\nabla u|^2 - \frac{W(u)}{\varepsilon} \right)_+ \\ &\quad + \int_{\varepsilon}^r \frac{d\tau}{\tau^n} \int_{B_{\tau}(x)} \varepsilon (\nabla u \cdot v)(\nabla u \cdot (y - x)). \end{aligned} \quad (3.21)$$

By (3.19) and $B_r(x) \subset U_s$, we have

$$\int_{\varepsilon}^r \frac{d\tau}{\tau^n} \int_{B_{\tau}(x)} \left(\frac{\varepsilon}{2} |\nabla u|^2 - \frac{W(u)}{\varepsilon} \right)_+ \leq \omega_n r \varepsilon^{-\beta_1}. \quad (3.22)$$

By (3.1) and (2.5), we have

$$\begin{aligned} \left| \int_{\varepsilon}^r \frac{d\tau}{\tau^n} \int_{B_{\tau}(x)} \varepsilon (\nabla u \cdot v)(\nabla u \cdot (y - x)) \right| &\leq \int_0^r \frac{d\tau}{\tau^{n-1}} \int_{B_{\tau}(x)} c_1^2 \varepsilon^{-1} |v| \\ &\leq c(\lambda_0, n, p, c_1) r^{3-\frac{n}{p}} \varepsilon^{-1}. \end{aligned} \quad (3.23)$$

Since $|u(x)| \leq \alpha$, using (3.1), we have $|u(y)| \leq \frac{\alpha+1}{2}$ for all $y \in B_{\frac{(1-\alpha)\varepsilon}{2c_1}}(x)$. By choosing a larger c_1 if necessary, we may assume $\frac{(1-\alpha)}{2c_1} \leq 1$. Define

$$c_4 := \frac{\omega_n}{2} \frac{(1-\alpha)^n}{(2c_1)^n} \min_{|t| \leq \frac{1+\alpha}{2}} W(t) > 0.$$

With this choice, we obtain

$$\begin{aligned} E(\varepsilon, x) &\geq \frac{1}{\varepsilon^{n-1}} \int_{B_{\frac{(1-\alpha)\varepsilon}{2c_1}}(x)} \frac{W(u)}{\varepsilon} \\ &\geq \omega_n \frac{(1-\alpha)^n}{(2c_1)^n} \min_{|t| \leq \frac{1+\alpha}{2}} W(t) = 2c_4. \end{aligned} \quad (3.24)$$

Since a larger β_1 satisfies (3.19) as well, we may assume $(3 - \frac{n}{p})\beta_1 - 1 > 0$ by choosing $\beta_1 < 1$ sufficiently close to 1. Then we may show that the sum of (3.22) and (3.23) may be bounded from above by c_4 for sufficiently small ε and c_5 if $r \leq c_5 \varepsilon^{\beta_1}$. Then, (3.20) follows from (3.21)–(3.24). $\square$

Theorem 3.6 *Given $0 < s < 1$, there exist constants $0 < \varepsilon_4, \beta_2 < 1$ and $0 < c_6$ which depend only on c_0, λ_0, W, n, p and s such that, if $B_r(x) \subset U_s, c_5 \varepsilon^{\beta_1} < r$ and $\varepsilon \leq \varepsilon_4$, then*

$$\frac{1}{r^n} \int_{B_r(x)} \left(\frac{\varepsilon}{2} |\nabla u|^2 - \frac{W(u)}{\varepsilon} \right)_+ \leq \frac{c_6}{r^{1-\beta_2}} (E(r, x) + 1). \quad (3.25)$$

Proof The proof is similar to [25, Proposition 3.4] with a minor modification. Define $\beta_2 := \frac{1-\beta_1}{2\beta_1}$ and $\beta_3 := \frac{1+\beta_1}{2}$. β_2 and β_3 are chosen so that

$$\beta_1 \beta_2 = \beta_3 - \beta_1, \quad (3.26)$$

$$0 < \beta_2 < 1, \quad 0 < \beta_1 < \beta_3 < 1. \quad (3.27)$$

We estimate the integral of (3.25) by separating $B_r(x)$ into three disjoint sets. Define

$$\begin{aligned} \mathcal{A} &:= B_r(x) \setminus B_{r-\varepsilon\beta_3}(x), \\ \mathcal{B} &:= \{y \in B_{r-\varepsilon\beta_3}(x) : \text{dist}(\{|u| \leq \alpha\}, y) < \varepsilon^{\beta_3}\}, \\ \mathcal{C} &:= \{y \in B_{r-\varepsilon\beta_3}(x) : \text{dist}(\{|u| \leq \alpha\}, y) \geq \varepsilon^{\beta_3}\}. \end{aligned}$$

Note that $r > c_5 \varepsilon^{\beta_1} > \varepsilon^{\beta_3}$ for small ε .

The estimates of the integral over $\mathcal{A}$ and $\mathcal{B}$ are exactly the same as in [25]. Namely, for $\mathcal{A}$, we use $\mathcal{L}^n(\mathcal{A}) \leq n\omega_n r^{n-1} \varepsilon^{\beta_3}$ and (3.19) as well as $r^{-1} \leq (c_5 \varepsilon^{\beta_1})^{-1}$. For $\mathcal{B}$, we use (3.20) and prove that $\mathcal{L}^n(\mathcal{B}) \leq c(n) \varepsilon^{n\beta_3} N$, where N is an integer satisfying $\mathcal{B} \subset \cup_{i=1}^N B_{5\varepsilon\beta_3}(x_i)$. Here we only consider the estimate on $\mathcal{C}$ and refer the reader to the proof of [25, Proposition 3.4]. Define

$$\phi(z) := \min \{1, \varepsilon^{-\beta_3} \text{dist}(\{y : |y - x| \geq r\} \cup \{|u| \leq \alpha\}, z)\}.$$

ϕ is a Lipschitz function and is 0 on $\{y : |y - x| > r\} \cup \{|u| < \alpha\}$, 1 on $\mathcal{C}$ and $|\nabla \phi| \leq \varepsilon^{-\beta_3}$. Differentiate (2.1) with respect to x_j , multiply it by $u_{x_j} \phi^2$ and sum over j . Then we have

$$\int \sum_j \varepsilon u_{x_j} \Delta u_{x_j} \phi^2 = \int \frac{W''}{\varepsilon} |\nabla u|^2 \phi^2 - \varepsilon \sum_j \int (v \cdot \nabla u)_{x_j} \phi^2 u_{x_j}. \quad (3.28)$$

By integration by parts, the Cauchy–Schwarz inequality and (3.1), we obtain

$$\begin{aligned} & \int \varepsilon |\nabla^2 u|^2 \phi^2 + \frac{W''}{\varepsilon} |\nabla u|^2 \phi^2 \\ &= \int - \sum_{i,j} 2\varepsilon u_{x_j} u_{x_i x_j} \phi \phi_{x_i} - \varepsilon \nabla u \cdot v (\Delta u \phi^2 + 2\phi \nabla u \cdot \nabla \phi) \\ &\leq \frac{1}{2} \int \varepsilon |\nabla^2 u|^2 \phi^2 + c_7 \int (\varepsilon |\nabla u|^2 |\nabla \phi|^2 + |v|^2 \phi^2 \varepsilon^{-1}), \end{aligned} \quad (3.29)$$

where c_7 depends on c_1 . Since $|u| \geq \alpha$ on the support of ϕ , we have $W'' \geq \kappa$. Thus

$$\int \frac{\varepsilon}{2} |\nabla^2 u|^2 \phi^2 + \frac{\kappa}{\varepsilon} |\nabla u|^2 \phi^2 \leq c_7 \int (\varepsilon |\nabla u|^2 |\nabla \phi|^2 + |v|^2 \phi^2 \varepsilon^{-1}). \quad (3.30)$$

By $|\nabla\phi| \leq \varepsilon^{-\beta_3}$ and (2.5), we have

$$\begin{aligned} \int \frac{\kappa}{\varepsilon} |\nabla u|^2 \phi^2 &\leq c_7 \left(\varepsilon^{-2\beta_3} \int_{B_r} \varepsilon |\nabla u|^2 + \varepsilon^{-1} \|v\|_{L^{\frac{np}{n-p}}}^2 (\omega_n r^n)^{\frac{np-2(n-p)}{np}} \right) \\ &\leq c_8 \left(\varepsilon^{-2\beta_3} \int_{B_r} \varepsilon |\nabla u|^2 + \varepsilon^{-1} r^{n-\frac{2(n-p)}{p}} \right), \end{aligned} \quad (3.31)$$

where $c_8 = c_7 + c(n, p)\lambda_0^2$. Since $\phi = 1$ on $\mathcal{C}$, multiplying (3.31) by $\frac{\varepsilon^2}{\kappa r^n}$,

$$\frac{1}{r^n} \int_{\mathcal{C}} \frac{\varepsilon}{2} |\nabla u|^2 \leq \frac{c_8}{\kappa} \left(\frac{\varepsilon^{2-2\beta_3}}{r} E(r, x) + \varepsilon r^{2-\frac{2n}{p}} \right). \quad (3.32)$$

By the definitions of $\beta_1, \beta_2, \beta_3$ and $r \geq c_5 \varepsilon^{\beta_1}$, we have

$$\frac{\varepsilon^{2-2\beta_3}}{r} \leq \frac{\varepsilon^{\beta_1\beta_2}}{r^{1-\beta_2} c_5^{\beta_2}}, \quad (3.33)$$

and using $\varepsilon \leq r$,

$$\varepsilon r^{2-\frac{2n}{p}} \leq \frac{1}{r^{\frac{2n}{p}-3}}, \quad (3.34)$$

where $\frac{2n}{p} - 3 < 1$. Hence, we obtain

$$\frac{1}{r^n} \int_{\mathcal{C}} \frac{\varepsilon}{2} |\nabla u|^2 \leq \frac{c_8}{\kappa} \left(\frac{\varepsilon^{\beta_1\beta_2}}{r^{1-\beta_2} c_5^{\beta_2}} E(r, x) + \frac{1}{r^{\frac{2n}{p}-3}} \right). \quad (3.35)$$

By re-defining $\beta_2 = \min\{\beta_2, 4 - \frac{2n}{p}\}$ and the estimates of integrals over $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, we proved (3.25). $\square$

To proceed, we need the following theorem (see [30, Theorem 5.12.4]).

Theorem 3.7 *Let μ be a positive Radon measure on $\mathbb{R}^n$ satisfying*

$$K(\mu) := \sup_{B_r(x) \subset \mathbb{R}^n} \frac{1}{r^{n-1}} \mu(B_r(x)) < \infty.$$

Then there exists a constant $c(n)$ such that

$$\left| \int_{\mathbb{R}^n} \phi \, d\mu \right| \leq c(n) K(\mu) \int_{\mathbb{R}^n} |\nabla \phi| \, d\mathcal{L}^n$$

for all $\phi \in C_c^1(\mathbb{R}^n)$.

Theorem 3.8 *There exists a constant $0 < c_9$ which depends only on $c_0, \lambda_0, W, E_0, n$ and p such that, if $0 < \varepsilon < 1/2$ and $U_{2r}(x) \subset U_{1-\varepsilon}$, then*

$$\text{dist}(x, \partial U_{1-\varepsilon})^{n-1} E(r, x) \leq c_9. \quad (3.36)$$

Proof Define

$$E_1 := \sup_{U_{2r}(x) \subset U_{1-\varepsilon}} \text{dist}(x, \partial U_{1-\varepsilon})^{n-1} E(r, x).$$

By Lemma 3.1, we have $\sup_{x \in U_{1-\varepsilon}} \varepsilon |\nabla u(x)| \leq c_1$. Thus for any $U_{2r}(x) \subset U_{1-\varepsilon}$, we have

$$E(r, x) \leq \omega_n r \left(\frac{c_1^2}{2\varepsilon} + \sup_{|t| \leq c_0} \frac{W(t)}{\varepsilon} \right) \leq \frac{c(n, c_1, W)}{\varepsilon}$$

and we have $E_1 < \infty$ for each ε . In the following, we give an estimate of E_1 depending only on c_0, λ_0, n, p, W and E_0 . Let $U_{2r_0}(x_0) \subset U_{1-\varepsilon}$ be fixed such that

$$\text{dist}(x_0, \partial U_{1-\varepsilon})^{n-1} E(r_0, x_0) > \frac{3}{4} E_1. \quad (3.37)$$

For simplicity, define

$$l := \text{dist}(x_0, \partial U_{1-\varepsilon}) = 1 - \varepsilon - |x_0|$$

and change variables by $\tilde{x} = (x - x_0)/l$, $\tilde{r} = r/l$, $\tilde{\varepsilon} = \varepsilon/l$, $\tilde{u}(\tilde{x}) = u(x)$ and $\tilde{v}(\tilde{x}) = lv(x)$. Note that $U_{l+\varepsilon}(x_0) \subset U_1$. In particular, we write

$$\tilde{r}_0 := r_0/l \leq 1/2.$$

By (2.1), (2.4) and (2.5), we have

$$-\tilde{\varepsilon} \Delta \tilde{u} + \frac{W'(\tilde{u})}{\tilde{\varepsilon}} = \tilde{\varepsilon} \tilde{v} \cdot \nabla \tilde{u} \quad \text{for } \tilde{x} \in U_{1+\tilde{\varepsilon}}, \quad (3.38)$$

$$\int_{U_{1+\tilde{\varepsilon}}} \left(\frac{\tilde{\varepsilon} |\nabla \tilde{u}|^2}{2} + \frac{W(\tilde{u})}{\tilde{\varepsilon}} \right) \leq l^{1-n} E_0, \quad (3.39)$$

$$\|\tilde{v}\|_{L^{\frac{np}{n-p}}(U_{1+\tilde{\varepsilon}})} + \|\nabla \tilde{v}\|_{L^p(U_{1+\tilde{\varepsilon}})} \leq l^{2-\frac{n}{p}} \lambda_0. \quad (3.40)$$

Define for $B_{\tilde{r}}(\tilde{x}) \subset U_{1+\tilde{\varepsilon}}$

$$\tilde{E}(\tilde{r}, \tilde{x}) := \frac{1}{\tilde{r}^{n-1}} \int_{B_{\tilde{r}}(\tilde{x})} \left(\frac{\tilde{\varepsilon} |\nabla \tilde{u}|^2}{2} + \frac{W(\tilde{u})}{\tilde{\varepsilon}} \right). \quad (3.41)$$

Under the above change of variables, note that we have $E(r, x) = \tilde{E}(\tilde{r}, \tilde{x})$. Next, for any $x \in B_{3l/4}(x_0)$, we have $\text{dist}(x, \partial U_{1-\varepsilon}) \geq l/4$. Hence for any $x \in B_{3l/4}(x_0)$ and $r < l/8 \leq \text{dist}(x, \partial U_{1-\varepsilon})/2$, by the definition of E_1 , we have

$$\text{dist}(x, \partial U_{1-\varepsilon})^{n-1} E(r, x) \leq E_1.$$

This shows (again using $\text{dist}(x, \partial U_{1-\varepsilon}) \geq l/4$)

$$\sup_{\tilde{x} \in B_{\frac{3}{4}}, 0 < \tilde{r} < \frac{1}{8}} \tilde{E}(\tilde{r}, \tilde{x}) \leq 4^{n-1} l^{1-n} E_1. \quad (3.42)$$

We next let $c_1, c_4, c_5, c_6, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \beta_1, \beta_2$ be constants obtained in Lemma 3.1–3.5 and Theorem 3.6 corresponding to the same c_0, λ_0, n, p, W and $s = 3/4$. Then note that the estimates up to Theorem 3.6 hold for $\tilde{u}$ and $\tilde{v}$ for $U_{3/4}$ and with respect to the new variables $(\tilde{x}, \tilde{r}, \tilde{\varepsilon}$ etc.) if

$$\tilde{\varepsilon} \leq \hat{\varepsilon} := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 1/2\} \quad (3.43)$$

due to (3.38) and (3.40). It is important to note that $\hat{\varepsilon}$ depends only on c_0, λ_0, n, p and W . Note that (3.40) yields an upper bound for $\|\tilde{v}\|_{L^{\frac{np}{n-p}}(U_{1+\tilde{\varepsilon}})} + \|\nabla \tilde{v}\|_{L^p(U_{1+\tilde{\varepsilon}})}$ independent of l ,

as $l < 1$ and $2 - \frac{n}{p} > 0$. A priori, we do not know if (3.43) holds or not and we prove the desired estimate for E_1 by exhausting all the possibilities.

First consider the case $\tilde{\varepsilon} \geq \hat{\varepsilon}$. We use (3.3) and (3.1), respectively, for $\tilde{\varepsilon} > 1/2$ and $1/2 \geq \tilde{\varepsilon} \geq \hat{\varepsilon}$. Suppose that $\tilde{\varepsilon} > 1/2$. By (3.37) and (3.3), we have

$$\frac{3}{4l^{n-1}} E_1 < \tilde{E}(\tilde{r}_0, 0) \leq \omega_n \tilde{r}_0 (\tilde{\varepsilon} c_1^2 + 2 \sup_{|x| \leq c_0} W(x))$$

and since $l\tilde{\varepsilon} = \varepsilon \leq 1$, $l < 1$ and $\tilde{r}_0 \leq 1/2$, we obtain

$$E_1 < \frac{4}{3} \omega_n (c_1^2 + 2 \sup_{|x| \leq c_0} W(x)). \quad (3.44)$$

If $1/2 \geq \tilde{\varepsilon} \geq \hat{\varepsilon}$, again by (3.37), (3.1) and $\tilde{r}_0 \leq 1/2$, we have

$$\begin{aligned} \frac{3}{4l^{n-1}} E_1 &< \tilde{E}(\tilde{r}_0, 0) \leq \omega_n (c_1^2 + \sup_{|x| \leq c_0} W(x)) \frac{\tilde{r}_0}{\tilde{\varepsilon}} \\ &\leq \omega_n (c_1^2 + \sup_{|x| \leq c_0} W(x)) \frac{1}{2\hat{\varepsilon}} \end{aligned}$$

and we obtain

$$E_1 < \omega_n (c_1^2 + \sup_{|x| \leq c_0} W(x)) \frac{2}{3\hat{\varepsilon}}. \quad (3.45)$$

Thus by (3.44) and (3.45), if $\tilde{\varepsilon} \geq \hat{\varepsilon}$, E_1 is bounded by a constant which depends only on c_0, λ_0, n, p and W .

For the rest of the proof, consider the case $\tilde{\varepsilon} < \hat{\varepsilon}$ and consider the following four cases (a)–(d) depending on how large $\tilde{r}_0 = r_0/l$ is relative to $\tilde{\varepsilon}$ and $\tilde{r}_1$, where $\tilde{r}_1$ will be determined shortly depending only on c_0, λ_0, n, p, W and E_0 :

$$(a) \tilde{r}_1 < \tilde{r}_0 \leq \frac{1}{2}, \quad (b) c_5 \tilde{\varepsilon}^{\beta_1} < \tilde{r}_0 \leq \tilde{r}_1, \quad (c) \tilde{\varepsilon} < \tilde{r}_0 \leq c_5 \tilde{\varepsilon}^{\beta_1}, \quad (d) 0 < \tilde{r}_0 \leq \tilde{\varepsilon}.$$

To control the term involving v in (3.17), define a Radon measure

$$\mu(A) := \int_{A \cap B_{\frac{3}{4}}} \left(\frac{\tilde{\varepsilon} |\nabla \tilde{u}|^2}{2} + \frac{W(\tilde{u})}{\tilde{\varepsilon}} \right).$$

By Theorem 3.7 and (3.42) (note that (3.42) has the restriction $\tilde{r} < 1/8$ but this can be dropped easily by replacing 4^{n-1} by a larger constant depending only on n), we have

$$\left| \int_{B_{\frac{3}{4}}} \phi \, d\mu \right| \leq c(n) l^{1-n} E_1 \int_{\mathbb{R}^n} |\nabla \phi| \, d\mathcal{L}^n \quad (3.46)$$

for all $\phi \in C_c^1(\mathbb{R}^n)$. By (3.17) and (3.25), if $c_5 \tilde{\varepsilon}^{\beta_1} < \tilde{r} \leq \frac{1}{2}$, we have

$$\begin{aligned} \frac{d}{d\tilde{r}} \tilde{E}(\tilde{r}, 0) &\geq -\frac{c_6}{\tilde{r}^{1-\beta_2}} (\tilde{E}(\tilde{r}, 0) + 1) - \frac{1}{\tilde{r}^{n-1}} \int_{B_{\tilde{r}}} \tilde{\varepsilon} |\tilde{v}| |\nabla \tilde{u}|^2 \\ &\quad + \frac{1}{\tilde{r}^n} \int_{B_{\tilde{r}}} \left(\frac{W(\tilde{u})}{\tilde{\varepsilon}} - \frac{\tilde{\varepsilon}}{2} |\nabla \tilde{u}|^2 \right). \end{aligned} \quad (3.47)$$

Let $\phi \in C_c^1(B_{\frac{3\tilde{r}}{2}})$ be such that $0 \leq \phi \leq 1$, $\phi(y) = 1$ in $B_{\tilde{r}}$ and $|\nabla \phi| \leq \frac{4}{\tilde{r}}$. We use (3.46) with (3.40) by smoothly approximating $|\tilde{v}|$ as

$$\begin{aligned} \int_{B_{\tilde{r}}} \tilde{\varepsilon} |\tilde{v}| |\nabla \tilde{u}|^2 &\leq \int_{B_{\frac{3\tilde{r}}{2}}} \tilde{\varepsilon} \phi |\tilde{v}| |\nabla \tilde{u}|^2 \leq c(n) l^{1-n} E_1 \int_{B_{\frac{3\tilde{r}}{2}}} |\nabla(\phi |\tilde{v}|)| \\ &\leq c(n) l^{1-n} E_1 \int_{B_{\frac{3\tilde{r}}{2}}} \frac{4}{\tilde{r}} |\tilde{v}| + |\nabla \tilde{v}| \leq c(n) l^{3-n-\frac{n}{p}} \lambda_0 \tilde{r}^{n-\frac{n}{p}} E_1. \end{aligned} \quad (3.48)$$

Hence, for $c_5 \tilde{\varepsilon}^{\beta_1} < \tilde{r} \leq \frac{1}{2}$, (3.47) with (3.48) and (3.42) give

$$\begin{aligned} \frac{d}{d\tilde{r}} \tilde{E}(\tilde{r}, 0) &\geq -c_6 \tilde{r}^{\beta_2-1} (c(n) l^{1-n} E_1 + 1) - c(n) l^{3-n-\frac{n}{p}} \lambda_0 \tilde{r}^{1-\frac{n}{p}} E_1 \\ &\quad + \frac{1}{\tilde{r}^n} \int_{B_{\tilde{r}}} \left(\frac{W(\tilde{u})}{\tilde{\varepsilon}} - \frac{\tilde{\varepsilon}}{2} |\nabla \tilde{u}|^2 \right)_+. \end{aligned} \quad (3.49)$$

By integrating (3.49) over $\tilde{r} \in (\tilde{s}_1, \tilde{s}_2)$ with $c_5 \tilde{\varepsilon}^{\beta_1} < \tilde{s}_1 < \tilde{s}_2 \leq \frac{1}{2}$, we obtain

$$\begin{aligned} \tilde{E}(\tilde{s}_2, 0) - \tilde{E}(\tilde{s}_1, 0) &\geq -c_{10} (\tilde{s}_2^{\beta_2} + l^{2-\frac{n}{p}} \tilde{s}_2^{2-\frac{n}{p}}) l^{1-n} E_1 - c_{10} \tilde{s}_2^{\beta_2} \\ &\quad + \int_{\tilde{s}_1}^{\tilde{s}_2} \frac{d\tilde{r}}{\tilde{r}^n} \int_{B_{\tilde{r}}} \left(\frac{W(\tilde{u})}{\tilde{\varepsilon}} - \frac{\tilde{\varepsilon}}{2} |\nabla \tilde{u}|^2 \right)_+, \end{aligned} \quad (3.50)$$

where c_{10} depends only on c_0, λ_0, n, p and W . At this point, we choose $\tilde{r}_1 < 1/2$ depending only on c_0, λ_0, n, p and W so that

$$c_{10} \left(\tilde{r}_1^{\beta_2} + \tilde{r}_1^{2-\frac{n}{p}} \right) < \frac{1}{4}.$$

This in particular implies from (3.50) that if $c_5 \tilde{\varepsilon}^{\beta_1} < \tilde{s}_1 < \tilde{s}_2 \leq \tilde{r}_1$, then

$$\tilde{E}(\tilde{s}_2, 0) - \tilde{E}(\tilde{s}_1, 0) \geq -c_{10} - \frac{1}{4} l^{1-n} E_1. \quad (3.51)$$

With this $\tilde{r}_1$ being fixed, we proceed to check that E_1 is bounded in terms of $c_0, \lambda_0, n, p, W, E_0$ in each case (a)–(d).

Case (a): By (3.37), (3.39) and $\tilde{r}_1 < \tilde{r}_0$, we have

$$\frac{3}{4} l^{1-n} E_1 \leq \tilde{E}(\tilde{r}_0, 0) \leq \tilde{r}_0^{1-n} l^{1-n} E_0 \leq \tilde{r}_1^{1-n} l^{1-n} E_0.$$

Hence

$$E_1 \leq \frac{4}{3} \tilde{r}_1^{1-n} E_0$$

and E_1 is bounded by a constant depending only on c_0, λ_0, n, p, W and E_0 .

Case (b): Since $c_5 \tilde{\varepsilon}^{\beta_1} < \tilde{r}_0 \leq \tilde{r}_1$, we may use (3.51) with $\tilde{s}_2 = \tilde{r}_1$ and $\tilde{s}_1 = \tilde{r}_0$. Then we obtain

$$\tilde{E}(\tilde{r}_1, 0) - \tilde{E}(\tilde{r}_0, 0) \geq -c_{10} - \frac{1}{4} l^{1-n} E_1.$$

Then, by (3.37) and (3.39), we obtain

$$E_1 \leq 4 \left(\tilde{E}(\tilde{r}_1, 0) + c_{10} \right) l^{n-1} \leq 4(\tilde{r}_1^{1-n} E_0 + c_{10}),$$

which depends only on c_0, λ_0, n, p, W and E_0 .

Case (c): By the same estimate used in the proof of Lemma 3.5, we have

$$\tilde{E}(c_5 \tilde{\varepsilon}^{\beta_1}, 0) - \tilde{E}(\tilde{r}_0, 0) \geq -c_4. \quad (3.52)$$

We use (3.51) with $\tilde{s}_1 = c_5 \tilde{\varepsilon}^{\beta_1}$ and $\tilde{s}_2 = \tilde{r}_1$ to obtain

$$\tilde{E}(\tilde{r}_1, 0) - \tilde{E}(c_5 \tilde{\varepsilon}^{\beta_1}, 0) \geq -c_{10} - \frac{1}{4} l^{1-n} E_1, \quad (3.53)$$

and (3.52) and (3.53) combined with (3.37) give

$$E_1 \leq 4l^{n-1} (\tilde{E}(\tilde{r}_1, 0) + c_4 + c_{10}) \leq 4\tilde{r}_1^{1-n} E_0 + 4(c_4 + c_{10}),$$

which depends only on c_0, λ_0, n, p, W and E_0 .

Case (d): Since $\tilde{r}_0 \leq \tilde{\varepsilon}$, we use (3.1) to obtain

$$\tilde{E}(\tilde{r}_0, 0) \leq \omega_n \left(c_1^2 + \sup_{|x| \leq c_0} W(x) \right) \frac{\tilde{r}_0}{\tilde{\varepsilon}} \leq \omega_n \left(c_1^2 + \sup_{|x| \leq c_0} W(x) \right). \quad (3.54)$$

Then (3.54) and (3.37) gives

$$E_1 < \frac{4}{3} \omega_n \left(c_1^2 + \sup_{|x| \leq c_0} W(x) \right) l^{n-1} \leq \omega_n \left(c_1^2 + \sup_{|x| \leq c_0} W(x) \right).$$

This completes the estimate for E_1 . $\square$

Once we obtain the upper density estimate, we may obtain the following monotonicity formula.

Theorem 3.9 *Given $0 < s < 1$, there exist constants $0 < c_{11}$ and $0 < \varepsilon_5 < 1$ depending only on $c_0, \lambda_0, n, p, W, E_0$ and s , such that, if $c_5 \varepsilon^{\beta_1} \leq s_1 < s_2$, $B_{s_2}(x) \subset U_s$ and $\varepsilon < \varepsilon_5$, then*

$$\begin{aligned} E(s_2, x) - E(s_1, x) &\geq -c_{11} \left(s_2^{\frac{2-n}{p}} + s_2^{\beta_2} \right) \\ &\quad + \int_{s_1}^{s_2} \frac{d\tau}{\tau^n} \int_{B_\tau(x)} \left(\frac{W(u)}{\varepsilon} - \frac{\varepsilon}{2} |\nabla u|^2 \right)_+. \end{aligned} \quad (3.55)$$

Proof Let $\varepsilon_5 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, (1-s)/2\}$ corresponding to the given s and suppose that $\varepsilon < \varepsilon_5$. For any $x \in U_s$ and $0 < r < (1-s)/2$, by Theorem 3.8, $E(r, x) \leq c_9(1-s-\varepsilon)^{1-n}$, where the right-hand side is bounded by a constant depending only on $c_0, \lambda_0, n, p, W, E_0$ and s . For $B_{s_2}(x) \subset U_s$, we have (3.25) and (3.17). Arguing as (3.46)–(3.50) without change of variables (so $l = 1$) and with μ restricted to B_s in place of $B_{3/4}$, we obtain (3.55). $\square$

Theorem 3.10 *Given $0 < s < 1$, there exist constants $0 < c_{12}$ depending only on $c_0, \lambda_0, n, p, W, E_0$ and s such that, if $\varepsilon < \varepsilon_5$, $|u(x)| \leq \alpha$, $\varepsilon \leq r$ and $B_r(x) \subset U_s$, then*

$$E(r, x) \geq c_{12}. \quad (3.56)$$

Proof By Lemma 3.5, we may assume $c_5 \varepsilon^{\beta_1} \leq r$ and $E(c_5 \varepsilon^{\beta_1}, x) \geq c_4$. In (3.55), let $s_1 = c_5 \varepsilon^{\beta_1}$ and $s_2 = r$. Fix $r_1 > 0$ depending only on $c_0, \lambda_0, n, p, W, E_0$ and s so that $c_{11}(r_1^{2-\frac{n}{p}} + r_1^{\beta_2}) \leq c_4/2$. Then for $c_5 \varepsilon^{\beta_1} \leq r \leq r_1$, (3.55) shows that $E(r, x) \geq c_4/2$. For $1 > r > r_1$, $E(r, x) \geq r_1^{n-1} E(r_1, x) \geq r_1^{n-1} c_4/2$. Thus, setting $c_{12} = r_1^{n-1} c_4/2$, we have (3.56). $\square$

For the rest of the present section, we finish the proof of Lemma 3.4. We use the following result proved in [25, Lemma 3.9].

Lemma 3.11 *Given $0 < \eta, \beta_4 < 1$, $\eta \leq \beta_4$, $0 < c_{13}$, there exist $\varepsilon_6 > 0$, $c_{14} > 0$ depending only on η, β_4, c_{13}, n and W with the following properties: Suppose $f \in C^3(U_{\varepsilon^{-\beta_4}})$, $g \in C^1(U_{\varepsilon^{-\beta_4}})$ and $0 < \varepsilon \leq \varepsilon_6$ satisfy*

$$-\Delta f + W'(f) = g$$

on $U_{\varepsilon^{-\beta_4}}$ and

$$\sup_{U_{\varepsilon^{-\beta_4}}} |f| \leq 1 + \varepsilon^\eta, \quad \sup_{U_{\varepsilon^{-\beta_4}}} \left(\frac{1}{2} |\nabla f|^2 - W(f) \right) \leq c_{13}.$$

Then

$$\sup_{B_{\frac{\varepsilon^{-\beta_4}}{2}}} \left(\frac{1}{2} |\nabla f|^2 - W(f) \right) \leq c_{13} (\varepsilon^{-\beta_4} \|g\|_{W^{1,n}(B_{\varepsilon^{-\beta_4}})} + \varepsilon^\eta). \quad (3.57)$$

We remark that the assumptions on W are essentially used for the proof of Lemma 3.11.

Proof of Lemma 3.4 As in the proof of Lemma 3.1, define $\tilde{u}(x) := u(\varepsilon x)$, $\tilde{v}(x) := \varepsilon v(\varepsilon x)$, and subsequently drop $\tilde{\cdot}$ for simplicity. We have

$$-\Delta u + W'(u) = \nabla u \cdot v$$

on $U_{\varepsilon^{-1}}$. With respect to the new variables, we need to prove

$$\sup_{U_{\varepsilon^{-1}}} \left(\frac{1}{2} |\nabla u|^2 - W(u) \right) \leq \varepsilon^{1-\beta_1} \quad (3.58)$$

for some $0 < \beta_1 < 1$ for all sufficiently small ε . Let ϕ_λ be the standard mollifier, namely, define

$$\phi(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where the constant $C > 0$ is selected so that $\int_{\mathbb{R}^n} \phi = 1$, and define $\phi_\lambda(x) := \frac{1}{\lambda^n} \phi(\frac{x}{\lambda})$. For $0 < \beta_5 < 1$ to be chosen depending only on n and p later, define for $x \in U_{\varepsilon^{-1-1}}$

$$f(x) := (u * \phi_{\varepsilon^{\beta_5}})(x) = \int u(x-y) \phi_{\varepsilon^{\beta_5}}(y) dy. \quad (3.59)$$

By (3.1) and (3.2), we have

$$\sup_{U_{\varepsilon^{-1-1}}} |f - u| \leq 2c_1 \varepsilon^{\beta_5}, \quad (3.60)$$

$$\sup_{U_{\varepsilon^{-1-1}}} |\nabla f - \nabla u| \leq 2c_1 \varepsilon^{\beta_5} \left(2 - \frac{n}{p} \right). \quad (3.61)$$

We next define g to be

$$g := (\nabla u \cdot v) * \phi_{\varepsilon\beta_5} + (W'(f) - W'(u) * \phi_{\varepsilon\beta_5}), \quad (3.62)$$

so that we have

$$-\Delta f + W'(f) = g. \quad (3.63)$$

To use Lemma 3.11, we next estimate the $W^{1,n}$ norm of g on $U_{\varepsilon^{-\beta_4}}(x)$ with $x \in U_{s\varepsilon^{-1}}$, where $0 < \beta_4 < 1$ will be chosen depending only on n and p . In the following, let us write $U_{\varepsilon^{-\beta_4}}(x)$ as $U_{\varepsilon^{-\beta_4}}$ and $U_{\frac{\varepsilon^{-\beta_4}}{2}}(x)$ as $U_{\frac{\varepsilon^{-\beta_4}}{2}}$ for simplicity. For sufficiently small ε depending on s and β_4 so that $U_{2\varepsilon^{-\beta_4}}(x) \subset U_{\varepsilon^{-1-1}}$ (and so that we may use (3.1)), the first term of (3.62) can be estimated as

$$\|(\nabla u \cdot v) * \phi_{\varepsilon\beta_5}\|_{W^{1,n}(U_{\varepsilon^{-\beta_4}})} \leq c_{15}(1 + \varepsilon^{-\beta_5})\|v\|_{L^n(U_{2\varepsilon^{-\beta_4}})} \quad (3.64)$$

where c_{15} depends only on ϕ , n and c_1 . By (3.5), we obtain

$$\begin{aligned} \|v\|_{L^n(U_{2\varepsilon^{-\beta_4}})} &\leq \|v\|_{L^{\frac{np}{n-p}}(U_{2\varepsilon^{-\beta_4}})} \{\omega_n(2\varepsilon^{-\beta_4})^n\}^{\frac{2p-n}{np}} \\ &\leq \lambda_0 \varepsilon^{(2-\frac{n}{p})(1-\beta_4)} (2^n \omega_n)^{\frac{2p-n}{np}}. \end{aligned} \quad (3.65)$$

Thus (3.64) and (3.65) show

$$\|(\nabla u \cdot v) * \phi_{\varepsilon\beta_5}\|_{W^{1,n}(U_{\varepsilon^{-\beta_4}})} \leq c_{16} \varepsilon^{(2-\frac{n}{p})(1-\beta_4)-\beta_5} \quad (3.66)$$

where c_{16} depends only on ϕ , n , p , λ_0 and c_1 . We next consider the second term of (3.62). By (3.60), (3.61) and

$$W'(f) - W'(u) * \phi_{\varepsilon\beta_5} = (W'(f) - W'(u)) + (W'(u) - W'(u) * \phi_{\varepsilon\beta_5}),$$

we compute

$$\sup |W'(f) - W'(u)| \leq \sup |W''| \sup |u - f| \leq c_{17} \varepsilon^{\beta_5}, \quad (3.67)$$

$$\begin{aligned} \sup |\nabla(W'(f) - W'(u))| &\leq \sup |W''| \sup |\nabla f - \nabla u| \\ &\quad + \sup |\nabla u| \sup |W'''| \sup |u - f| \\ &\leq c_{17} \varepsilon^{\beta_5} (2 - \frac{n}{p}), \end{aligned} \quad (3.68)$$

$$\sup |W'(u) - W'(u) * \phi_{\varepsilon\beta_5}| \leq c_{17} \varepsilon^{\beta_5}, \quad (3.69)$$

$$\sup |\nabla(W'(u) - W'(u) * \phi_{\varepsilon\beta_5})| \leq c_{17} \varepsilon^{\beta_5} (2 - \frac{n}{p}). \quad (3.70)$$

Here c_{17} depends only on ϕ , n , λ_0 , c_1 and W . Hence, (3.67)–(3.70) show

$$\|W'(f) - W'(u) * \phi_{\varepsilon\beta_5}\|_{W^{1,n}(U_{\varepsilon^{-\beta_4}})} \leq 4c_{17} \varepsilon^{\beta_5} (2 - \frac{n}{p}) - \beta_4. \quad (3.71)$$

By (3.62), (3.66) and (3.71), we have

$$\|g\|_{W^{1,n}(U_{\varepsilon^{-\beta_4}})} \leq c_{16} \varepsilon^{(2-\frac{n}{p})(1-\beta_4)-\beta_5} + 4c_{17} \varepsilon^{\beta_5} (2 - \frac{n}{p}) - \beta_4. \quad (3.72)$$

We use Lemma 3.11 to f and g . Due to Lemma 3.2, we have $\sup |f| \leq \sup |u| \leq 1 + \varepsilon^\eta$ on $U_{\varepsilon^{-\beta_4}}$ and we may choose smaller η if necessary. Because of (3.1), we have $c_{13} \geq \sup_{U_{\varepsilon^{-\beta_4}}} (\frac{1}{2} |\nabla f|^2 - W(f))$ for a constant depending only on c_1 and W (here again we

restrict ε small so that $U_{\varepsilon^{-\beta_4}}(x) \subset U_{\varepsilon^{-1-1}} \cap U_{\varepsilon^{-1}(1+s)/2}$. Then we have all the assumptions for Lemma 3.11 and obtain

$$\begin{aligned} \sup_{U_{\frac{\varepsilon^{-\beta_4}}{2}}} \left(\frac{1}{2} |\nabla f|^2 - W(f) \right) \\ \leq c_{14} (\varepsilon^{(2-\frac{n}{p})(1-\beta_4)-\beta_4-\beta_5} + \varepsilon^{\beta_5} (2 - \frac{n}{p}) - 2\beta_4 + \varepsilon^\eta). \end{aligned} \quad (3.73)$$

At this point, we fix sufficiently small $0 < \beta_4, \beta_5 < 1$ depending only on n and p such that

$$\left(2 - \frac{n}{p} \right) (1 - \beta_4) - \beta_4 - \beta_5 > 0, \quad \beta_5 \left(2 - \frac{n}{p} \right) - 2\beta_4 > 0.$$

This shows that we may choose a $0 < \beta_1 < 1$ such that

$$\sup_{U_{\frac{\varepsilon^{-\beta_4}}{2}}} \left(\frac{1}{2} |\nabla f|^2 - W(f) \right) \leq \varepsilon^{1-\beta_1} \quad (3.74)$$

for all sufficiently small $\varepsilon > 0$. We may take the center of $U_{\frac{\varepsilon^{-\beta_4}}{2}} (= U_{\frac{\varepsilon^{-\beta_4}}{2}}(x))$ to be any $x \in U_{s\varepsilon^{-1}}$ so that we have the estimate on $U_{s\varepsilon^{-1}}$. By (3.61), (3.74) and

$$\sup |W(f) - W(u)| \leq \sup |W'| \sup |u - f| \leq c_{18} \varepsilon^{\beta_5},$$

we may also replace f by u in (3.74) by choosing a larger $0 < \beta_1 < 1$ if necessary. This proves the desired estimate. $\square$

4 Rectifiability and integrality of the limit varifold

In this section, we recover the index i and assume that $\{u_i\}_{i=1}^\infty$, $\{v_i\}_{i=1}^\infty$ and $\{\varepsilon_i\}_{i=1}^n$ satisfy (2.1)–(2.5). Define μ_i and V_i as in (2.7) and (2.8). By the standard weak compactness theorem of Radon measures, there exists a subsequence (denoted by the same index) and a Radon measure μ and a varifold V such that

$$\mu_i \rightarrow \mu, \quad V_i \rightarrow V.$$

Lemma 4.1 *For $x \in \text{spt } \mu$, there exists a subsequence $x_i \in U_1$ such that $u_i(x_i) \in [-\alpha, \alpha]$ and $\lim_{i \rightarrow \infty} x_i = x$.*

Proof We prove this by contradiction and assume that there exists some $r > 0$ such that $|u_i| \geq \alpha$ on $U_r(x)$ for all large i . Without loss of generality, assume $u_i \geq \alpha$ on $U_r(x)$. Then we repeat the same argument leading to (3.30) with ϕ there replaced by $C_c^1(U_r(x))$. The argument shows that $\lim_{i \rightarrow \infty} \int \frac{\varepsilon_i}{2} |\nabla u_i|^2 \phi^2 = 0$. Next, multiplying $u_i - 1$ to the equation (2.1) and using $W'(u_i)(u_i - 1) \geq \frac{\kappa}{2} (u_i - 1)^2$, we obtain

$$\begin{aligned} \int \frac{W}{\varepsilon_i} \phi^2 &\leq c(W) \int \frac{(u_i - 1)^2}{\varepsilon_i} \phi^2 \leq \frac{2c(W)}{\kappa} \int \frac{W'(u_i)(u_i - 1)}{\varepsilon_i} \phi^2 \\ &= \frac{2c(W)}{\kappa} \int (\varepsilon_i \Delta u_i (u_i - 1) + \varepsilon_i (v_i \cdot \nabla u_i) (u_i - 1)) \phi^2. \end{aligned} \quad (4.1)$$

By integration by parts and (2.5), the right-hand side of (4.1) converges to 0. This shows that $\mu(U_r(x)) = 0$ and contradicts $x \in \text{spt } \mu$. $\square$

Theorem 4.2 *There exist constants $0 < D_1 \leq D_2 < \infty$ which depend only on $c_0, \lambda_0, n, p, W, E_0$ and s such that, for $x \in \text{spt } \mu \cap U_s$ and $B_r(x) \subset U_s$, we have*

$$D_1 r^{n-1} \leq \mu(B_r(x)) \leq D_2 r^{n-1}. \quad (4.2)$$

Proof This follows immediately from Theorem 3.8, 3.10 and Lemma 4.1. $\square$

For the subsequent use, define

$$\xi_i := \frac{\varepsilon_i |\nabla u_i|^2}{2} - \frac{W(u_i)}{\varepsilon_i}.$$

Once we have the monotonicity formula (3.55), we may prove the following “equi-partition of energy” by the same proof as in [22, Proposition 4.3]:

Theorem 4.3 $\xi_i, \frac{\varepsilon_i}{2} |\nabla u_i|^2 - |\nabla w_i|$ and $\frac{W(u_i)}{\varepsilon_i} - |\nabla w_i|$ all converge to zero in $L^1_{loc}(U_1)$.

Proof of Theorem 2.1 Recall that $\|V_i\| = \mu_i \llcorner_{\{|\nabla u_i| \neq 0\}}$. Since

$$\sigma \mu_i \llcorner_{\{|\nabla u_i| = 0\}} \leq |\xi_i| d\mathcal{L}^n \rightarrow 0$$

by Theorem 4.3, $\mu_i \llcorner_{\{|\nabla u_i| \neq 0\}}$ converges to μ . We also know that $\|V_i\|$ converges to $\|V\|$ by definition, thus we have $\|V\| = \mu$. This proves (1). The claims (2) and (3) follows from Theorem 3.8, 3.10 and Lemma 4.1 (see also [22, Proposition 4.2]). Next, by (2.9), (3.18) and Theorem 4.3,

$$\begin{aligned} \delta V_i(g) &= \int_{\{|\nabla u_i| \neq 0\}} \text{div } g \, d\mu_i \\ &\quad - \frac{1}{\sigma} \int_{\{|\nabla u_i| \neq 0\}} \nabla g \cdot \left(\frac{\nabla u_i}{|\nabla u_i|} \otimes \frac{\nabla u_i}{|\nabla u_i|} \right) (\varepsilon_i |\nabla u_i|^2 - \xi_i) \\ &= -\frac{1}{\sigma} \int \varepsilon_i (v_i \cdot \nabla u_i) (\nabla u_i \cdot g) + o(1) \end{aligned} \quad (4.3)$$

for $g \in C_c^1(U_1; \mathbb{R}^n)$, where $\lim_{i \rightarrow \infty} o(1) = 0$. By Theorem 3.8 and $\text{spt } g \subset U_s$ for some $0 < s < 1$, we have a uniform bound on $E(r, x)$ (corresponding to u_i) for $B_r(x) \subset U_s$. Hence, by Theorem 3.7, we have

$$\begin{aligned} \int \varepsilon_i ((v_i - v) \cdot \nabla u_i) (\nabla u_i \cdot g) &\leq c \left(\int |v_i - v|^p |g|^p \varepsilon_i |\nabla u_i|^2 \right)^{\frac{1}{p}} \\ &\leq c \left(\int |\nabla(|v_i - v|^p |g|^p)| \right)^{\frac{1}{p}} \\ &\leq c \left(\|\nabla v_i - \nabla v\|_{L^p} \|v_i - v\|_{L^p}^{p-1} + \|v_i - v\|_{L^p}^p \right)^{\frac{1}{p}} \end{aligned} \quad (4.4)$$

where the integrations are over $\text{spt } g$. The above converges to 0 since we may choose a further subsequence of v_i which converges to v strongly in L^p_{loc} . Thus in the right-hand side of (4.3), we may replace v_i by v . Let $\epsilon > 0$ be arbitrary and let $\tilde{v}$ be a smooth vector field such that $\|v - \tilde{v}\|_{W^{1,p}(U_s)} < \epsilon$. By the varifold convergence $V_i \rightarrow V$, we have

$$\begin{aligned} \frac{1}{\sigma} \int \varepsilon_i (\tilde{v} \cdot \nabla u_i) (\nabla u_i \cdot g) &= \int S^\perp(\tilde{v}) \cdot g \, dV_i(x, S) + o(1) \\ &\rightarrow \int S^\perp(\tilde{v}) \cdot g \, dV(x, S). \end{aligned} \quad (4.5)$$

We may arbitrarily approximate the quantities in (4.5) by v by the same argument in (4.4), hence by (4.3)–(4.5), we obtain

$$\delta V(g) = - \int S^\perp(v) \cdot g \, dV(x, S). \quad (4.6)$$

Hence, $\|\delta V\|$ is a Radon measure on U_1 . By (4.2) and Allard's rectifiability theorem [1, 5.5.(1)], V is rectifiable. Since V is rectifiable, there exist an $\mathcal{H}^{n-1}$ measurable and countably $n-1$ -rectifiable set Γ such that

$$\int S^\perp(v) \cdot g \, dV(x, S) = - \int (T_x \Gamma)^\perp(v(x)) \cdot g(x) \, d\|V\|(x). \quad (4.7)$$

The set Γ is the measure-theoretic support of $\|V\|$ and $T_x \Gamma$ is the approximate tangent space of Γ which exists for $\mathcal{H}^{n-1}$ a.e. on Γ . Next from (4.6), $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ and the generalized mean curvature H_V exists. By (4.6) and (4.7), we have $H_V(x) = (T_x \Gamma)^\perp(v(x))$ holds for $\|V\|$ a.e. for x . This proves (5), except that we do not yet take $\Gamma = \text{spt } \|V\|$. The proof of (4) is the same as [25] for the following reason. We may set $f = \varepsilon \nabla u \cdot v$ in [25] and we have $\|\varepsilon \nabla u \cdot v\|_{L^{\frac{np}{n-p}}(U_\delta)} \leq c_1 \lambda_0$ due to Lemma 3.1. In the proof, as long as we have the monotonicity formula (3.55) and the estimate Lemma 3.4, all the argument goes through. The point is that we do not need to take a derivative of f for the proof of integrality and we only need the control of $L^{\frac{np}{n-p}}$ norm as well as the estimate (3.2). See the comment in the proof of [22, Proposition 4.8] where it is explained that (3.2) is necessary. We should also point out that the L^n control of ∇f is not needed in the proof. Finally, by arguing as in (4.4) and the Hölder inequality, we have

$$\int_{U_\delta} \phi^{\frac{p(n-1)}{n-p}} \, d\|V\| \leq c \|\nabla \phi\|_{L^p} \|\phi\|_{L^{\frac{np(p-1)}{np/(n-p)}}}^{n(p-1)/(n-p)}$$

for any function $\phi \in C_c^1(U_\delta; \mathbb{R}^+)$ and we have the same inequality for $v \in W^{1,p}(U_1)$ by the density argument. Thus we have (6). By the well-known property of varifold having the generalized mean curvature in L^q with $q > n-1$ (see [19, 17.9(1)]), $\text{spt } \|V\|$ coincide with the measure-theoretic support Γ , so that we have (5) with $\Gamma = \text{spt } \|V\|$. This concludes the proof of Theorem 2.1. $\square$

5 Concluding remarks

In [11,21], we studied the singular perturbation problem for

$$\partial_t u_\varepsilon + v_\varepsilon \cdot \nabla u_\varepsilon = \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon^2} \quad (5.1)$$

and proved that the time-parametrized family of limit varifolds satisfies the motion law of “normal velocity = mean curvature vector + $v^\perp$ ” in a weak formulation (see [9,23] for the case of $v_\varepsilon = 0$). In these works, we assumed that the prescribed initial data satisfies a boundedness of the upper density ratio. Part of the difficulty was to show that the upper density ratio bound can be controlled locally in time and uniformly with respect to ε . For the equilibrium problem, it is certainly not natural to assume such an upper density ratio estimate. It is interesting to see if one can drop the upper density ratio assumption for the initial data in the proof of [11,21].

The vectorial prescribed mean curvature problem as in Theorem 2.2 seems, as far as we know, little studied so far. Traditionally, the prescription is the scalar version, i.e. given a

scalar function (or constant) f , one looks for a hypersurface satisfying $H \cdot \nu = f$, where ν is the normal unit vector. The vectorial version is physically natural from the view point of force balance, in that the problem seeks the equality between the surface tension force and an external force acting on the surface. It must be said that the prescribed vector field in Theorem 2.2 is the gradient of a potential ρ , and not a general vector field. This is rather restrictive for applications and it is interesting to know if there can be a remedy for generalizations. If there may not exist a variational framework such as the min–max method to find solutions of (1.2), it should be still useful to have this diffused interface approach since the functional is well-behaved functional-analytically. As a further question, it is also interesting to investigate the asymptotic behavior of stable critical points of F_ε in the proof of Theorem 2.2, since we have a very successful analogy in [24,26]. In this direction, we mention that a construction of prescribed scalar mean curvature hypersurfaces along the lines suggested here has been announced recently [3] (see also [2]).

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