Correction to: Interpolation and optimal hitting for complete minimal surfaces with finite total curvature

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Throughout this note we use the notation from [1].

Franc Forstnerič pointed out that [1, Proposition 2.3] is incorrect by giving us a counterexample; we are grateful to him for reporting and for helpful discussions. We overlooked the possible contribution of the cusp points of the curve γ in the result to the absolute value of its winding number. To fix this, we provide a corrected version of [1, Proposition 2.3] which measures the possible influence of the cusp points on the winding number of the curve (see Proposition 2.3). We then use this result to obtain corrected versions of [1, Theorems 1.3 and 1.4] on optimal hitting theory for complete minimal surfaces (see Theorems 1.3 and 1.4). The other main results in [1], namely [1, Theorems 1.1 and 3.1], are not affected by the error.

Here are the corrected statements; see [1, Sects. 1 and 2.5] for background.

Proposition 2.3 *Let $\gamma(t)$ be a real analytic, piecewise regular, closed curve in $\mathbb{R}^2$ admitting a regular normal field. Denote by t_γ and m_γ the turning number and the number of cusp points of γ , respectively. Fix $p \in \mathbb{R}^2 \setminus \gamma$ and let $w_\gamma(p)$ denote the winding number of γ with respect to p . Then, we have $|w_\gamma(p)| \leq 2t_\gamma + \frac{1}{2}m_\gamma$.*

The addend $\frac{1}{2}m_\gamma$ is missing in the bound stated in [1, Proposition 2.3].

Theorem 1.3 *Let $r \geq 1$ be an integer. For any integer m with $2 - 2r \leq m \leq 1$ there is a set $A_{r;m} \subset \mathbb{R}^3$ which is against the family $\bigcup_{k \leq m} \mathcal{Z}_{r;k}$ and consists of $12r + 50r^3 + 2m + 1$ points*

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whose affine span is a plane. In particular, the set $A_{r;1}$, which consists of $12r + 50r^3 + 3$ points, is against the family $\mathcal{X}_r$.

Thus, if $X: M \rightarrow \mathbb{R}^3$ is a complete nonflat orientable immersed minimal surface with empty boundary and the Euler characteristic $\chi(M) \leq m$, and if $A_{r;m} \subset X(M)$, then the total curvature $\text{TC}(X) < -4\pi r$. In particular, no complete nonflat orientable immersed minimal surface X with $|\text{TC}(X)| \leq 4\pi r$ contains $A_{r;1}$.

Theorem 1.4 *Let $X: M \rightarrow \mathbb{R}^3$ be a complete orientable immersed minimal surface of finite total curvature and empty boundary. If $L \subset \mathbb{R}^3$ is a straight line not contained in $X(M)$, then*

$$\#(X^{-1}(L)) \leq 6\text{Deg}(N) + 25\text{Deg}(N)^3 + \chi(M)$$

where $\text{Deg}(N)$ is the degree of the Gauss map N of X and $\chi(M)$ is the Euler characteristic of M .

The terms $50r^3$ and $25\text{Deg}(N)^3$ are not present in the statements of [1, Theorems 1.3 and 1.4], respectively. They come from the term $\frac{1}{2}m_\gamma$ in Proposition 2.3.

Proof of Proposition 2.3. We follow [1, proof of Proposition 2.3] until the very end. Note that each curve γ_j there, $j = 1, \dots, 2t_\gamma$, has precisely $m_j - 1$ cusp points, and hence $m_\gamma = \sum_{j=1}^{2t_\gamma} (m_j - 1)$. Since every two consecutive subarcs $\gamma_{j,i}, \gamma_{j,i+1}$ of γ_j have the opposite character (i.e., one is positive and the other one negative) and γ_j consists of precisely m_j arcs $\gamma_{j,i}$, it turns out that the absolute value of the signed number of crossings of γ_j with ℓ is at most $1 + E(\frac{m_j-1}{2})$, where $E(\cdot)$ denotes the integer part, and hence $|w_\gamma(p)| \leq \sum_{j=1}^{2t_\gamma} (1 + E(\frac{m_j-1}{2})) \leq 2t_\gamma + \frac{1}{2}m_\gamma$. $\square$

Proof of Theorem 1.4. We assume without loss of generality that X is primitive in the sense that there is no nontrivial (finite) holomorphic covering $\rho: M \rightarrow M_0$ into an open Riemann surface M_0 admitting a conformal minimal immersion $X_0: M_0 \rightarrow \mathbb{R}^3$ such that $X = X_0 \circ \rho$. Also, we assume that X is neither a plane nor a catenoid; otherwise, the conclusion of the theorem is obvious. The proof then follows word by word that of [1, Theorem 1.4] until the top of page 18 where [1, Proposition 2.3] is applied. Instead, we apply Proposition 2.3 and obtain $|\sum_{j=1}^m f_*(c_j)| \leq \sum_{j=1}^m |w_j| \leq \sum_{j=1}^m (2t_j + \frac{1}{2}m_j) = 2\sum_{j=1}^m t_j + \frac{1}{2}\sum_{j=1}^m m_j$, where m_j denotes the number of cusp points of the curve α_j (this replaces (4.5) in [1]). As in the proof in [1], we have $\sum_{j=1}^m t_j = \text{Deg}(N|_{\overline{\Omega}})$, and so if we set $m_\Omega = \sum_{j=1}^m m_j$ then $|\sum_{j=1}^m f_*(c_j)| \leq 2\text{Deg}(N|_{\overline{\Omega}}) + \frac{1}{2}m_\Omega$ (this replaces (4.6) in [1]). Combining this with equations (4.2), (4.3), and (4.4) in [1], we get that $\#(X^{-1}(L) \cap \overline{\Omega}) = \#(X^{-1}(L) \cap \Omega) \leq 2\text{Deg}(N|_{\overline{\Omega}}) + \frac{1}{2}m_\Omega + \sum_{j=1}^r I_{q_j}$. Denote by $\Omega_1, \dots, \Omega_a$ ($a \in \mathbb{N}$) the connected components of $\Sigma \setminus Q$. Joining together the above information for all the Ω_i 's, and taking into account that each Jordan curve in Q lies in the boundary of exactly two of them, we obtain that $\#(X^{-1}(L)) \leq 4\text{Deg}(N) + \frac{1}{2}\sum_{i=1}^a m_{\Omega_i} + \sum_{q \in E} I_q$ (this replaces (4.7) in [1]). The Jorge–Meeks formula (see (2.5) in [1]) then gives

$$\#(X^{-1}(L)) \leq 6\text{Deg}(N) + \chi(M) + \frac{1}{2}\sum_{i=1}^a m_{\Omega_i}. \quad (1)$$

To complete the proof, we shall now provide an upper bound for $\sum_{i=1}^a m_{\Omega_i}$ in terms of the degree of the Gauss map N of X . This term is precisely what is missing in [1, proof of Theorem 1.4] and what makes the addend $25\text{Deg}(N)^3$ to appear in the statement of Theorem 1.4. Indeed, we claim that

$$\sum_{i=1}^a m_{\Omega_i} \leq 50 \operatorname{Deg}(N)^3. \quad (2)$$

To prove this, denote by g the complex Gauss map of $X = (X_1, X_2, X_3)$. Since the affine line $L \subset \mathbb{R}^3$ is assumed to be the x_3 -axis and to lie in $\mathcal{G}_0$ (see conditions (a), (b), and (c) in [1, p. 15]), we have that $Q = Q_L = |g|^{-1}(1) = \bigcup_{i=1}^a b\Omega_i$. Denote by C the set of points in $|g|^{-1}(1)$ whose image by the map $(X_1, X_2): |g|^{-1}(1) \rightarrow \mathbb{R}^2$ is a singular point. Note that C is finite and contains all cusp points of $(X_1, X_2)|_{|g|^{-1}(1)}$, and hence to prove (2) it suffices to show that

$$\#C \leq 25 \operatorname{Deg}(g)^3; \quad (3)$$

note that $\operatorname{Deg}(g) = \operatorname{Deg}(N)$ and that each Jordan curve in $|g|^{-1}(1)$ lies in the boundary of exactly two domains Ω_i . For, observe that the meromorphic function $F = g \frac{\partial X_3}{\partial g}$ on $\Sigma = M \cup E$ satisfies

$$C = \{p \in |g|^{-1}(1) : \Re F(p) = 0\}. \quad (4)$$

Choose $\theta \in \mathbb{C}$ such that $|\theta| = 1$ and $g^{-1}(\theta)$ is disjoint from C , consider the meromorphic function $u = i \frac{g+\theta}{g-\theta}$ on Σ , and notice that

$$|g|^{-1}(1) = u^{-1}(\mathbb{R} \cup \{\infty\}) \quad \text{and} \quad C \subset u^{-1}(\mathbb{R}). \quad (5)$$

Since X is not a plane or a catenoid, we have that u and F are nonconstant, and hence [2, Proposition IV.11.6] provides a nonconstant irreducible complex polynomial P in two variables such that $P(u, F) = 0$ everywhere on Σ . Since the Weierstrass data of $X: M = \Sigma \setminus E \rightarrow \mathbb{R}^3$ are determined by g , F , and dg and the immersion X is primitive, it is not difficult to see that (u, F) is a primitive pair on Σ in the sense of [2, p. 248], and hence Σ is identified with the algebraic curve associated with $\{P = 0\}$ via the biholomorphism induced by (u, F) . Since $\operatorname{Deg}(u) = \operatorname{Deg}(g)$ and $\operatorname{Deg}(F) \leq 4 \operatorname{Deg}(g)$, [2, Proposition IV.11.9] ensures that the total algebraic degree of P satisfies $\operatorname{Deg}(P) \leq 5 \operatorname{Deg}(g)$. Setting $F_1 = \Re F$ and $F_2 = \Im F$, (4) and (5) give

$$C = \{(u, iF_2) \in \mathbb{R} \times i\mathbb{R} : P(u, iF_2) = 0\}. \quad (6)$$

Write $Q_1(u, F_2) = \Re P(u, iF_2)$ and $Q_2(u, F_2) = \Im P(u, iF_2)$. Note that Q_1 and Q_2 are real polynomials in two real variables with the total algebraic degree $\operatorname{Deg}(Q_j) \leq \operatorname{Deg}(P) \leq 5 \operatorname{Deg}(g)$, $j = 1, 2$. Thus, [3, Theorem 8.1] ensures that the algebraic system $Q_1(u, F_2) = Q_2(u, F_2) = 0$ has at most $25 \operatorname{Deg}(g)^2$ solutions in $\mathbb{R}^2$. Since $\operatorname{Deg}(u) = \operatorname{Deg}(g)$, this and (6) imply (3), which proves (2).

Finally, (1) and (2) give $\#(X^{-1}(L)) \leq 6 \operatorname{Deg}(N) + \chi(M) + 25 \operatorname{Deg}(N)^3$. $\square$

Proof of Theorem 1.3. Follow [1, proof of Theorem 1.3] but choosing each set C_j ($j = 1, 2$) consisting of $m + 6r + 25r^3 + 1$ points and applying Theorem 1.4. $\square$

Note that [1, Corollary 4.5] remains to hold true; just replace [1, Theorem 1.4] by Theorem 1.4 in its proof. Finally, the statement of [1, Corollary 4.4] has to be corrected by adding $50r^3$ points to the set $A_{r,m}^*$; here is its correct formulation.

Corollary 4.4 *Let r and m be as in Theorem 1.3. There is a set $A_{r,m}^* \subset \mathbb{R}^3$, consisting of $12r + 50r^3 + 2m + 2$ points, such that if $X: M \rightarrow \mathbb{R}^3$ is a complete orientable immersed minimal surface with $\chi(M) \leq m$ and $A_{r,m}^* \subset X(M)$, then the absolute value of the total curvature $|\operatorname{TC}(X)| > 4\pi r$.*

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