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Calculus of Variations

Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger equation

Chao Ji[1] · Vicenţiu D. Rădulescu[2,3,4]

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Abstract

In this paper, we study the following nonlinear magnetic Schrödinger equation

$$\begin{cases} \left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N \ (N \geq 2), \\ u \in H^1(\mathbb{R}^N, \mathbb{C}), \end{cases}$$

where ϵ is a positive parameter, and $V : \mathbb{R}^N \to \mathbb{R}$, $A : \mathbb{R}^N \to \mathbb{R}^N$ are continuous potentials. Under a local assumption on the potential V, by combining variational methods, penalization techniques, and the Ljusternik–Schnirelmann theory, we prove multiplicity and concentration properties of solutions for $\varepsilon > 0$ small. In our problem, the function f is only continuous, which allows to consider larger classes of nonlinearities in the reaction.

Mathematics Subject Classification 35J60 (Primary); 35B33 (Secondary)

1 Introduction and main results

The Schrödinger equation is central in quantum mechanics and it plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamical system. It is striking to point out that talking about his celebrated equation,

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✉ Vicenţiu D. Rădulescu
radulescu@inf.ucv.ro

Chao Ji
jichao@ecust.edu.cn

1 Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China

2 Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland

3 Department of Mathematics, University of Craiova, 200585 Craiova, Romania

4 School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, Sichuan, China

Erwin Schrödinger said: “I don’t like it, and I’m sorry I ever had anything to do with it”. The linear Schrödinger equation is a central tool of quantum mechanics, which provides a thorough description of a particle in a non-relativistic setting. Schrödinger’s linear equation is

$$\nabla^2\psi + \frac{8\pi^2 m}{\hbar^2}\,(E - V(x))\,\psi = 0\,,$$

where ψ is the Schrödinger wave function, m is the mass of the particle, $\hbar$ denotes Planck’s renormalized constant, E is the energy, and V stands for the potential energy.

Schrödinger also established the classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to de Broglie’s ideas. He developed a perturbation method, inspired by the work of Lord Rayleigh in acoustics, proved the equivalence between his wave mechanics and Heisenberg’s matrix, and introduced the time dependent Schrödinger’s equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\,\nabla^2\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi \qquad x \in \mathbb{R}^N\ (N \geq 2), \tag{1.1}$$

where $p < 2N/(N-2)$ if $N \geq 3$ and $p < +\infty$ if $N = 2$.

In physical problems, a cubic nonlinearity corresponding to $p = 3$ in equation (1.1) is common; in this case, problem (1.1) is called the Gross-Pitaevskii equation. In the study of equation (1.1), Floer, Weinstein [22] and Oh [30,31] supposed that the potential V is bounded and possesses a non-degenerate critical point at $x = 0$. More precisely, it is assumed that V belongs to the class (V_a) (for some real number a) introduced by Kato [27]. Taking $\gamma > 0$ and $\hbar > 0$ sufficiently small and using a Lyapunov-Schmidt type reduction, Oh [30,31] proved the existence of bound state solutions of problem (1.1), that is, solutions of the form

$$\psi(x,t) = e^{-iEt/\hbar}u(x)\,. \tag{1.2}$$

Using the Ansatz (1.2), we reduce the nonlinear Schrödinger equation (1.1) to the semilinear elliptic equation

$$-\frac{\hbar^2}{2m}\,\nabla^2 u + (V(x) - E)\,u = |u|^{p-1}u\,.$$

The change of variable $y = \hbar^{-1}x$ (and replacing y by x) yields

$$-\nabla^2 u + 2m\,(V_\hbar(x) - E)\,u = |u|^{p-1}u \qquad x \in \mathbb{R}^N\,, \tag{1.3}$$

where $V_\hbar(x) = V(\hbar x)$.

1.1 Related results

In this paper, we are concerned with multiplicity and concentration results for the following nonlinear magnetic Schrödinger equation

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N\ (N \geq 2), \tag{1.4}$$

where $u \in H^1(\mathbb{R}^N, \mathbb{C})$, $\varepsilon > 0$ is a parameter, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, $f \in C(\mathbb{R}, \mathbb{R})$, and the magnetic potential $A : \mathbb{R}^N \to \mathbb{R}^N$ is Hölder continuous with exponent $\alpha \in (0, 1]$.

Problem (1.4) arises when one looks for standing wave solutions $\psi(x,t) := e^{-iEt/\hbar}u(x)$ (with $E \in \mathbb{R}$) of the nonlinear evolution system

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{\hbar}{i}\nabla - A(x)\right)^2 \psi + U(x)\psi - f(|\psi|^2)\psi \quad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

From a physical point of view, the existence of such solutions and the study of their shape in the semiclassical limit, namely, as $\hbar \to 0^+$ (or, equivalently, as $\varepsilon \to 0^+$ in (1.4)), is of the greatest importance, since the transition from quantum mechanics to classical mechanics can be formally performed by sending to zero the Planck constant $\hbar$.

For problem (1.4), there is a vast literature concerning the existence and the multiplicity of bound state solutions for the case without magnetic field, namely if $A \equiv 0$. The first result in this direction was given by Floer and Weinstein [22], who considered the case $N = 1$ and $f = i_{\mathbb{R}}$. Later on, several authors generalized this result to larger values of N, using different methods. For instance, del Pino and Felmer [20] studied the existence and concentration of solutions to the following problem

$$\begin{cases} -\varepsilon^2\nabla^2 u + V(x)u = f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u > 0 \text{ in } \Omega, \end{cases}$$

where Ω is a possibly unbounded domain in $\mathbb{R}^N$ $(N \geq 3)$, the potential V is locally Hölder continuous, bounded from below away from zero, there exists a bounded open set $\Lambda \subset \Omega$ such that

$$\inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x), \tag{1.5}$$

and the nonlinearity f satisfies some subcritical growth conditions. In [1], Alves and Figueiredo considered the following quasilinear elliptic equation

$$\begin{cases} -\varepsilon^p\Delta_p u + V(x)|u|^{p-2}u = f(u) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \end{cases}$$

where V is a positive continuous function and satisfies the local assumption (1.5), $f \in C^1$ is a function having subcritical and superlinear growth. By using the Nehari manifold method and the Ljusternik–Schnirelmann category theory, the authors obtained the multiplicity of positive solutions. In order to apply the Nehari manifold method, the authors assumed that $f \in C^1$, which ensures that the Nehari manifold is a C^1-manifold. If f is only continuous, then the Nehari manifold is only a topological manifold, thus the arguments developed in [1] collapse. We notice that Szulkin and Weth in [34] considered the multiple solutions for the nonlinear stationary Schrödinger equation $-\Delta u + V(x)u = f(x,u)$ in $\mathbb{R}^N$, here f is superlinear, subcritical and continuous. In order to use the method of Nehari manifold, they developed a new approach. For further results about the existence, multiplicity and qualitative properties of semiclassical states with various types of concentration behaviors, which have been established under various assumptions on the potential V and on the nonlinearity f, see [2,4,6–8,12,14,15,19,30,31,36] the references therein (see also [5,23] for the fractional case).

On the other hand, the magnetic nonlinear Schrödinger equation (1.4) has been extensively investigated by many authors applying suitable variational and topological methods (see [3,10,11,16–18,21,25,26,28] and references therein). To the best of our knowledge, the first result involving the magnetic field was obtained by Esteban and Lions [21]. They used the concentration-compactness principle and minimization arguments to obtain solutions for $\varepsilon > 0$ fixed and $N = 2, 3$. In particular, due to our scope, we want to mention [3] where the authors used the method of the Nehari manifold, the penalization method, and the Ljusternik–Schnirelmann category theory for a subcritical nonlinearity $f \in C^1$. We point out that if f is only continuous, then the arguments developed in [3] fail. Moreover, as we will see later, due to the presence of the magnetic field $A(x)$, problem (1.4) cannot be changed into a pure real-valued problem, hence we must deal directly with a complex-valued problem, which causes several new difficulties in employing the methods to deal with our problem. Our problem is more complicated than the problem without magnetic field and we need additional technical estimates.

1.2 Main result

In this paper, motivated by [3,24,34], for the case where f is only continuous, we establish multiplicity and concentration properties of nontrivial solutions to problem (1.4).

Throughout the paper, we make the following assumptions on the potential V.

$(V1)$ There exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$;
$(V2)$ There exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 = \min_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Moreover, let the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying the following hypotheses.

$(f1)$ $f(t) = 0$ if $t \leq 0$;
$(f2)$ There exists $q \in (2, 2^*)$ such that

$$\lim_{t\to+\infty} \frac{f(t^2)t}{t^{q-1}} = 0,$$

where $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 2$;
$(f3)$ There is a positive constant $\theta > 2$ such that

$$0 < \frac{\theta}{2}F(t) \leq tf(t), \qquad \forall t > 0, \quad \text{where } F(t) = \int_0^t f(s)ds;$$

$(f4)$ $f(t)$ is strictly increasing in $(0, \infty)$.

The main result of this paper is the following.

Theorem 1.1 *Assume that V satisfies $(V1)$, $(V2)$ and f satisfies $(f1)$–$(f4)$. Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^N : \text{dist}\,(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $0 < \varepsilon < \varepsilon_\delta$, problem (1.4) has at least $cat_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, if we denote by u_{ε_n} one of these solutions of problem (1.4) for $\varepsilon = \varepsilon_n$ and $\eta_{\varepsilon_n} \in \mathbb{R}^N$ is the global maximum point of $|u_{\varepsilon_n}|$, then

$$\lim_n V(\eta_{\varepsilon_n}) = V_0.$$

The paper is organized as follows. In Sect. 2, we introduce the functional setting and give some preliminaries. In Sect. 3, we study the modified problem. We prove the Palais-Smale condition for the modified energy functional and provide some tools which are useful to establish a multiplicity result. In Sect. 4, we study the autonomous problem associated. This allows us to show that the modified problem has multiple solutions. Finally, in Sect. 5, we complete the paper with the proof of Theorem 1.1. We refer to the recent monograph [32] for some of the main abstract methods used in this paper.

Notation

- $C, C_1, C_2, \ldots$ denote positive constants whose exact values are inessential and can change from line to line;
- $B_R(y)$ denotes the open disk centered at $y \in \mathbb{R}^N$ with radius $R > 0$ and $B_R^c(y)$ denotes the complement of $B_R(y)$ in $\mathbb{R}^N$;
- $\|\cdot\|$, $\|\cdot\|_q$, and $\|\cdot\|_{L^\infty(\Omega)}$ denote the usual norms of the spaces $H^1(\mathbb{R}^N, \mathbb{R})$, $L^q(\mathbb{R}^N, \mathbb{R})$, and $L^\infty(\Omega, \mathbb{R})$, respectively, where $\Omega \subset \mathbb{R}^N$. $\langle \cdot, \cdot \rangle_0$ denotes the inner product of the space $H^1(\mathbb{R}^N, \mathbb{R})$.

2 Abstract setting and preliminary results

In this section, we present the functional spaces and some useful preliminary remarks which will be useful for our arguments. We also introduce a classical equivalent version of problem (1.4).

For $u : \mathbb{R}^N \to \mathbb{C}$, let us denote by

$$\nabla_A u := \left(\frac{\nabla}{i} - A \right) u,$$

and

$$H_A^1(\mathbb{R}^N, \mathbb{C}) := \{u \in L^2(\mathbb{R}^N, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^N, \mathbb{R})\}.$$

The space $H_A^1(\mathbb{R}^N, \mathbb{C})$ is an Hilbert space endowed with the scalar product

$$\langle u, v \rangle := \mathrm{Re} \int_{\mathbb{R}^2} \left(\nabla_A u \overline{\nabla_A v} + u \overline{v} \right) dx, \quad \text{for any } u, v \in H_A^1(\mathbb{R}^N, \mathbb{C}),$$

where Re and the bar denote the real part of a complex number and the complex conjugation, respectively. Moreover we denote by $\|u\|_A$ the norm induced by this inner product.

On $H_A^1(\mathbb{R}^N, \mathbb{C})$ we will frequently use the following diamagnetic inequality (see, e.g., Lieb and Loss [29, Theorem 7.21])

$$|\nabla_A u(x)| \geq |\nabla |u(x)||. \tag{2.1}$$

Moreover, making a simple change of variables, we can see that (1.4) is equivalent to

$$\left(\frac{1}{i} \nabla - A_\varepsilon(x) \right)^2 u + V_\varepsilon(x) u = f(|u|^2) u \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$.

Let H_ε be the Hilbert space obtained as the closure of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to the scalar product

$$\langle u, v\rangle_\epsilon := \operatorname{Re}\int_{\mathbb{R}^N}\left(\nabla_{A_\varepsilon}u\overline{\nabla_{A_\varepsilon}v} + V_\varepsilon(x)u\overline{v}\right)dx$$

and let us denote by $\|\cdot\|_\varepsilon$ the norm induced by this inner product.

The diamagnetic inequality (2.1) implies that if $u \in H^1_{A_\varepsilon}(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$ and $\|u\| \le C\|u\|_\varepsilon$. Therefore, the embedding $H_\varepsilon \hookrightarrow L^r(\mathbb{R}^N, \mathbb{C})$ is continuous for $2 \le r \le 2^*$ and the embedding $H_\varepsilon \hookrightarrow L^r_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$ is compact for $1 \le r < 2^*$.

3 The modified problem

As in [20], to study system (1.4), or equivalently, problem (2.2) by variational methods, we modify suitably the nonlinearity f so that, for $\varepsilon > 0$ small enough, the solutions of the modified problem are also solutions of the original one. More precisely, we choose $K > 2$. By $(f4)$ there exists a unique number $a > 0$ verifying $Kf(a) = V_0$, where V_0 is given in $(V1)$. Hence we consider the function

$$\tilde{f}(t) := \begin{cases} f(t), & t \le a, \\ V_0/K, & t > a. \end{cases}$$

Now we introduce the penalized nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$

$$g(x,t) := \chi_\Lambda(x)f(t) + (1-\chi_\Lambda(x))\tilde{f}(t), \tag{3.1}$$

where χ_Λ is the characteristic function on Λ and $G(x,t) := \int_0^t g(x,s)ds$.

In view of $(f1)$–$(f4)$, we deduce that g is a Carathéodory function satisfying the following properties:

- $(g1)$ $g(x,t) = 0$ for each $t \le 0$;
- $(g2)$ $\lim\limits_{t\to 0^+} g(x,t) = 0$ uniformly in $x \in \mathbb{R}^N$;
- $(g3)$ $g(x,t) \le f(t)$ for all $t \ge 0$ and uniformly in $x \in \mathbb{R}^N$;
- $(g4)$ $0 < \theta G(x,t) \le 2g(x,t)t$, for each $x \in \Lambda$, $t > 0$;
- $(g5)$ $0 < G(x,t) \le g(x,t)t \le V_0 t/K$, for each $x \in \Lambda^c$, $t > 0$;
- $(g6)$ for each $x \in \Lambda$, the function $t \mapsto g(x,t)$ is strictly increasing in $t \in (0,+\infty)$ and for each $x \in \Lambda^c$, the function $t \mapsto g(x,t)$ is strictly increasing in $(0,a)$.

Then we consider the modified problem

$$\left(\frac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u = g(\varepsilon x, |u|^2)u \quad \text{in } \mathbb{R}^N. \tag{3.2}$$

Note that if u is a solution of problem (3.2) with

$$|u(x)|^2 \le a \quad \text{for all } x \in \Lambda^c_\varepsilon, \quad \Lambda_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\},$$

then u is a solution of problem (2.2).

The energy functional associated to problem (3.2) is

$$J_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_{A_\varepsilon}u|^2 + V_\varepsilon(x)|u|^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} G(\varepsilon x, |u|^2)dx \quad \text{for all } u \in H_\varepsilon.$$

It is standard to prove that $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ and its critical points are the weak solutions of the modified problem (3.2).

We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold of J_ε, that is,

$$\mathcal{N}_\varepsilon := \{u \in H_\varepsilon \backslash \{0\} : J'_\varepsilon(u)[u] = 0\},$$

and define the number c_ε by

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Let H_ε^+ be the open subset of H_ε given by

$$H_\varepsilon^+ = \{u \in H_\varepsilon : |\text{supp}(u) \cap \Lambda_\varepsilon| > 0\},$$

and $S_\varepsilon^+ = S_\varepsilon \cap H_\varepsilon^+$, where S_ε is the unit sphere of H_ε. Note that S_ε^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_ε and contained in H_ε^+. Therefore, $H_\varepsilon = T_u S_\varepsilon^+ \bigoplus \mathbb{R}u$ for each $u \in T_u S_\varepsilon^+$, where $T_u S_\varepsilon^+ = \{v \in H_\varepsilon : \langle u, v\rangle_\epsilon = 0\}$.

Now we show that the functional J_ε satisfies the mountain pass geometry (see [9,33,37]).

Lemma 3.1 *For any fixed $\varepsilon > 0$, the functional J_ε satisfies the following properties:*

(i) *there exist $\beta, r > 0$ such that $J_\varepsilon(u) \geq \beta$ if $\|u\|_\varepsilon = r$;*
(ii) *there exists $e \in H_\varepsilon$ with $\|e\|_\varepsilon > r$ such that $J_\varepsilon(e) < 0$.*

Proof (i) By $(g3)$, $(f1)$ and $(f2)$, for any $\zeta > 0$ small, there exists $C_\zeta > 0$ such that

$$G(\varepsilon x, |u|^2) \leq \zeta|u|^2 + C_\zeta|u|^q \quad \text{for all } x \in \mathbb{R}^N.$$

By the Sobolev embedding it follows that

$$\begin{aligned} J_\varepsilon(u) &\geq \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla_{A_\varepsilon}u|^2 + V_\varepsilon(x)|u|^2)dx - \frac{\zeta}{2}\int_{\mathbb{R}^N}|u|^2dx - \frac{C_\zeta}{2}\int_{\mathbb{R}^N}|u|^q dx \\ &\geq \frac{1}{4}\|u_n\|_\varepsilon^2 - CC_\zeta\|u_n\|_\varepsilon^q. \end{aligned}$$

Hence we can choose some $\beta, r > 0$ such that $J_\varepsilon(u) \geq \beta$ if $\|u\|_\varepsilon = r$ since $q > 2$.
(ii) For each $u \in H_\varepsilon^+$ and $t > 0$, by the definition of g and $(f3)$, one has

$$\begin{aligned} J_\varepsilon(tu) &\leq \frac{t^2}{2}\int_{\mathbb{R}^N}(|\nabla_{A_\varepsilon}u|^2 + V_\varepsilon(x)|u|^2)dx - \frac{1}{2}\int_{\Lambda_\varepsilon} G(\varepsilon x, t^2|u|^2)dx, \\ &\leq \frac{t^2}{2}\|u\|_\varepsilon^2 - C_1 t^\theta \int_{\Lambda_\varepsilon}|u|^\theta dx + C_2|\text{supp}(u) \cap \Lambda_\varepsilon|. \end{aligned}$$

Since $\theta > 2$, we get the conclusion. □

Since f is only continuous, the next results are very important because they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of S_ε^+.

Lemma 3.2 *Assume that $(V1)$–$(V2)$ and $(f1)$–$(f4)$ are satisfied, then the following properties hold.*

$(A1)$ *For any $u \in H_\varepsilon^+$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = J_\varepsilon(tu)$. Then there exists a unique $t_u > 0$ such that $g'_u(t) > 0$ in $(0, t_u)$ and $g'_u(t) < 0$ in (t_u, ∞).*
$(A2)$ *There exists $\tau > 0$ independent on u such that $t_u \geq \tau$ for all $u \in S_\varepsilon^+$. Moreover, for each compact $\mathcal{W} \subset S_\varepsilon^+$ there is $C_\mathcal{W}$ such that $t_u \leq C_\mathcal{W}$, for all $u \in \mathcal{W}$.*

$(A3)$ *The map* $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ *given by* $\widehat{m}_\varepsilon(u) = t_u u$ *is continuous and* $m_\varepsilon = \widehat{m}_\varepsilon|_{S_\varepsilon^+}$ *is a homeomorphism between* S_ε^+ *and* $\mathcal{N}_\varepsilon$. *Moreover,* $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$.

$(A4)$ *If there is a sequence* $\{u_n\} \subset S_\varepsilon^+$ *such that* $dist(u_n, \partial S_\varepsilon^+) \to 0$, *then* $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ *and* $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$.

Proof $(A1)$ As in the proof of Lemma 3.1, we have $g_u(0) = 0$, $g_u(t) > 0$ for $t > 0$ small and $g_u(t) < 0$ for $t > 0$ large. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a global maximum point $t = t_u$ verifying $g_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. From $(f4)$, the definition of g and $|\text{supp}(u)\cap\Lambda_\varepsilon| > 0$, we may obtain the uniqueness of t_u. Therefore, $\max_{t\geq 0} g_u(t)$ is achieved at a unique $t = t_u$ so that $g_u'(t) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$.

$(A2)$ For $\forall\, u \in S_\varepsilon^+$, we have

$$t_u = \int_{\mathbb{R}^N} g(\varepsilon x, t_u^2|u|^2)t_u|u|^2dx.$$

From $(g2)$, $(g3)$, the Sobolev embeddings and $q > 2$, we get

$$t_u \leq \zeta t_u \int_{\mathbb{R}^N} |u|^2dx + C_\zeta t_u^{q-1}\int_{\mathbb{R}^N} |u|^q dx \leq C_1\zeta t_u + C_2C_\zeta t_u^{q-1},$$

which implies that $t_u \geq \tau$ for some $\tau > 0$. If $\mathcal{W} \subset S_\varepsilon^+$ is compact, and suppose by contradiction that there is $\{u_n\} \subset \mathcal{W}$ with $t_n := t_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there exists $u \in \mathcal{W}$ such that $u_n \to u$ in H_ε. Moreover, using the proof of Lemma 3.1(ii), we have that $J_\varepsilon(t_n u_n) \to -\infty$.

On the other hand, let $v_n := t_n u_n \in \mathcal{N}_\varepsilon$, from $(g4)$, $(g5)$ and $(g6)$, it yields that

$$\begin{aligned}
J_\varepsilon(v_n) &= J_\varepsilon(v_n) - \frac{1}{\theta}J_\varepsilon'(v_n)[v_n]\\
&\geq \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\|v_n\|_\varepsilon^2 + \int_{\Lambda_\varepsilon^c}\Big(\frac{1}{\theta}g(\varepsilon x, |v_n|^2)|v_n|^2 - \frac{1}{2}G(\varepsilon x, |v_n|^2)\Big)dx\\
&\geq \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\Big(\|v_n\|_\varepsilon^2 - \frac{1}{K}\int_{\mathbb{R}^N} V(\varepsilon x)|v_n|^2dx\Big)\\
&\geq \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\left(1-\frac{1}{K}\right)\|v_n\|_\varepsilon^2.
\end{aligned}$$

Thus, substituting $v_n := t_n u_n$ and $\|v_n\|_\varepsilon = t_n$, we obtain

$$0 < \Big(\frac{1}{2}-\frac{1}{\theta}\Big)\left(1-\frac{1}{K}\right) \leq \frac{J_\varepsilon(v_n)}{t_n^2} \leq 0$$

as $n \to \infty$, which yields a contradiction. This proves $(A2)$.

$(A3)$ First of all, we note that $\widehat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by $(A2)$, for each $u \in H_\varepsilon^+$, there is a unique $\widehat{m}_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$, then $u \in H_\varepsilon^+$. Otherwise, we have $|\text{supp}(u) \cap \Lambda_\varepsilon| = 0$ and by $(g5)$ we have

$$\begin{aligned}
\|u\|_\varepsilon^2 &= \int_{\mathbb{R}^N} g(\varepsilon x, |u|^2)|u|^2dx = \int_{\Lambda_\varepsilon^c} g(\varepsilon x, |u|^2)|u|^2dx \leq \frac{1}{K}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^2dx\\
&\leq \frac{1}{K}\|u\|_\varepsilon^2
\end{aligned}$$

which is impossible since $K > 1$ and $u \neq 0$. Therefore, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in S_\varepsilon^+$ is well defined and continuous. From

$$m_\varepsilon^{-1}\big(m_\varepsilon(u)\big) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{t_u\|u\|_\varepsilon} = u, \quad \forall u \in S_\varepsilon^+,$$

we conclude that m_ε is a bijection. Now we prove $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous, let $\{u_n\} \subset H_\varepsilon^+$ and $u \in H_\varepsilon^+$ such that $u_n \to u$ in H_ε. By $(A2)$, there is a $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Using $t_n u_n \in \mathcal{N}_\varepsilon$, i.e.,

$$t_n^2 \|u_n\|_\varepsilon^2 = \int_{\mathbb{R}^N} g(\varepsilon x, t_n^2 |u_n|^2) t_n^2 |u_n|^2 dx, \quad \forall n \in N,$$

and passing to the limit as $n \to \infty$ in the last equality, we obtain

$$t_0^2 \|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} g(\varepsilon x, t_0^2 |u|^2) t_0^2 |u|^2 dx,$$

which implies that $t_0 u \in \mathcal{N}_\varepsilon$ and $t_u = t_0$. This proves $\widehat{m}_\varepsilon(u_n) \to \widehat{m}_\varepsilon(u)$ in H_ε^+. Thus, $\widehat{m}_\varepsilon$ and m_ε are continuous functions and $(A3)$ is proved.

$(A4)$ Let $\{u_n\} \subset S_\varepsilon^+$ be a subsequence such that $\operatorname{dist}(u_n, \partial S_\varepsilon^+) \to 0$, then for each $v \in \partial S_\varepsilon^+$ and $n \in N$, we have $|u_n| = |u_n - v|$ a.e. in Λ_ε. Therefore, by $(V1)$, $(V2)$ and the Sobolev embedding, there exists a constant $C_t > 0$ such that

$$\begin{aligned} \|u_n\|_{L^t(\Lambda_\varepsilon)} &\le \inf_{v \in \partial S_\varepsilon^+} \|u_n - v\|_{L^t(\Lambda_\varepsilon)} \\ &\le C_t \left(\inf_{v \in \partial S_\varepsilon^+} \int_{\Lambda_\varepsilon} (|\nabla_{A_\varepsilon} u_n - v|^2 + V_\varepsilon(x)|u_n - v|^2) dx \right)^{\frac{1}{2}} \\ &\le C_t \operatorname{dist}(u_n, \partial S_\varepsilon^+) \end{aligned}$$

for all $n \in N$, $t \in [2, 2^*]$. By $(g2)$, $(g3)$ and $(g5)$, for each $t > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} G(\varepsilon x, t^2 |u_n|^2) dx &\le \int_{\Lambda_\varepsilon} F(t^2 |u_n|^2) dx + \frac{t^2}{K} \int_{\Lambda_\varepsilon^c} V(\varepsilon x) |u_n|^2 dx \\ &\le C_1 t^2 \int_{\Lambda_\varepsilon} |u_n|^2 dx + C_2 t^q \int_{\Lambda_\varepsilon} |u_n|^q dx + \frac{t^2}{K} \|u_n\|_\varepsilon^2 \\ &\le C_3 t^2 \operatorname{dist}(u_n, \partial S_\varepsilon^+)^2 + C_4 t^q \operatorname{dist}(u_n, \partial S_\varepsilon^+)^q + \frac{t^2}{K}. \end{aligned}$$

Therefore,

$$\limsup_n \int_{\mathbb{R}^N} G(\varepsilon x, t^2 |u_n|^2) dx \le \frac{t^2}{K}, \quad \forall t > 0.$$

On the other hand, from the definition of m_ε and the last inequality, for all $t > 0$, one has

$$\begin{aligned} \liminf_n J_\varepsilon(m_\varepsilon(u_n)) &\ge \liminf_n J_\varepsilon(t u_n) \\ &\ge \liminf_n \frac{t^2}{2} \|u_n\|_\varepsilon^2 - \frac{t^2}{K} \\ &= \frac{K-2}{2K} t^2, \end{aligned}$$

this implies that

$$\liminf_n \frac{1}{2} \|m_\varepsilon(u_n)\|_\varepsilon^2 \ge \liminf_n J_\varepsilon(m_\varepsilon(u_n)) \ge \frac{K-2}{2K} t^2, \quad \forall t > 0.$$

From the arbitrariness of $t > 0$, it is easy to see that $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$ as $n \to \infty$. This completes the proof of Lemma 3.2. □

Now we define the function

$$\widehat{\Psi}_\varepsilon : H_\varepsilon^+ \to \mathbb{R},$$

by $\widehat{\Psi}_\varepsilon(u) = J_\varepsilon(\widehat{m}_\varepsilon(u))$ and denote by $\Psi_\varepsilon := (\widehat{\Psi}_\varepsilon)|_{S_\varepsilon^+}$.

From Lemma 3.2, arguing as Corollary 10 in [35], we may obtain the following result.

Lemma 3.3 *Assume that* $(V1)$–$(V2)$ *and* $(f1)$–$(f4)$ *are satisfied, then*

$(B1)$ $\widehat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_\varepsilon'(u)v = \frac{\|\widehat{m}_\varepsilon(u)\|_\epsilon}{\|u\|_\epsilon} J_\varepsilon'(\widehat{m}_\varepsilon(u))[v], \quad \forall\, u \in H_\varepsilon^+ \text{ and } \forall\, v \in H_\varepsilon;$$

$(B2)$ $\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ *and*

$$\Psi_\varepsilon'(u)v = \|m_\varepsilon(u)\|_\epsilon J_\varepsilon'(\widehat{m}_\varepsilon(u))[v], \quad \forall v \in T_u S_\varepsilon^+;$$

$(B3)$ *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* Ψ_ε*, then* $\{m_\varepsilon(u_n)\}$ *is a* $(PS)_c$ *sequence of* J_ε*. If* $\{u_n\} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(PS)_c$ *sequence of* J_ε*, then* $\{m_\varepsilon^{-1}(u_n)\}$ *is a* $(PS)_c$ *sequence of* Ψ_ε*;*

$(B4)$ u *is a critical point of* Ψ_ε *if and only if* $m_\varepsilon(u)$ *is a critical point of* J_ε*. Moreover, the corresponding critical values coincide and*

$$\inf_{S_\varepsilon^+} \Psi_\varepsilon. = \inf_{\mathcal{N}_\varepsilon} J_\varepsilon.$$

As in [35], we have the following variational characterization of the infimum of J_ε over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u \in H_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu) = \inf_{u \in S_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu).$$

Lemma 3.4 *Let* $c > 0$ *and* $\{u_n\}$ *is a* $(PS)_c$ *sequence for* J_ε*, then* $\{u_n\}$ *is bounded in* H_ε*.*

Proof Assume that $\{u_n\} \subset H_\varepsilon$ is a $(PS)_c$ sequence for J_ε, that is, $J_\varepsilon(u_n) \to c$ and $J_\varepsilon'(u_n) \to 0$. By using $(g4)$ and $(g5)$, we have

$$\begin{aligned}
c + o_n(1) + o_n(1)\|u_n\|_\varepsilon &\geq J_\varepsilon(u_n) - \frac{1}{\theta} J_\varepsilon'(u_n)[u_n] \\
&= \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \int_{\mathbb{R}^N} \Big(\frac{1}{\theta} g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |u_n|^2)\Big) dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_\varepsilon^2 + \int_{\Lambda_\varepsilon^c} \Big(\frac{1}{\theta} g(\varepsilon x, |u_n|^2)|u_n|^2 - \frac{1}{2} G(\varepsilon x, |u_n|^2)\Big) dx \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(\|u_n\|_\varepsilon^2 - \int_{\Lambda_\varepsilon^c} g(\varepsilon x, |u_n|^2)|u_n|^2 dx\Big) \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\Big(\|u_n\|_\varepsilon^2 - \frac{1}{K}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2 dx\Big) \\
&\geq \Big(\frac{1}{2} - \frac{1}{\theta}\Big)(1 - \frac{1}{K})\|u_n\|_\varepsilon^2.
\end{aligned}$$

Since $K > 2$, from the above inequalities we obtain that $\{u_n\}$ is bounded in H_ε. □

The following result is important to prove the $(PS)_{c_\varepsilon}$ condition for the functional J_ε.

Lemma 3.5 *The functional* J_ε *satisfies the* $(PS)_c$ *condition at any level* $c > 0$*.*

Proof Let $(u_n) \subset H_\varepsilon$ be a $(PS)_c$ sequence for J_ε. By Lemma 3.4, (u_n) is bounded in H_ε. Thus, up to a subsequence, $u_n \rightharpoonup u$ in H_ε and $u_n \to u$ in $L^r_{\text{loc}}(\mathbb{R}^N, \mathbb{C})$ for all $1 \le r < 2^*$ as $n \to +\infty$. Moreover, the subcritical growth of g imply that $J'_\varepsilon(u) = 0$, and

$$\|u\|^2_\varepsilon = \int_{\mathbb{R}^N} g(\varepsilon x, |u|^2)|u|^2 dx.$$

Let $R > 0$ be such that $\Lambda_\varepsilon \subset B_{R/2}(0)$. We show that for any given $\zeta > 0$, for R large enough,

$$\limsup_n \int_{B^c_R(0)} (|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)dx \le \zeta. \tag{3.3}$$

Let $\phi_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be a cut-off function such that

$$\phi_R = 0 \quad x \in B_{R/2}(0), \quad \phi_R = 1 \quad x \in B^c_R(0), \quad 0 \le \phi_R \le 1, \quad \text{and} \quad |\nabla \phi_R| \le C/R$$

where $C > 0$ is a constant independent of R. Since the sequence $(\phi_R u_n)$ is bounded in H_ε, we have

$$J'_\varepsilon(u_n)[\phi_R u_n] = o_n(1),$$

that is

$$\text{Re} \int_{\mathbb{R}^N} \nabla_{A_\varepsilon} u_n \overline{\nabla_{A_\varepsilon}(\phi_R u_n)} dx + \int_{\mathbb{R}^N} V_\varepsilon(x)|u_n|^2 \phi_R dx = \int_{\mathbb{R}^N} g(\varepsilon x, |u_n|^2)|u_n|^2 \phi_R dx + o_n(1).$$

Since $\overline{\nabla_{A_\varepsilon}(u_n \phi_R)} = i\overline{u_n}\nabla\phi_R + \phi_R \overline{\nabla_{A_\varepsilon} u_n}$, using $(g5)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)\phi_R dx \\
&\quad = \int_{\mathbb{R}^N} g(\varepsilon x, |u_n|^2)|u_n|^2 \phi_R dx - \text{Re} \int_{\mathbb{R}^N} i\overline{u_n} \nabla_{A_\varepsilon} u_n \nabla \phi_R dx + o_n(1) \\
&\quad \le \frac{1}{K} \int_{\mathbb{R}^N} V_\varepsilon(x)|u_n|^2 \phi_R dx - \text{Re} \int_{\mathbb{R}^N} i\overline{u_n} \nabla_{A_\varepsilon} u_n \nabla \phi_R dx + o_n(1).
\end{aligned}$$

By the definition of ϕ_R, the Hölder inequality and the boundedness of (u_n) in H_ε, we obtain

$$\left(1 - \frac{1}{K}\right) \int_{\mathbb{R}^N} (|\nabla_{A_\varepsilon} u_n|^2 + V_\varepsilon(x)|u_n|^2)\phi_R dx \le \frac{C}{R}\|u_n\|_2 \|\nabla_{A_\varepsilon} u_n\|_2 + o_n(1) \le \frac{C_1}{R} + o_n(1)$$

and so (3.3) holds.

Using $u_n \to u$ in $L^r_{\text{loc}}(\mathbb{R}^N)$, for all $1 \le r < 2^*$ again, up to a subsequence, we have that

$$|u_n| \to |u| \text{ a.e. in } \mathbb{R}^N \text{ as } n \to +\infty,$$

then

$$g(\varepsilon x, |u_n|^2)|u_n|^2 \to g(\varepsilon x, |u|^2)|u|^2 \text{ a.e. in } \mathbb{R}^N \text{ as } n \to +\infty.$$

Moreover, from the subcritical growth of g and and the Lebesgue Dominated Convergence Theorem, we can infer

$$\lim_n \int_{B_R(0)} \left| g(\varepsilon x, |u_n|^2)|u_n|^2 - g(\varepsilon x, |u|^2)|u|^2 \right| dx = 0.$$

Now, by $(g5)$ and (3.3) we have

$$\int_{B_R^c(0)} \left|g(\varepsilon x, |u_n|^2)|u_n|^2 - g(\varepsilon x, |u|^2)|u|^2\right| dx \le \frac{2}{K}\int_{B_R^c(0)} (|\nabla_{A_\varepsilon} u_n|^2 + V(\varepsilon x)|u_n|^2)dx < \frac{2\zeta}{K}$$

for every $\zeta > 0$.

Therefore

$$\int_{\mathbb{R}^N} g(\varepsilon x, |u_n|^2)|u_n|^2 dx \to \int_{\mathbb{R}^N} g(\varepsilon x, |u|^2)|u|^2 dx \text{ as } n \to +\infty.$$

Finally, since $J_\varepsilon'(u) = 0$, we have

$$o_n(1) = J_\varepsilon'(u_n)[u_n] = \|u_n\|_\varepsilon^2 - \int_{\mathbb{R}^N} g(\varepsilon x, |u_n|^2)|u_n|^2 dx = \|u_n\|_\varepsilon^2 - \|u\|_\varepsilon^2 + o_n(1).$$

Thus, the sequence (u_n) strong converges to u in H_ε. □

Since f is only assumed to be continuous, the following result is required for the multiplicity result in the next section.

Corollary 3.1 *The functional Ψ_ε satisfies the $(PS)_c$ condition on S_ε^+ at any level $c > 0$.*

Proof Let $\{u_n\} \subset S_\varepsilon^+$ be a $(PS)_c$ sequence for Ψ_ε. Then $\Psi_\varepsilon(u_n) \to c$ and $\|\Psi_\varepsilon'(u_n)\|_* \to 0$, where $\|\cdot\|_*$ is the norm in the dual space $(T_{u_n} S_\varepsilon^+)^*$. By Lemma 3.3$(B3)$, we know that $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence for J_ε in H_ε. From Lemma 3.5, we know that there exists a $u \in S_\varepsilon^+$ such that, up to a subsequence, $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in H_ε. By Lemma 3.2$(A3)$, we obtain

$$u_n \to u \text{ in } S_\varepsilon^+,$$

and the proof is complete. □

Proposition 3.1 *Assume that $(V1)$–$(V2)$ and $(f1)$–$(f4)$ hold, then problem (3.2) has a ground state solution for any $\epsilon > 0$.*

Proof Since

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u \in H_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu) = \inf_{u \in S_\varepsilon^+} \sup_{t>0} J_\varepsilon(tu),$$

by the Ekeland variational principle [37], we obtain a minimizing $(PS)_{c_\varepsilon}$ sequence on S_ε^+ for the functional Ψ_ε. Moreover, by Corollary 3.1, we deduce the existence of a ground state $u \in H_\varepsilon$ for problem (3.2). □

4 Multiple solutions for the modified problem

4.1 The autonomous problem

For our scope, we need also to study the following *limit* problem

$$-\Delta u + V_0 u = f(u^2)u, \quad u : \mathbb{R}^N \to \mathbb{R}, \tag{4.1}$$

whose associated C^1-functional, defined in $H^1(\mathbb{R}^N, \mathbb{R})$, is

$$I_0(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_0 u^2)dx - \frac{1}{2}\int_{\mathbb{R}^N} F(u^2)dx.$$

Let

$$\mathcal{N}_0 := \{u \in H^1(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} : I_0'(u)[u] = 0\}$$

and

$$c_{V_0} := \inf_{u \in \mathcal{N}_0} I_0(u).$$

Let S_0 be the unit sphere of $H_0 := H^1(\mathbb{R}^N, \mathbb{R})$. Note that S_0 is a complete smooth manifold of codimension 1, therefore, $H_0 = T_u S_0 \bigoplus \mathbb{R}u$ for each $u \in T_u S_0$, where $T_u S_0 = \{v \in H_0 : \langle u, v\rangle_0 = 0\}$. Arguing as in Lemma 3.2, we have the following result.

Lemma 4.1 *Let V_0 be given in $(V1)$ and suppose that $(f1)$–$(f4)$ are satisfied, then the following properties hold.*

- $(a1)$ *For any $u \in H_0\backslash\{0\}$, let $g_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $g_u(t) = I_0(tu)$. Then there exists a unique $t_u > 0$ such that $g_u'(t) > 0$ in $(0, t_u)$ and $g_u'(t) < 0$ in (t_u, ∞);*
- $(a2)$ *There exists $\tau > 0$ independent on u such that $t_u > \tau$ for all $u \in S_0$. Moreover, for each compact set $\mathcal{W} \subset S_0$ there is $C_{\mathcal{W}}$ such that $t_u \le C_{\mathcal{W}}$, for all $u \in \mathcal{W}$;*
- $(a3)$ *The map $\widehat{m} : H_0\backslash\{0\} \to \mathcal{N}_0$ given by $\widehat{m}(u) = t_u u$ is continuous and $m_0 = \widehat{m}_0|_{S_0}$ is a homeomorphism between S_0 and $\mathcal{N}_0$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|_0}$.*

We shall consider the functional defined by

$$\widehat{\Psi}_0(u) = I_0(\widehat{m}(u)) \quad \text{and} \quad \Psi_0 := \widehat{\Psi}_0|_{S_0}.$$

Arguing as Proposition 9 and Corollary 10 in [35], we have that

Lemma 4.2 *Let V_0 be given in $(V1)$ and suppose that $(f1)$–$(f4)$ are satisfied, then*

- $(b1)$ $\widehat{\Psi}_0 \in C^1(H_0\backslash\{0\}, \mathbb{R})$ *and*

$$\widehat{\Psi}_0'(u)v = \frac{\|\widehat{m}(u)\|_0}{\|u\|_0} I_0'(\widehat{m}(u))[v], \quad \forall u \in H_0\backslash\{0\} \text{ and } \forall v \in H_0;$$

- $(b2)$ $\Psi_0 \in C^1(S_0, \mathbb{R})$ *and*

$$\Psi_0'(u)v = \|m(u)\|_0 I_0'(\widehat{m}(u))[v], \quad \forall v \in T_u S_0;$$

- $(b3)$ *If $\{u_n\}$ is a $(PS)_c$ sequence of Ψ_0, then $\{m(u_n)\}$ is a $(PS)_c$ sequence of I_0. If $\{u_n\} \subset \mathcal{N}_0$ is a bounded $(PS)_c$ sequence of I_0, then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence of Ψ_0;*
- $(b4)$ *u is a critical point of Ψ_0 if and only if $m(u)$ is a critical point of I_0. Moreover, the corresponding critical values coincide and*

$$\inf_{S_0} \Psi_0 = \inf_{\mathcal{N}_0} I_0.$$

Similar to the previous argument, we have the following variational characterization of the infimum of I_0 over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u \in \mathcal{N}_0} I_0(u) = \inf_{u \in H_0\backslash\{0\}} \sup_{t>0} I_0(tu) = \inf_{u \in S_0} \sup_{t>0} I_0(tu).$$

The next result is useful in later arguments.

Lemma 4.3 *Let $\{u_n\} \subset H_0$ be a $(PS)_c$ sequence for I_0 such that $u_n \rightharpoonup 0$. Then one of the following alternatives occurs:*

(i) $u_n \to 0$ *in* H_0 *as* $n \to +\infty$;
(ii) *there are a sequence* $\{y_n\} \subset \mathbb{R}^N$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 dx \geq \beta.$$

Proof Assume that (ii) does not hold. Then, for every $R > 0$, we have

$$\lim_n \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0.$$

Since $\{u_n\}$ is bounded in H_0, by Lions' lemma [37], it follows that

$$u_n \to 0 \text{ in } L^r(\mathbb{R}^N),\ 2 < r < 2^*.$$

From the subcritical growth of f, we have

$$\int_{\mathbb{R}^N} F(u_n^2) dx = o_n(1) = \int_{\mathbb{R}^N} f(u_n^2) u_n^2 dx.$$

Moreover, from $I_0'(u_n)[u_n] \to 0$, it follows that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V_0 u_n^2) dx = \int_{\mathbb{R}^N} f(u_n^2) u_n^2 dx + o_n(1) = o_n(1).$$

Thus, property (i) holds. □

Remark 4.1 From Lemma 4.3 we see that if u is the weak limit of $(PS)_{c_{V_0}}$ sequence $\{u_n\}$ of the functional I_0, then we have $u \neq 0$. Otherwise we have that $u_n \rightharpoonup 0$ and if $u_n \nrightarrow 0$, from Lemma 4.3 it follows that there are a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_n \int_{B_R(y_n)} |u_n|^2 dx \geq \beta > 0.$$

Then set $v_n(x) = u_n(x + z_n)$, it is easy to see that $\{v_n\}$ is also a $(PS)_{c_{V_0}}$ sequence for the functional I_0, it is bounded, and there exists $v \in H_0$ such that $v_n \rightharpoonup v$ in H_0 with $v \neq 0$.

Lemma 4.4 *Assume that* V *satisfies* $(V1)$, $(V2)$ *and* f *satisfies* $(f1)$–$(f4)$, *then problem* (4.1) *has a positive ground state solution.*

Proof First of all, it is easy to show that $c_{V_0} > 0$. Moreover, if $u_0 \in \mathcal{N}_0$ satisfies $I_0(u_0) = c_{V_0}$, then $m^{-1}(u_0) \in S_0$ is a minimizer of Ψ_0, so that u_0 is a critical point of I_0 by Lemma 4.2. Now, we show that there exists a minimizer $u \in \mathcal{N}_0$ of $I_0|_{\mathcal{N}_0}$. Since $\inf_{S_0} \Psi_0 = \inf_{\mathcal{N}_0} I_0 = c_{V_0}$ and S_0 is a C^1 manifold, by Ekeland's variational principle, there exists a sequence $\omega_n \subset S_0$ with $\Psi_0(\omega_n) \to c_{V_0}$ and $\Psi_0'(\omega_n) \to 0$ as $n \to \infty$. Put $u_n = m(\omega_n) \in \mathcal{N}_0$ for $n \in N$. Then $I_0(u_n) \to c_{V_0}$ and $I_0'(u_n) \to 0$ as $n \to \infty$ by Lemma 4.2$(b3)$. Similar to the proof of Lemma 3.4, it is easy to know that $\{u_n\}$ is bounded in H_0. Thus, we have $u_n \rightharpoonup u$ in H_0, $u_n \to u$ in $L^r_{\text{loc}}(\mathbb{R}^N)$, $1 \leq r < 2^*$ and $u_n \to u$ a.e. in $\mathbb{R}^N$, thus $I_0'(u) = 0$. From Remark 4.1, we know that $u \neq 0$. Moreover,

$$\begin{aligned} c_{V_0} &\leq I_0(u) = I_0(u) - \frac{1}{\theta} I_0'(u)[u] \\ &= \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u\|_0^2 + \int_{\mathbb{R}^N} \Big(\frac{1}{\theta} f(u^2)u^2 - \frac{1}{2}F(u^2)\Big) dx \\ &\leq \liminf_n \Big\{\Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_n\|_0^2 + \int_{\mathbb{R}^N} \Big(\frac{1}{\theta} f(u_n)u_n^2 - \frac{1}{2}F(u_n^2)\Big) dx\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \liminf_{n} \left\{ I_0(u_n) - \frac{1}{\theta} I_0'(u_n)[u_n] \right\} \\
&= c_{V_0},
\end{aligned}
$$

thus, u is a ground state solution. From the assumption of f, $u \geq 0$, moreover, by [13, Proposition 6 and Proposition 7], we know that $u(x) > 0$ for $x \in \mathbb{R}^N$. The proof is complete. □

Note that, by [13, Proposition 3 and Proposition 4], the ground state solution of problem (4.1) is radially symmetric, which implies that every ground state solution decays exponentially at infinity with its gradient, and is $C^2(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N, \mathbb{R})$.

Lemma 4.5 *Let $(u_n) \subset \mathcal{N}_0$ be such that $I_0(u_n) \to c_{V_0}$. Then (u_n) has a convergent subsequence in H_0.*

Proof Since $(u_n) \subset \mathcal{N}_0$, from Lemma 4.1($a3$), Lemma 4.2($b4$) and the definition of c_{V_0}, we have

$$
v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0, \quad \forall n \in N,
$$

and

$$
\Psi_0(v_n) = I_0(u_n) \to c_{V_0} = \inf_{u \in S_0} \Psi_0(u).
$$

Since S_0 is a complete C^1 manifold, by Ekeland's variational principle, there exists a sequence $\{\tilde{v}_n\} \subset S_0$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_{V_0}}$ sequence for Ψ_0 on S_0 and

$$
\|\tilde{v}_n - v_n\|_0 = o_n(1).
$$

Similar to the proof of Lemma 4.4, we may obtain the conclusion of this lemma. □

4.2 The technical results

In this subsection, we prove a multiplicity result for the modified problem (3.2) using the Ljusternik–Schnirelmann category theory. In order to get it, we first provide some useful preliminaries.

Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, $\omega \in H^1(\mathbb{R}^N, \mathbb{R})$ be a positive ground state solution of the limit problem (4.1), and $\eta \in C^\infty(\mathbb{R}^+, [0, 1])$ be a nonincreasing cut-off function defined in $[0, +\infty)$ such that $\eta(t) = 1$ if $0 \leq t \leq \delta/2$ and $\eta(t) = 0$ if $t \geq \delta$.

For any $y \in M$, let us introduce the function

$$
\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\omega\Big(\frac{\varepsilon x - y}{\varepsilon}\Big) \exp\Big(i\tau_y\Big(\frac{\varepsilon x - y}{\varepsilon}\Big)\Big),
$$

where

$$
\tau_y(x) := \sum_i^N A_i(y) x_i.
$$

Let $t_\varepsilon > 0$ be the unique positive number such that

$$
\max_{t \geq 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}).
$$

Note that $t_\varepsilon \Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$.

Let us define $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ as

$$\Phi_\varepsilon(y) := t_\varepsilon \Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$. Moreover, the energy of the above functions has the following behavior as $\varepsilon \to 0^+$.

Lemma 4.6 *The limit*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}$$

holds uniformly in $y \in M$.

Proof Assume by contradiction that the statement is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0^+$ satisfying

$$\left| J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{V_0} \right| \geq \delta_0.$$

For simplicity, we write Φ_n, Ψ_n and t_n for $\Phi_{\varepsilon_n}(y_n)$, $\Psi_{\varepsilon_n, y_n}$ and t_{ε_n}, respectively.

We can check that

$$\|\Psi_n\|^2_{\varepsilon_n} \to \int_{\mathbb{R}^N} (|\nabla\omega|^2 + V_0\omega^2)dx \text{ as } n \to +\infty. \tag{4.2}$$

Indeed, by a change of variable of $z = (\varepsilon_n x - y_n)/\varepsilon_n$, the Lebesgue Dominated Convergence Theorem, the continuity of V and $y_n \in M \subset \Lambda$(which is bounded), we deduce that

$$\int_{\mathbb{R}^N} V(\varepsilon_n x)|\Psi_n|^2 dx = \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)|\eta(|\varepsilon_n z|)\omega(z)|^2 dx \to V_0 \int_{\mathbb{R}^N} \omega^2 dx \text{ as } n \to +\infty.$$

Moreover, by the same change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we also have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla_{A_{\varepsilon_n}} \Psi_n|^2 dx &= \varepsilon_n^2 \int_{\mathbb{R}^N} |\eta'(|\varepsilon_n z|)\omega(z)|^2 dz + \int_{\mathbb{R}^N} |\eta(|\varepsilon_n z|)\nabla\omega(z)|^2 dz \\
&\quad + \int_{\mathbb{R}^N} \left|\eta(|\varepsilon_n z|)\Big(A(y_n) - A(\varepsilon_n z + y_n)\Big)\omega(z)\right|^2 dz \\
&\quad + 2\varepsilon_n \int_{\mathbb{R}^N} \eta(|\varepsilon_n z|)\eta'(|\varepsilon_n z|)\omega(z)\nabla\omega(z) \cdot \frac{z}{|z|} dz.
\end{aligned}$$

It is clear that

$$\lim_n \int_{\mathbb{R}^N} |\eta(|\varepsilon_n z|)\nabla\omega(z)|^2 dz = \int_{\mathbb{R}^N} |\nabla\omega(z)|^2 dz.$$

Moreover, using the definition of η, the Hölder continuity with exponent $\alpha \in (0, 1]$ of A, the exponential decay of ω, and the Lebesgue Dominated Convergence Theorem, we can infer

$$\int_{\mathbb{R}^N} |\eta'(|\varepsilon_n z|)\omega(z)|^2 dz = o_n(1),$$

$$\int_{\mathbb{R}^N} |\eta(|\varepsilon_n z|)\eta'(|\varepsilon_n z|)\omega(z)\nabla\omega(z)| dz = o_n(1),$$

and

$$\int_{\mathbb{R}^N} \left|\eta(|\varepsilon_n z|)\Big(A(y_n) - A(\varepsilon_n z + y_n)\Big)\omega(z)\right|^2 dz \leq C\varepsilon_n^{2\alpha} \int_{|\varepsilon_n z| \leq \delta} \omega^2(z)|z|^{2\alpha} dz = o_n(1),$$

obtaining (4.2).

On the other hand, since $J'_{\varepsilon_n}(t_n\Psi_n)(t_n\Psi_n) = 0$, by the change of variables $z = (\varepsilon_n x - y_n)/\varepsilon_n$, observe that, if $z \in B_{\delta/\varepsilon_n}(0)$, then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$, we have

$$\begin{aligned}
\|\Psi_n\|^2_{\varepsilon_n} &= \int_{\mathbb{R}^N} g(\varepsilon_n z + y_n, t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \\
&= \int_{\mathbb{R}^N} f(t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \\
&\geq \int_{B_{\delta/(2\varepsilon_n)}(0)} f(t_n^2\omega^2(z))\omega^2(z)dz \\
&\geq \int_{B_{\delta/2}(0)} f(t_n^2\omega^2(z))\omega^2(z)dz \\
&\geq f(t_n^2\gamma^2)\int_{B_{\delta/2}(0)} \omega^2(z)dz
\end{aligned}$$

for all n large enough and where $\gamma = \min\{\omega(z) : |z| \leq \delta/2\}$.

If $t_n \to +\infty$, by $(f4)$ we deduce that $\|\Psi_n\|^2_{\varepsilon_n} \to +\infty$ which contradicts (4.2). Therefore, up to a subsequence, we may assume that $t_n \to t_0 \geq 0$.

If $t_n \to 0$, using the fact that f is increasing and the Lebesgue Dominated Convergence Theorem, we obtain that

$$\|\Psi_n\|^2_{\varepsilon_n} = \int_{\mathbb{R}^N} f(t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z))\eta^2(|\varepsilon_n z|)\omega^2(z)dz \to 0, \text{ as } n \to +\infty,$$

which contradicts (4.2). Thus, we have $t_0 > 0$ and

$$\int_{\mathbb{R}^N} (|\nabla\omega|^2 + V_0\omega^2)dx = \int_{\mathbb{R}^N} f(t_0\omega^2)\omega^2 dx,$$

so that $t_0\omega \in \mathcal{N}_{V_0}$. Since $\omega \in \mathcal{N}_{V_0}$, we obtain that $t_0 = 1$ and so, using the Lebesgue Dominated Convergence Theorem, we get

$$\lim_n \int_{\mathbb{R}^N} F(|t_n\Psi_n|^2)dx = \int_{\mathbb{R}^N} F(\omega^2)dx.$$

Hence

$$\lim_n J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_0(\omega) = c_{V_0}$$

which is a contradiction and the proof is complete. □

Now we define the barycenter map.

Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and consider $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ defined by setting

$$\Upsilon(x) := \begin{cases} x, & \text{if } |x| < \rho, \\ \rho x/|x|, & \text{if } |x| \geq \rho. \end{cases}$$

The barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ is defined by

$$\beta_\varepsilon(u) := \frac{1}{\|u\|_2^2}\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)|u(x)|^2 dx.$$

We have the following lemma.

Lemma 4.7 *The limit*

$$\lim_{\varepsilon\to 0^+}\beta_\varepsilon(\Phi_\varepsilon(y))=y$$

holds uniformly in $y\in M$.

Proof Assume by contradiction that there exist $\kappa>0$, $(y_n)\subset M$ and $\varepsilon_n\to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))-y_n|\ge\kappa. \tag{4.3}$$

Using the change of variable $z=(\varepsilon_n x-y_n)/\varepsilon_n$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))=y_n+\frac{\displaystyle\int_{\mathbb{R}^N}(\Upsilon(\varepsilon_n z+y_n)-y_n)\eta^2(|\varepsilon_n z|)\omega^2(z)dz}{\displaystyle\int_{\mathbb{R}^N}\eta^2(|\varepsilon_n z|)\omega^2(z)dz}.$$

Taking into account $(y_n)\subset M\subset M_\delta\subset B_\rho$ and the Lebesgue Dominated Convergence Theorem, we can obtain that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n))-y_n|=o_n(1),$$

which contradicts (4.3). □

Now, we prove the following useful compactness result.

Proposition 4.1 *Let* $\varepsilon_n\to 0^+$ *and* $(u_n)\subset\mathcal{N}_{\varepsilon_n}$ *be such that* $J_{\varepsilon_n}(u_n)\to c_{V_0}$. *Then there exists* $(\tilde{y}_n)\subset\mathbb{R}^N$ *such that the sequence* $(|v_n|)\subset H^1(\mathbb{R}^N,\mathbb{R})$, *where* $v_n(x):=u_n(x+\tilde{y}_n)$, *has a convergent subsequence in* $H^1(\mathbb{R}^N,\mathbb{R})$. *Moreover, up to a subsequence,* $y_n:=\varepsilon_n\tilde{y}_n\to y\in M$ *as* $n\to+\infty$.

Proof The proof of this proposition can be found in [3]. However, for the reader's convenience, we give in what follows the details of the proof. Since $J'_{\varepsilon_n}(u_n)[u_n]=0$ and $J_{\varepsilon_n}(u_n)\to c_{V_0}$, arguing as in the proof of Lemma 3.4, we can prove that there exists $C>0$ such that $\|u_n\|_{\varepsilon_n}\le C$ for all $n\in\mathbb{N}$.

Arguing as in the proof of Lemma 3.2 and recalling that $c_{V_0}>0$, we have that there exist a sequence $\{\tilde{y}_n\}\subset\mathbb{R}^N$ and constants $R,\beta>0$ such that

$$\liminf_n\int_{B_R(\tilde{y}_n)}|u_n|^2dx\ge\beta. \tag{4.4}$$

Now, let us consider the sequence $\{|v_n|\}\subset H^1(\mathbb{R}^N,\mathbb{R})$, where $v_n(x):=u_n(x+\tilde{y}_n)$. By the diamagnetic inequality (2.1), we get that $\{|v_n|\}$ is bounded in $H^1(\mathbb{R}^N,\mathbb{R})$, and using (4.4), we may assume that $|v_n|\rightharpoonup v$ in $H^1(\mathbb{R}^N,\mathbb{R})$ for some $v\neq 0$.

Let $t_n>0$ be such that $\tilde{v}_n:=t_n|v_n|\in\mathcal{N}_{V_0}$, and set $y_n:=\varepsilon_n\tilde{y}_n$.

By the diamagnetic inequality (2.1), we have

$$c_{V_0}\le I_0(\tilde{v}_n)\le\max_{t\ge 0}J_{\varepsilon_n}(tu_n)=J_{\varepsilon_n}(u_n)=c_{V_0}+o_n(1),$$

which yields $I_0(\tilde{v}_n)\to c_{V_0}$ as $n\to+\infty$.

Since the sequences $\{|v_n|\}$ and $\{\tilde{v}_n\}$ are bounded in $H^1(\mathbb{R}^N,\mathbb{R})$ and $|v_n|\not\to 0$ in $H^1(\mathbb{R}^N,\mathbb{R})$, then (t_n) is also bounded and so, up to a subsequence, we may assume that $t_n\to t_0\ge 0$.

We claim that $t_0>0$. Indeed, if $t_0=0$, then, since $(|v_n|)$ is bounded, we have $\tilde{v}_n\to 0$ in $H^1(\mathbb{R}^N,\mathbb{R})$, that is $I_0(\tilde{v}_n)\to 0$, which contradicts $c_{V_0}>0$. Thus, up to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$, and, by Lemma 4.5, we can deduce that $\tilde{v}_n \to \tilde{v}$ in $H^1(\mathbb{R}^N, \mathbb{R})$, which gives $|v_n| \to v$ in $H^1(\mathbb{R}^N, \mathbb{R})$.

Now we show the final part, namely that $\{y_n\}$ has a subsequence such that $y_n \to y \in M$. Assume by contradiction that $\{y_n\}$ is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then for n large enough, we have $|y_n| > 2R$, and, for any $x \in B_{R/\varepsilon_n}(0)$,

$$|\varepsilon_n x + y_n| \geq |y_n| - \varepsilon_n |x| > R.$$

Since $u_n \in \mathcal{N}_{\varepsilon_n}$, using $(V1)$ and the diamagnetic inequality (2.1), we get that

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla |v_n||^2 + V_0 |v_n|^2) dx &\leq \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, |v_n|^2)|v_n|^2 dx \\ &\leq \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(|v_n|^2)|v_n|^2 dx + \int_{B^c_{R/\varepsilon_n}(0)} f(|v_n|^2)|v_n|^2 dx.\end{aligned} \tag{4.5}$$

Since $|v_n| \to v$ in $H^1(\mathbb{R}^N, \mathbb{R})$ and $\tilde{f}(t) \leq V_0/K$, we can see that (4.5) yields

$$\min\left\{1, V_0\left(1 - \frac{1}{K}\right)\right\} \int_{\mathbb{R}^N} (|\nabla |v_n||^2 + |v_n|^2) dx = o_n(1),$$

that is $|v_n| \to 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$, which contradicts to $v \not\equiv 0$.

Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^N$. Assume by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $r > 0$ such that for every n large enough we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \overline{\Lambda}^c$. Then, if $x \in B_{r/\varepsilon_n}(0)$, we have that $|\varepsilon_n x + y_n - y_0| < 2r$ so that $\varepsilon_n x + y_n \in \overline{\Lambda}^c$ and so, arguing as before, we reach a contradiction. Thus, $y_0 \in \overline{\Lambda}$.

To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using the Fatou's lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t \geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n)$, we obtain

$$\begin{aligned}c_{V_0} = I_0(\tilde{v}) &< \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \tilde{v}|^2 + V(y_0)|\tilde{v}|^2) dx - \frac{1}{2}\int_{\mathbb{R}^N} F(|\tilde{v}|^2) dx \\ &\leq \liminf_n \left(\frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \tilde{v}_n|^2 + V(\varepsilon_n x + y_n)|\tilde{v}_n|^2) dx - \frac{1}{2}\int_{\mathbb{R}^N} F(|\tilde{v}_n|^2) dx\right) \\ &= \liminf_n \left(\frac{t_n^2}{2}\int_{\mathbb{R}^N} (|\nabla |u_n||^2 + V(\varepsilon_n z)|u_n|^2) dz - \frac{1}{2}\int_{\mathbb{R}^N} F(|t_n u_n|^2) dz\right) \\ &\leq \liminf_n J_{\varepsilon_n}(t_n u_n) \leq \liminf_n J_{\varepsilon_n}(u_n) = c_{V_0}\end{aligned}$$

which is impossible and the proof is complete. □

Let now

$$\tilde{\mathcal{N}}_\varepsilon := \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq c_{V_0} + h(\varepsilon)\},$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Fixed $y \in M$, since, by Lemma 4.6, $|J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$, we get that $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$ small enough.

We have the following relation between $\tilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

Lemma 4.8 *We have*

$$\lim_{\varepsilon \to 0^+} \sup_{u \in \tilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof Let $\varepsilon_n \to 0^+$ as $n \to +\infty$. For any $n \in \mathbb{N}$, there exists $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u \in \tilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y \in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$

Therefore, it is enough to prove that there exists $(y_n) \subset M_\delta$ such that

$$\lim_n |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

By the diamagnetic inequality (2.1), we can see that $I_0(t|u_n|) \leq J_{\varepsilon_n}(tu_n)$ for any $t \geq 0$. Therefore, recalling that $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we can deduce that

$$c_{V_0} \leq \max_{t \geq 0} I_0(t|u_n|) \leq \max_{t \geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) \leq c_{V_0} + h(\varepsilon_n) \tag{4.6}$$

which implies that $J_{\varepsilon_n}(u_n) \to c_{V_0}$ as $n \to +\infty$. Then Proposition 4.1 implies that there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n large enough. Thus, making the change of variable $z = x - \tilde{y}_n$, we get

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} (\Upsilon(\varepsilon_n z + y_n) - y_n)|u_n(z + \tilde{y}_n)|^2 dz}{\int_{\mathbb{R}^N} |u_n(z + \tilde{y}_n)|^2 dz}.$$

Since, up to a subsequence, $|u_n|(\cdot + \tilde{y}_n)$ converges strongly in $H^1(\mathbb{R}^N, \mathbb{R})$ and $\varepsilon_n z + y_n \to y \in M$ for any $z \in \mathbb{R}^N$, we conclude the proof. □

4.3 Multiplicity of solutions for problem (3.2)

Finally, we present a relation between the topology of M and the number of solutions of the modified problem (3.2).

Theorem 4.1 *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, problem (3.2) has at least $cat_{M_\delta}(M)$ nontrivial solutions.*

Proof For any $\epsilon > 0$, we define the function $\pi_\epsilon : M \to S_\varepsilon^+$ by

$$\pi_\epsilon(y) = m_\varepsilon^{-1}(\Phi_\epsilon(y)), \ \forall y \in M.$$

By Lemma 4.6 and Lemma 3.3($B4$), we obtain

$$\lim_{\epsilon \to 0} \Psi_\epsilon(\pi_\epsilon(y)) = \lim_{\epsilon \to 0} J_\epsilon(\Phi_\epsilon(y)) = c_{V_0}, \text{ uniformly in } y \in M.$$

Hence, there is a number $\hat{\epsilon} > 0$ such that the set $\tilde{S}_\varepsilon^+ := \{u \in S_\varepsilon^+ : \Psi_\varepsilon(u) \leq c_{V_0} + h(\varepsilon)\}$ is nonempty, for all $\epsilon \in (0, \hat{\epsilon})$, since $\pi_\epsilon(M) \subset \tilde{S}_\varepsilon^+$. Here h is given in the definition of $\tilde{\mathcal{N}}_\varepsilon$.

Given $\delta > 0$, by Lemma 4.6, Lemma 3.2($A3$), Lemma 4.7, and Lemma 4.8, we can find $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, the following diagram

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \pi_\epsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and continuous. From Lemma 4.7, we can choose a function $\Theta(\epsilon, z)$ with $|\Theta(\epsilon, z)| < \frac{\delta}{2}$ uniformly in $z \in M$, for all $\epsilon \in (0, \hat{\epsilon})$ such that $\beta_\varepsilon(\Phi_\varepsilon(z)) = z + \Theta(\epsilon, z)$ for all $z \in M$. Define $H(t, z) = z + (1 - t)\Theta(\epsilon, z)$. Then $H : [0, 1] \times M \to M_\delta$ is continuous. Clearly, $H(0, z) = \beta_\varepsilon(\Phi_\varepsilon(z))$, $H(1, z) = z$ for all $z \in M$. That is, $H(t, z)$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ \pi_\epsilon$ and the embedding $\iota : M \to M_\delta$. Thus, this fact implies that

$$\text{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M)) \geq \text{cat}_{M_\delta}(M). \tag{4.7}$$

By Corollary 3.1 and the abstract category theorem [35], Ψ_ε has at least $\mathrm{cat}_{\pi_\epsilon(M)}(\pi_\epsilon(M))$ critical points on S_ε^+. Therefore, from Lemma 3.3$(B4)$ and (4.7), we have that J_ε has at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $\tilde{\mathcal{N}}_\varepsilon$ which implies that problem (3.2) has at least $\mathrm{cat}_{M_\delta}(M)$ solutions. □

5 Proof of Theorem 1.1

In this section we prove our main result. The idea is to show that the solutions u_ε obtained in Theorem 4.1 satisfy

$$|u_\varepsilon(x)|^2 \le a \text{ for } x \in \Lambda_\varepsilon^c$$

for ε small. The key ingredient is the following result.

Lemma 5.1 *Let $\varepsilon_n \to 0^+$ and $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of problem (3.2) for $\varepsilon = \varepsilon_n$. Then $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Moreover, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that, if $v_n(x) := u_n(x + \tilde{y}_n)$, we have that $\{|v_n|\}$ is bounded in $L^\infty(\mathbb{R}^N, \mathbb{R})$ and*

$$\lim_{|x|\to+\infty} |v_n(x)| = 0 \quad \textit{uniformly in } n \in \mathbb{N}.$$

Proof The proof of this lemma can be found in [3], for the convenience of the readers, we give the proof here. Since $J_{\varepsilon_n}(u_n) \le c_{V_0} + h(\varepsilon_n)$ with $\lim_n h(\varepsilon_n) = 0$, we can argue as in the proof of Lemma 4.8 (see (4.6)) to conclude that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Thus, by Proposition 4.1, we obtain the existence of a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $\{|v_n|\} \subset H^1(\mathbb{R}^N, \mathbb{R})$, where $v_n(x) := u_n(x + \tilde{y}_n)$, has a convergent subsequence in $H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$ as $n \to +\infty$.

For any $R > 0$ and $0 < r \le R/2$, let $\eta \in C^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$ with $\eta(x) = 1$ if $|x| \ge R$ and $\eta(x) = 0$ if $|x| \le R - r$ and $|\nabla \eta| \le 2/r$.

For each $n \in \mathbb{N}$ and $L > 0$, we consider the functions

$$v_{L,n}(x) := \begin{cases} |v_n(x)| & \text{if } |v_n(x)| \le L, \\ L & \text{if } |v_n(x)| > L, \end{cases} \qquad z_{L,n} := \eta^2 v_{L,n}^{2(\beta-1)} v_n, \quad \text{and} \quad w_{L,n} := \eta v_{L,n}^{\beta-1}|v_n|,$$

where $\beta > 1$ will be determined later.

Since, by the diamagnetic inequality (2.1) we have that

$$\begin{aligned} \mathrm{Re}(\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)} v_n \cdot \overline{\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)} z_{L,n}}) &= \eta^2 v_{L,n}^{2(\beta-1)} |\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)} v_n|^2 + \mathrm{Re}(\nabla v_n \overline{v_n}) \nabla\left(\eta^2 v_{L,n}^{2(\beta-1)}\right) \\ &= \eta^2 v_{L,n}^{2(\beta-1)} |\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)} v_n|^2 + |v_n| \nabla |v_n| \nabla\left(\eta^2 v_{L,n}^{2(\beta-1)}\right) \\ &\ge \eta^2 v_{L,n}^{2(\beta-1)} |\nabla |v_n||^2 + 2\eta \nabla \eta v_{L,n}^{2(\beta-1)} |v_n| \nabla |v_n|, \end{aligned}$$

taking $z_{L,n}$ as a test function, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla |v_n||^2 \eta^2 v_{L,n}^{2(\beta-1)} dx + 2\int_{\mathbb{R}^N} \eta \nabla \eta v_{L,n}^{2(\beta-1)} |v_n| \nabla |v_n| dx \\ &\quad + \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) \eta^2 v_{L,n}^{2(\beta-1)} |v_n|^2 dx \le \mathrm{Re} \int_{\mathbb{R}^N} (\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)} v_n \cdot \overline{\nabla_{A_{\varepsilon_n}(\cdot+\tilde{y}_n)} z_{L,n}}) dx \\ &\quad + \mathrm{Re} \int_{\mathbb{R}^N} V(\varepsilon_n x + \varepsilon_n \tilde{y}_n) v_n \overline{z_{L,n}} dx \\ &= \int_{\mathbb{R}^N} g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, |v_n|^2) \eta^2 v_{L,n}^{2(\beta-1)} |v_n|^2 dx. \end{aligned} \tag{5.1}$$

We first deal with the case $N \geq 3$. From the definition of g, for any $0 < \zeta < V_0$ small, there exists $C_\zeta > 0$ such that

$$g(x, t^2)t^2 \leq \zeta t^2 + C_\zeta |t|^{2^*} \quad \text{for all } x \in \mathbb{R}^N. \tag{5.2}$$

Using (5.1) and (5.2), we can obtain that

$$\int_{\mathbb{R}^N} |\nabla |v_n||^2 \eta^2 v_{L,n}^{2(\beta-1)} dx \leq 2 \int_{\mathbb{R}^N} \eta v_{L,n}^{2(\beta-1)} |v_n||\nabla |v_n|||\nabla \eta| dx + C \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} |v_n|^{2^*} dx. \tag{5.3}$$

For each $\delta > 0$, using Young's inequality, we have from (5.3) that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla |v_n||^2 \eta^2 v_{L,n}^{2(\beta-1)} dx \leq\ & 2\delta \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} |\nabla |v_n||^2 dx + 2C_\delta \int_{\mathbb{R}^N} v_{L,n}^{2(\beta-1)} |v_n|^2 |\nabla \eta|^2 dx \\ & + C \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} |v_n|^{2^*} dx. \end{aligned}$$

Choosing $\delta \in (0, \frac{1}{4})$, it yields

$$\int_{\mathbb{R}^N} |\nabla |v_n||^2 \eta^2 v_{L,n}^{2(\beta-1)} dx \leq C \int_{\mathbb{R}^N} v_{L,n}^{2(\beta-1)} |v_n|^2 |\nabla \eta|^2 dx + C \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} |v_n|^{2^*} dx. \tag{5.4}$$

On the other hand, by the Sobolev and Hölder inequalities, we have

$$\begin{aligned} |\omega_{L,n}|_{2^*}^2 &\leq C \int_{\mathbb{R}^N} |\nabla \omega_{L,n}|^2 dx = C \int_{\mathbb{R}^N} |\nabla \left(\eta |v_n| v_{L,n}^{\beta-1} \right)|^2 dx \\ &\leq C\beta^2 \Big(\int_{\mathbb{R}^N} v_{L,n}^{2(\beta-1)} |v_n|^2 |\nabla \eta|^2 dx + \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} |\nabla |v_n||^2 dx \Big). \end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5), we have

$$|\omega_{L,n}|_{2^*}^2 \leq C\beta^2 \Big(\int_{\mathbb{R}^N} v_{L,n}^{2(\beta-1)} |v_n|^2 |\nabla \eta|^2 dx + \int_{\mathbb{R}^N} \eta^2 v_{L,n}^{2(\beta-1)} |v_n|^{2^*} dx \Big). \tag{5.6}$$

Let $\beta = \frac{2^*}{2}$, by the definition of $\omega_{L,n}$ and (5.6), we rewrite the last inequality as

$$\begin{aligned} & \Big(\int_{\mathbb{R}^N} (\eta |v_n| v_{L,n}^{(2^*-2)/2})^{2^*} \Big)^{2/2^*} \\ & \leq C(N,2) \Big\{ \Big(\int_{\mathbb{R}^N} (\eta |v_n| v_{L,n}^{(2^*-2)/2})^{2^*} dx \Big)^{2/2^*} \Big(\int_{|x| \geq R-r} |v_n|^{2^*} \Big)^{(2^*-2)/2} \\ & \quad + \int_{\mathbb{R}^N} v_{L,n}^{2^*-2} |v_n|^2 |\nabla \eta|^2 dx \Big\} \\ & \leq C(N,2) \Big\{ \Big(\int_{\mathbb{R}^N} (\eta |v_n| v_{L,n}^{(2^*-2)/2})^{2^*} dx \Big)^{2/2^*} |v_n|_{2^*(|x| \geq R/2)}^{2^*-2} \\ & \quad + \int_{\mathbb{R}^N} v_{L,n}^{2^*-2} |v_n|^2 |\nabla \eta|^2 dx \Big\}. \end{aligned}$$

From Lemma 4.5, $|v_n| \to |v|$ in $H^1(\mathbb{R}^N)$, for R large enough, we conclude that

$$|v_n|_{2^*(|x| \geq R/2)}^{2^*-2} \leq \frac{1}{2C(N,2)} \quad \text{uniformly in } n \in \mathbb{N}.$$

Hence we obtain

$$\Big(\int_{|x|\geq R}(|v_n|v_{L,n}^{(2^*-2)/2})^{2^*}\Big)^{2/2^*} \leq 2C(N,2)\int_{\mathbb{R}^N} v_{L,n}^{2^*-2}|v_n|^2|\nabla\eta|^2dx$$
$$\leq \frac{C}{r^2}\int_{\mathbb{R}^N}|v_n|^{2^*}dx.$$

Using the Fatou's lemma in the variable L, we have

$$|v_n|\in L^{2^{*2}/2}(|x|\geq R) \quad \text{for } R \text{ large enough.} \tag{5.7}$$

Next, we note that if $\beta = 2^*(t-1)/2t$ with $t = 2^{*2}/2(2^*-2)$, then $\beta > 1$ and $2t/(t-1) < 2^*$. Now suppose that $|v_n| \in L^{2\beta t/(t-1)}(|x| \geq R-r)$ for some $\beta \geq 1$. Using the Hölder inequality with exponent $t/(t-1)$ and t, then (5.7) gives that

$$|\omega_{L,n}|_{2^*}^2 \leq C\beta^2\Big\{\Big(\int_{|x|\geq R-r}(\eta^2|v_n|^{2\beta})^{t/(t-1)}dx\Big)^{1-1/t}\Big(\int_{|x|\geq R-r}|v_n|^{(2^*-2)t}\Big)^{1/t}$$
$$+\frac{(R^N-(R-r)^N)^{1/t}}{r^2}\Big(\int_{|x|\geq R-r}|v_n|^{2\beta t/(t-1)}dx\Big)^{1-1/t}\Big\}$$
$$\leq C\beta^2\Big(1+\frac{R^{N/t}}{r^2}\Big)\Big(\int_{|x|\geq R-r}|v_n|^{2\beta t/(t-1)}dx\Big)^{1-1/t}. \tag{5.8}$$

Letting $L\to+\infty$ in (5.8), we obtain

$$|v_n|^{2\beta}_{2^*\beta(|x|\geq R)} \leq C\beta^2\Big(1+\frac{R^{N/t}}{r^2}\Big)|v_n|^{2\beta}_{2\beta t/(t-1)(|x|\geq R-r)}.$$

If we set $\chi := 2^*(t-1)/(2t)$, $s := 2t/(t-1)$, then

$$|v_n|_{\beta\chi s(|x|\geq R)} \leq C^{1/\beta}\beta^{1/\beta}\Big(1+\frac{R^{N/t}}{r^2}\Big)^{1/(2\beta)}|v_n|_{\beta s(|x|\geq R-r)}. \tag{5.9}$$

Let $\beta=\chi^m (m=1,2,\ldots)$, we obtain

$$|v_n|_{\chi^{m+1}s(|x|\geq R)} \leq C^{\chi^{-m}}\chi^{m\chi^{-m}}\Big(1+\frac{R^{N/t}}{r^2}\Big)^{1/(2\beta)}|v_n|_{\chi^m s(|x|\geq R-r)}.$$

It is clear that $2 > N/t$. So if we take $r_m = 2^{-(m+1)}R$, then (5.9) implies

$$|v_n|_{\chi^{m+1}s(|x|\geq R)} \leq |v_n|_{\chi^{m+1}s(|x|\geq R-r_{m+1})}$$
$$\leq C^{\sum_{i=1}^m\chi^{-i}}\chi^{\sum_{i=1}^m i\chi^{-i}}\exp\Big(\sum_{i=1}^m\frac{\ln(1+2^{2(i+1)})}{2\chi^i}\Big)|v_n|_{\chi s(|x|\geq R-r_1)}$$
$$\leq C|v_n|_{2^*(|x|\geq R/2)}.$$

Letting $m\to\infty$ in the last inequality, we get

$$|v_n|_{L^\infty(|x|\geq R)} \leq C|v_n|_{2^*(|x|\geq R/2)}. \tag{5.10}$$

Using $|v_n| \to |v|$ in $H^1(\mathbb{R}^N)$ again, for any fixed $a > 0$, there exists $R > 0$ such that $|v_n|_{L^\infty(|x|\geq R)} \leq a$ for all $n\in\mathbb{N}$. Therefore, $\lim_{|x|\to\infty}|v_n(x)| = 0$ uniformly in n.

To show that $|v_n|_{L^\infty(\mathbb{R}^N)} < +\infty$, we need only show that for any $x_0\in\mathbb{N}$, there is a ball $B_R(x_0) = \{x\in\mathbb{R}^N) : |x-x_0|\leq R\}$ such that $|v_n|_{L^\infty(B_R(x_0))} < +\infty$. We can use the same arguments and take $\eta \in C^\infty(\mathbb{R}^N)$, $0 \le \eta \le 1$ with $\eta(x) = 1$ if $|x - x_0| \le \rho'$ and $\eta(x) = 0$ if $|x - x_0| > 2\rho'$ and $|\nabla \eta| \le \frac{2}{\rho'}$, to prove that

$$|v_n|_{L^\infty(|x-x_0|\le\rho')} \le C|v_n|_{2(|x|\ge 2\rho')}. \tag{5.11}$$

From (5.10) and (5.11), using a standard covering argument it follows that

$$|v_n|_{L^\infty(\mathbb{R}^N)} \le C$$

for some positive constant C.

For the case $N = 2$, similar with the proof for the case $N \ge 3$, we also let $z_{L,n} := \eta^2 v_{L,n}^{2(\beta-1)} v_n$ and $w_{L,n} := \eta v_{L,n}^{\beta-1}|v_n|$ with $\beta > 1$ to be determined later. Taking $z_{L,n}$ as a test function, we also have (5.1). Moreover, from the definition of g, for any $0 < \zeta < V_0$ small, there exists $C_\zeta > 0$ such that

$$g(x, t^2)t^2 \le \zeta t^2 + C_\zeta |t|^q \quad \text{for all } x \in \mathbb{R}^N. \tag{5.12}$$

where $2 < q < \infty$.

By (5.1) and (5.12), we obtain that

$$\int_{\mathbb{R}^2} |\nabla|v_n||^2 \eta^2 v_{L,n}^{2(\beta-1)} dx \le 2\int_{\mathbb{R}^2} \eta v_{L,n}^{2(\beta-1)} |v_n||\nabla|v_n|||\nabla\eta| dx + C\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\beta-1)}|v_n|^q dx. \tag{5.13}$$

For any $\delta > 0$, using Young's inequality, we have from (5.13) that

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla|v_n||^2 \eta^2 v_{L,n}^{2(\beta-1)} dx \le{}& 2\delta\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\beta-1)}|\nabla|v_n||^2 dx + 2C_\delta\int_{\mathbb{R}^2} v_{L,n}^{2(\beta-1)}|v_n|^2|\nabla\eta|^2 dx\\ &+C\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\beta-1)}|v_n|^q dx.\end{aligned}$$

Choosing $\delta \in (0, \frac{1}{4})$, it yields

$$\int_{\mathbb{R}^2} |\nabla|v_n||^2 \eta^2 v_{L,n}^{2(\beta-1)} dx \le C\int_{\mathbb{R}^2} v_{L,n}^{2(\beta-1)}|v_n|^2|\nabla\eta|^2 dx + C\int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\beta-1)}|v_n|^q dx. \tag{5.14}$$

On the other hand, by the Sobolev embedding,

$$|\omega_{L,n}|_q^2 \le C\beta^2\Big(\int_{\mathbb{R}^2} v_{L,n}^{2(\beta-1)}|v_n|^2|\nabla\eta|^2 dx + \int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\beta-1)}|\nabla|v_n||^2 dx\Big). \tag{5.15}$$

Using (5.14) and (5.15), we have

$$|\omega_{L,n}|_q^2 \le C\beta^2\Big(\int_{\mathbb{R}^2} v_{L,n}^{2(\beta-1)}|v_n|^2|\nabla\eta|^2 dx + \int_{\mathbb{R}^2} \eta^2 v_{L,n}^{2(\beta-1)}|v_n|^q dx\Big). \tag{5.16}$$

Let $\beta = \frac{q}{2}$, by the definition of $\omega_{L,n}$ and (5.6), we rewrite the last inequality as

$$\begin{aligned}\Big(\int_{\mathbb{R}^2}(\eta|v_n|v_{L,n}^{(q-2)/2})^q\Big)^{2/q} \le{}& C(2,2)\Big\{\Big(\int_{\mathbb{R}^2}(\eta|v_n|v_{L,n}^{(q-2)/2})^q dx\Big)^{2/q}\Big(\int_{|x|\ge R-r}|v_n|^q\Big)^{(q-2)/q}\\ &+\int_{\mathbb{R}^2} v_{L,n}^{q-2}|v_n|^2|\nabla\eta|^2 dx\Big\}\end{aligned}$$

$$\leq C(2,2)\Big\{\Big(\int_{\mathbb{R}^2}(\eta|v_n|v_{L,n}^{(q-2)/2})^q dx\Big)^{2/q}|v_n|^{q-2}_{q(|x|\geq R/2)} + \int_{\mathbb{R}^2} v_{L,n}^{q-2}|v_n|^2|\nabla\eta|^2 dx\Big\}.$$

From Lemma 4.5, $|v_n| \to |v|$ in $H^1(\mathbb{R}^2)$, we have $|v_n| \to |v|$ in $L^q(\mathbb{R}^2)$. Thus, for R large enough, we conclude that

$$|v_n|^{q-2}_{q(|x|\geq R/2)} \leq \frac{1}{2C(2,2)} \quad \text{uniformly in } n \in \mathbb{N}.$$

Hence we obtain

$$\begin{aligned}\Big(\int_{|x|\geq R}(|v_n|v_{L,n}^{(q-2)/2})^q\Big)^{2/q} &\leq 2C(2,2)\Big(\int_{\mathbb{R}^2} v_{L,n}^{q-2}|v_n|^2|\nabla\eta|^2 dx + \int_{\mathbb{R}^2}\eta^2 v_{L,n}^{q-2}|v_n|^2 dx\Big)\\ &\leq \frac{C}{r^2}\int_{\mathbb{R}^2}|v_n|^q dx.\end{aligned}$$

Using the Fatou's lemma in the variable L, we have

$$|v_n| \in L^{q^2/2}(|x| \geq R) \quad \text{for } R \text{ large enough.} \tag{5.17}$$

Next, we note that if $\beta = q(t-1)/2t$ with $t = q^2/2(q-2)$, then $\beta > 1$ and $2t/(t-1) < q$. Now suppose that $|v_n| \in L^{2\beta t/(t-1)}(|x| \geq R-r)$ for some $\beta \geq 1$. Using the Hölder inequality with exponent $t/(t-1)$ and t, then (5.16) gives that

$$\begin{aligned}|\omega_{L,n}|_q^2 &\leq C\beta^2\Big\{\Big(\int_{|x|\geq R-r}(|v_n|^{2\beta})^{t/(t-1)}dx\Big)^{1-1/t}\Big(\int_{|x|\geq R-r}|v_n|^{(q-2)t}\Big)^{1/t}\\ &\quad+\frac{(R^2-(R-r)^2)^{1/t}}{r^2}\Big(\int_{|x|\geq R-r}|v_n|^{2\beta t/(t-1)}dx\Big)^{1-1/t}\Big\}\\ &\leq C\beta^2\Big(1+\frac{R^{2/t}}{r^2}\Big)\Big(\int_{|x|\geq R-r}|v_n|^{2\beta t/(t-1)}dx\Big)^{1-1/t}.\end{aligned} \tag{5.18}$$

Letting $L \to +\infty$ in (5.18), we obtain

$$|v_n|^{2\beta}_{q\beta(|x|\geq R)} \leq C\beta^2\Big(1+\frac{R^{2/t}}{r^2}\Big)|v_n|^{2\beta}_{2\beta t/(t-1)(|x|\geq R-r)}.$$

If we set $\chi := q(t-1)/(2t)$, $s := 2t/(t-1)$, then

$$|v_n|_{\beta\chi s(|x|\geq R)} \leq C^{1/\beta}\beta^{1/\beta}\Big(1+\frac{R^{2/t}}{r^2}\Big)^{1/(2\beta)}|v_n|_{\beta s(|x|\geq R-r)}. \tag{5.19}$$

Let $\beta = \chi^m (m = 1, 2, \ldots)$, we obtain

$$|v_n|_{\chi^{m+1}s(|x|\geq R)} \leq C^{\chi^{-m}}\chi^{m\chi^{-m}}\Big(1+\frac{R^{2/t}}{r^2}\Big)^{1/(2\beta)}|v_n|_{\chi^m s(|x|\geq R-r)}.$$

It is clear that $2 > 2/t$. So if we take $r_m = 2^{-(m+1)}R$, then (5.19) implies

$$\begin{aligned}|v_n|_{\chi^{m+1}s(|x|\geq R)} &\leq |v_n|_{\chi^{m+1}s(|x|\geq R-r_{m+1})}\\ &\leq C^{\sum_{i=1}^m \chi^{-i}}\chi^{\sum_{i=1}^m i\chi^{-i}}\exp\Big(\sum_{i=1}^m\frac{\ln(1+2^{2(i+1)})}{2\chi^i}\Big)|v_n|_{\chi s(|x|\geq R-r_1)}\\ &\leq C|v_n|_{q(|x|\geq R/2)}.\end{aligned}$$

Letting $m \to \infty$ in the last inequality, we get

$$|v_n|_{L^\infty(|x|\geq R)} \leq C|v_n|_{q(|x|\geq R/2)}. \tag{5.20}$$

Using $|v_n| \to |v|$ in $H^1(\mathbb{R}^2)$ again, for any fixed $a > 0$, there exists $R > 0$ such that $|v_n|_{L^\infty(|x|\geq R)} \leq a$ for all $n \in \mathbb{N}$. Therefore, $\lim\limits_{|x|\to\infty} |v_n(x)| = 0$ uniformly in n.

Similarly, in order to show that $|v_n|_{L^\infty(\mathbb{R}^2)} < +\infty$, we need only show that for any $x_0 \in \mathbb{R}^2$, there is a ball $B_R(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| \leq R\}$ such that $|\psi_n|_{L^\infty(B_R(x_0))} < +\infty$. We can use the same arguments and take $\eta \in C^\infty(\mathbb{R}^2)$, $0 \leq \eta \leq 1$ with $\eta(x) = 1$ if $|x - x_0| \leq \rho'$ and $\eta(x) = 0$ if $|x - x_0| > 2\rho'$ and $|\nabla\eta| \leq \frac{2}{\rho'}$, to prove that

$$|v_n|_{L^\infty(|x-x_0|\leq\rho')} \leq C|v_n|_{2(|x|\geq 2\rho')}. \tag{5.21}$$

From (5.20) and (5.21), using a standard covering argument it follows that

$$|v_n|_{L^\infty(\mathbb{R}^2)} \leq C$$

for some positive constant C and the proof is complete. □

Now, we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. We want to show that there exists $\hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$ and any $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ solution of problem (3.2), it holds

$$\|u_\varepsilon\|^2_{L^\infty(\Lambda^c_\varepsilon)} \leq a. \tag{5.22}$$

We argue by contradiction and assume that there is a sequence $\varepsilon_n \to 0$ such that for every n there exists $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ which satisfies $J'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|^2_{L^\infty(\Lambda^c_{\varepsilon_n})} > a. \tag{5.23}$$

Arguing as in Lemma 5.1, we have that $J_{\varepsilon_n}(u_n) \to c_{V_0}$, and therefore we can use Proposition 4.1 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $y_n := \varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$. Then, we can find $r > 0$, such that $B_r(y_n) \subset \Lambda$, and so $B_{r/\varepsilon_n}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$ for all n large enough.

Using Lemma 5.1, there exists $R > 0$ such that $|v_n|^2 \leq a$ in $B^c_R(0)$ and n large enough, where $v_n = u_n(\cdot + \tilde{y}_n)$. Hence $|u_n|^2 \leq a$ in $B^c_R(\tilde{y}_n)$ and n large enough. Moreover, if n is so large that $r/\varepsilon_n > R$, then $\Lambda^c_{\varepsilon_n} \subset B^c_{r/\varepsilon_n}(\tilde{y}_n) \subset B^c_R(\tilde{y}_n)$, which gives $|u_n|^2 \leq a$ for any $x \in \Lambda^c_{\varepsilon_n}$. This contradicts (5.23) and proves the claim.

Let now $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \tilde{\varepsilon}_\delta\}$, where $\tilde{\varepsilon}_\delta > 0$ is given by Theorem 4.1. Then we have $\mathrm{cat}_{M_\delta}(M)$ nontrivial solutions to problem (3.2). If $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ is one of these solutions, then, by (5.22) and the definition of g, we conclude that u_ε is also a solution to problem (2.2).

Finally, we study the behavior of the maximum points of $|\hat{u}_\varepsilon|$, where $\hat{u}_\varepsilon(x) := u_\varepsilon(x/\varepsilon)$ is a solution to problem (1.4), as $\varepsilon \to 0^+$.

Take $\varepsilon_n \to 0^+$ and the sequence (u_n) where each u_n is a solution of (3.2) for $\varepsilon = \varepsilon_n$. From the definition of g, there exists $\gamma \in (0, a)$ such that

$$g(\varepsilon x, t^2)t^2 \leq \frac{V_0}{K}t^2, \quad \text{for all } x \in \mathbb{R}^N,\ |t| \leq \gamma.$$

Arguing as above we can take $R > 0$ such that, for n large enough,

$$\|u_n\|_{L^\infty(B^c_R(\tilde{y}_n))} < \gamma. \tag{5.24}$$

Up to a subsequence, we may also assume that for n large enough

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{5.25}$$

Indeed, if (5.25) does not hold, up to a subsequence, if necessary, we have $\|u_n\|_\infty < \gamma$. Thus, since $J'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$, using $(g5)$ and the diamagnetic inequality (2.1) that

$$\int_{\mathbb{R}^N} (|\nabla |u_n||^2 + V_0|u_n|^2)dx \le \int_{\mathbb{R}^N} g(\varepsilon_n x, |u_n|^2)|u_n|^2 dx \le \frac{V_0}{K} \int_{\mathbb{R}^N} |u_n|^2 dx$$

and, being $K > 2$, $\|u_n\| = 0$, which is a contradiction.

Taking into account (5.24) and (5.25), we can infer that the global maximum points p_n of $|u_{\varepsilon_n}|$ belongs to $B_R(\tilde{y}_n)$, that is $p_n = q_n + \tilde{y}_n$ for some $q_n \in B_R$. Recalling that the associated solution of problem (1.4) is $\hat{u}_n(x) = u_n(x/\varepsilon_n)$, we can see that a maximum point η_{ε_n} of $|\hat{u}_n|$ is $\eta_{\varepsilon_n} = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$. Since $q_n \in B_R$, $\varepsilon_n \tilde{y}_n \to y_0$ and $V(y_0) = V_0$, the continuity of V allows to conclude that

$$\lim_n V(\eta_{\varepsilon_n}) = V_0.$$

The proof is now complete. □

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