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Calculus of Variations

Local regularity for an anisotropic elliptic equation

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Abstract
We establish the interior Hölder continuity for locally bounded solutions, and the Harnack inequality for non-negative continuous solutions to a class of anisotropic elliptic equations with bounded and measurable coefficients, whose prototype equation is

$$u_{xx} + \Delta_{q,y} u = 0 \quad \text{locally in } \mathbb{R} \times \mathbb{R}^{N-1}, \quad \text{for } q < 2,$$

via ideas and tools originating from the regularity theory for degenerate and singular parabolic equations.

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1 Introduction

1.1 Notation and the main results

Let E be an open set in $\mathbb{R}^N$ with $N \geq 2$. We denote a general point in E by $z = (x, y) \in \mathbb{R} \times \mathbb{R}^{N-1}$. For a function u defined in E, the symbols $D_x u$ (or u_x) and $D_{y_i} u$ (or u_{y_i}) represent the differentiation of u with respect to x and y_i variables. Accordingly we also set

$$D_y = (D_{y_1}, \ldots, D_{y_{N-1}}), \qquad D = (D_x, D_y).$$

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For $1 < q < 2$, we shall consider the elliptic partial differential equation

$$u_{xx} + \sum_{i=1}^{N-1} D_{y_i} A_i(z, u, Du) = 0 \quad \text{weakly in } E, \tag{1}$$

where the functions $A_i(z, u, \xi) : E \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, i.e. they are measurable in (u, ξ) for all $z \in E$ and continuous in z for a.e. $(u, \xi) \in \mathbb{R}^{N+1}$. Moreover, they are subject to the following structure conditions a.e. in E

$$\begin{cases} \sum_{i=1}^{N-1} A_i(z, u, Du) \cdot D_{y_i} u \geq C_o |D_y u|^q \\ |A_i(z, u, Du)| \leq C_1 |D_y u|^{q-1} \quad i = 1, \ldots, N-1 \end{cases} \tag{2}$$

with given positive constants C_o and C_1. The prototype equation is

$$u_{xx} + \operatorname{div}_y(|D_y u|^{q-2} D_y u) = 0.$$

Before stating the main result, let us recall that the anisotropic elliptic partial differential Eq. (1) is a special case of the more general equation

$$\sum_{i=1}^{N} D_{x_i} A_i(x, u, Du) = 0 \quad \text{weakly in } E, \tag{3}$$

where the functions $A_i(x, u, \xi) : E \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions, and subject to the structure conditions

$$\begin{cases} A_i(x, u, Du) \cdot u_{x_i} \geq C_o |u_{x_i}|^{p_i}, \\ |A_i(x, u, Du)| \leq C_1 |u_{x_i}|^{p_i - 1}, \end{cases} \tag{4}$$

for some constants $p_i > 1$, $C_o > 0$ and $C_1 > 0$. The prototype equation is

$$\sum_{i=1}^{N} \left(a_i(x) |u_{x_i}|^{p_i - 2} u_{x_i} \right)_{x_i} = 0 \quad \text{weakly in } E.$$

Here $a_i(x)$, $i = 1, \ldots, N$ are measurable functions, satisfying $C_o \leq a_i(x) \leq C_1$ for some positive C_o and C_1. Note also we slightly abused the symbols x and D in (3) and (4), which represent a vector in $\mathbb{R}^N$ and the gradient in x.

When $p_1 = \cdots = p_N = p > 1$, the general Eq. (3) reduces to the standard p-Laplacian type equation whose local regularity theory, such as Hölder regularity and Harnack estimates, is well studied. See [3,13,21,22,24], for example.

When p_i's are potentially different, the study of local regularity was initiated in [17–20]. It has been shown that solutions are Lipschitz continuous, provided the coefficients of the operator are differentiable and satisfy some proper structure conditions. See also [14,15]. However, much less is known about the local regularity for (3) with merely bounded and measurable coefficients.

It has been proved that the local boundedness of local solutions to (3) is inherent in the notion of weak solution in [10,12], provided

$$\max\{p_1, \ldots, p_N\} \leq \frac{N \bar{p}}{N - \bar{p}} \quad \text{where} \quad \frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}.$$

The above condition indicates that the indices p_i's are not too far apart. In fact, examples are constructed in [11,17] to show that solutions could be unbounded, if the indices are too far apart. Recently, a rather detailed discussion on this issue has been carried out in [1].

Concerning the continuity of locally bounded solutions, a first step was made in [16]. It has been shown that locally bounded solutions to (1) are Hölder continuous for $q > 2$. We also mention that the Hölder regularity for more general anisotropic operators is considered in [8,9].

In order to define the notion of weak solution formally, we introduce the following anisotropic Sobolev spaces

$$W^{1,[2,q]}(E) = \left\{u \in L^1(E) : u_x \in L^2(E),\ u_{y_i} \in L^q(E),\ i = 1, \dots, N-1\right\},$$
$$W_o^{1,[2,q]}(E) = W_o^{1,1}(E) \cap W^{1,[2,q]}(E).$$

A function $u \in W^{1,[2,q]}_{\text{loc}}(E)$ is called a local, weak sub(super)-solution to (1), if for every compact set $K \subset E$ it satisfies

$$\int_K u_x \phi_x \, dz + \int_K \sum_{i=1}^{N-1} A_i(z, u, Du)\phi_{y_i} \, dz \leq (\geq)\, 0,$$

for all non-negative test functions $\phi \in W_o^{1,[2,q]}(K)$.

In this note, by sub(super)-solutions we always refer to the weak ones defined above. When we speak of structural *data*, we refer to the set of parameters $\{q,\ N,\ C_o,\ C_1\}$. We also write γ as a generic positive constant that can be quantitatively determined a priori only in terms of the data and it may change line by line. When we write $\gamma(b)$ we mean to emphasize the dependence on the quantity b.

For $\rho > 0$ let $K_\rho(y)$ be the cube of center at y in $\mathbb{R}^{N-1}$ and edge 2ρ. When $y = 0$ we simply write K_ρ. For $z_o = (x_o, y_o) \in E$, we define the cylinders scaled by a positive parameter θ:

$$\begin{cases} \text{centered cylinders:} & z_o + Q_\rho(\theta) = \left[x_o - \theta\rho^{\frac{q}{2}}, x_o + \theta\rho^{\frac{q}{2}}\right] \times K_\rho(y_o), \\ \text{forward cylinders:} & z_o + Q_\rho^+(\theta) = \left[x_o, x_o + \theta\rho^{\frac{q}{2}}\right] \times K_\rho(y_o), \\ \text{backward cylinders:} & z_o + Q_\rho^-(\theta) = \left[x_o - \theta\rho^{\frac{q}{2}}, x_o\right] \times K_\rho(y_o). \end{cases}$$

Concerning the parameter θ, it will be a quantity that restores the homogeneity of the Eq. (1) depending on the structure it is subject to. More precisely, we will set $\theta = [u]^{\frac{2-q}{2}}$, where $[u]$ represents a quantity that is of the "dimension" of a given solution u. When $q = 2$, the cylinder $z_o + Q_\rho(\theta)$ formally recovers the standard cube in $\mathbb{R}^N$ of center at z_o and edge 2ρ. The exact form of θ hinges upon the context and will be specified in the following. Thus, such cylinders are *intrinsically* scaled. When $\theta = 1$, we simply write $z_o + Q_\rho^\pm$.

Suppose that $u \in L^\infty(E)$ is a local weak solution to (1) and (2). For a compact set $K \subset E$ introduce the *intrinsic* distance from K to ∂E by

$$(2, q) - \text{dist}(K;\ \partial E) \stackrel{\text{def}}{=} \inf_{\substack{z \in K \\ \bar{z} \in \partial E}} \left\{ |y - \bar{y}| + \|u\|_{\infty,E}^{\frac{q-2}{q}} |x - \bar{x}|^{\frac{2}{q}} \right\}.$$

Fix $z_o \in E$ and let $R > 0$ be so small that

$$Q_o \stackrel{\text{def}}{=} [x_o - R^{\frac{q}{2}}, x_o + R^{\frac{q}{2}}] \times K_R(y_o) \subset E$$

for a small parameter $\varepsilon > 0$; we may assume that z_o coincides with the origin. We set

$$\mu^+ = \operatorname*{ess\,sup}_{Q_o} u, \quad \mu^- = \operatorname*{ess\,inf}_{Q_o} u, \quad \omega = \operatorname*{ess\,osc}_{Q_o} u = \mu^+ - \mu^-.$$

Construct the cylinder

$$Q_R(\theta) = \left[-\theta R^{\frac{q}{2}}, \theta R^{\frac{q}{2}}\right] \times K_R \quad \text{with} \quad \theta = \left(\frac{\omega}{\Lambda}\right)^{\frac{2-q}{2}},$$

where $\Lambda > 1$ is a parameter to be determined in terms of the data only. Without loss of generality, we may assume that $\omega \le 1$, such that

$$Q_R(\theta) \subset Q_o \quad \text{and} \quad \operatorname*{ess\,osc}_{Q_R(\theta)} u \le \omega.$$

This $Q_R(\theta)$ is the starting cylinder for the reduction of oscillation.

Now we state our theorem concerning the interior Hölder regularity which holds for all $1 < q < 2$.

Theorem 1 *Let u be a bounded, local, weak solution to (1) and (2) in E. Then u is locally Hölder continuous in E. More precisely, there exist constants $\gamma,\ \Lambda > 1$ and $\alpha \in (0, 1)$ that can be determined a priori only in terms of the data, such that for any $0 < \rho < R$, we have*

$$\operatorname*{ess\,osc}_{z_o + Q_\rho(\theta)} u \le \gamma\omega\left(\frac{\rho}{R}\right)^{\alpha}.$$

In other words, for every compact set $K \subset E$,

$$|u(z_1) - u(z_2)| \le \gamma \|u\|_{\infty,E} \left(\frac{|y_1 - y_2| + \|u\|_{\infty,E}^{\frac{q-2}{q}} |x_1 - x_2|^{\frac{2}{q}}}{(2, q) - \operatorname{dist}(K; \partial E)}\right)^{\alpha}$$

for every pair of points $z_1,\ z_2 \in K$.

In order to state Harnack's inequality, introduce the following intrinsically scaled cylinder centered at $z_o \in E$:

$$z_o + Q_{4\rho}(\theta) \quad \text{where } \theta = [\bar{c}u(z_o)]^{\frac{2-q}{2}}, \tag{5}$$

where $\bar{c} \in (0, 1)$ will be determined in terms of the data and

$$\frac{2(N-1)}{N} \overset{\text{def}}{=} q_* < q < 2. \tag{6}$$

It is worth mentioning that (6) ensures $\kappa \overset{\text{def}}{=} Nq - 2(N-1) > 0$.

Since every locally bounded, local, weak solution to (1) and (2) has a continuous representative by Theorem 1, we may only deal with continuous weak solutions.

Theorem 2 *Let u be a continuous, non-negative, local, weak solution to (1) and (2) in E. Assume (6) holds and the cylinder (5) is contained in E. There exist constants $c,\ \bar{c} \in (0, 1)$ depending only on the data, such that*

$$c \operatorname*{sup}_{z_o + Q_\rho(\theta)} u \le u(z_o) \le c^{-1} \operatorname*{inf}_{z_o + Q_\rho(\theta)} u.$$

Theorem 2 has been stated for continuous solutions, to give meaning to $u(z_o)$. However, it continues to hold for non-negative weak solutions to (1) and (2) for almost all $z_o \in E$ and for the corresponding intrinsic cylinders.

The proofs of Theorems 1 and 2 are based on the following *expansion of positivity*. To this end, we introduce another intrinsically scaled cylinder centered at $z_o \in E$:

$$z_o + Q_{16\rho}(\theta) \quad \text{where } \theta = \delta M^{\frac{2-q}{2}}, \tag{7}$$

for some positive number M and a parameter $\delta \in (0, 1)$ to be determined in terms of the data only.

Proposition 1 *Let u be a non-negative, local, weak super-solution to (1) and (2) in E. Assume the cylinder (7) is contained in E and*

$$|[u(x_o, \cdot) \geq M] \cap K_\rho(y_o)| \geq \beta |K_\rho|$$

for some $\beta \in (0, 1)$ and $M > 0$. There exist constants $\eta,\ \delta \in (0, 1)$ depending only upon the data and β, such that

$$u \geq \eta M \quad \text{a.e. in } z_o + Q_{2\rho}(\theta) \quad \text{with } \theta = \delta M^{\frac{2-q}{2}}.$$

The proof of Theorem 2 will also use the following $L^1_{\text{loc}} - L^\infty_{\text{loc}}$ estimate.

Theorem 3 *Let u be a non-negative, local, weak solution to (1) and (2) in E and assume (6) holds. There exists a constant $\gamma > 0$ depending only on the data, such that for all cylinders*

$$[x_o - 2t, x_o + 2t] \times K_{2\rho}(y_o) \subset E,$$

we have

$$\operatorname*{ess\,sup}_{[x_o - t, x_o + t] \times K_\rho(y_o)} u \leq \gamma \left(\frac{\rho^q}{t^2}\right)^{\frac{N-1}{\kappa}} \left(\fint_{K_{2\rho}(y_o)} u(x_o, y)\, dy\right)^{\frac{q}{\kappa}} + \gamma \left(\frac{t^2}{\rho^q}\right)^{\frac{1}{2-q}},$$

where $\kappa = Nq - 2(N - 1)$.

Remark 1 Our main results continue to hold if we multiply u_{xx} in (1) by a measurable coefficient $a(y)$ satisfying $C_o \leq a(y) \leq C_1$ for some positive C_o and C_1. Nevertheless we do not know how to demonstrate them for more general operators at this moment. See (8)–(10) for the technical obstruction.

1.2 Novelty and significance

The main contribution of this work is to present a local regularity theory regarding (1) in the case $q < 2$. In Theorem 1, we obtain that locally bounded solutions are Hölder continuous for all $1 < q < 2$. Moreover, we establish an intrinsic Harnack inequality for non-negative solutions to (1) in Theorem 2, which holds only for $q > q_*$, where $q_* > 1$ is defined in (6). The main tool of proving Theorems 1 and 2, i.e. the *expansion of positivity* for non-negative, super-solutions, is presented in Proposition 1. Such a property plays a central role in any kind of Harnack estimates for elliptic and parabolic equations. Moreover, Theorem 3 serves as another main component in the proof of Theorem 2. It may be viewed as a Harnack inequality in $L^1_{\text{loc}} - L^\infty_{\text{loc}}$ topology.

The main idea of treating the local regularity issues for (1) is the so-called *intrinsic scaling*, which was originally formulated in the theory of degenerate and singular parabolic

equations. We refer to [2,4,7] for an account of the theory. To realize such a key idea, we also owe technical tools to the parabolic theory. The exponential shift technique in our proof of Proposition 1 is borrowed from [6], which was used to handle the singular parabolic equations. Theorem 3 also has a counterpart in the theory of singular parabolic equations, though we need some new input to prove it. See [5] and [7, Appendix A]. However, it is not immediately clear whether the same kind of approach can be applied to more general anisotropic p-Laplacian operators. We think this intriguing interplay between the theories of elliptic equations and parabolic equations deserves a deeper understanding in the future investigation.

As for the organization of this note, we first collect some preliminary tools in Sect. 2, including energy estimates, DeGiorgi-type lemmas, a logarithmic lemma and its consequences. In Sect. 3, we give a proof of Proposition 1, concerning the expansion of positivity. Section 4 is devoted to the proof of Theorem 1. To streamline the presentation, we show Theorem 2 in Sect. 5 assuming Theorem 3, which will be proven in Sect. 6.

2 Preliminaries

For a multi-index $\mathbf{p} = (p_1, \dots, p_N)$, $p_i \geq 1$, introduce anisotropic Sobolev spaces

$$W^{1,\mathbf{p}}(E) = \left\{u \in L^1(E) : u_{x_i} \in L^{p_i}(E),\ i = 1, \dots, N\right\},$$
$$W_o^{1,\mathbf{p}}(E) = W_o^{1,1}(E) \cap W^{1,\mathbf{p}}(E).$$

We state here the Sobolev–Troisi inequality [23].

Lemma 1 *Let $E \subset \mathbb{R}^N$ be a bounded open set and consider $u \in W_o^{1,\mathbf{p}}(E)$, $p_i \geq 1$ for all $i = 1, \dots, N$. Set*

$$\frac{1}{\bar{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}, \qquad \bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}.$$

If $\bar{p} < N$, then there exists a positive constant γ depending only on the set of parameters $\{N,\ \mathbf{p}\}$, such that

$$\|u\|_{\bar{p}^*}^N \leq \gamma \prod_{i=1}^{N} \|u_{x_i}\|_{p_i}.$$

According to our notion of weak solution for (1), the index we will use is $\mathbf{p} = (2, q, \dots, q)$.

2.1 Energy estimates

For $k \in \mathbb{R}$, we set

$$(u-k)_+ = \max\{u-k,\ 0\}, \qquad (u-k)_- = \max\{-(u-k),\ 0\}.$$

The first proposition concerns energy estimates in the interior.

Proposition 2 *Let u be a local, weak sub(super)-solution to (1) and (2) in E. Then there exists a constant $\gamma > 0$ depending only on the data, such that for any cube $Q \subset E$ and for*

every non-negative, piecewise smooth cutoff function ζ vanishing on ∂Q, we have for any $k \in \mathbb{R}$ that

$$\int_Q |D_x(u-k)_\pm|^2\zeta^2\,dz + \int_Q |D_y(u-k)_\pm|^q\zeta^2\,dz \le \gamma\int_Q (u-k)_\pm^2|\zeta_x|^2\,dz + \gamma\int_Q (u-k)_\pm^q|D_y\zeta|^q\,dz.$$

Proof We only show the case of super-solutions, the other case being similar. In the weak formulation of (1), we take the test function $-(u-k)_-\zeta^2$; a standard calculation yields that

$$-\int_Q \zeta^2 u_x D_x(u-k)_-\,dz - \int_Q \zeta^2\sum_{i=1}^{N-1} A_i(z,u,Du)D_{y_i}(u-k)_-\,dz \le 2\int_Q \zeta|u_x|(u-k)_-|\zeta_x|\,dz + 2\int_Q \zeta\sum_{i=1}^{N-1}|A_i(z,u,Du)|(u-k)_-|D_{y_i}\zeta|\,dz.$$

We will employ the structure conditions and Young's inequality repeatedly in estimating the various terms. We first estimate the terms on the left-hand side. The first term is

$$-\int_Q \zeta^2 u_x D_x(u-k)_-dz = \int_Q |D_x(u-k)_-|^2\zeta^2\,dz,$$

while the second term is estimated by

$$-\int_Q \zeta^2\sum_{i=1}^{N-1} A_i(z,u,Du)D_{y_i}(u-k)_-\,dz \ge C_o\int_Q |D_y(u-k)_-|^q\zeta^2\,dz.$$

Now we estimate the terms on the right-hand side. The first term is estimated by

$$2\int_Q \zeta|u_x|(u-k)_-|\zeta_x|\,dz \le \frac{1}{2}\int_Q \zeta^2|D_x(u-k)_-|^2\,dz + 2\int_Q (u-k)_-^2|\zeta_x|^2\,dz.$$

Similarly, the second term is estimated by

$$2\int_Q \zeta\sum_{i=1}^{N-1}|A_i(z,u,Du)|(u-k)_-|D_{y_i}\zeta|\,dz \le \frac{C_o}{2}\int_Q \zeta^2|D_y(u-k)_-|^q\,dz + \gamma\int_Q (u-k)_-^q|D_y\zeta|^q\,dz.$$

Collecting all these estimate gives the desired result. □

The second proposition concerns energy estimates involving boundary information. We state it in the case of super-solutions, while sub-solutions have an analogous statement. Let θ and M be positive parameters.

Proposition 3 *Let u be a local, weak super-solution to (1) and (2) in E. Suppose $z_o+Q_\rho^\pm(\theta) \subset E$ and*

$$u(x_o,\cdot) \ge M \quad a.e.\ in\ K_\rho(y_o).$$

Then there exists a constant $\gamma > 0$ depending only on the data, such that for any $k \le M$ and for every non-negative, piecewise smooth cutoff function ζ vanishing on $\partial Q_\rho(\theta)$, we have

$$\int_{z_o+Q_\rho^\pm(\theta)} |D_x(u-k)_-|^2\zeta^2\,dz + \int_{z_o+Q_\rho^\pm(\theta)} |D_y(u-k)_-|^q\zeta^2\,dz$$

$$\leq \gamma \int_{z_o+Q_\rho^{\pm}(\theta)} (u-k)_-^2 |\zeta_x|^2 \, dz + \gamma \int_{z_o+Q_\rho^{\pm}(\theta)} (u-k)_-^q |D_y \zeta|^q \, dz.$$

Proof Assume $z_o = (0,0)$. The function $-(u-k)_-\zeta^2$ vanishes on the boundary of $Q_\rho^{\pm}(\theta)$, since $u(0,\cdot) \geq M$ on the set K_ρ and $k \leq M$, and thus it is an admissible test function. Using this test function in the weak formulation in $Q_\rho^{\pm}(\theta)$, the remaining calculation runs similar to the proof of Proposition 2. □

2.2 DeGiorgi-type Lemmas

Suppose θ and M are positive parameters.

Lemma 2 *Let u be a non-negative, local, weak super-solution to (1) and (2) in E. There exists a number $\nu > 0$ depending only upon the parameters θ, M and the data, such that if*

$$|[u \leq M] \cap [z_o + Q_{2\rho}(\theta)]| \leq \nu |Q_{2\rho}(\theta)|,$$

then

$$u \geq \frac{M}{2} \quad a.e.\ in\ z_o + Q_\rho(\theta).$$

Proof We may assume $z_o = (0,0)$. For $n = 0, 1, \ldots$, we set

$$k_n = \frac{M}{2} + \frac{M}{2^{n+1}}, \quad \rho_n = \rho + \frac{\rho}{2^n}, \quad \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2},$$
$$Q_n = Q_{\rho_n}(\theta), \quad \tilde{Q}_n = Q_{\tilde{\rho}_n}(\theta).$$

Introduce the cutoff function ζ vanishing on ∂Q_n and equal to identity in $\tilde{Q}_n$, such that

$$|D_y \zeta| \leq \gamma \frac{2^n}{\rho} \quad \text{and} \quad |\zeta_x| \leq \frac{\gamma}{\theta} \left(\frac{2^n}{\rho}\right)^{\frac{q}{2}}.$$

In this setting the energy estimate in Proposition 2 yields

$$\begin{aligned}
&\int_{\tilde{Q}_n} |D_x (u-k_n)_-|^2 \, dz + \int_{\tilde{Q}_n} |D_y (u-k_n)_-|^q \, dz \\
&\leq \gamma \frac{2^{qn}}{\theta^2 \rho^q} \int_{Q_n} (u-k_n)_-^2 \, dz + \gamma \frac{2^{qn}}{\rho^q} \int_{Q_n} (u-k_n)_-^q \, dz \\
&\leq \gamma \frac{2^{qn} M^q}{\rho^q} \left[1 + \frac{M^{2-q}}{\theta^2}\right] |A_n|,
\end{aligned}$$

where we have set $A_n = [u < k_n] \cap Q_n$. Now setting ζ to be a cutoff function which vanishes on the boundary of $\tilde{Q}_n$ and equals identity in Q_{n+1}, an application of Lemma 1 gives that

$$\begin{aligned}
\left(\frac{M}{2^{n+2}}\right)^{\bar{q}} |A_{n+1}| &\leq \int_{\tilde{Q}_n} (u-k_n)_-^{\bar{q}} \zeta^{\bar{q}} \, dz \\
&\leq \left(\int_{\tilde{Q}_n} [(u-k_n)_- \zeta]^{\frac{N\bar{q}}{N-\bar{q}}} \, dz \right)^{\frac{N-\bar{q}}{N}} |A_n|^{\frac{\bar{q}}{N}} \\
&\leq \gamma \left(\|D_x (u-k_n)_-\|_2 \prod_{i=1}^{N-1} \|D_{y_i} (u-k_n)_-\|_q \right)^{\frac{\bar{q}}{N}} |A_n|^{\frac{\bar{q}}{N}}
\end{aligned}$$

$$\leq \gamma \frac{2^{qn} M^q}{\rho^q} \left(1 + \frac{M^{2-q}}{\theta^2}\right) |A_n|^{1+\frac{\bar{q}}{N}},$$

where $\bar{q}$ satisfies

$$\frac{1}{\bar{q}} = \frac{1}{N}\left(\frac{1}{2} + \frac{N-1}{q}\right).$$

In terms of $Y_n = |A_n|/|Q_n|$, this can be rewritten as

$$Y_{n+1} \leq \gamma \gamma_1^n \left(\frac{\theta^2}{M^{2-q}}\right)^{\frac{\bar{q}}{2N}} \left(1 + \frac{M^{2-q}}{\theta^2}\right) Y_n^{1+\frac{\bar{q}}{N}},$$

where γ_1 and γ depend only on the data. Hence by [2, Chapter I, Lemma 4.1], $Y_n \to 0$ if we require that

$$Y_o \leq \frac{1}{\gamma}\left(\frac{M^{2-q}}{\theta^2}\right)^{\frac{1}{2}} \left(1 + \frac{M^{2-q}}{\theta^2}\right)^{-\frac{N}{\bar{q}}} \stackrel{\text{def}}{=} \nu.$$

□

We also have a version involving boundary data.

Lemma 3 *Let u be a non-negative, local weak super-solution to (1) and (2) in E. Suppose $z_o + Q_{2\rho}^{\pm}(\theta) \subset E$ and $u(x_o, \cdot) \geq M$ on the set $K_{2\rho}(y_o)$. There exists a number $\nu > 0$ depending only upon the parameters θ, M and the data, such that if*

$$|[u \leq M] \cap [z_o + Q_{2\rho}^{\pm}(\theta)]| \leq \nu |Q_{2\rho}^{\pm}(\theta)|,$$

then

$$u(x, y) \geq \frac{M}{2} \quad \text{a.e. in } z_o + Q_{\rho}^{\pm}(\theta).$$

Proof The proof is similar to that of Lemma 2. This time, we may employ the energy estimate in Proposition 3, since $k_n \leq M$. The constant ν presents the same form as in Lemma 2. □

2.3 A logarithmic estimate

Let Q be a cylinder in E. Suppose u is a non-negative, weak super-solution to (1) and (2) in E. For $a \in (0, 1)$ and $M > 0$, we introduce the following function in Q:

$$G(u(z)) = \left[\frac{1}{u(z) + aM} - \frac{1}{M}\right]_+.$$

Note that $G(u) = 0$ if $u \geq (1-a)M \stackrel{\text{def}}{=} k$.

Lemma 4 *Let u be a non-negative, local, weak super-solution to (1) and (2) in E. There exists a constant $\gamma > 0$ depending only on the data, such that for every piecewise smooth cutoff function ζ in Q vanishing on ∂Q, there holds*

$$\int_Q \left| D_x \ln_+ \left(\frac{M}{u + aM}\right)\right|^2 \zeta^2 \, dz$$
$$\leq \gamma \int_{Q \cap [u<k]} |D_y \zeta|^q (u + aM)^{q-2} \, dz + \gamma \int_{Q \cap [u<k]} |\zeta_x|^2 \, dz.$$

Proof Use the test function $\phi = -G(u)\zeta^2 \in W_o^{1,[2,q]}$ in the weak formulation of u to get

$$\int_{Q\cap[u<k]} u_x \frac{\zeta^2 u_x}{(u+aM)^2}\,dz + \int_{Q\cap[u<k]} \sum_{i=1}^{N-1} A_i(z,u,Du)\frac{\zeta^2 D_{y_i}u}{(u+aM)^2}\,dz$$
$$\le 2\int_Q \zeta\zeta_x u_x G(u)\,dz + 2\int_Q \zeta G(u)\sum_{i=1}^{N-1} A_i(z,u,Du)D_{y_i}\zeta\,dz.$$

We use the structure conditions (2) and Young's inequality repeatedly to estimate various terms separately. For the left-hand side, the first term is

$$\int_{Q\cap[u<k]} |u_x|^2\frac{\zeta^2}{(u+aM)^2}\,dz = \int_Q \left|D_x \ln_+\left(\frac{M}{u+aM}\right)\right|^2\zeta^2\,dz.$$

The second term on the left is treated similarly:

$$\int_{Q\cap[u<k]} A_i(z,u,Du)\frac{\zeta^2 D_{y_i}u}{(u+aM)^2}\,dz \ge C_o\int_Q \frac{|D_y u|^q\zeta^2\chi_{[u<k]}}{(u+aM)^2}\,dz.$$

For the right-hand side, the first term is estimated by

$$2\int_Q \zeta\zeta_x u_x G(u)\,dz \le 2\int_Q \zeta|\zeta_x|G(u)|u_x|\,dz$$
$$\le 2\int_Q \left|D_x\ln_+\left(\frac{M}{u+aM}\right)\right|\zeta|\zeta_x|dz$$
$$\le \frac{1}{2}\int_Q \left|D_x\ln_+\left(\frac{M}{u+aM}\right)\right|^2\zeta^2 dz + 2\int_Q |\zeta_x|^2\chi_{[u<k]}\,dz.$$

The second term is estimated similarly:

$$2\int_Q \zeta G(u)\sum_{i=1}^{N-1} A_i(z,u,Du)D_{y_i}\zeta\,dz \le \frac{C_o}{4}\int_Q \frac{|D_y u|^q\zeta^2\chi_{[u<k]}}{(u+aM)^2}\,dz$$
$$+\gamma\int_{Q\cap[u<k]} |D_y\zeta|^q(u+aM)^{q-2}\,dz.$$

Collecting all above estimates yields the desired result. □

2.4 Consequences of the logarithmic estimate

Suppose u is a non-negative, local, weak super-solution to (1) and (2) in E. Let $z_o \in E$, $\rho > 0$ and $M > 0$, such that $z_o + Q_{2\rho}(\theta) \subset E$, where for some $j_* > 0$ to be determined we have defined

$$\theta = (2^{-j_*}M)^{\frac{2-q}{2}}.$$

2.4.1 Propagation of measure theoretical information

Lemma 5 *Assume for some $\beta \in (0,1)$ there holds*

$$|[u(x_o,\cdot) \ge M]\cap K_\rho(y_o)| \ge \beta|K_\rho|.$$

Then there exists $j_* > 0$ depending only on the data and β, such that

$$\left|\left[u(x,\cdot) \geq \frac{M}{2^{j_*}}\right] \cap K_\rho(y_o)\right| \geq \frac{\beta}{2}|K_\rho| \quad \text{for all } x \in \left[x_o - \theta\rho^{\frac{q}{2}}, x_o + \theta\rho^{\frac{q}{2}}\right].$$

Proof Assume $z_o = (0,0)$. We first apply Lemma 4 in $Q_{2\rho}(\theta)$, choosing $a = 2^{-j}$ for some $j \in \{1, 2, \ldots, j_*\}$ and a cutoff function ζ that equals 1 in $Q_\rho(\theta)$ and vanishes on $\partial Q_{2\rho}(\theta)$, such that

$$|\zeta_x| \leq \frac{\gamma}{\theta\rho^{\frac{q}{2}}}, \qquad |D_y\zeta| \leq \frac{\gamma}{\rho}.$$

In this way, we obtain

$$\begin{aligned}
&\int_{Q_\rho(\theta)} \left|D_x \ln_+\left(\frac{M}{u+aM}\right)\right|^2 dz \\
&\leq \gamma \int_{Q_{2\rho}(\theta)} |D_y\zeta|^q (u+aM)^{q-2} \chi_{[u<(1-a)M]}\, dz + \gamma \int_{Q_{2\rho}(\theta)} |\zeta_x|^2\, dz \\
&\leq \frac{\gamma}{\theta^2\rho^q}|Q_\rho(\theta)|.
\end{aligned}$$

From this, a straightforward application of Hölder's inequality yields

$$\int_{Q_\rho(\theta)} |\psi_x|\, dz \leq \frac{\gamma}{\theta\rho^{\frac{q}{2}}}|Q_\rho(\theta)|,$$

where, for ease of notation, we have set

$$\psi(x,y) = \ln_+\left[\frac{M}{u(x,y)+aM}\right].$$

By the measure theoretical information known at $x = 0$, we estimate

$$\int_{K_\rho} \psi(0,y)\, dy \leq (1-\beta)\, j \ln 2.$$

These estimates joint with the mean value theorem give us

$$\begin{aligned}
\int_{K_\rho} \psi(x,y)\, dy &\leq \int_{K_\rho} \psi(0,y)\, dy + \int_{K_\rho}\int_{-\theta\rho^{\frac{q}{2}}}^{\theta\rho^{\frac{q}{2}}} |\psi_x|\, dxdy \\
&\leq (1-\beta)\, j \ln 2 + \frac{\gamma}{\theta\rho^{\frac{q}{2}}}|Q_\rho(\theta)| \\
&\leq (1-\beta)\, j \ln 2 + \gamma|K_\rho|,
\end{aligned}$$

for all $x \in \left[-\theta\rho^{\frac{q}{2}}, \theta\rho^{\frac{q}{2}}\right]$. The left-hand side is estimated from below over a smaller set

$$\left[u(x,\cdot) \leq \frac{M}{2^j}\right] \cap K_\rho$$

to obtain

$$\int_{K_\rho} \psi(x,y)\, dy \geq (j-1)\ln 2 \left|\left[u(x,\cdot) \leq \frac{M}{2^j}\right] \cap K_\rho\right|.$$

Collecting the above estimates, we have for all $x \in \left[-\theta\rho^{\frac{q}{2}}, \theta\rho^{\frac{q}{2}}\right]$,

$$\left|\left[u(x,\cdot) \le \frac{M}{2^j}\right] \cap K_\rho\right| \le \frac{j}{j-1}(1-\beta)|K_\rho| + \frac{\gamma}{j-1}|K_\rho|.$$

To conclude, we may choose j_* large enough, such that

$$\frac{\gamma}{j_*-1} \le \frac{\beta}{4}, \quad \frac{j_*}{j_*-1}(1-\beta) \le 1 - \frac{3}{4}\beta.$$

□

2.4.2 A shrinking lemma

Lemma 6 *Assume*

$$u(x_o,\cdot) \ge M \quad a.e.\ in\ K_\rho(y_o).$$

Then there exists a constant $\gamma > 0$ depending only on the data, such that for any $j_* > 0$

$$\left|\left[u(x,\cdot) \le \frac{M}{2^{j_*}}\right] \cap K_\rho(y_o)\right| \le \frac{\gamma}{j_*}|K_\rho|,$$

for all $x \in \left[x_o - \theta\rho^{\frac{q}{2}}, x_o + \theta\rho^{\frac{q}{2}}\right]$.

Proof Assume $z_o = (0,0)$. The proof is similar to that of Lemma 5. However, the quantitative information $u(0,\cdot) \ge M$ in K_ρ yields

$$\int_{K_\rho} \psi(0,y)\,dy = 0.$$

Hence we have for all $x \in \left[-\theta\rho^{\frac{q}{2}}, \theta\rho^{\frac{q}{2}}\right]$,

$$\left|\left[u(x,\cdot) \le \frac{M}{2^j}\right] \cap K_\rho\right| \le \frac{\gamma}{j-1}|K_\rho|.$$

The proof is concluded by redefining γ properly. □

2.4.3 Propagation of pointwise information

A combination of Lemmas 3 and 6 shows that the positivity of a non-negative, local supersolution at x_o over a y-cube spreads in x direction over a smaller y-cube.

Lemma 7 *Assume*

$$u(x_o,\cdot) \ge M \quad a.e.\ in\ K_{2\rho}(y_o).$$

Then there exists $j_* > 0$ depending only on the data, such that

$$u(x,\cdot) \ge \frac{M}{2^{j_*+1}} \quad a.e.\ in\ K_\rho(y_o),$$

for all $x \in \left[x_o - \theta\rho^{\frac{q}{2}}, x_o + \theta\rho^{\frac{q}{2}}\right]$ where $\theta = (2^{-j_*}M)^{\frac{2-q}{2}}$.

Proof Assume $z_o = (0,0)$. By Lemma 3 in the cylinder $Q^{\pm}_{2\rho}(\theta)$ with $\theta = (2^{-j_*}M)^{\frac{2-q}{2}}$ for some $j_* > 0$ to be chosen and $a = \frac{1}{2}$, there exists a positive constant ν depending only on the data, such that if

$$\left|\left[u \le \frac{M}{2^{j_*}}\right] \cap Q^{\pm}_{2\rho}(\theta)\right| \le \nu |Q^{\pm}_{2\rho}(\theta)|$$

then

$$u \ge \frac{M}{2^{j_*+1}} \quad \text{a.e. in } Q^{\pm}_{\rho}(\theta).$$

Now we choose the constant j_* according to Lemma 6, such that $\gamma\, j_*^{-1} \le \nu$. □

3 Proof of Proposition 1

The proof of Proposition 1 hinges upon the following preliminary version.

Proposition 4 *Suppose the hypothesis in Proposition 1 holds. If for some $\beta \in (0,1)$ and $M > 0$ there holds*

$$|[u(x,\cdot) \ge M] \cap K_\rho(y_o)| \ge \beta |K_\rho|$$

for all

$$|x - x_o| \le M^{\frac{2-q}{2}} \rho^{\frac{q}{2}}.$$

Then there exist constants $\eta,\ \delta \in (0,1)$ depending only upon the data and β, such that

$$u(x,\cdot) \ge \eta M \quad \text{a.e. in } K_{2\rho}(y_o)$$

for all

$$(1-\delta) M^{\frac{2-q}{2}} \rho^{\frac{q}{2}} \le |x - x_o| \le M^{\frac{2-q}{2}} \rho^{\frac{q}{2}}.$$

We will assume without loss of generality that z_o coincides with the origin. To simplify the presentation, we deal with the half space $[x > 0]$ only, since the case of $[x < 0]$ is similar.

3.1 Change of variables

Introduce the new variable $\tilde{z} = (\tilde{x}, \tilde{y})$ and the unknown function v defined by

$$\tilde{y} = \frac{y}{\rho}, \quad -e^{-\tilde{x}} = \frac{x - M^{\frac{2-q}{2}} \rho^{\frac{q}{2}}}{M^{\frac{2-q}{2}} \rho^{\frac{q}{2}}}, \quad v(\tilde{z}) = \frac{1}{M} u(z) e^{\frac{2}{2-q}\tilde{x}}. \tag{8}$$

It maps the cylinder $\left[0, M^{\frac{2-q}{2}} \rho^{\frac{q}{2}}\right] \times K_{16\rho}$ into $(0,\infty) \times K_{16}$. Now we proceed to show that v satisfies a differential inequality with a similar type of structure conditions as (2). To start with, we have by differentiation that

$$u_x = \frac{M e^{(1-\lambda)\tilde{x}}}{M^{\frac{1}{\lambda}} \rho^{\frac{q}{2}}} (v_{\tilde{x}} - \lambda v), \quad \lambda \stackrel{\text{def}}{=} \frac{2}{2-q},$$

$$u_{xx} = \frac{Me^{(2-\lambda)\tilde{x}}}{\left(M^{\frac{1}{\lambda}}\rho^{\frac{q}{2}}\right)^2}\left[v_{\tilde{x}\tilde{x}} - (2\lambda-1)v_{\tilde{x}} + \lambda(\lambda-1)v\right],$$
$$D_y u = \frac{M}{\rho}e^{-\lambda\tilde{x}}D_{\tilde{y}}v.$$

As a result, v will satisfy the following differential inequality weakly:

$$v_{\tilde{x}\tilde{x}} - (2\lambda-1)v_{\tilde{x}} + \lambda(\lambda-1)v + \sum_{i=1}^{N-1} D_{\tilde{y}_i}\tilde{A}_i(\tilde{z}, v, D_{\tilde{z}}v) \le 0, \tag{9}$$

where we have defined

$$\tilde{A}_i(\tilde{z}, v, D_{\tilde{z}}v) = \frac{\left(M^{\frac{1}{\lambda}}\rho^{\frac{q}{2}}\right)^2}{\rho M e^{(2-\lambda)\tilde{x}}}A_i(z, u, Du).$$

Moreover, we have

$$\begin{cases} \sum_{i=1}^{N-1}\tilde{A}_i(\tilde{z}, v, D_{\tilde{z}}v)\cdot D_{\tilde{y}_i}v \ge C_o|D_{\tilde{y}}v|^q \\ |\tilde{A}_i(\tilde{z}, v, D_{\tilde{z}}v)| \le C_1|D_{\tilde{y}}v|^{q-1} \quad i = 1, \dots, N. \end{cases}$$

3.2 Energy estimates for *v*

Proposition 5 *Suppose $v \ge 0$ satisfies (9) weakly in a cube Q in $\mathbb{R}^N$. There exists a constant $\gamma > 0$ depending only on the data, such that for every non-negative, piecewise smooth cutoff function ζ vanishing on ∂Q, we have for any $k \ge 0$ that*

$$\begin{aligned} &\int_Q |D_{\tilde{x}}(v-k)_-|^2\zeta^2\,d\tilde{z} + \int_Q |D_{\tilde{y}}(v-k)_-|^q\zeta^2\,d\tilde{z} \\ &\le \gamma\int_Q (v-k)_-^2(|\zeta_{\tilde{x}}| + |\zeta_{\tilde{x}}|^2)\,d\tilde{z} + \gamma\int_Q (v-k)_-^q|D_{\tilde{y}}\zeta|^q\,d\tilde{z}. \end{aligned} \tag{10}$$

Proof For simplicity of notation, we still denote the variable of v by $z = (x, y)$ and $\tilde{A}_i$ by A_i. In the weak formulation of (9), we take the test function $(v-k)_-\zeta^2$; discarding the non-negative contribution of v, a standard calculation yields that

$$\begin{aligned} &-\int_Q \zeta^2 v_x D_x(v-k)_-\,dz - (2\lambda-1)\int_Q \zeta^2 v_x(v-k)_-\,dz \\ &\quad -\int_Q \zeta^2\sum_{i=1}^{N-1} A_i(z, v, Dv)D_{y_i}(v-k)_-\,dz \\ &\le 2\int_K \zeta|v_x|(v-k)_-|\zeta_x|\,dz \\ &\quad + 2\int_Q \zeta\sum_{i=1}^{N-1}|A_i(z, v, Dv)|(v-k)_-|D_{y_i}\zeta|\,dz. \end{aligned}$$

We will employ the structure conditions and Young's inequality repeatedly in estimating the various terms. We first estimate the terms on the left-hand side. The first term is estimated by

$$-\int_Q \zeta^2 v_x D_x(v-k)_- dz \geq \int_K |D_x(v-k)_-|^2 \zeta^2\, dz,$$

while the second integral is

$$\begin{aligned} -\int_Q \zeta^2 v_x (v-k)_-\, dz &= \frac{1}{2}\int_K \zeta^2 D_x |(v-k)_-|^2\, dz \\ &= -\int_K \zeta\zeta_x |(v-k)_-|^2\, dz. \end{aligned}$$

The third term is estimated by

$$-\int_Q \zeta^2 \sum_{i=1}^{N-1} A_i(z, v, Dv) D_{y_i}(v-k)_-\, dz \geq C_o \int_K |D_y(v-k)_-|^q \zeta^2\, dz.$$

Now we estimate the terms on the right-hand side. The first term is estimated by

$$\begin{aligned} &2\int_Q \zeta |v_x| (v-k)_- |\zeta_x|\, dz \\ &\quad \leq \frac{1}{2}\int_Q \zeta^2 |D_x(v-k)_-|^2\, dz + 2\int_Q (v-k)_-^2 |\zeta_x|^2\, dz. \end{aligned}$$

Similarly, the second term is estimated by

$$\begin{aligned} &2\int_Q \zeta \sum_{i=1}^{N-1} |A_i(z, v, Dv)| (v-k)_- |D_{y_i}\zeta|\, dz \\ &\quad \leq \frac{C_o}{2}\int_Q \zeta^2 |D_y(v-k)_-|^q\, dz + \gamma \int_Q (v-k)_-^q |D_y\zeta|^q\, dz. \end{aligned}$$

Collecting all these estimates gives the desired result. □

3.3 Shrinking the measure of the set [$v \approx 0$]

The measure theoretical information in Lemma 5 gives that for some positive integer j_o depending only on the data, such that

$$\left|\left[v(\tilde{x}, \cdot) \geq 2^{-j_o} e^{\frac{2}{2-q}\tilde{x}}\right] \cap K_1\right| \geq \frac{\beta}{2}|K_1| \quad \text{for all } \tilde{x} \geq 0.$$

Let $\tilde{x}_o,\ n_* > 0$ to be chosen and set

$$\bar{\epsilon} = 2^{-j_o}, \quad k_o = \bar{\epsilon} e^{\frac{2}{2-q}\tilde{x}_o}, \quad k_n = \frac{k_o}{2^n} \quad \text{for } n = 0, 1, \ldots, n_*. \tag{11}$$

With this stipulation the measure theoretical information above implies

$$|[v(\tilde{x}, \cdot) \geq k_n] \cap K_8| \geq \frac{\beta}{2}\frac{1}{8^{N-1}}|K_8| \quad \text{for all } \tilde{x} \in (\tilde{x}_o, \infty), \tag{12}$$

and for all $n > 0$. Assume momentarily that $\tilde{x}_o$ has been chosen. Introduce the pair of cylinders

$$\mathcal{Q} = \left[\tilde{x}_o + k_o^{2-q}, \tilde{x}_o + 2k_o^{2-q}\right] \times K_8, \quad \mathcal{Q}' = \left[\tilde{x}_o, \tilde{x}_o + 3k_o^{2-q}\right] \times K_{16}.$$

Lemma 8 *There exists $n_* > 0$ depending only on the data, such that for any $\nu \in (0,1)$, we have*

$$\left|\left[v \le \frac{k_o}{2^{n_*}}\right] \cap \mathcal{Q}\right| \le \nu|\mathcal{Q}|.$$

Proof We employ the energy estimate (10) in the cylinder $\mathcal{Q}'$, with levels k_n and a non-negative, piecewise smooth, cutoff function ζ in $\mathcal{Q}'$, which equals 1 in $\mathcal{Q}$ and vanishes on $\partial\mathcal{Q}'$, satisfying

$$|D_{\tilde{y}}\zeta| \le \frac{1}{8}, \qquad |\zeta_{\tilde{x}}| \le k_o^{q-2}.$$

Since j_o and thus $\bar{\epsilon}$ are fixed, one verifies that if we take $\tilde{x}_o$ large enough then we have

$$k_o^{q-2} = \bar{\epsilon}^{q-2}e^{-2\tilde{x}_o} \le 1,$$

and as a result

$$|\zeta_{\tilde{x}}| + |\zeta_{\tilde{x}}|^2 \le 2k_o^{q-2}.$$

From this observation, the energy estimate (10) gives us that

$$\begin{aligned}\int_{\mathcal{Q}} |D_{\tilde{y}}(v-k_n)_+|^q \, d\tilde{z} &\le \gamma k_n^q \left(1 + \frac{k_n^{2-q}}{k_o^{2-q}}\right) |[v < k_n] \cap \mathcal{Q}| \\ &\le \gamma k_n^q |[v < k_n] \cap \mathcal{Q}|,\end{aligned}$$

where we have used the fact that $k_n^{2-q} \le k_o^{2-q}$.

Next, we apply [2, Chapter I, Lemma 2.2] to $v(\tilde{x},\cdot)$ for

$$\tilde{x} \in I \stackrel{\text{def}}{=} \left[\tilde{x}_o + k_o^{2-q}, \tilde{x}_o + 2k_o^{2-q}\right]$$

over K_8, with levels $k_n > k_{n+1}$. Taking into account the measure theoretical information in (12), this gives

$$\begin{aligned}&\frac{k_o}{2^{n+1}} |[v(\tilde{x},\cdot) < k_{n+1}] \cap K_8| \\ &\quad\le \frac{\gamma}{|[v(\tilde{x},\cdot) > k_n] \cap K_8|} \int_{[k_{n+1} < v(\tilde{x},\cdot) < k_n] \cap K_8} |D_{\tilde{y}} v| \, d\tilde{y} \\ &\quad\le \frac{\gamma}{\beta} \left(\int_{[k_{n+1} < v(\tilde{x},\cdot) < k_n] \cap K_8} |D_{\tilde{y}} v|^q \, d\tilde{y}\right)^{\frac{1}{q}} \\ &\qquad\times |([v(\tilde{x},\cdot) < k_n] - [v(\tilde{x},\cdot) < k_{n+1}]) \cap K_8|^{1-\frac{1}{q}}.\end{aligned}$$

Set

$$A_n = [v < k_n] \cap \mathcal{Q}$$

and integrate the above estimate in $d\tilde{x}$ over I; we obtain by using the energy estimate

$$\begin{aligned}\frac{k_o}{2^{n+1}} |A_{n+1}| &\le \frac{\gamma}{\beta} \left(\int_{\mathcal{Q}} |D_{\tilde{y}}(v-k_n)_+|^q \, d\tilde{y}\right)^{\frac{1}{q}} (|A_n| - |A_{n+1}|)^{1-\frac{1}{q}} \\ &\le \gamma \frac{k_o}{2^n} |\mathcal{Q}|^{\frac{1}{q}} (|A_n| - |A_{n+1}|)^{1-\frac{1}{q}}.\end{aligned}$$

Now take the power $\frac{q}{q-1}$ on both sides to obtain

$$|A_{n+1}|^{\frac{q}{q-1}} \le \gamma |\mathcal{Q}|^{\frac{1}{q-1}} (|A_n| - |A_{n+1}|).$$

Add these inequalities from 0 to $n_* - 1$ to obtain

$$n_* |A_{n_*}|^{\frac{q}{q-1}} \le \sum_{n=0}^{n_*-1} |A_{n+1}|^{\frac{q}{q-1}} \le \gamma |\mathcal{Q}|^{\frac{q}{q-1}}.$$

From this we conclude that

$$|A_{n_*}| \le \frac{\gamma}{n_*^{\frac{q-1}{q}}} |\mathcal{Q}|.$$

Thus we may fix n_* by choosing $\gamma n_*^{-\frac{q-1}{q}} \le \nu$. □

3.4 A DeGiorgi-type lemma for *v*

Suppose for the moment $\tilde{x}_o > 0$ has been chosen in terms of the data. Assume for some $b > 0$ to be determined only in terms of the data, there holds the set inclusion:

$$(x_1, 0) + Q_8(\theta) \subset \mathcal{Q} \subset [0, b] \times K_{16}, \quad \text{where } \theta = (2^{-n_*} k_o)^{\frac{2-q}{2}}, \ k_o = \bar{\epsilon} e^{\frac{2}{2-q}\tilde{x}_o}.$$

We may employ the energy estimate (10) in $(x_1, 0) + Q_8(\theta)$ to obtain the following.

Lemma 9 *There exists a number $\nu > 0$ depending only upon the data, such that if*

$$\left| \left[v \le \frac{k_o}{2^{n_*}} \right] \cap [(x_1, 0) + Q_8(\theta)] \right| \le \nu |Q_8(\theta)|,$$

then

$$v \ge \frac{k_o}{2^{n_*+1}} \quad \text{a.e. in } (x_1, 0) + Q_4(\theta).$$

Proof We may assume $(x_1, 0) = (0, 0)$. For $n = 0, 1, \ldots$, we set

$$l_n = \frac{M_1}{2} + \frac{M_1}{2^{n+1}}, \quad M_1 = \frac{k_o}{2^{n_*}},$$
$$\rho_n = 4 + \frac{8}{2^n}, \quad \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2},$$
$$Q_n = Q_{\rho_n}(\theta), \quad \tilde{Q}_n = Q_{\tilde{\rho}_n}(\theta), \quad \theta = M_1^{\frac{2-q}{2}}.$$

Introduce a non-negative, piecewise smooth, cutoff function ζ vanishing on ∂Q_n and equal to identity in $\tilde{Q}_n$, such that

$$|D_{\tilde{y}} \zeta| \le \gamma 2^n \quad \text{and} \quad |\zeta_{\tilde{x}}| \le \gamma \frac{2^{\frac{q}{2}n}}{\theta}.$$

The constant n_* will be chosen in (14) first and then $\tilde{x}_o$ in (17), such that

$$\begin{cases} M_1 = \dfrac{e^{\frac{2}{2-q}\tilde{x}_o}}{2^{n_*}} \overset{\text{def}}{=} \Lambda_o > 1 \text{ and } \theta = M_1^{\frac{2-q}{2}} \overset{\text{def}}{=} \Lambda_o^{\frac{2-q}{2}} \\ \text{are absolute constants depending only on the data.} \end{cases} \tag{13}$$

Assuming this for the moment, we would have

$$|\zeta_{\tilde{x}}| + |\zeta_{\tilde{x}}|^2 \le \gamma \left(\Lambda_o^{\frac{2-q}{2}} + \Lambda_o^{2-q} \right) 2^{qn} \le \gamma \Lambda_o^{2-q} 2^{qn}.$$

Consequently, the energy estimate (10) yields

$$\begin{aligned}
&\int_{\tilde{Q}_n} |D_{\tilde{x}}(v - l_n)_-|^2 \, d\tilde{z} + \int_{\tilde{Q}_n} |D_{\tilde{y}}(v - l_n)_-|^q \, d\tilde{z} \\
&\le \gamma 2^{qn} \Lambda_o^{2-q} \int_{Q_n} (v - l_n)_-^2 \, d\tilde{z} + \gamma 2^{qn} \int_{Q_n} (v - l_n)_-^q \, d\tilde{z} \\
&\le \gamma 2^{qn} \Lambda_o^{2-q} (M_1^2 + M_1^q) |A_n| \\
&\le \gamma 2^{qn} \Lambda_o^{4-q} |A_n|,
\end{aligned}$$

where we have set $A_n = [v < l_n] \cap Q_n$.

Next, we may proceed exactly as in Lemma 2 to obtain the recurrence inequality

$$Y_{n+1} \le \gamma \Lambda_o^d \gamma_1^n Y_n^{1+\frac{\bar{q}}{N}} \quad \text{for } n \ge 0,$$

where d, γ_1 and γ are positive constants depending only on the data, and $Y_n = |A_n|/|Q_n|$.

Hence by [2, Chapter I, Lemma 4.1], $Y_n \to 0$ if we require that

$$Y_o \le \gamma \Lambda_o^{-\frac{dN}{\bar{q}}} \stackrel{\text{def}}{=} \nu.$$

□

3.5 Expanding the positivity of *v*

Assume momentarily that n_* and $\tilde{x}_o$ have been determined. We may also assume $\bar{\epsilon}^{\frac{2-q}{2}} e^{\tilde{x}_o} 2^{\frac{2-q}{2} n_*}$ is an integer. Then recalling the definition of k_o and k_{n_*} in (11), we slice $\mathcal{Q}$ into $\bar{\epsilon}^{\frac{2-q}{2}} e^{\tilde{x}_o} 2^{\frac{2-q}{2} n_*}$ cylinders, each of length $k_{n_*}^{\frac{2-q}{2}}$, by setting

$$\begin{cases} Q_n = \left[\tilde{x}_o + k_o^{2-q} + n k_{n_*}^{\frac{2-q}{2}}, \tilde{x}_o + k_o^{2-q} + (n+1) k_{n_*}^{\frac{2-q}{2}} \right] \times K_8 \\ \text{for } n = 0, 1, \ldots, \bar{\epsilon}^{\frac{2-q}{2}} e^{\tilde{x}_o} 2^{\frac{2-q}{2} n_*} - 1. \end{cases}$$

For at least one of these cylinders, say Q_n, there must hold

$$|[v < k_{n_*}] \cap Q_n| \le \nu |Q_n|.$$

Having ν fixed in terms of the data as in Lemma 9, we may choose n_* according to Lemma 8, such that

$$\gamma n_*^{-1} \le \nu. \tag{14}$$

Apply Lemma 9 to v over Q_n with $\theta = k_{n_*}^{\frac{2-q}{2}}$ to obtain that

$$v \left(\tilde{x}_o + k_o^{2-q} + \left(n + \tfrac{1}{2}\right) k_{n_*}^{\frac{2-q}{2}}, \cdot \right) \ge \frac{k_{n_*}}{2} \quad \text{a.e. in } K_4.$$

Consequently, there exists some $\tilde{x}_1$ in the range

$$\tilde{x}_o + k_o^{2-q} < \tilde{x}_1 < \tilde{x}_o + 2k_o^{2-q},$$

such that

$$v(\tilde{x}_1, \cdot) \geq \sigma_o e^{\frac{2}{2-q}\tilde{x}_o} \quad \text{a.e. in } K_4$$

where

$$\sigma_o = \frac{\bar{\epsilon}}{2^{n_*+1}}. \tag{15}$$

3.6 Returning to the original coordinates

In terms of the original coordinates and the function u, we arrive at

$$u(x_1, \cdot) \geq \sigma_o M e^{\frac{-2}{2-q}(\tilde{x}_1 - \tilde{x}_o)} \stackrel{\text{def}}{=} M_o \quad \text{a.e. in } K_{4\rho},$$

where x_1 corresponds to $\tilde{x}_1$ according to the change of variables (8). Apply now Lemma 7 with M replaced by M_o to obtain that there exists $j_* > 0$ depending only on the data, such that, setting $\epsilon = 2^{-j_*}$,

$$\begin{aligned} u(x, \cdot) &\geq \epsilon M_o = \epsilon \sigma_o M e^{\frac{-2}{2-q}(\tilde{x}_1 - \tilde{x}_o)} \\ &\geq \epsilon \sigma_o e^{\frac{-4}{2-q}e^{2\tilde{x}_o}} M \quad \text{a.e. in } K_{2\rho} \end{aligned} \tag{16}$$

for all

$$x_1 \leq x \leq x_1 + (\epsilon M_o)^{\frac{2-q}{2}} (2\rho)^{\frac{q}{2}}.$$

The parameter $\tilde{x}_o$ is still to be chosen. Now we choose it such that the right-hand side of the above interval equals $M^{\frac{2-q}{2}} \rho^{\frac{q}{2}}$, i.e.,

$$M^{\frac{2-q}{2}} \rho^{\frac{q}{2}} e^{-\tilde{x}_1} = M^{\frac{2-q}{2}} \rho^{\frac{q}{2}} - x_1 = (\epsilon \sigma_o)^{\frac{2-q}{2}} M^{\frac{2-q}{2}} e^{\tilde{x}_o - \tilde{x}_1} (2\rho)^{\frac{q}{2}},$$

which implies the choice of $\tilde{x}_o$:

$$e^{\tilde{x}_o} = \frac{1}{2^{\frac{q}{2}} (\epsilon \sigma_o)^{\frac{2-q}{2}}}.$$

Taking into consideration (15), this choice of $\tilde{x}_o$ implies that

$$\frac{e^{\tilde{x}_o}}{2^{\frac{2-q}{2} n_*}} = \frac{2^{1-q}}{\epsilon^{\frac{2-q}{2}}} \stackrel{\text{def}}{=} \Lambda_o^{\frac{2-q}{2}} > 1, \tag{17}$$

which guarantees that (13) holds. Therefore, (16) holds for

$$x_1 = \rho^{\frac{q}{2}} M^{\frac{2-q}{2}} - (\epsilon M_o)^{\frac{2-q}{2}} (2\rho)^{\frac{q}{2}} \leq x \leq \rho^{\frac{q}{2}} M^{\frac{2-q}{2}}.$$

From the definition of $\tilde{x}_o$ and the change of variable (8) one estimates

$$x_1 \leq (1 - \delta) M^{\frac{2-q}{2}} \rho^{\frac{q}{2}} \quad \text{where } \delta = e^{-\tilde{x}_o - 2e^{2\tilde{x}_o}}.$$

3.7 Proof of Proposition 1 concluded

Since

$$|[u(\cdot,0)\geq M]\cap K_\rho|\geq\beta|K_\rho|,$$

according to Lemma 5, there exists $j_o>0$, such that

$$\left|\left[u(x,\cdot)\geq\frac{M}{2^{j_o}}\right]\cap K_\rho\right|\geq\frac{\beta}{2}|K_\rho|$$

for all

$$-\left(\frac{M}{2^{j_o}}\right)^{\frac{2-q}{2}}\rho^{\frac{q}{2}}\leq x\leq\left(\frac{M}{2^{j_o}}\right)^{\frac{2-q}{2}}\rho^{\frac{q}{2}}.$$

We apply Proposition 4 with M replaced by $M_1=\bar{\delta}M=2^{-j_o}M$ and $x_o=\pm M_1^{\frac{2-q}{2}}\rho^{\frac{q}{2}}$. As a result, there exist $\eta,\ \delta\in(0,1)$, such that

$$u(x,\cdot)\geq\eta\bar{\delta}M\quad\text{a.e. in } K_{2\rho}$$

for all

$$|x|\leq\delta(\bar{\delta}M)^{\frac{2-q}{2}}\rho^{\frac{q}{2}}.$$

The proof of Proposition 1 is completed by properly redefining constants η and δ.

4 Proof of Theorem 1

Let the parameters δ and η be fixed as in Proposition 1 with $\beta=\frac{1}{2}$. In order to apply Proposition 1, we set

$$\Lambda=4\delta^{\frac{-2}{2-q}}.$$

Then one of the following two alternatives must hold:

$$\left|\left[\mu^+-u(0,\cdot)\geq\frac{\omega}{4}\right]\cap K_R\right|\geq\frac{1}{2}|K_R|\quad\text{or}$$
$$\left|\left[u(0,\cdot)-\mu^-\geq\frac{\omega}{4}\right]\cap K_R\right|\geq\frac{1}{2}|K_R|.$$

According to Proposition 1 with $M=\frac{1}{4}\omega$, we have

$$\pm(\mu^\pm-u)\geq\frac{\eta}{4}\omega\quad\text{a.e. in } Q_{\frac{R}{2}}(\theta)\text{ with }\theta=\left(\frac{\omega}{\Lambda}\right)^{\frac{2-q}{2}}.$$

In either case, we obtain a reduction of oscillation. More precisely,

$$\operatorname*{ess\,osc}_{Q_{\frac{R}{2}}(\theta)}u\leq\left(1-\frac{\eta}{4}\right)\omega.$$

Once we have this reduction of oscillation, the rest of the proof is quite standard. We refer to p. 45 of [2] for details.

5 Proof of Theorem 2 assuming Theorem 3

Fix $z_o \in E$, assume $u(z_o) > 0$, and construct cylinders

$$z_o + Q_{4\rho}(\theta) \subset E, \quad \text{where } \theta = [\bar{c}u(z_o)]^{\frac{2-q}{2}}.$$

Introduce the new variables $\tilde{z} = (\tilde{x}, \tilde{y})$ defined by

$$\tilde{y} = \frac{y - y_o}{\rho}, \qquad \tilde{x} = \frac{x - x_o}{\rho^{\frac{q}{2}}[u(z_o)]^{\frac{2-q}{2}}}.$$

Under this mapping, the cylinder $z_o + Q_{4\rho}(\theta)$ is transformed into

$$\tilde{Q} = \left[-4^{\frac{q}{2}}\bar{c}^{\frac{2-q}{2}}, 4^{\frac{q}{2}}\bar{c}^{\frac{2-q}{2}}\right] \times K_4, \quad q < 2.$$

Consider the new function in $\tilde{Q}$:

$$v(\tilde{x}, \tilde{y}) = \frac{1}{u(z_o)} u\left(x_o + \tilde{x}\rho^{\frac{q}{2}}[u(z_o)]^{\frac{2-q}{2}}, y_o + \rho\tilde{y}\right).$$

Then $v(\tilde{z})$ is a bounded non-negative, weak solution to

$$v_{\tilde{x}\tilde{x}} + \sum_{i=1}^{N-1} D_{\tilde{y}_i} \tilde{A}_i(\tilde{z}, v, Dv) = 0, \tag{18}$$

where the function $\tilde{A}_i(\tilde{z}, v, Dv) : E \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is defined by

$$\tilde{A}_i(\tilde{z}, v, Dv) = A_i(z, u, Du)[u(z_o)]^{1-q}\rho^{q-1},$$

subject to the following structure conditions a.e. in $\tilde{Q}$:

$$\begin{cases} \displaystyle\sum_{i=1}^{N-1} \tilde{A}_i(\tilde{z}, v, Dv) \cdot D_{\tilde{y}_i} v \geq C_o |D_{\tilde{y}} v|^q \\ |\tilde{A}_i(\tilde{z}, v, Dv)| \leq C_1 |D_{\tilde{y}} v|^{q-1} \quad i = 1, \ldots, N-1. \end{cases}$$

Therefore Proposition 1 holds for v. In what follows, for simplicity of notation, let us still denote the variables of v by $z = (x, y)$.

The proof of the right-hand side inequality in Theorem 2 is a consequence of the following.

Proposition 6 *There exist constants c, $\bar{c} \in (0, 1)$, which can be determined only in terms of the data, such that*

$$v \geq c \quad \text{in } Q_1(\bar{c}).$$

5.1 Proof of Proposition 6

For $\tau \in (0, 1)$, introduce the family of nested cubes $\{K_\tau\}$ and the families of non-negative numbers $\{M_\tau\}$ and $\{N_\tau\}$ as follows:

$$M_\tau = \sup_{K_\tau} v(0, \cdot), \quad N_\tau = (1 - \tau)^{-\sigma},$$

where $\sigma > 1$ is to be chosen. The two functions $[0, 1) \ni \tau \to M_\tau$, N_τ are increasing, and $M_o = N_o = 1$ since $v(0, 0) = 1$. Moreover, $N_\tau \to \infty$ as $\tau \to 1$ whereas M_τ is bounded

since v is locally bounded. Therefore the equation $M_\tau = N_\tau$ has roots and we denote the largest one as τ_*. By the continuity of v, there exists $\bar{y} \in K_{\tau_*}$, such that

$$v(0, \bar{y}) = M_{\tau_*} = N_{\tau_*} = (1 - \tau_*)^{-\sigma}.$$

Moreover,

$$K_{2R}(\bar{y}) \subset K_{\frac{1+\tau_*}{2}} \subset K_1, \quad \text{where } 2R \overset{\text{def}}{=} \frac{1 - \tau_*}{2}.$$

Therefore by the definition of τ_*,

$$\sup_{K_{2R}(\bar{y})} v(0, \cdot) \leq \sup_{K_{\frac{1+\tau_*}{2}}} v(0, \cdot) \leq 2^\sigma (1 - \tau_*)^{-\sigma} \overset{\text{def}}{=} M_*.$$

Now we apply Theorem 3 to conclude that there exists $\bar{\gamma} \in (0, 1)$ depending on the data, such that

$$\sup_{(0,\bar{y})+Q_R(\theta)} v \leq \bar{\gamma} M_* \quad \text{where } \theta = M_*^{\frac{2-q}{2}}.$$

Next, we may take Λ in the proof of Theorem 1 even larger, such that

$$(0, \bar{y}) + Q_R(\bar{\theta}) \subset (0, \bar{y}) + Q_R(\theta) \quad \text{and} \quad \operatorname*{osc}_{(0,\bar{y})+Q_R(\bar{\theta})} v \leq \bar{\gamma} M_*,$$

where

$$\bar{\theta} = \left(\frac{\bar{\gamma} M_*}{\Lambda}\right)^{\frac{2-q}{2}}.$$

Then $(0, \bar{y}) + Q_R(\bar{\theta})$ serves as the starting cylinder in Theorem 1.

Let $\epsilon_* \in (0, 1)$ and set $r = \epsilon_* R$. By Theorem 1, for all $r < R$ and for all $z \in (0, \bar{y}) + Q_r(\bar{\theta})$

$$v(z) - v(0, \bar{y}) \geq -\gamma[\bar{\gamma} 2^\sigma (1 - \tau_*)^{-\sigma}] \left(\frac{r}{R}\right)^\alpha \geq -\frac{1}{2}(1 - \tau_*)^{-\sigma},$$

provided we choose ϵ_* so small that

$$\gamma \bar{\gamma} 2^\sigma \epsilon_*^\alpha \leq \frac{1}{2}.$$

This in turn gives

$$v(z) \geq \frac{1}{2}(1 - \tau_*)^{-\sigma} \overset{\text{def}}{=} M \quad \text{for all } z \in (0, \bar{y}) + Q_r(\bar{\theta}).$$

From this we may start employing Proposition 1 to conclude that there exist positive constants η and δ as indicated, such that

$$v(x, \cdot) \geq \eta M \quad \text{in } K_{2r}(\bar{y})$$

for all

$$-\delta \bar{\theta} r^{\frac{q}{2}} \leq x \leq \delta \bar{\theta} r^{\frac{q}{2}}.$$

Repeated applications of Proposition 1 then yield positive constants η and δ as indicated, such that

$$v(x, \cdot) \geq \eta^n M \quad \text{in } K_{2^n r}(\bar{y})$$

for all

$$-\delta\bar{\theta}r^{\frac{q}{2}} \le x \le \delta\bar{\theta}r^{\frac{q}{2}}.$$

We may assume $\epsilon_*(1-\tau_*)$ is a negative, integral power of 2. Then choose n such that $2^n r = 2$. In this way, we calculate

$$\eta^n M = \frac{1}{2}(1-\tau_*)^{-\sigma}\eta^n = \frac{1}{2}\left(\frac{8}{\epsilon_* 2^n}\right)^{-\sigma}\eta^n = \frac{1}{2}(2^\sigma\eta)^n\left(\frac{\epsilon_*}{8}\right)^\sigma.$$

Finally, we may choose σ such that $2^\sigma\eta = 1$. As a result, setting $c = 2^{-3\sigma-1}\epsilon_*^\sigma$, we have

$$v(x,\cdot) \ge c \quad \text{in } K_1$$

for all

$$-\delta\bar{\theta}r^{\frac{q}{2}} \le x \le \delta\bar{\theta}r^{\frac{q}{2}}.$$

On the other hand, one estimates

$$\begin{aligned}\delta\bar{\theta}r^{\frac{q}{2}} &= \delta[\bar{\gamma}\Lambda^{-1}2^\sigma(1-\tau_*)^{-\sigma}]^{\frac{2-q}{2}}[2^{-2}\epsilon_*(1-\tau_*)]^{\frac{q}{2}}\\ &= \delta^{\frac{2-q}{2}}\bar{\gamma}^{\frac{2-q}{2}}\Lambda^{\frac{q-2}{2}}2^{\sigma\frac{2-q}{2}-q}\epsilon_*^{\frac{q}{2}}(1-\tau_*)^{-\sigma\frac{2-q}{2}+\frac{q}{2}}\\ &\ge \delta^{\frac{2-q}{2}}\bar{\gamma}^{\frac{2-q}{2}}\Lambda^{\frac{q-2}{2}}2^{\sigma\frac{2-q}{2}-q}\epsilon_*^{\frac{q}{2}} \stackrel{\text{def}}{=} \bar{c}^{\frac{2-q}{2}},\end{aligned}$$

provided $\sigma \ge q/(2-q)$, which may be assumed by possibly taking η smaller if necessary. In conclusion,

$$v \ge c \quad \text{in } Q_1(\bar{c}).$$

5.2 Proof of Theorem 2 concluded

We have shown in the last section that

$$u(z_o) \le c^{-1}\inf_{z_o+Q_\rho(\theta)} u \quad \text{where} \quad \theta = [\bar{c}u(z_o)]^{\frac{2-q}{2}}.$$

Now we claim that

$$\sup_{z_o+Q_\rho(\theta)} u \le 2c^{-1}u(z_o).$$

Indeed, if not, by continuity of u, there would exist $z_* \in z_o + Q_\rho(\theta)$ such that

$$u(z_*) = 2c^{-1}u(z_o).$$

The membership of z_* in $z_o + Q_\rho(\theta)$ implies that

$$|x_o - x_*| \le \rho^{\frac{q}{2}}[\bar{c}u(z_o)]^{\frac{2-q}{2}} = \rho^{\frac{q}{2}}\left[\frac{c\bar{c}}{2}u(z_*)\right]^{\frac{2-q}{2}},$$

which in turn gives that

$$z_o \in z_* + Q_{\bar{\rho}}(\theta_*) \quad \text{where } \bar{\rho} = \rho\left(\frac{c}{2}\right)^{\frac{2-q}{q}} \text{ and } \theta_* = [\bar{c}u(z_*)]^{\frac{2-q}{2}}.$$

However, this leads to a contradiction:

$$2c^{-1}u(z_o) = u(z_*) \le c^{-1} \inf_{z_* + Q_{\tilde{\rho}}(\theta_*)} u \le c^{-1}u(z_o).$$

The proof of Theorem 2 is now concluded by properly redefining c.

6 Proof of Theorem 3

We first present two propositions from which Theorem 3 follows.

Proposition 7 *Let u be a non-negative, local, weak super-solution to (1) and (2) in E. There exists a constant $\gamma > 0$ depending only on the data, such that for all cylinders*

$$[x_o - T, x_o + T] \times K_{2\rho}(y_o) \subset E,$$

we have

$$\sup_{0<t<T} \frac{1}{t\rho^{N-1}} \int_{x_o - t}^{x_o + t} \int_{K_\rho(y_o)} u \, dy dx \le \frac{\gamma}{\rho^{N-1}} \int_{K_{2\rho}(y_o)} u(x_o, y) \, dy + \gamma \left(\frac{T^2}{\rho^q} \right)^{\frac{1}{2-q}}.$$

Proposition 8 *Let u be a locally bounded, local, weak sub(super)-solution to (1) and (2) in E. For $r \in (0, 2]$, let*

$$\kappa_r \overset{def}{=} (N + r - 1)q - 2(N - 1) > 0.$$

There exists a constant $\gamma > 0$ depending only on the data, such that for all cylinders

$$[x_o - 2t, x_o + 2t] \times K_{2\rho}(y_o) \subset E,$$

we have

$$\operatorname*{ess\,sup}_{[x_o - t, x_o + t] \times K_\rho(y_o)} u_\pm \le \gamma \left(\frac{\rho^q}{t^2} \right)^{\frac{N-1}{\kappa_r}} \left(\fint_{K_{2\rho}(y_o)} \fint_{x_o - 2t}^{x_o + 2t} u_\pm^r \, dx dy \right)^{\frac{q}{\kappa_r}} + \gamma \left(\frac{t^2}{\rho^q} \right)^{\frac{1}{2-q}}.$$

6.1 Proof of Proposition 7

Assuming $z_o = (0, 0)$, let us consider the cylinder $[-T, T] \times K_\rho$. In what follows we denote by $\zeta_2(y)$ a piecewise smooth function that vanishes on K_ρ and equals 1 on $K_{\sigma\rho}$, such that

$$|D_y \zeta_2| \le \frac{\gamma}{(1 - \sigma)\rho},$$

whereas for $0 < t < T$ we define

$$\zeta_1(x) = \left(1 - \frac{x^2}{t^2} \right)_+.$$

Let us introduce a constant α that satisfies

$$1 < \alpha < \min \left\{ q, \frac{1}{q - 1} \right\},$$

such that the quantities $2-\alpha, q-\alpha$ and $\alpha(q-1)$ are all in $(0,1)$. The proof of Proposition 7 replies on the following two lemmas.

6.1.1 Auxiliary lemmas

Lemma 10 *Suppose the hypothesis in Proposition 7 holds. There exists a constant $\gamma > 0$ depending only on the data and α, such that*

$$\int_{-t}^{t}\int_{K_\rho} |D_y u|^q u^{-\alpha}\zeta_1\zeta_2^q\, dydx \le \frac{\gamma}{t^2}\int_{-t}^{t}\int_{K_\rho} u^{2-\alpha}\, dydx + \frac{\gamma}{(1-\sigma)^q\rho^q}\int_{-t}^{t}\int_{K_\rho} u^{q-\alpha}\, dydx.$$

Proof We may assume that $u \ge \epsilon > 0$ for otherwise we may work with $u+\epsilon$ and then let $\epsilon \to 0$. Now we use $u^{1-\alpha}\zeta_1\zeta_2^q$ as a test function in the weak formulation of (1). Formally, we have

$$\int_{-t}^{t}\int_{K_\rho} u_{xx}u^{1-\alpha}\zeta_1\zeta_2^q\, dydx + \sum_{i=1}^{N-1}\int_{-t}^{t}\int_{K_\rho} D_{y_i}A_i(z,u,Du)u^{1-\alpha}\zeta_1\zeta_2^q\, dydx \le 0.$$

We estimate the two terms on the left separately. First of all,

$$\begin{aligned}
&\int_{-t}^{t}\int_{K_\rho} u_{xx}u^{1-\alpha}\zeta_1\zeta_2^q\, dydx\\
&= (\alpha-1)\int_{-t}^{t}\int_{K_\rho} |u_x|^2u^{-\alpha}\zeta_1\zeta_2^q\, dydx + \frac{2}{t^2}\int_{-t}^{t}\int_{K_\rho} xu_xu^{1-\alpha}\zeta_2^q\, dydx\\
&= (\alpha-1)\int_{-t}^{t}\int_{K_\rho} |u_x|^2u^{-\alpha}\zeta_1\zeta_2^q\, dydx + \frac{2}{(2-\alpha)t^2}\int_{-t}^{t}\int_{K_\rho} xD_x(u^{2-\alpha})\zeta_2^q\, dydx\\
&\ge (\alpha-1)\int_{-t}^{t}\int_{K_\rho} |u_x|^2u^{-\alpha}\zeta_1\zeta_2^q\, dydx - \frac{2}{(2-\alpha)t^2}\int_{-t}^{t}\int_{K_\rho} u^{2-\alpha}\zeta_2^q\, dydx.
\end{aligned}$$

Next,

$$\begin{aligned}
&\sum_{i=1}^{N-1}\int_{-t}^{t}\int_{K_\rho} D_{y_i}A_i(z,u,Du)u^{1-\alpha}\zeta_1\zeta_2^q\, dydx\\
&\ge C_o(\alpha-1)\int_{-t}^{t}\int_{K_\rho} |D_y u|^q u^{-\alpha}\zeta_1\zeta_2^q\, dydx\\
&\quad - qC_1\int_{-t}^{t}\int_{K_\rho} |D_y u|^{q-1}u^{1-\alpha}\zeta_1\zeta_2^{q-1}|D_y\zeta_2|\, dydx\\
&\ge \frac{C_o}{2}(\alpha-1)\int_{-t}^{t}\int_{K_\rho} |D_y u|^q u^{-\alpha}\zeta_1\zeta_2^q\, dydx\\
&\quad - \gamma\int_{-t}^{t}\int_{K_\rho} u^{q-\alpha}|D_y\zeta_2|^q\, dydx.
\end{aligned}$$

Combining them we obtain the desired conclusion. □

Lemma 11 *Suppose the hypothesis in Proposition 7 holds. There exists a constant $\gamma > 0$ depending only on the data and α, such that*

$$\frac{1}{(1-\sigma)\rho^N}\int_{-t}^{t}\int_{K_\rho}|D_y u|^{q-1}\zeta_1\zeta_2^{q-1}\,dydx \le \frac{\gamma t^{\frac{2-q}{q}}}{(1-\sigma)\rho}\mathcal{S}^{2\frac{q-1}{q}} + \frac{\gamma t}{(1-\sigma)^q\rho^q}\mathcal{S}^{q-1},$$

where

$$\mathcal{S} = \sup_{0<t<T}\frac{1}{t\rho^{N-1}}\int_{-t}^{t}\int_{K_\rho} u\,dydx.$$

Proof Notice that by our choice of α, the quantities $2-\alpha$, $q-\alpha$ and $\alpha(q-1)$ are all in $(0,1)$. By Hölder's inequality and Lemma 10, we estimate

$$\begin{aligned}
&\frac{1}{(1-\sigma)\rho^N}\int_{-t}^{t}\int_{K_\rho}|D_y u|^{q-1}\zeta_1\zeta_2^{q-1}\,dydx\\
&\le \frac{1}{(1-\sigma)\rho^N}\left[\int_{-t}^{t}\int_{K_\rho}|D_y u|^q u^{-\alpha}\zeta_1\zeta_2^q\,dydx\right]^{\frac{q-1}{q}}\left[\int_{-t}^{t}\int_{K_\rho}u^{\alpha(q-1)}\,dydx\right]^{\frac{1}{q}}\\
&\le \frac{1}{(1-\sigma)\rho^N}\left[\frac{\gamma}{(1-\sigma)^q\rho^q}\int_{-t}^{t}\int_{K_\rho}u^{q-\alpha}\,dydx\right]^{\frac{q-1}{q}}\left[\int_{-t}^{t}\int_{K_\rho}u^{\alpha(q-1)}\,dydx\right]^{\frac{1}{q}}\\
&\quad+\frac{1}{(1-\sigma)\rho^N}\left[\frac{\gamma}{t^2}\int_{-t}^{t}\int_{K_\rho}u^{2-\alpha}\,dydx\right]^{\frac{q-1}{q}}\left[\int_{-t}^{t}\int_{K_\rho}u^{\alpha(q-1)}\,dydx\right]^{\frac{1}{q}}\\
&\le \frac{\gamma t^{\frac{2-q}{q}}}{(1-\sigma)\rho}\mathcal{S}^{2\frac{q-1}{q}} + \frac{\gamma t}{(1-\sigma)^q\rho^q}\mathcal{S}^{q-1}.
\end{aligned}$$

□

6.1.2 Proof of Proposition 7 concluded

We use $\zeta_1\zeta_2^q$ as a test function in the weak formulation of (1) for super-solutions. Formally, we have

$$\int_{-t}^{t}\int_{K_\rho}u_{xx}\zeta_1\zeta_2^q\,dydx + \sum_{i=1}^{N-1}\int_{-t}^{t}\int_{K_\rho}D_{y_i}A_i(z,u,Du)\zeta_1\zeta_2^q\,dydx \le 0.$$

We estimate the two terms on the left separately. First of all,

$$\begin{aligned}
\int_{-t}^{t}\int_{K_\rho}u_{xx}\zeta_1\zeta_2^q\,dydx &= \frac{2}{t^2}\int_{-t}^{t}\int_{K_\rho}xu_x\zeta_2^q\,dydx\\
&= \frac{2}{t}\int_{K_\rho}u(t,\cdot)\zeta_2^q\,dy + \frac{2}{t}\int_{K_\rho}u(-t,\cdot)\zeta_2^q\,dy - \frac{2}{t^2}\int_{-t}^{t}\int_{K_\rho}u\zeta_2^q\,dydx\\
&= 2\rho^{N-1}h'(t),
\end{aligned}$$

where

$$h(t) = \frac{1}{t\rho^{N-1}}\int_{-t}^{t}\int_{K_\rho}u\zeta_2^q\,dydx.$$

On the other hand, we estimate

$$\int_{-t}^{t}\int_{K_\rho} D_{y_i}A_i(z,u,Du)\zeta_1\zeta_2^q\,dydx = -q\int_{-t}^{t}\int_{K_\rho} A_i(z,u,Du)\zeta_1 D_{y_i}\zeta_2\zeta_2^{q-1}\,dydx$$
$$\geq -\frac{qC_1}{(1-\sigma)\rho}\int_{-t}^{t}\int_{K_\rho}|D_y u|^{q-1}\zeta_1\zeta_2^{q-1}\,dydx.$$

Combining the above two estimates we obtain

$$h'(t) \leq \frac{\gamma}{(1-\sigma)\rho^N}\int_{-t}^{t}\int_{K_\rho}|D_y u|^{q-1}\zeta_1\zeta_2^{q-1}\,dydx.$$

We integrate the above inequality in dt over $(0,\tau)$ for $\tau < T$, use Lemma 11 and apply Young's inequality to obtain for an arbitrary $\delta \in (0,1)$,

$$h(\tau)-h(0) \leq \gamma\frac{\tau^{\frac{2}{q}}}{(1-\sigma)\rho}\mathcal{S}^{2\frac{q-1}{q}} + \gamma\frac{\tau^2}{(1-\sigma)^q\rho^q}\mathcal{S}^{q-1}$$
$$\leq \delta\mathcal{S} + \frac{\gamma}{(1-\sigma)^{\frac{q}{2-q}}\delta^{\frac{q-1}{2-q}}}\left(\frac{T^2}{\rho^q}\right)^{\frac{1}{2-q}}.$$

As τ ranges over $(0,T)$, we arrive at

$$\sup_{0<t<T}\frac{1}{t\rho^{N-1}}\int_{-t}^{t}\int_{K_{\sigma\rho}} u\,dydx \leq \delta\sup_{0<t<T}\frac{1}{t\rho^{N-1}}\int_{-t}^{t}\int_{K_\rho} u\,dydx$$
$$+\frac{2}{\rho^{N-1}}\int_{K_\rho}u(0,y)\,dy + \frac{\gamma}{(1-\sigma)^{\frac{q}{2-q}}\delta^{\frac{q-1}{2-q}}}\left(\frac{T^2}{\rho^q}\right)^{\frac{1}{2-q}}.$$

An interpolation argument ([2, Chapter I, Lemma 4.3]) yields that

$$\sup_{0<t<T}\frac{1}{t\rho^{N-1}}\int_{-t}^{t}\int_{K_\rho} u\,dydx \leq \frac{\gamma}{\rho^{N-1}}\int_{K_{2\rho}}u(0,y)\,dy + \gamma\left(\frac{T^2}{\rho^q}\right)^{\frac{1}{2-q}}.$$

6.2 Proof of Proposition 8

For parameters $k,\ \rho,\ t > 0$ and $n = 0, 1, \ldots$, we set

$$k_n = k - \frac{k}{2^n},\quad \rho_n = \sigma\rho + \frac{(1-\sigma)\rho}{2^n},\quad \tilde{\rho}_n = \frac{\rho_n+\rho_{n+1}}{2},$$
$$t_n = \sigma t + \frac{(1-\sigma)t}{2^n},\quad \tilde{t}_n = \frac{t_n+t_{n+1}}{2},$$
$$Q_n = [-t_n,t_n]\times K_{\rho_n},\quad \tilde{Q}_n = [-\tilde{t}_n,\tilde{t}_n]\times K_{\tilde{\rho}_n}.$$

By construction $Q_\infty = [-\sigma t,\sigma t]\times K_{\sigma\rho}$. Assuming u is locally bounded in E, we may define

$$M = \operatorname*{ess\,sup}_{Q_o} u_+,\qquad M_\sigma = \operatorname*{ess\,sup}_{Q_\infty} u_+.$$

Introduce the cutoff function ζ vanishing on ∂Q_n and equal to identity in $\tilde{Q}_n$, such that

$$|D_y\zeta| \leq \frac{\gamma 2^n}{(1-\sigma)\rho}\quad\text{and}\quad |D_x\zeta| \leq \frac{\gamma 2^n}{(1-\sigma)t}.$$

In this setting the energy estimate in Proposition 2 yields

$$\int_{\tilde{Q}_n} |D_x(u-k_{n+1})_+|^2\,dz + \int_{\tilde{Q}_n} |D_y(u-k_{n+1})_+|^q\,dz \le \frac{\gamma 2^{2n}}{(1-\sigma)^2 t^2}\int_{Q_n}(u-k_{n+1})_+^2\,dz + \frac{\gamma 2^{qn}}{(1-\sigma)^q\rho^q}\int_{Q_n}(u-k_{n+1})_+^q\,dz.$$

Next setting $A_n = [u > k_{n+1}] \cap Q_n$, we observe that for any $r > 0$

$$\int_{Q_n}(u-k_n)_+^r\,dz \ge (k_{n+1}-k_n)^r |[u>k_{n+1}]\cap Q_n| = \frac{k^r}{2^{r(n+1)}}|A_n|.$$

As a result, applying Hölder's inequality and the above observation with $r = 2$, we estimate the second integral on the right-hand side of the energy estimate by

$$\int_{Q_n}(u-k_{n+1})_+^q\,dz \le \left(\int_{Q_n}(u-k_{n+1})_+^2\,dz\right)^{\frac{q}{2}} |[u>k_{n+1}]\cap Q_n|^{1-\frac{q}{2}} \le \frac{\gamma 2^{2n-qn}}{k^{2-q}}\int_{Q_n}(u-k_n)_+^2\,dz.$$

Putting this back to the energy estimate gives us

$$\int_{\tilde{Q}_n} |D_x(u-k_{n+1})_+|^2\,dz + \int_{\tilde{Q}_n} |D_y(u-k_{n+1})_+|^q\,dz \le \frac{\gamma 2^{2n}k^{q-2}}{(1-\sigma)^2\rho^q}\left(1+k^{2-q}\frac{\rho^q}{t^2}\right)\int_{Q_n}(u-k_n)_+^2\,dz.$$

Now setting ζ to be a cutoff function which vanishes on the boundary of $\tilde{Q}_n$ and equals identity in Q_{n+1}, an application of Lemma 1 and the energy estimate gives that

$$\begin{aligned}
\int_{Q_{n+1}}(u-k_{n+1})_+^2\,dz &\le \int_{\tilde{Q}_n}(u-k_{n+1})_+^2\zeta^2\,dz \\
&\le \left(\int_{\tilde{Q}_n}[(u-k_{n+1})_+\zeta]^{\frac{N\bar{q}}{N-\bar{q}}}\,dz\right)^{2\frac{N-\bar{q}}{N\bar{q}}} |A_n|^{1-\frac{2}{q}+\frac{2}{N}} \\
&\le \gamma\left(\|D_x(u-k_{n+1})_+\|_2\prod_{i=1}^{N-1}\|D_{y_i}(u-k_{n+1})_+\|_q\right)^{\frac{2}{N}}|A_n|^{1-\frac{2}{q}+\frac{2}{N}} \\
&\le \left[\frac{\gamma 2^{2n}k^{q-2}}{(1-\sigma)^2\rho^q}\left(1+k^{2-q}\frac{\rho^q}{t^2}\right)\right]^{\frac{2}{q}}\left(\frac{2^{2n}}{k^2}\right)^{1-\frac{2}{q}+\frac{2}{N}}\left[\int_{Q_n}(u-k_n)_+^2\,dz\right]^{1+\frac{2}{N}},
\end{aligned}$$

where $\bar{q}$ satisfies

$$\frac{1}{\bar{q}} = \frac{1}{N}\left(\frac{1}{2}+\frac{N-1}{q}\right).$$

Hence by setting

$$Y_n = \fint_{Q_n}(u-k_n)_+^2\,dz,$$

we arrive at the recurrence inequality

$$Y_{n+1} \le \frac{\gamma\gamma_1^n}{(1-\sigma)^{\frac{4}{q}}k^{\frac{4}{N}}}\left(\frac{1}{k^{2-q}}\frac{t^2}{\rho^q}\right)^{\frac{1}{N}}\left(1+k^{2-q}\frac{\rho^q}{t^2}\right)^{\frac{2}{q}}Y_n^{1+\frac{2}{N}},$$

where $\gamma,\ \gamma_1 > 0$ are absolute constants.

Now let us stipulate that

$$k \ge \left(\frac{t^2}{\rho^q}\right)^{\frac{1}{2-q}},$$

such that the above recurrence inequality becomes

$$\begin{aligned} Y_{n+1} &\le \frac{\gamma\gamma_1^n}{(1-\sigma)^{\frac{4}{q}}k^{\frac{4}{N}}}\left(\frac{1}{k^{2-q}}\frac{t^2}{\rho^q}\right)^{\frac{1}{N}}\left(k^{2-q}\frac{\rho^q}{t^2}\right)^{\frac{2}{q}}Y_n^{1+\frac{2}{N}} \\ &\le \frac{\gamma\gamma_1^n}{(1-\sigma)^{\frac{4}{q}}k^{\frac{4}{N}}}\left(k^{2-q}\frac{\rho^q}{t^2}\right)^{\frac{2}{q}-\frac{1}{N}}Y_n^{1+\frac{2}{N}} \\ &\le \frac{\gamma\gamma_1^n}{(1-\sigma)^{\frac{4}{q}}k^{\frac{4}{N}}}\left(k^{2-q}\frac{\rho^q}{t^2}\right)^{\frac{2(N-1)}{Nq}}Y_n^{1+\frac{2}{N}}. \end{aligned}$$

Hence by [2, Chapter I, Lemma 4.1], $Y_n \to 0$, i.e. $M_\sigma \le k$, if we require that

$$Y_o \le \gamma(1-\sigma)^{\frac{2N}{q}}k^2\left(k^{2-q}\frac{\rho^q}{t^2}\right)^{-\frac{N-1}{q}} = \gamma(1-\sigma)^{\frac{2N}{q}}k^{\frac{\kappa_2}{q}}\left(\frac{t^2}{\rho^q}\right)^{\frac{N-1}{q}}.$$

This is fulfilled if we require $\kappa_2 = (N+1)q - 2(N-1) > 0$ and take

$$k = \frac{\gamma Y_o^{\frac{q}{\kappa_2}}}{(1-\sigma)^{\frac{2qN}{\kappa_2 q}}}\left(\frac{\rho^q}{t^2}\right)^{\frac{N-1}{\kappa_2}} + \left(\frac{t^2}{\rho^q}\right)^{\frac{1}{2-q}}.$$

As a result, we arrive at

$$\begin{aligned} M_\sigma &\le \frac{\gamma}{(1-\sigma)^{\frac{2qN}{\kappa_2 q}}}\left(\frac{\rho^q}{t^2}\right)^{\frac{N-1}{\kappa_2}}\left(\fint_{Q_o}u_+^2\,dz\right)^{\frac{q}{\kappa_2}} + \left(\frac{t^2}{\rho^q}\right)^{\frac{1}{2-q}} \\ &\le \frac{\gamma M^{\frac{q(2-r)}{\kappa_2}}}{(1-\sigma)^{\frac{2qN}{\kappa_2 q}}}\left(\frac{\rho^q}{t^2}\right)^{\frac{N-1}{\kappa_2}}\left(\fint_{Q_o}u_+^r\,dz\right)^{\frac{q}{\kappa_2}} + \left(\frac{t^2}{\rho^q}\right)^{\frac{1}{2-q}}, \end{aligned}$$

for any r satisfying $r < 2$ and $\kappa_r = (N+r-1)q - 2(N-1) > 0$. Thus we have

$$0 < \frac{q(2-r)}{\kappa_2} = 1 - \frac{\kappa_r}{\kappa_2} < 1,$$

and an interpolation argument ([2, Chapter I, Lemma 4.3]) gives that

$$\operatorname*{ess\,sup}_{[-t,t]\times K_\rho} u_+ \le \gamma\left(\frac{\rho^q}{t^2}\right)^{\frac{N-1}{\kappa_r}}\left(\fint_{K_{2\rho}}\int_{-2t}^{2t}u_+^r\,dxdy\right)^{\frac{q}{\kappa_r}} + \gamma\left(\frac{t^2}{\rho^q}\right)^{\frac{1}{2-q}}.$$

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