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Calculus of Variations

Quasilinear problems under local Landesman–Lazer condition

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Abstract

This paper presents results on the existence and multiplicity of solutions for quasilinear problems in bounded domains involving the p-Laplacian operator under local versions of the Landesman–Lazer condition. The main results do not require any growth restriction at infinity on the nonlinear term which may change sign. The existence of solutions is established by combining variational methods, truncation arguments and approximation techniques based on a compactness result for the inverse of the p-Laplacian operator. These results also establish the intervals of the projection of the solution on the direction of the first eigenfunction of the p-Laplacian operator. This fact is used to provide the existence of multiple solutions when the local Landesman–Lazer condition is satisfied on disjoint intervals.

Mathematics Subject Classification 35J20 · 35J92 · 47J30

1 Introduction and main results

This paper deals with the study of weak solutions for a class of nonlinear problems involving the p-Laplacian operator. More specifically, we are concerned with the quasilinear problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \mu h_\mu(x,u) \text{ in } \Omega, \\ \qquad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded regular domain in $\mathbb{R}^N$, $p > 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\lambda > 0$, $\mu \neq 0$ are real parameters and $h_\mu : \Omega \times \mathbb{R} \to \mathbb{R}$ is a family of Carathéodory functions depending on μ.

Our main objective is to provide local hypotheses on the family of functions h_μ that guarantee the existence and multiplicity of solutions for problem (1.1) when the parameters μ and λ are close, respectively, to zero and λ_1, the principal eigenvalue of the operator $-\Delta_p$ with zero boundary conditions (see [6]).

When $p = 2$, problem (1.1) becomes the semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda u + \mu h_\mu(x, u) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

Since the seminal work of Landesman and Lazer [23], problem (1.2) under resonant conditions has been extensively studied (see e.g. [9,11,28,29,31] and their references). Considering, for example, $\lambda = \lambda_1$ and $h_\mu(x, s) = g(s) - f(x)$, with $f \in L^2(\Omega)$ and $g : \mathbb{R} \to \mathbb{R}$ a bounded continuous function with finite limits $g^\pm = \lim_{s\to\pm\infty} g(s)$, a solution of (1.2) exists whenever h satisfies the Landesman–Lazer condition.

$$\left[\int_\Omega (g^- - f)\varphi_1 dx\right]\left[\int_\Omega (g^+ - f)\varphi_1 dx\right] < 0,$$

where φ_1 is a positive eigenfunction associated with λ_1. This result has been extended to the quasilinear problem by Arcoya and Orsina [10] (see also [4,7,9,13] for related results).

In a recent article Rezende et al. [30], considering $h_\mu(x, s) = h_0(x, s)$, for every $\mu \neq 0$, established the existence of a weak solution for problem (1.2), whenever $\mu > 0$ is close to zero and $|\lambda - \lambda_1|/\mu$ is sufficiently small, by supposing a local Landesman–Lazer condition on the interval $(t_1, t_2) \subset \mathbb{R}$:

(H_0^+) $$\int_\Omega h_0(x, t_1\varphi_1)\varphi_1 dx > 0 > \int_\Omega h_0(x, t_2\varphi_1)\varphi_1 dx,$$

or

(H_0^-) $$\int_\Omega h_0(x, t_1\varphi_1)\varphi_1 dx < 0 < \int_\Omega h_0(x, t_2\varphi_1)\varphi_1 dx.$$

We note that one of the characteristics of the results in [30] is that it is not assumed any global growth restriction on the nonlinear term h_0. For the existence of solution under the hypothesis (H_0^+), these authors suppose that h_0 is locally $L^\sigma(\Omega)$-bounded, $\sigma > \max\{N/2, 1\}$. Using an approximation technique, the existence of a solution for the original problem is obtained by finding a local minimum for the functional associated with an appropriated truncation of the nonlinear term h_0. Under the hypothesis (H_0^-), further assuming that h_0 is locally $L^\sigma(\Omega)$-Lipschitz, the solution (derived via Lyapunov–Schmidt reduction method [14,15,24]) is a saddle point of the functional associated with the truncated problem.

We emphasize that the projection on direction of φ_1 of the solution derived in [30] is located in the interval $(t_1\varphi_1, t_2\varphi_1)$. Based on this fact, the existence of multiple solutions for the semilinear problem is established in the mentioned work [30] when the local Landesman–Lazer conditions $(H_0^\pm)$ hold on disjoint open intervals.

In this paper we provide versions of the results established in [30] for the quasilinear problem (1.1) when the family of functions h_μ is uniformly locally $L^\sigma(\Omega)$-bounded:

(H_1) Given $S > 0$, there are $\mu_1 > 0$ and $\eta_S \in L^\sigma(\Omega)$, $\sigma > \max\{N/p, 1\}$, such that

$$|h_\mu(x, s)| \leq \eta_S(x), \text{ for every } |s| \leq S, \text{ a.e. in } \Omega, \text{ for every } \mu \in (0, \mu_1).$$

We also suppose h_μ satisfies the following versions of conditions $(H_0^\pm)$:

Definition 1.1 We shall say that the family of functions h_μ satisfies the local Landesman–Lazer condition (H_μ^+) [respectively, (H_μ^-)] on the interval (t_1, t_2) if there exists a Carathéodory function $h_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

(i) h_0 satisfies (H_0^+) [respectively, (H_0^-)];
(ii) $h_\mu(x, s) \to h_0(x, s_0)$, as $(\mu, s) \to (0, s_0)$, for every $s_0 \in \mathbb{R}$, a.e. in Ω.

Note that hypothesis (H_1) implies that the integrals in (H_0^+) [respectively (H_0^-)] are well defined whenever the family of functions h_μ satisfies (H_μ^+) [respectively (H_μ^-)]. Furthermore, we have that h_μ satisfies (H_0^+) [respectively (H_0^-)] for $\mu > 0$ sufficiently small.

Since there is no global growth restriction on the nonlinearities h_μ, the associated functional may not be well defined in $W_0^{1,p}(\Omega)$. Following the argument employed by Rezende et al. [30], we overcome this difficulty by applying an approximation argument combined with an appropriated truncation of the functions h_μ. We emphasize that hypothesis (H_1) plays an important role in the approximation method used in this article.

In order to state our results, we consider $X = \{v \in W_0^{1,p}(\Omega); \int_\Omega |\nabla\varphi_1|^{p-2}\nabla\varphi_1 \cdot \nabla v dx = 0\}$. We note that X is a topological complement in $W_0^{1,p}(\Omega)$ of the space generated by φ_1. Indeed, take $T_{\varphi_1^{p-1}} : W_0^{1,p}(\Omega) \to \mathbb{R}$, defined by $T_{\varphi_1^{p-1}}(v) = \int_\Omega \varphi_1^{p-1} v dx$, $v \in W_0^{1,p}(\Omega)$, the continuous linear functional in $W_0^{1,p}(\Omega)$ associated with φ_1^{p-1}. By the variational characterization of the eigenvalue λ_1, X is the closed subspace of $W_0^{1,p}(\Omega)$ orthogonal to the functional $T_{\varphi_1^{p-1}}$.

Theorem 1.2 *If h_μ satisfies (H_1) and (H_μ^+) on the interval (t_1, t_2), then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, problem (1.1) has a weak solution $u_\mu = t\varphi_1 + v$, with $t \in (t_1, t_2)$ and $v \in X$.*

We emphasize that, as a direct consequence of Theorem 1.2, we may establish a multiplicity result for (1.1) when (H_μ^+) is satisfied on disjoint open intervals:

Corollary 1.3 *If h_μ satisfies (H_1) and (H_μ^+) on each one of the intervals (t_{2j-1}, t_{2j}), $1 \le j \le k$, $k \ge 2$, with $t_1 < \cdots < t_{2k}$, then there exist positive constants μ^*, ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, problem (1.1) has k solutions $\{u_\mu^1, \cdots, u_\mu^k\}$ such that $u_\mu^j = \tau_j\varphi_1 + v_j$, $\tau_j \in (t_{2j-1}, t_{2j})$, $v_j \in X$, $1 \le j \le k$.*

It is important to observe that it is possible to prove Theorem 1.2 and Corollary 1.3 using, in Definition 1.1, the weaker condition $h_\mu(x, s) \to h_0(x, s)$, as $\mu \to 0$, for every $s \in \mathbb{R}$, a.e. in Ω, instead of (ii). See Remark 2.7.

It is clear that, under the hypotheses of Corollary 1.3, the family of functions h_μ satisfies the condition (H_μ^-) on each interval (t_{2j}, t_{2j+1}), $1 \le j \le k-1$. Hence, based on the results by Rezende et al. [30], we may expect to obtain $(k-1)$ more solutions for problem (1.1) which projections on the φ_1-axis are in the intervals $(t_{2j}\varphi_1, t_{2j+1}\varphi_1)$, $1 \le j \le k-1$.

It is worthwhile mentioning that when dealing with the hypothesis (H_μ^-) for $p \ne 2$, unlike in [30], we may not rely on the Lyapunov–Schmidt reduction method since problem (1.1) involves the quasilinear p-Laplacian operator. In this article the existence of solutions mentioned in the above paragraph are derived by applying the mountain pass theorem [5,29] for functionals associated with aproppiated truncated problems. We note that one of the most important difficulties we face when applying minimax methods is exactly to establish the region where the minimax critical point is located.

In our first result on the direction of obtaining minimax solutions for problem (1.1), we consider that h_μ satisfies (H_μ^+) on two open intervals, one of them contained in $(-\infty, 0)$ and the other in $(0, \infty)$:

Theorem 1.4 *If h_μ satisfies (H_1) and (H_μ^+) on the intervals (t_1, t_2) and (t_3, t_4), with $t_2 < 0 < t_3$, then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, problem (1.1) has three distinct weak solutions $u_\mu^i = \tau_i\varphi_1 + v_i$, with $v_i \in X$, $i = 1, 2, 3$, and $\tau_1 \in (t_1, t_2)$, $\tau_2 \in (t_3, t_4)$ and $\tau_3 \in (t_1, t_4)$.*

We note that the first two solutions u_μ^1 and u_μ^2 are a consequence of Theorem 1.2. The novelty in Theorem 1.4 is the existence of the third solution u_μ^3, which is derived via the mountain pass theorem.

Next we deal with the existence of minimax solutions for problem (1.1) when the local Landesman–Lazer condition is satisfied in one of the semi-axes $(-\infty, 0)$ or $(0, \infty)$.

In our next result we establish the existence of two nonnegative nonzero solutions for problem (1.1):

Theorem 1.5 *If h_μ satisfies $h_\mu(x, 0) \geq 0$ a.e. in Ω, (H_1), (H_μ^-) on the interval (t_1, t_2), with $t_1 > 0$, and (H_μ^+) on the interval (t_2, t_3), then there exist positive constants μ^*, ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, problem (1.1) has two nonnegative nonzero weak solutions $u_\mu^i = \tau_i\varphi_1 + v_i$, with $v_i \in X$, $i = 1, 2$, and $\tau_1 \in (t_1, t_3)$, $\tau_2 \in (t_2, t_3)$.*

We remark that we have a related result providing the existence of two nonpositive solutions for problem (1.1) when h_μ satisfies (H_μ^+) and (H_μ^-) on the intervals (t_1, t_2) and (t_2, t_3), respectively, with $t_3 < 0$.

As an application of the above results we may establish the existence of k nonnegative nonzero solutions for problem (1.1) when the hypotheses (H_μ^-) and (H_μ^+) are satisfied on consecutive open intervals. For example, supposing

$(H_\mu)_k$ h_μ satisfies the item (ii) of Definition 1.1 and there exist $k \in \mathbb{N}$ and $0 < t_1 < t_2 < \cdots < t_k < t_{k+1}$ such that

$$\Big[\int_\Omega h_0(x, t_j\varphi_1)\varphi_1 dx\Big]\Big[\int_\Omega h_0(x, t_{j+1}\varphi_1)\varphi_1 dx\Big] < 0, \quad 1 \leq j \leq k;$$
$$\int_\Omega h_0(x, t_{k+1}\varphi_1)\varphi_1 dx < 0,$$

as a consequence of Theorems 1.2 and 1.5, we may state:

Corollary 1.6 *If h_μ satisfies $h_\mu(x, 0) \geq 0$, a.e. in Ω, (H_1) and $(H_\mu)_k$, then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \nu^*\mu$, problem (1.1) has k nonnegative nonzero solutions $\{u_\mu^1, \cdots, u_\mu^k\}$.*

This paper is organized in the following way: in Sect. 2 we use the Sect. 2.1 to establish a version of Theorem 1.2 under the hypothesis that the family of functions h_μ is $L^\sigma(\Omega)$-bounded. We begin the Sect. 2.2 stating a regularity result—Theorem 2.6—used in the proofs of our main results. After that, we present the proof Theorem 1.2. In the Sect. 3, we reserve the Sect. 3.1 to prove Theorem 1.4. In Sect. 3.2, supposing $h_\mu(x, 0) = 0$ a.e. in Ω and an additional hypothesis, we provide the existence of multiple nontrivial solutions for problem 1.1 (see Theorem 3.2 and Corollary 3.3). The proof of Theorem 1.5 is presented in Sect. 4. Section 5 is reserved for applications of our main results. For the sake of completeness, we present the proof of Theorem 2.6 in the "Appendix".

Throughout this work, for $p > 1$ and $q \in [1, \infty)$, we denote by

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}, \quad \|u\|_q = \left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}} \quad \text{and} \quad \|u\|_\infty = \sup \text{ess}_\Omega |u|,$$

the norms of the spaces $W_0^{1,p}(\Omega)$, $L^q(\Omega)$ and $L^\infty(\Omega)$, respectively. Moreover, for $\alpha \in (0, 1)$, we consider $C^{0,\alpha}(\overline{\Omega})$ equipped with its usual norm $\|\cdot\|_{0,\alpha}$. For $p > 1$ we consider its conjugate $p' = p/(p-1)$ and the Sobolev exponent $p^* = Np/(N-p)$.

2 Existence of a minimum solution

2.1 A version of Theorem 1.2

Before proving Theorem 1.2 we shall first present a version for a related problem with a function f replacing the power $|s|^{p-2}s$ and h_μ satisfying (H_μ^+) and a stronger version of (H_1).

We consider the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) + \mu h_\mu(x, u) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

with the family of functions h_μ satisfying

$(\hat{H}_1)$ there exist $\eta \in L^\sigma(\Omega)$, $\sigma > \max\{N/p, 1\}$ and $\mu_1 > 0$ such that, for every $\mu \in (0, \mu_1)$,

$$|h_\mu(x, s)| \leq \eta(x), \text{ for every } s \in \mathbb{R}, \text{ a. e. in } \Omega.$$

Furthermore we assume that (H_μ^+) holds on $(t_1, t_2) \subset \mathbb{R}$ and that f satisfies

(F_1) there exists an interval $[T_1, T_2] \subset \mathbb{R}$ such that $\{t\varphi_1(x); t_1 \leq t \leq t_2, x \in \Omega\} \subset [T_1, T_2]$ and

(i) $f(s) = |s|^{p-2}s, \ \forall s \in [T_1, T_2]$;
(ii) $|f(s)| \leq |s|^{p-1}, \ \forall s \in \mathbb{R}$.

The functional $I_{\lambda,\mu}$ associated with problem (2.1) is given by

$$I_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega F(u)dx - \mu \int_\Omega H_\mu(x, u)dx, \ \forall u \in W_0^{1,p}(\Omega),$$

where $F(s) = \int_0^s f(\tau)d\tau$ and $H_\mu(x, s) = \int_0^s h_\mu(x, \tau)d\tau$. We note that under the hypotheses $(\hat{H}_1)$ and (F_1), $I_{\lambda,\mu} \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ for every $0 \leq \mu < \mu_1$. Furthermore the critical points of $I_{\lambda,\mu}$ are weak solutions of problem (2.1).

Now we may state a version of Theorem 1.2 for problem (2.1):

Theorem 2.1 *Suppose (H_μ^+), $(\hat{H}_1)$ and (F_1) are satisfied. Then there exist $[\hat{t}_1, \hat{t}_2] \subset (t_1, t_2)$, $\mu^* \in (0, \mu_1)$ and $\nu^* > 0$ such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \nu^*\mu$, problem (2.1) has a weak solution $u_\mu = t_\mu\varphi_1 + v_\mu$, with $t_\mu \in (\hat{t}_1, \hat{t}_2)$ and $v_\mu \in X$.*

As observed in the introduction, for proving Theorem 2.1 we follow a minimization argument based on the following lemma:

Lemma 2.2 *Assume that conditions $(\hat{H}_1)$ and (F_1) are satisfied.*

(*i*) *If* $0 < \mu < \mu_1$ *and* $\lambda \in \mathbb{R}$, *then there is* $u_\mu \in C := \{u = t\varphi_1 + v \,:\, t_1 \le t \le t_2, v \in X\}$ *such that*

$$I_{\lambda,\mu}(u_\mu) = m_C := \inf\{I_{\lambda,\mu}(u) : u \in C\}.$$

Moreover, given $t \in [t_1, t_2]$, *there exists* $v_\mu \in X$ *such that*

$$I_{\lambda,\mu}(t\varphi_1 + v_\mu) = m_t := \inf\{I_{\lambda,\mu}(u) : u = t\varphi_1 + v, v \in X\}.$$

(*ii*) *Given* $\delta > 0$, *there exists* $\mu_0 \in (0, \mu_1)$ *and* $\epsilon_0 > 0$ *such that, if* $0 < \mu < \mu_0$ *and* $|\lambda - \lambda_1| < \epsilon_0$, *then* $\|v\| < \delta$, *for every* $v \in S_t(\mu, \lambda) := \{v \in X : I_{\lambda,\mu}(t\varphi_1 + v) = m_t\}$, *with* $t \in [t_1, t_2]$.

Remark 2.3 By (ii) of the above lemma, $v \in S_t(\mu, \lambda)$ converges strongly to the origin in $W_0^{1,p}(\Omega)$ when $\mu \to 0$ and $\lambda \to \lambda_1$ uniformly in the interval $[t_1, t_2]$.

Proof (i) For every $\mu \in [0, \mu_1)$, by $(\hat{H}_1)$ we have

$$|H_\mu(x, s)| \le \eta(x)|s|, \ \forall\, s \in \mathbb{R}, \ \text{a. e. in } \Omega. \tag{2.2}$$

As a direct consequence of (2.2), $\sigma > \max\{N/p, 1\}$ and the Sobolev imbedding theorem, we obtain $c_1 > 0$ such that

$$\left| \int_\Omega H_\mu(x, u)dx \right| \le c_1\|u\|, \ \forall\, u \in W_0^{1,p}(\Omega), \tag{2.3}$$

whenever $\mu \in [0, \mu_1)$. Similarly, by the boundedness of f it is also possible to find $c_2 > 0$ such that

$$\left| \int_\Omega F(u)dx \right| \le c_2\|u\|, \ \forall\, u \in X. \tag{2.4}$$

By inequalities (2.3) and (2.4), $I_{\lambda,\mu}$ is coercive and bounded from below on $W_0^{1,p}(\Omega)$ for every $0 < \mu < \mu_1$ and $\lambda \in \mathbb{R}$. This together to $(\hat{H}_1)$, the Sobolev imbedding theorem and the fact that $W_0^{1,p}(\Omega)$ is uniformly convex, implies that $I_{\lambda,\mu}$ has a point of minimum in the set C and in the set $t\varphi_1 + X$ for every $t \in [t_1, t_2]$.

(ii) Arguing by contradiction, we suppose that there exist $\delta > 0$ and sequences $(\mu_m) \subset (0, \mu_1)$, $(\lambda_n) \subset \mathbb{R}$, $(t_n) \subset [t_1, t_2]$ and $(v_n) \subset X$ such that

$$\begin{cases} \mu_n \to 0 \text{ and } \lambda_n - \lambda_1 \to 0 \text{ as } n \to \infty, \\ \|v_n\| \ge \delta > 0, \ v_n \in S_{t_n}(\mu_n, \lambda_n), \ \forall\, n \in \mathbb{N}. \end{cases} \tag{2.5}$$

Without loss of generality, we may suppose that $t_n \to t \in [t_1, t_2]$. Furthermore, in view of (2.3), (2.4) and (2.5), we may also assume that $v_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$. Considering Sobolev imbedding theorem and taking a subsequence if necessary, we have

$$\begin{cases} v_n(x) \to v(x) \text{ strongly in } L^r(\Omega), 1 \le r < p^*, \\ v_n(x) \to v(x) \text{ a. e. in } \Omega, \\ |v_n(x)| \le \psi_r(x) \in L^r(\Omega), 1 \le r < p^*, \ \text{a. e. in } \Omega. \end{cases} \tag{2.6}$$

Setting $u_n = t_n\varphi_1 + v_n$, $n \in \mathbb{N}$, from the definition of $S_t(\mu, \lambda)$, we have

$$\langle I'_{\lambda_n,\mu_n}(u_n), v_m - v_n\rangle = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(v_m - v_n)dx - \lambda_n \int_\Omega f(u_n)(v_m - v_n)dx$$
$$-\mu_n \int_\Omega h_{\mu_n}(x, u_n)(v_m - v_n)dx = 0 \tag{2.7}$$

and

$$\langle I_{\lambda_n,\mu_m}(u_m), v_m - v_n\rangle = \int_\Omega |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla(v_m - v_n)dx - \lambda_m \int_\Omega f(u_m)(v_m - v_n)dx$$
$$-\mu_m \int_\Omega h_{\mu_m}(x, u_m)(v_m - v_n)dx = 0. \tag{2.8}$$

The boundedness of f, (2.5) and (2.6) imply that

$$\lambda_n \int_\Omega f(u_n)(v_m - v_n)dx \to 0, \text{ and } \lambda_m \int_\Omega f(u_m)(v_m - v_n)dx \to 0, \text{ as } m, n \to \infty. \tag{2.9}$$

From $(\hat{H}_1)$ and Hölder inequality, we get

$$\Big|\int_\Omega h_{\mu_n}(u_n)(v_m - v_n)dx\Big| \le \int_\Omega \eta|v_m - v_n|dx \le \|\eta\|_{L^\sigma(\Omega)}\|v_m - v_n\|_{L^{\sigma'}(\Omega)}.$$

Since $1 < \sigma' < p^*$, by (2.5) and (2.6) we conclude that

$$\mu_n \int_\Omega h_{\mu_n}(u_n)(v_m - v_n)dx \to 0 \text{ and } \mu_m \int_\Omega h_{\mu_m}(u_m)(v_m - v_n)dx \to 0, \text{ as } m, n \to \infty. \tag{2.10}$$

Subtracting (2.7) of (2.8) and invoking (2.9)–(2.10), we obtain

$$\int_\Omega \left[|\nabla u_m|^{p-2}\nabla u_m - |\nabla u_n|^{p-2}\nabla u_n\right] \cdot \nabla(v_m - v_n)dx \to 0 \text{ as } m, n \to \infty.$$

From the above result, the fact that $(u_m) \subset W_0^{1,p}(\Omega)$ is a bounded sequence and $t_m - t_n \to 0$ as $m, n \to \infty$, we get

$$\begin{aligned}
&\Big|\int_\Omega \left[|\nabla u_m|^{p-2}\nabla u_m - |\nabla u_n|^{p-2}\nabla u_n\right] \cdot \nabla(u_m - u_n)dx\Big| \\
&\le |t_m - t_n|[\|u_m\|^{p-1} + \|u_n\|^{p-1}]\|\varphi_1\| \\
&\quad + \Big|\int_\Omega \left[|\nabla u_m|^{p-2}\nabla u_m - |\nabla u_n|^{p-2}\nabla u_n\right] \cdot \nabla\big(v_m \\
&\quad - v_n\big)dx\Big| \to 0 \text{ as } m, n \to \infty.
\end{aligned}$$

Supposing $p \ge 2$ and using the estimate (see [27])

$$c_p|a - b|^p \le [|a|^{p-2}a - |b|^{p-2}b] \cdot (a - b), \ \forall\, a, b \in \mathbb{R}^N,$$

where c_p denotes positive constant depending on p, we get

$$\int_\Omega |\nabla(u_m - u_n)|^p dx \to 0, \text{ as } m, n \to \infty. \tag{2.11}$$

On the other hand, if $1 < p < 2$, we invoke the estimate (see [27])

$$\frac{c_p|a - b|^2}{(1 + |a| + |b|)^{2-p}} \le (|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b), \ \forall\, a, b \in \mathbb{R}^N,$$

to obtain

$$\lim_{m,n\to\infty}\int_\Omega \frac{|\nabla(u_m-u_n)|^2}{(1+|\nabla u_m|+|\nabla u_n|)^{2-p}}dx=0. \tag{2.12}$$

Hence, applying Hölder inequality with exponents $2/p$ and $2/(2-p)$, we have

$$\int_\Omega |\nabla(u_m-u_n)|^p dx=\int_\Omega \Big[\frac{|\nabla(u_m-u_n)|^p}{(1+|\nabla u_m|+|\nabla u_n|)^{2-p}}\Big]^{\frac{p}{2}}\cdot(1+|\nabla u_m|+|\nabla u_n|)^{\frac{p(2-p)}{2}}dx$$
$$\le\Big[\int_\Omega \frac{|\nabla(u_m-u_n)|^2}{(1+|\nabla u_m|+|\nabla u_n|)^{2-p}}dx\Big]^{\frac{2}{p}}\Big[\int(1+|\nabla u_m|+|\nabla u_n|)^p dx\Big]^{\frac{2-p}{2}}.$$

As a direct consequence of (2.12) and the boundedness of the sequence $(u_n)\subset W_0^{1,p}(\Omega)$, we may conclude that (2.11) also holds for $1<p<2$. From (2.11) and the convergence of (t_n) to t, we may assert the strong convergence in $W_0^{1,p}(\Omega)$ of the sequence (u_n) to $u=t\varphi_1+v$. Using this convergence, (2.2), (2.5) and the fact that the function f is bounded, we obtain that $\|v\|\ge\delta>0$ and

$$I_{\lambda_n,\mu_n}(u_n)=\frac{1}{p}\|u_n\|^p-\lambda_n\int_\Omega F(u_n)dx$$
$$-\mu_n\int_\Omega H_{\mu_n}(x,u_n)dx\longrightarrow\frac{1}{p}\|u\|^p-\lambda_1\int_\Omega F(u)dx,$$

as $n\to\infty$. Since $\|v\|\ge\delta>0$, from (F_1)-(ii), we deduce that

$$\frac{1}{p}\|u\|^p-\lambda_1\int_\Omega F(u)dx\ge\frac{1}{p}\|u\|^p-\frac{\lambda_1}{p}\|u\|_p^p>0,$$

and that $I_{\lambda_n,\mu_n}(t_n\varphi_1+v_n)>0$, for n sufficiently large.

On the other hand, from (2.2), $(t_n)\subset[t_1,t_2]$ and (F_1)-(i), we get

$$I_{\lambda_n,\mu_n}(t_n\varphi_1)=\frac{1}{p}\|t_n\varphi_1\|^p-\frac{\lambda_n}{p}\|t_n\varphi_1\|_p^p$$
$$-\mu_n\int_\Omega H_{\mu_n}(x,t_n\varphi_1)dx\to\frac{1}{p}\|t\varphi_1\|^p-\frac{\lambda_1}{p}\|t\varphi_1\|_p^p=0,$$

and we conclude that $I_{\lambda_n,\mu_n}(t_n\varphi_1)<I_{\lambda_n,\mu_n}(t_n\varphi_1+v_n)$, for n sufficiently large. However, this contradicts $v_n\in S_{t_n}(\mu_n,\lambda_n)$ for $n\in\mathbb{N}$. The proof of Lemma 2.2 is complete. □

In our proof of Theorem 2.1, we shall use the following technical result:

Lemma 2.4 *Suppose the family of functions h_μ satisfies (H_μ) and $(\hat{H}_1)$. Let (u_n) be a sequence in $W_0^{1,p}(\Omega)$ such that $u_n\rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$, as $n\to\infty$. Then*

$$\begin{cases}\int_\Omega\big[H_0(x,u_n(x))-H_0(x,u(x))\big]dx\to0, \text{ as } n\to\infty,\\ \int_\Omega\big[H_\mu(x,u_n(x))-H_0(x,u(x))\big]dx\to0, \text{ as } \mu\to0, n\to\infty.\end{cases} \tag{2.13}$$

Proof First of all we note that without loss of generality we may suppose

$$\begin{cases}u_n(x)\to u(x), \text{ a. e. in } \Omega, \text{ as } n\to\infty,\\ |u_n(x)|\le\psi_r(x) \text{ a. e. in } \Omega, \text{ with } \psi_r\in L^r(\Omega),\ 1\le r<p^*,\ \forall n\in\mathbb{N}.\end{cases} \tag{2.14}$$

The first limit in (2.13) is a consequence of (2.2), (2.14), with $r = \sigma'$, and the Lebesgue dominated convergence theorem. Next, we shall verify the second limit in (2.13). Let (μ_n) any sequence in $(0, \mu_1)$ converging to zero. We claim that

$$H_{\mu_n}(x, u_n(x)) \to H_0(x, u(x)), \text{ as } n \to \infty, \text{ a. e. in } \Omega. \tag{2.15}$$

By the definitions of H_μ and H_0, we have

$$H_{\mu_n}(x, u_n(x)) - H_0(x, u(x)) = \int_0^{u(x)} [h_{\mu_n}(x, s) - h_0(x, s)]ds + \int_{u(x)}^{u_n(x)} h_{\mu_n}(x, s)ds. \tag{2.16}$$

Next we fix $x \in \Omega$ such that $|u(x)| < \infty$ and $\eta(x) < \infty$, with η given by $(\hat{H}_1)$. Hence, by $(\hat{H}_1)$ and (H_μ), for every $n \in \mathbb{N}$,

$$|h_{\mu_n}(x, s) - h_0(x, s)| \le |h_{\mu_n}(x, s)| + |h_0(x, s)| \le 2\eta(x) < \infty, \ \forall\, s \in \mathbb{R}. \tag{2.17}$$

Invoking (H_μ) one more time, the above estimate and the Lebesgue dominated convergence theorem, we obtain

$$\int_0^{u(x)} [h_{\mu_n}(x, s) - h_0(x, s)]ds \to 0, \text{ as } n \to \infty. \tag{2.18}$$

On the other hand, from $(\hat{H}_1)$ and (2.14), we have

$$\left| \int_{u(x)}^{u_n(x)} h_{\mu_n}(x, s)ds \right| \le \eta(x)|u_n(x) - u(x)| \to 0, \text{ as } n \to \infty.$$

Consequently, it follows from (2.16) and (2.18) that (2.15) holds almost everywhere in Ω. The claim is proved.

In view of (2.2) and (2.14), there exists $\psi_{\sigma'} \in L^{\sigma'}(\Omega)$ such that, for every $n \in \mathbb{N}$,

$$\left|H_{\mu_n}(x, u_n(x)) - H_0(x, u(x))\right| \le \eta(x)(|u_n(x)| + |u(x)|) \le 2\eta(x)\psi_{\sigma'}(x) \in L^1(\Omega), \text{ a. e. in } \Omega.$$

By the above estimate, (2.15) and the Lebesgue dominated convergence theorem we have that

$$\int_\Omega [H_{\mu_n}(x, u_n(x)) - H_0(x, u(x))]dx \to 0, \text{ as } n \to \infty.$$

The proof of Lemma 2.4 is complete. □

Now we may present:

Proof of Theorem 2.1: In view of the hypothesis (H_μ^+), the function h_0 satisfies (H_0^+), which together to (2.2) implies that there exist $a, \delta > 0$ and $t_0 \in (t_1 + \delta, t_2 - \delta)$ such that

$$\int_\Omega H_0(x, t_0\varphi_1)dx - \int_\Omega H_0(x, t\varphi_1)dx \ge a > 0, \ \forall\, t \in [t_1, t_1 + \delta] \cup [t_2 - \delta, t_2]. \tag{2.19}$$

We shall verify that there exist $\mu^* \in (0, \mu_1)$ and $\nu^* > 0$ such that, for every $0 < \mu < \mu^*$ and $|\lambda - \lambda_1| < \nu^* \mu$, we obtain

$$I_{\lambda,\mu}(t_0\varphi_1) < I_{\lambda,\mu}(t\varphi_1 + v), \ \forall\, t \in [t_1, t_1 + \delta] \cup [t_2 - \delta, t_2], \ v \in S_t(\mu, \lambda). \tag{2.20}$$

From the definition of the functional $\hat{I}_{\lambda,\mu}$ and (F_1), we have

$$I_{\lambda,\mu}(t_0\varphi_1) = \frac{(\lambda_1 - \lambda)}{p\lambda_1}|t_0|^p - \mu \int_\Omega H_\mu(x, t_0\varphi_1)dx \tag{2.21}$$

and

$$I_{\lambda,\mu}(t\varphi_1+v) \geq \frac{(\lambda_1-\lambda)}{p\lambda_1}\|t\varphi_1+v\|^p - \mu\int_\Omega H_\mu(x,t\varphi_1+v)dx,\ \forall\, t\in[t_1,t_2],\ v\in S_t(\mu,\lambda). \tag{2.22}$$

Considering $a>0$ given by (2.19), we take $0<\epsilon<a/4$ and $0<\nu^*<ap\lambda_1 2^{2(p-1)}/(4T+a)^p$, $T=\max\{|t_1|,|t_2|\}$. □

In view of Lemma 2.2-(ii) and Lemma 2.4, we may find $0<\mu^*<\mu_1$ such that, for every $0<\mu<\mu^*$ and $|\lambda-\lambda_1|<\nu^*\mu$, we have

$$\begin{cases} \left|\int_\Omega \big[H_\mu(x,t\varphi_1+v)-H_0(x,t\varphi_1)\big]dx\right| < \epsilon,\ \forall\, t\in[t_1,t_2],\ v\in S_t(\mu,\lambda), \\ \left|\int_\Omega \big[H_\mu(x,t_0\varphi)-H_0(x,t_0\varphi_1)\big]dx\right| < \epsilon, \\ \|v\|<\epsilon,\ \forall\, v\in S_t(\mu,\lambda),\ t\in[t_1,t_2]. \end{cases}$$

From the above relations, (2.21) and (2.22), we obtain

$$\begin{aligned} I_{\lambda,\mu}(t\varphi_1+v)-I_{\lambda,\mu}(t_0\varphi_1) &\geq \frac{(\lambda_1-\lambda)}{p\lambda_1}\big[\|t\varphi_1+v\|^p-|t_0|^p\big] \\ &\quad -\mu\int_\Omega \big[H_\mu(x,t\varphi_1+v)-H_\mu(x,t_0\varphi_1)\big]dx \\ &\geq -\nu^*\mu\frac{[(|t|+\epsilon)^p+|t_0|^p]}{p\lambda_1} \\ &\quad -\mu\int_\Omega \big[H_\mu(x,t\varphi_1+v)-H_0(x,t\varphi_1)\big]dx \\ &\quad -\mu\int_\Omega \big[H_0(x,t_0\varphi_1)-H_\mu(x,t_0\varphi_1)\big]dx \\ &\quad +\mu\int_\Omega \big[H_0(x,t_0\varphi_1)-H_0(x,t\varphi_1)\big]dx \\ &\geq -\nu^*\mu\frac{[(|t|+\epsilon)^p+|t_0|^p]}{p\lambda_1} - 2\mu\epsilon \\ &\quad +\mu\int_\Omega \big[H_0(x,t_0\varphi_1)-H_0(x,t\varphi_1)\big]dx. \end{aligned}$$

In view of (2.19), $0<\epsilon<a/4$ and $0<\nu^*<ap\lambda_1 2^{2(p-1)}/(4T+a)^p$, if $0<\mu<\mu^*$, $|\lambda-\lambda_1|<\nu^*\mu$, we get

$$I_{\lambda,\mu}(t\varphi_1+v)-I_{\lambda,\mu}(t_0\varphi_1) \geq -\nu^*\frac{4T+a)^p}{p\lambda_1 2^{2(p-1)}}\mu+\mu\frac{a}{2} = \Big[\frac{a}{2}-\frac{\nu^*(4T+a)^p}{p\lambda_1 2^{2p-1}}\Big]\mu>0,$$

whenever $t\in[t_1,t_1+\delta]\cup[t_2-\delta,t_2]$ and $v\in S_t(\mu,\lambda)$. That concludes the verification of (2.20).

Considering $0<\mu<\mu^*$ and $|\lambda-\lambda_1|<\nu^*\mu$, by Lemma 2.2-$(i)$, there exist u_μ such that $I_{\lambda,\mu}(u_\mu)=m_C=m_C(\mu,\lambda)=\inf\{I_{\lambda,\mu}(u);\, u\in C\}$. From (2.20) and the definitions of m_C and m_t in Lemma 2.2-(ii), we get

$$I_{\lambda,\mu}(u_\mu) \leq I_{\lambda,\mu}(t_0\varphi_1) < m_t(\mu,\lambda),\ \forall t\in[t_1,t_1+\delta]\cup[t_2-\delta,t_2].$$

Hence, writing $u_\mu = t_\mu\varphi_1 + v_\mu, v_\mu \in X$, we have that $t_1 + \delta < t_\mu < t_2 - \delta$. This implies that u_μ is a point of minimum local of $I_{\lambda,\mu}$ and, consequently, a weak solution of problem (2.1). The proof of Theorem 2.1 is complete with $\hat{t}_1 = t_1 + \delta$ and $\hat{t}_2 = t_2 - \delta$.

Remark 2.5 We emphasize that the solution u_μ of problem (2.1) given by Theorem 2.1 is actually a point of minimum of the functional $I_{\lambda,\mu}$ in the interior of the cylinder $C = \{u = t\varphi_1 + v; t \in [t_1, t_2], v \in X\}$.

2.2 Proof of Theorem 1.2

Before proving Theorem 1.2, we state a regularity result for the weak solution of the boundary value problem

$$\begin{cases} -\Delta_p u = g(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.23}$$

where Ω is a bounded regular domain of $\mathbb{R}^N$, $N \geq 2$, $1 < p \leq N$ and $g \in L^\sigma(\Omega)$, with $\sigma > N/p$.

Theorem 2.6 *Suppose Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with $\partial\Omega$ satisfying the exterior uniform cone condition. If $1 < p \leq N$ and $g \in L^\sigma(\Omega)$, $\sigma > N/p$, then the weak solution $u \in W_0^{1,p}(\Omega)$ of problem (2.23) is in $C^{0,\alpha}(\overline{\Omega})$, for some exponent $\alpha \in (0, 1)$. Moreover there is a constant $c > 0$ such that $\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq c$, with the exponent α and the constant c depending on Ω and $\|g\|_\sigma$.*

For the convenience of the reader, we present the proof of this result in the "Appendix". It is based on the argument of the proof of Theorem 1.1 of chapter 4 by Ladyzenskaya and Uralt'seva [22] and Remark 2.6 by Arcoya et al. [8]. We also observe that, for $p > N$, Theorem 2.6 follows directly from the Sobolev imbedding $W_0^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha < 1 - N/p$.

Proof of Theorem 1.2: Considering $T = \max\{|t_1|, |t_2|\}$ and $R > T\|\varphi_1\|_\infty > 0$, we take a function $\chi \in C(\mathbb{R}, [0, 1])$ such that $\chi(s) \equiv 1$, if $|s| \leq R$, and $\chi(s) = 0$, if $|s| \geq R + 2$. Next we define the truncated functions f_R and $h_{\mu,R}$ for $0 \leq \mu < \mu_1$, by

$$\begin{cases} f_R(s) = |s|^{p-2}s\chi(s), & \text{for every } s \in \mathbb{R} \\ h_{\mu,R}(x, s) = h_\mu(x, s)\chi(s), & \text{for every } (x, s) \in \Omega \times \mathbb{R}. \end{cases} \tag{2.24}$$

Associated with f_R and $h_{\mu,R}$, we have the problem

$$\begin{cases} -\Delta_p u = \lambda f_R(u) + \mu h_{\mu,R}(x, u), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.25}$$

From (H_1) and the definition (2.24) we may assert that the family of functions $h_{\mu,R}$ satisfies $(\hat{H}_1)$. Moreover, as $\|t_i\varphi_1\|_\infty < R, i = 1, 2$, it follows from (2.24) and $({H_\mu}^+)$ that $h_{\mu,R}$ satisfies $({H_\mu}^+)$ on the interval (t_1, t_2). We also note that f_R satisfies (F_1) with $[T_1, T_2] = [-R, R]$. Applying Theorem 2.1 we find $[\hat{t}_1, \hat{t}_2] \subset (t_1, t_2)$, $\mu^* \in (0, \mu_1)$ and $\nu^* > 0$ such that, for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \mu\nu^*$, problem (2.25) has a weak solution $u_\mu = t_\mu\varphi_1 + v_\mu$, with $t_\mu \in (\hat{t}_1, \hat{t}_2)$ and $v_\mu \in X$.

To conclude the proof of Theorem 1.2 we verify that $\lim_{\mu\to 0} \|u_\mu\|_\infty < R$. Indeed, let $(\mu_n) \subset (0, \mu^*)$ and $(\lambda_n) \subset \mathbb{R}$ be sequences such that $\mu_n \to 0$, as $n \to \infty$ and $|\lambda_n - \lambda_1| < \mu_n\nu^*$, for every $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, we write $u_n := u_{\mu_n} = t_{\mu_n}\varphi_1 + v_{\mu_n}, v_{\mu_n} \in X$

and $t_{\mu_n} \in (\hat{t}_1, \hat{t}_2)$. Setting $g_n(x) = \lambda_n f_R(u_n(x)) + \mu_n h_{\mu_n,R}(x, u_n(x))$ for $x \in \Omega$, by the boundedness of f_R and $(\hat{H}_1)$, we may find $\psi \in L^\sigma(\Omega), \sigma > \max\{N/p, 1\}$, such that, for every $n \in \mathbb{N}$, $|g_n(x)| \leq \psi(x)$, a.e. in Ω.

Hence, by the compactness of the inverse of the p-Laplacian operator $(-\Delta_p)^{-1}$: $L^\sigma(\Omega) \to W_0^{1,p}(\Omega)$, we may find a subsequence (u_{n_k}) of (u_n) such that $u_{n_k} \to u$ strongly in $W_0^{1,p}(\Omega)$, as $k \to \infty$. Furthermore, invoking Theorem 2.6 when $p \leq N$ and the Sobolev imbedding theorem for $p > N$, we may assume that $u_{n_k} \to u$ strongly in $C(\overline{\Omega})$, as $n \to \infty$. Using this last result, $(\hat{H}_1)$ and that $\mu_n \to 0$ and $\lambda_n \to \lambda_1$, as $n \to \infty$, we obtain that u is a solution of the problem

$$\begin{cases} -\Delta_p u = \lambda_1 f_R(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Since $|f_R(s)| \leq \lambda_1 |s|^{p-1}$, we get that actually $u = t\varphi_1$. Moreover, observing that $(t_{\mu n}) \subset [\hat{t}_1, \hat{t}_2] \subset (t_1, t_2)$, we have $t \in (t_1, t_2)$ and, consequently, $\lim_{k\to\infty} \|u_{n_k}\|_\infty < R$. The proof of Theorem 1.2 is complete. □

Remark 2.7 Note that Theorem 1.2 holds if we suppose

$$(\hat{ii})\ h_\mu(x, s) \to h_0(x, s), \text{ for every } s \in \mathbb{R}, \text{ a.e. in } \Omega, \text{ as } \mu \to 0,$$

instead of condition $(H_\mu) - (ii)$. Indeed, this last condition has been used in the proof of the Lemma 2.4 to verify (2.17) and (2.18), which are satisfied if we assume $(\hat{ii})$.

Remark 2.8 Note that to prove Theorem 1.2 we could apply Theorem 2.1 to any function f satisfying (F_1) and to any family of functions $\hat{h}_\mu$ satisfying $(\hat{H}_1)$ and

$$\hat{h}_\mu(x, s) = h_\mu(x, s), \text{ a.e. in } \Omega, \text{ for } 0 \leq \mu < \mu_1, \tag{2.26}$$

for every $s \in [T_1, T_2]$.

Moreover, we observe that the solution $u_\mu \in W_0^{1,p}(\Omega)$ of problem (1.1) provided by Theorem 2.1 satisfies $u_\mu = t_\mu\varphi_1 + v_\mu$, $t_\mu \in (\hat{t}_1, \hat{t}_2)$, $v_\mu \in X$ and

$$I_{\lambda,\mu}(u_\mu) < \min\{I_{\lambda,\mu}(t_1\varphi_1), I_{\lambda,\mu}(t_2\varphi_1)\},$$

where $I_{\lambda,\mu}$ is the associated functional.

3 A first result on the existence of a minimax solution

3.1 Proof of Theorem 1.4

We begin the proof of Theorem 1.4 by considering appropriated truncations of the family of functions h_μ and the power $|s|^{p-2}s$: considering $T_1 := t_1\|\varphi_1\|_\infty < 0 < T_2 := t_4\|\varphi_1\|_\infty$, we take $\chi \in C(\mathbb{R}, [0, 1])$ to be a function satisfying $\chi(s) = 1$, if $s \in [T_1, T_2]$, $\chi(s) = 0$, if $s \in (-\infty, T_1 - 1] \cup [T_2 + 1, \infty)$, and $0 < \chi(s) < 1$, otherwise.

Next, given $\mu \in [0, \mu_1)$, we define $\hat{f} : \mathbb{R} \to \mathbb{R}$ and $\hat{h}_\mu : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{cases} \hat{f}(s) = |s|^{p-2}s\chi(s), & \text{for every } s \in \mathbb{R} \\ \hat{h}_\mu(x, s) = h_\mu(x, s)\chi(s), & \text{for every } (x, s) \in \Omega \times \mathbb{R}. \end{cases} \tag{3.1}$$

Associated with the family of functions $\hat{h}_\mu$ and the function $\hat{f}$, we consider the quasilinear problem

$$\begin{cases} -\triangle_p u = \lambda \hat{f}(u) + \mu \hat{h}_\mu(x,u), \text{ in } \Omega, \\ \quad u = 0, \text{ on } \partial\Omega. \end{cases} \tag{3.2}$$

In view of the definitions (3.1), we get that $\hat{h}_\mu$ satisfies $(\hat{H}_1)$ and

$$\begin{cases} \hat{f}(s) = |s|^{p-2}s, \text{ for every } s \in [T_1, T_2] \\ |\hat{f}(s)| < |s|^{p-1}, \text{ for every } s \in (-\infty, T_1) \cup (T_2, \infty). \end{cases} \tag{3.3}$$

Furthermore, noting that $\{t\varphi_1(x); t_1 \le t \le t_4, x \in \Omega\} \subset [T_1, T_2]$, we have that $\hat{f}$ satisfies (F_1) and $\hat{h}_\mu$ satisfies (2.26) with respect to the intervals (t_1, t_2) and (t_3, t_4). Consequently, by Theorem 1.2 and Remark 2.8, we find $[\hat{t}_1, \hat{t}_2] \subset (t_1, t_2)$, $[\hat{t}_3, \hat{t}_4] \subset (t_3, t_4)$, $\hat{\mu}^* \in (0, \mu_1)$ and $\hat{\nu}^* > 0$ such that, for every $\mu \in (0, \hat{\mu}^*)$ and $|\lambda - \lambda_1| < \hat{\nu}^*\mu$, problem (1.1) has two weak solutions $u_\mu^i = \tau_i\varphi_1 + v_i$, $v_i \in X$, $i = 1, 2$, and $\tau_1 \in (t_1, t_2)$, $\tau_2 \in (t_3, t_4)$. Moreover, considering the functional associated with problem (3.2), $\hat{I}_{\lambda,\mu} \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, defined by

$$\hat{I}_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega \hat{F}(u)dx - \mu \int_\Omega \hat{H}_\mu(x,u)dx, \ \forall\, u \in W_0^{1,p}(\Omega), \tag{3.4}$$

where $\hat{F}(s) := \int_0^s f(\tau)d\tau$ and $\hat{H}_\mu(x,s) := \int_0^s \hat{h}_\mu(x,\tau)d\tau$, we have that u_μ^1 and u_μ^2 are point of minimum of the functional $\hat{I}_{\lambda,\mu}$ and satisfy

$$\begin{cases} \hat{I}_{\lambda,\mu}(u_\mu^1) < \min\{\hat{I}_{\lambda,\mu}(t_1\varphi_1), \hat{I}_{\lambda,\mu}(t_2\varphi_1)\} \\ \hat{I}_{\lambda,\mu}(u_\mu^2) < \min\{\hat{I}_{\lambda,\mu}(t_3\varphi_1), \hat{I}_{\lambda,\mu}(t_4\varphi_1)\}. \end{cases} \tag{3.5}$$

Our next task is to derive a third solution u_μ^3 for problem (3.2) via a minimax theorem. We first verify the geometric properties that are necessary for applying the mountain pass theorem:

Lemma 3.1 *Suppose h_μ satisfies the hypotheses of Theorem 1.4. Then there exist $t_0 \in (t_2, t_3)$, $\mu_1^* \in (0, \hat{\mu}^*)$ and $\nu_1^* \in (0, \hat{\nu}^*)$ such that $\min_{v\in X} \hat{I}_{\lambda,\mu}(t_0\varphi_1 + v) > \max\{\hat{I}_{\lambda,\mu}(t_2\varphi_1), \hat{I}_{\lambda,\mu}(t_3\varphi_1)\}$, for every $\mu \in (0, \mu_1^*)$ and $|\lambda - \lambda_1| < \nu_1^*\mu$.*

Proof Without loss of generality, we suppose that $\hat{I}_{\lambda,\mu}(t_2\varphi_1) = \max\{\hat{I}_{\lambda,\mu}(t_2\varphi_1), \hat{I}_{\lambda,\mu}(t_3\varphi_1)\}$. In view of the hypothesis (H_μ^-) on the interval (t_2, t_3) and $(\hat{H}_1)$, we find $\hat{a}, \delta > 0$ and $t_0 \in (t_2 + \delta, t_3 - \delta)$ such that

$$\int_\Omega \hat{H}_0(x, t\varphi_1)dx - \int_\Omega \hat{H}_0(x, t_0\varphi_1)dx > \hat{a} > 0, \ \forall\, t \in [t_2, t_2 + \delta] \cup [t_3 - \delta, t_3]. \tag{3.6}$$

Setting $\hat{S}_t(\mu, \lambda) = \{v \in X; \hat{I}_{\lambda,\mu}(t\varphi_1 + v) = \inf_{z\in X} \hat{I}_{\lambda,\mu}(t\varphi_1 + z)\}$, $t \in [t_2, t_3]$, by (3.3) we have

$$\hat{I}_{\lambda,\mu}(t_0\varphi_1+v) \ge \frac{1}{p}\frac{(\lambda_1-\lambda)}{\lambda_1}\|t_0\varphi_1+v\|^p - \mu\int_\Omega \hat{H}_\mu(x, t_0\varphi_1+v)dx, \text{ for every } v \in \hat{S}_{t_0}(\mu, \lambda), \tag{3.7}$$

and

$$\hat{I}_{\lambda,\mu}(t_2\varphi_1) = \frac{1}{p}\frac{(\lambda_1-\lambda)}{\lambda_1}|t_2|^p - \mu\int_\Omega \hat{H}_\mu(x, t_2\varphi_1)dx. \tag{3.8}$$

Now, given $0 < \epsilon < \hat{a}/4$, we may apply Lemma 2.4 to find $\mu_1^* \in (0, \hat{\mu}^*)$ and $\nu_1^* \in (0, \hat{\nu}^*)$ such that, for every $0 < \mu < \mu_1^*$ and $|\lambda - \lambda_1| < \nu_1^* \mu$, we obtain

$$\begin{cases} \|v\| < \epsilon, \ \forall\, v \in \hat{S}_{t_0}(\mu, \lambda), \\ \Big| \int_\Omega [\hat{H}_\mu(x, t_0\varphi_1 + v) - \hat{H}_0(x, t_0\varphi_1)]dx \Big| < \epsilon, \ \forall\, v \in \hat{S}_{t_0}(\mu, \lambda), \\ \Big| \int_\Omega [\hat{H}_\mu(x, t_2\varphi_1) - \hat{H}_0(x, t_2\varphi_1)]dx \Big| < \epsilon. \end{cases}$$

Consequently, from (3.6)–(3.8), for every $0 < \mu < \mu_1^*$, $|\lambda - \lambda_1| < \nu^* \mu$ and $v \in \hat{S}_{t_0}(\mu, \lambda)$, we get

$$\begin{aligned} \hat{I}_{\lambda,\mu}(t_0\varphi_1 + v) - \hat{I}_{\lambda,'\mu}(t_2\varphi_1) &\geq \frac{1}{p}\frac{(\lambda_1 - \lambda)}{\lambda_1}\|t_0\varphi_1 + v\|^p \\ &\quad -\mu\Big[\int_\Omega [\hat{H}_\mu(x, t_0\varphi_1 + v) - \hat{H}_0(x, t_0\varphi_1)]dx\Big] \\ &\quad -\frac{1}{p}\frac{(\lambda_1 - \lambda)}{\lambda_1}|t_2|^p + \mu\Big[\int_\Omega [\hat{H}_\mu(x, t_2\varphi_1) - \hat{H}_0(x, t_2\varphi_1)dx\Big] \\ &\quad +\mu\Big[\int_\Omega [\hat{H}_0(x, t_2\varphi_1) - \hat{H}_0(x, t_0\varphi_1)dx\Big] \\ &\geq -\frac{1}{p}\frac{\nu_1^*}{\lambda_1}\Big[2\big(|\hat{T}| + \frac{\hat{a}}{4}\big)^p\Big]\mu - 2\frac{\hat{a}}{4}\mu + \mu\hat{a} \\ &= \Big[-\frac{1}{p}\frac{\nu_1^*}{\lambda_1}\frac{(4\hat{T} + \hat{a})^p}{2^{2p-1}} + \frac{\hat{a}}{2}\Big]\mu, \end{aligned}$$

where $\hat{T} = \max\{|t_2|, |t_3|\}$. Assuming further that $0 < \nu_1^* < p\lambda_1\hat{a}2^{2(p-1)}/(4\hat{T} + \hat{a})^p$, we obtain $\hat{I}_{\lambda,\mu}(t_0\varphi_1 + v) > \hat{I}_{\lambda,\mu}(t_2\varphi_1) = \max\{\hat{I}_{\lambda,\mu}(t_2\varphi_1), \hat{I}_{\lambda,\mu}(t_3\varphi_1)\}$. The proof of Lemma 3.1 is complete. □

Next we present:

Proof of Theorem 1.4: Since $\hat{h}_\mu$ satisfies (H_μ^+) on (t_1, t_2) and (t_3, t_4), we find $a > 0$ and $0 < \delta < \min\{t_2 - t_1, t_4 - t_3\}$ such that

$$\begin{cases} \int_\Omega \hat{h}_0(x, t\varphi_1)\varphi_1 dx < -a < 0, \ \forall\, t \in [t_4 - \delta, t_4], \\ \int_\Omega \hat{h}_0(x, t\varphi_1)\varphi_1 dx > a > 0, \ \forall\, t \in [t_1, t_1 + \delta]. \end{cases} \tag{3.9}$$

Given $\mu \in (0, \mu_1^*)$ and $|\lambda - \lambda_1| < \nu_1^* \mu$, μ_1^* and ν_1^* given by Lemma 3.1, we define

$$c_{\lambda,\mu} = \inf_{\gamma \in \Gamma} \max_{0 \leq s \leq 1} \hat{I}_{\lambda,\mu}(\gamma(s)), \tag{3.10}$$

where

$$\Gamma = \{\gamma \in C([0, 1], W_0^{1,p}(\Omega));\ \gamma(0) = t_2\varphi_1, \gamma(1) = t_3\varphi_1\}. \tag{3.11}$$

Since $\hat{f}$ is a bounded function and $(\hat{h}_\mu)$ satisfies $(\hat{H}_1)$, we may assert that $\hat{I}_{\lambda,\mu}$ satisfies the Palais-Smale condition. Hence, invoking Lemma 3.1 and the mountain pass theorem [5] (see also [29]), $\hat{I}_{\lambda,\mu}$ has a critical point $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\hat{I}_{\lambda,\mu}(u_\mu) = c_{\lambda,\mu} > \max\{\hat{I}_{\lambda,\mu}(t_2\varphi_1), \hat{I}_{\lambda,\mu}(t_3\varphi_1)\}. \tag{3.12}$$

Take $0 < \nu^* < \min\{\nu_1^*, \lambda_1 a/T^{p-1}\}$, with $T = \max\{|t_1|, |t_2|\}$ and a given by (3.9). We claim that there exists $\mu_1^* \in (0, \mu^*)$ such that, for every $\mu \in (0, \mu^*)$, $|\lambda - \lambda_1| < \nu^*\mu$, we must have

$$\begin{cases} u_\mu = t_\mu\varphi_1 + v_\mu, \text{ with } t_1 + \delta < t_\mu < t_4 - \delta, \; v_\mu \in X; \\ T_1 < u(x) < T_2, \text{ a.e. in } \Omega. \end{cases}$$

For proving such claim it suffices to verify that given sequences $(\mu_n) \subset (0, \mu_1^*)$ and $(\lambda_n) \subset \mathbb{R}$ such that $\mu_n \to 0$, as $n \to \infty$, and $|\lambda_n - \lambda_1| < \nu^*\mu_n$, for every $n \in \mathbb{N}$, we may find, up to a subsequence, $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$\begin{cases} u_n := u_{\mu_n} = t_{\mu_n}\varphi_1 + v_{\mu_n}, \text{ with } t_{\mu_n} \in (t_1 + \delta, t_4 - \delta), \; v_{\mu_n} \in X; \\ T_1 < u_{\mu_n}(x) < T_2, \text{ a.e. in } \Omega. \end{cases} \tag{3.13}$$

Setting $g_n(x) = \lambda_n \hat{f}(u_n(x)) + \mu_n \hat{h}_{\mu_n}(x, u_n(x))$, for every $x \in \Omega$, and arguing as in the proof of Theorem 1.2, we may suppose that there exists $t \in \mathbb{R}$ such that

$$\begin{cases} u_n \to t\varphi_1, \text{ strongly in } W_0^{1,p}(\Omega), \text{ as } n \to \infty; \\ u_n \to t\varphi_1, \text{ strongly in } C(\overline{\Omega}), \text{ as } n \to \infty. \end{cases} \tag{3.14}$$

Moreover $-\triangle_p(t\varphi_1) = \lambda_1 \hat{f}(t\varphi_1)$ in Ω. Hence

$$\lambda_1 \int_\Omega \hat{f}(t\varphi_1)t\varphi_1 dx = \lambda_1 \int_\Omega |t|^p \varphi_1^p dx.$$

Note that the above relation implies that $t \in [t_1, t_4]$. Effectively, if $t \notin [t_1, t_4]$, by (3.3) and our choices of T_1 and T_2, we have that the set $E = \{x \in \Omega; |\hat{f}(t\varphi_1(x))| < |t|^{p-1}|\varphi_1(x)|^{p-1}\}$ has positive measure and, consequently,

$$\lambda_1 \int_\Omega \hat{f}(t\varphi_1)t\varphi_1 dx < \lambda_1 \int_\Omega |t|^p \varphi_1^p dx.$$

Next we assert that actually $t \in (t_1 + \delta, t_4 - \delta)$. Indeed, if we suppose otherwise, by (3.9) and the first limit in (3.14), we obtain

$$-\int_\Omega \hat{h}_{\mu_n}(x, u_n)u_n dx \to -t \int_\Omega \hat{h}_0(x, t\varphi_1)\varphi_1 dx > |t|a > 0. \tag{3.15}$$

From (3.3) and $|\lambda_n - \lambda_1| < \nu^*\mu_n$, we may write

$$\begin{aligned} 0 = \langle \hat{I}'_{\lambda_n,\mu_n}(u_n), u_n\rangle &\geq \|u_n\|^p - \lambda_n\|u_n\|_p^p - \mu_n \int_\Omega \hat{h}_{\mu_n}(x, u_n)u_n dx \\ &\geq \Big[-\frac{\nu^*\|u_n\|^p}{\lambda_1} - \int_\Omega \hat{h}_{\mu_n}(x, u_n)u_n dx \Big]\mu_n. \end{aligned}$$

The above inequality, (3.14) and (3.15) imply that $0 \geq -\nu^*|t|^p/\lambda_1 + |t|a$. However, this inequality contradicts $t \neq 0$ and $0 < \nu^* < \lambda_1 a/T^{p-1}$.

Using the above assertion and (3.14), we conclude that, up to subsequence, there exists $n_0 \in \mathbb{N}$ such that (3.13) holds for every $n \geq n_0$. The claim is proved.

From the above claim and definitions (3.1), we have that for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \nu^*\mu$, $u_\mu = t_\mu\varphi_1 + v_\mu$ is a solution of problem (1.1) such that $t_1 < t_\mu < t_4$. Furthermore, by (3.5) and (3.12), $u_\mu \notin \{u_\mu^1, u_\mu^2\}$, where u_μ^1 and u_μ^2 are the solutions of problem (1.1) found by applying Theorem 1.2 in the intervals (t_1, t_2) and (t_3, t_4), respectively. The proof of Theorem 1.4 is complete. □

3.2 Existence of nontrivial solutions

In this subsection we suppose that the function h_μ satisfies the following hypothesis:

(H_2) there exist $\eta_1 \in L^{\sigma_1}(\Omega)$, $\sigma_1 > \max\{\frac{N}{p}, 1\}$, such that

$$h_\mu(x,s) = o(\eta_1 |s|^{p-1}), \text{ as } s \to 0 \text{ uniformly a.e. in } \Omega \text{ and } \mu \in [0, \mu_1).$$

Note that condition (H_2) implies that $u = 0$ in Ω is a trivial solution of problem (1.1). In our first result we establish the existence of an additional nontrivial solution under the above condition and the hypotheses of Theorem 1.2.

Theorem 3.2 *Suppose h_μ satisfies (H_1), (H_2) and (H_μ^+) on the interval (t_1, t_2), with $t_1 > 0$. Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $\lambda \in (\lambda_1 - \nu^*\mu, \lambda_1)$, problem (1.1) has two nonnegative nontrivial solutions $u_\mu^i = \tau_i\varphi_1 + v_i$, $v_i \in X$, $i = 1, 2$, $\tau_1 \in (t_1, t_2)$ and $\tau_2 \in (0, t_2)$.*

As a direct consequence of the above result, we may state:

Corollary 3.3 *Suppose h_μ satisfies the hypotheses of Theorem 1.4 and (H_2). Then there exist positive constants μ^* and ν^* such that, for every $\mu \in (0, \mu^*)$ and $\lambda \in (\lambda_1 - \nu^*\mu, \lambda_1)$, problem (1.1) has four nontrivial solutions $u_\mu^i = \tau_i\varphi_1 + v_i$, $v_i \in X$, $i = 1, 2, 3, 4$, $\tau_1 \in (t_3, t_4)$, $\tau_2 \in (0, t_4)$, $\tau_3 \in (t_1, t_2)$ and $\tau_4 \in (t_1, 0)$*

Proof of Theorem 3.2: Since we are looking for nonnegative solutions we use a slightly different argument of the one employed in the proof of Theorem 1.5. Consider $T_1 = 0$, $T_2 = t_2\|\varphi_1\|_\infty$ and $\chi \in C(\mathbb{R}, [0,1])$ such that $\chi(s) = 1$ if $s \le T_2$, $\chi(s) = 0$ if $s \ge T_2 + 1$, and $0 < \chi(s) < 1$ otherwise. Then we define

$$\begin{cases} \hat{f}(s) = (s^+)^{p-1}\chi(s), \text{ for every } s \in \mathbb{R} \\ \hat{h}_\mu(x,s) = h_\mu(x, s^+)\chi(s), \text{ for every } (x,s) \in \Omega \times \mathbb{R}. \end{cases} \tag{3.16}$$

We also consider the associated quasilinear problem

$$\begin{cases} -\Delta_p u = \lambda \hat{f}(u) + \mu \hat{h}_\mu(x,u), \text{ in } \Omega, \\ \quad u = 0, \text{ on } \partial\Omega. \end{cases} \tag{3.17}$$

By definition (3.16) and (H_1) we have that h_μ satiusfies $(\hat{H}_1)$. We also have that $\hat{f}$ satisfies (F_1) and $\hat{h}_\mu$ satisfies (2.26) with respect to the interval (t_1, t_2) since $\{t\varphi_1(x); t_1 \le t \le t_2, x \in \Omega\} \subset [T_1, T_2]$. Applying Theorem 2.1 and Remark 2.8, we find $(\hat{t}_1, \hat{t}_2) \subset (t_1, t_2)$, $\hat{\mu}^* \in (0, \mu_1)$ and $\hat{\nu}^* > 0$ such that, for every $\mu \in (0, \hat{\mu}^*)$ and $|\lambda - \lambda_1| < \hat{\nu}^*\mu$, problem (1.1) has a weak nonnegative solution $u_\mu^1 = \tau_1\varphi_1 + v_1$, $v_1 \in X$ and $\tau_1 \in (t_1, t_2)$. Moreover,

$$\hat{I}_{\lambda,\mu}(u_\mu^1) < \min\{\hat{I}_{\lambda,\mu}(t_1\varphi_1 + v); v \in X\}, \tag{3.18}$$

where $\hat{I}_{\lambda,\mu} \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ is the functional

$$\hat{I}_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega \hat{F}(u)dx - \mu \int_\Omega \hat{H}_\mu(x,u)dx, \ \forall\, u \in W_0^{1,p}(\Omega), \tag{3.19}$$

with $\hat{F}(s) := \int_0^s \hat{f}(\tau)d\tau$ and $\hat{H}_\mu(x,s) := \int_0^s \hat{h}_\mu(x,\tau)d\tau$, for every $(x,s) \in \Omega \times \mathbb{R}$.

As in the proof of Theorem 1.5, the second nonnegative solution u_μ^2 for problem (1.1) is derived by applying the mountain pass theorem. Fixing $\mu \in (0, \hat{\mu}^*)$ and $\lambda \in (\lambda_1 - \hat{\nu}^*\mu, \lambda_1)$, we claim that there exist $\alpha, \rho > 0$ such that

$$\hat{I}_{\lambda,\mu}(u) \geq \alpha\|u\|^p, \ \forall\, u \in B_\rho(0). \tag{3.20}$$

Given $\epsilon > 0$, by (H_2), $(\hat{H}_1)$ and using the fact that $p^*/\sigma' > p$, there is $A > 0$ such that

$$|\hat{H}_\mu(x,s)| \leq \frac{\epsilon}{p}|\eta_1||s|^p + A\eta(x)|s|^{\frac{p^*}{\sigma'}}, \ \forall\, s \in \mathbb{R}, \text{ a.e. in } \Omega,$$

with η and η_1 given by $(\hat{H}_1)$ and (H_2), respectively.

From the above inequality and (3.16), we get

$$\hat{I}_{\lambda,\mu}(u) \geq \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\|u\|_p^p - \frac{\mu}{p}\epsilon\int_\Omega |\eta_1||s|^p dx - A\int_\Omega \eta(x)|u|^{\frac{p^*}{\sigma'}}dx.$$

By Hölder inequality, $\sigma, \sigma_1 > N/p$ and the Sobolev imbedding theorem, we find $c_1, c_2 > 0$ such that

$$\hat{I}_{\lambda,\mu}(u) \geq \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\|u\|_p^p - \frac{\mu}{p}\epsilon c_1\|\eta_1\|_{\sigma_1}\|u\|^p - Ac_2\|\eta\|_\sigma\|u\|^{\frac{p^*}{\sigma'}}.$$

Hence, using that $\|u\|_p^p < 1/\lambda_1\|u\|^p$, we get

$$\hat{I}_{\lambda,\mu}(u) \geq \frac{1}{p}\Big[\frac{(\lambda_1-\lambda)}{\lambda_1} - \mu\epsilon c_1\|\eta_1\|_{\sigma_1} - Apc_2\|\eta\|_\sigma\|u\|^{(\frac{p^*}{\sigma'}-p)}\Big]\|u\|^p.$$

Noting that $(\lambda_1 - \lambda) > 0$, we may choose $\epsilon > 0$ such that $(\lambda_1-\lambda)/\lambda_1 - \mu\epsilon c_1\|\eta_1\|_{\sigma_1} > 0$. Next, since $p^*/\sigma' - p > 0$, we may find $\alpha, \rho > 0$ such that $\hat{I}_{\lambda,\mu}$ satisfies (3.20). The claim is proved.

Now we define

$$\hat{c}_{\lambda,\mu} = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1}\hat{I}_{\lambda,\mu}(\gamma(t)), \tag{3.21}$$

where

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega));\ \gamma(0) = 0, \gamma(1) = u_\mu^1\}. \tag{3.22}$$

By (3.18) and (3.20),

$$\hat{c}_{\lambda,\mu} > \max\{\hat{I}_{\lambda,\mu}(0), \hat{I}_{\lambda,\mu}(u_\mu^1)\}. \tag{3.23}$$

Since $\hat{I}_{\lambda,\mu}$ satisfies the (PS) condition, there exist u_μ, a critical point of $\hat{I}_{\lambda,\mu}$, such that $\hat{I}_{\lambda,\mu}(u_\mu) = \hat{c}_{\lambda,\mu}$. From (3.23), $u_\mu \neq \{0, u_\mu^1\}$. Moreover, by (3.16) and (3.17), $u_\mu \geq 0$ a.e. in Ω.

Since h_μ satisfies (H_μ^+) on (t_1, t_2), we find $a > 0$ and $0 < \delta < t_2 - t_1$ such that

$$\int_\Omega \hat{h}_0(x, t\varphi_1)\varphi_1 dx < -a < 0, \ \forall\, t \in [t_2 - \delta, t_2]. \tag{3.24}$$

Take $0 < \nu^* < \min\{\hat{\nu}^*, \lambda_1 a/t_2\}$. Let $(\mu_n) \subset (0, \hat{\mu}^*)$ and $(\lambda_n) \in \mathbb{R}$ such that $\mu_n \to 0$ as $n \to \infty$ and $|\lambda_n - \lambda_1| < \nu^*\mu_n$, for every $n \in \mathbb{N}$. For proving Theorem 3.2, we must verify that, up to a subsequence, there is $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$\begin{cases} u_n := u_{\mu_n} = \tau_n\varphi_1 + v_n, \tau_n \in (0, t_2),\ v_n \in X; \\ 0 \leq u_n(x) < T_2, \text{ a.e. in } \Omega. \end{cases} \tag{3.25}$$

Setting $g_n(x,s) = \lambda_n \hat{f}(s) + \mu_n \hat{h}_{\mu_n}(x,s)$, for every $(x,s) \in \Omega \times \mathbb{R}$, we may argue as in the proof of Theorem 1.5 to conclude that (3.14) holds with $t \in [0, t_2]$. Actually we note that $t \in [0, t_2 - \delta)$. Indeed, if we suppose otherwise, by (3.24) and (3.14), we obtain

$$-\int_\Omega \hat{h}_{\mu_n}(x, u_n) u_n dx \to -t \int_\Omega \hat{h}_0(x, t\varphi_1)\varphi_1 dx > ta > 0.$$

However this may not occur since, by (3.16) and $\lambda_n < \lambda_1$,

$$\int_\Omega \hat{h}_{\mu_n}(x, u_n) u_n dx \leq \frac{1}{\mu_n} \langle \hat{I}'_{\lambda_n, \mu_n}(u_n), u_n \rangle = 0.$$

The fact that $\tau_n < t_2$ if n is sufficiently large, follows from $t < t_2 - \delta$ and the first relation in (3.14). We also note that $u_n \geq 0$, $u_n \not\equiv 0$ in Ω implies that $\tau_n > 0$.

The second relation in (3.25) follows by the second limit in (3.14). The Theorem 3.2 is proved. □

4 A second result on the existence of a minimax solution

4.1 Proof of Theorem 1.5

Following the argument used in Sects. 2 and 3, we begin the proof of Theorem 1.5 by choosing appropriated perturbations of the functions $|s|^{p-1}s$ and $h_\mu(x,s)$. The main difficult is that the type of truncation employed in our proof of Theorem 1.5 does not provide the localization of the projection of the solution on the direction of φ_1. Here we consider a local perturbation of $h_\mu(x,s)$. We also remark that a key ingredient in the proof of Theorem 1.5 is the compactness of the inverse of the p-Laplacian operator from $L^\sigma(\Omega)$ to $C(\bar{\Omega})$.

Invoking the conditions (H_μ^-) on (t_1, t_2) and (H_μ^+) on (t_2, t_3), we find $a > 0$ and $0 < \delta < t_1$ such that

$$\int_\Omega h_0(x, t\varphi_1)\varphi_1 dx < -a < 0, \text{ for every } |t - t_i| < \delta, \ i = 1, 3. \tag{4.1}$$

Fixed $x_0 \in \Omega$, we use the continuity of the eigenfunction φ_1 to find $r > 0$ such that $\bar{B}_r(x_0) \subset \Omega$ and

$$(t_1 - \delta) \max\{\varphi_1(x); x \in \bar{B}_r(x_0)\} < (t_1 - \delta/2)\varphi_1(x_0) < t_1 \min\{\varphi_1(x); x \in \bar{B}_r(x_0)\}, \tag{4.2}$$

and we consider $\psi \in C_c(B_r(x_0), [0,1])$ such that $\psi(x) = 1$, for every $x \in \bar{B}_{r/2}(x_0)$.

Setting $T_0 = (t_1 - \delta/2)\varphi_1(x_0)$, $T_1 = t_1 \min\{\varphi_1(x); x \in \bar{B}_r(x_0)\}$ and $T_2 = t_3\|\varphi_1\|_\infty$, from (4.2) and $t_1 < t_3$, we have that $0 < T_0 < T_1 < T_2$. Next we take $0 < \delta_1 < (T_1 - T_0)/2$ and $\chi_1 \in C(\mathbb{R}, [0,1])$ such that $\chi_1(s) = 0$, if $s \leq T_0$ or $s \geq T_1$; $\chi_1(s) = 1$, if $T_0 + \delta_1 \leq s \leq T_1 - \delta_1$. Furthermore we consider $\chi_2 \in C(\mathbb{R}, [0,1])$ satisfying $\chi_2(s) = 1$, if $s \leq T_2$; $\chi_2(s) = 0$, if $s \geq T_2 + 1$; and $0 < \chi_2(s) < 1$, if $T_2 < s < T_2 + 1$.

Now, for every $\mu \in [0, \mu_1)$ and $A > 0$ to be chosen posteriorly (see Lemma 4.2), we define

$$\begin{cases} \hat{f}(s) = (s^+)^{p-1}\chi_2(s), \text{ for every } s \in \mathbb{R}; \\ \hat{h}_\mu(x,s) = h_\mu(x, s^+)\chi_2(s) - A\chi_1(s)\psi(x), \text{ for every } (x,s) \in \Omega \times \mathbb{R}, \end{cases} \tag{4.3}$$

and we consider the associated problem

$$\begin{cases} -\triangle_p u = \lambda \hat{f}(u) + \mu \hat{h}_\mu(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases} \tag{4.4}$$

In the following lemma we present a list with the properties of the function $\hat{h}\mu$:

Lemma 4.1 *Suppose h_μ satisfies the hypotheses of Theorem 1.5. Then the family of functions $\hat{h}_\mu$, defined in (4.3), satisfies, for every $\mu \in [0, \mu_1)$,*

- (P_1) $\hat{h}_\mu(x,s) = h_\mu(x,s^+)$, *if*
  - (i) $x \in \Omega$ *and* $s \in (-\infty, T_0] \cup [T_1, T_2]$ *or*
  - (ii) $x \in \Omega - \bar{B}_r(x_0)$ *and* $s \in (-\infty, T_2]$.
- (P_2) $\hat{h}_\mu(x,s) = h_\mu(x,0) \geq 0$, *for every* $s \leq 0$, *a.e in* Ω.
- (P_3) $\hat{h}_\mu(x, t\varphi_1(x)) = h_\mu(x, t\varphi_1(x))$, *for every* $t_1 \leq t \leq t_3$, $x \in \Omega$.
- (P_4) $\hat{h}_\mu$ *satisfies* $(\hat{H}_1)$, (H_μ^-) *and* (H_μ^+) *on* (t_1,t_2) *and* (t_2,t_3), *respectively.*
- (P_5)

$$\begin{cases} \hat{h}_\mu(x,s) \leq h_\mu(x,s^+), \text{ for every } s \leq T_2, \text{ a.e. in } \Omega; \\ \hat{H}_\mu(x,s) \leq H_\mu(x,s^+) \text{ for every } s \leq T_2, \text{ a.e. in } \Omega, \end{cases}$$

where $\hat{H}_\mu(x,s) = \int_0^s \hat{h}_\mu(x,t)dt$, $H_\mu(x,s) = \int_0^s h_\mu(x,t)dt$, *for every* $s \in \mathbb{R}$, $x \in \Omega$.

Proof The property (P_1) is a direct consequence of (4.2), (4.3) and the properties of the function ψ, χ_1 and χ_2. The property (P_2) follows from property (P_1) and $h_\mu(x,0) \geq 0$, for almost every $x \in \Omega$. Given $t_1 \leq t \leq t_3$, we have that $0 < t\varphi_1(x) \leq t_3\|\varphi_1\|_\infty = T_2$, for every $x \in \Omega$. Moreover, by (4.2), $t\varphi_1(x) \geq t_1\varphi_1(x) \geq T_1$, for every $x \in \bar{B}_r(x_0)$. In view of these facts, we may apply property (P_1) to assert that property (P_3) holds. The property (P_4) follows from the hypotheses of Theorem 1.5, the definition (4.3) and (P_3). Finally we observe that (P_5) is a direct consequence of (P_2), the definition of $\hat{h}_\mu$, $\chi_2(s) = 1$ if $s \leq T_2$ and $\psi(x)\chi_1(s) \geq 0$ for every $s \in \mathbb{R}$, $x \in \Omega$. The lemma is proved. □

We also observe that $\hat{f}$, defined in (4.3), is a bounded function satisfying

$$\begin{cases} \hat{f}(s) = 0, \text{ for every } s \leq 0; \\ \hat{f}(s) = s^{p-1}, \text{ for every } s \in [0, T_2]; \\ |\hat{f}(s)| < |s|^{p-1}, \text{ for every } s \in (-\infty, 0) \cup (T_2, \infty). \end{cases} \tag{4.5}$$

Considering $\hat{F}(s) = \int_0^s \hat{f}(t)dt$, for every $s \in \mathbb{R}$, we may invoke the property (P_4) in Lemma 4.1 and (4.5) to conclude that the functional $\hat{I}_{\lambda,\mu} : W_0^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$\hat{I}_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega \hat{F}(u)dx - \mu \int_\Omega \hat{H}_\mu(x,u)dx, \ \forall\, u \in W_0^{1,p}(\Omega), \tag{4.6}$$

is of class C^1. Furthermore critical points of $\hat{I}_{\lambda,\mu}$ are weak solutions of problem (4.4).

We also remark that property (P_2) in Lemma 4.1 and (4.5) imply that any critical point $u \in W_0^{1,p}(\Omega)$ of $\hat{I}_{\lambda,\mu}$ is nonnegative almost everywhere in Ω.

Next result provides the appropriated value of A to be considered in definition (4.3). Setting $D_{\lambda,\mu} = \{u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega);\ \hat{I}'_{\lambda,\mu}(u) = 0,\ \|u\|_\infty \leq T_2, \text{ and } u(x) \leq T_0, \text{ for almost every } x \in B_r(x_0)\}$, we state:

Lemma 4.2 *Suppose h_μ satisfies the hypotheses of Theorem 1.5. Then there exists $A > 0$ such that the functional $\hat{I}_{\lambda,\mu}$, defined by (4.6), satisfies $\hat{I}_{\lambda,\mu}(t_1\varphi_1) > \max\{\hat{I}_{\lambda,\mu}(u);\ u \in D_{\lambda,\mu}\}$, for every $\mu \in (0, \mu_1)$ and $|\lambda - \lambda_1| \le \mu$.*

Proof First of all we note that by (H_1) we have $\eta \in L^\sigma(\Omega)$ such that

$$\begin{cases} |h_\mu(x,s)| \le \eta(x), \text{ for every } 0 \le s \le T_2, \text{ a.e. in } \in \Omega; \\ |H_\mu(x,s)| \le \eta(x)s, \text{ for every } 0 \le s \le T_2, \text{ a.e. in } \in \Omega. \end{cases} \tag{4.7}$$

Recalling that any critical point of $\hat{I}_{\lambda,\mu}$ is nonnegative, given $u \in D_{\lambda,\mu}$, by (4.7), (4.5) and property (P_1) of Lemma 4.1, we obtain

$$\begin{aligned} \hat{I}_{\lambda,\mu}(u) &= \hat{I}_{\lambda,\mu}(u) - \frac{1}{p}\langle \hat{I}_{\lambda,\mu}(u), u\rangle \\ &= \mu \int_\Omega \left[\frac{1}{p} h_\mu(x,u)u - H_\mu(x,u)\right]dx \le \mu\frac{(p+1)}{p}T_2\|\eta\|_1. \end{aligned} \tag{4.8}$$

On the other hand, from (4.3), (4.5) and (4.6), we have that

$$\hat{I}_{\lambda,\mu}(t_1\varphi_1) = \frac{(\lambda_1-\lambda)}{p\lambda_1}t_1{}^p - \mu\int_{B_{\frac{r}{2}}(x_0)} \hat{H}_\mu(x,t_1\varphi_1)dx - \mu\int_{\Omega - B_{\frac{r}{2}}(x_0)} \hat{H}_\mu(x,t_1\varphi_1)dx. \tag{4.9}$$

By property (P_5) of Lemma 4.1, $t_1\|\varphi_1\|_\infty < T_2$ and (4.7), we obtain

$$\hat{H}_\mu(x,t_1\varphi_1(x)) \le H_\mu(x,t_1\varphi_1(x)) \le \eta(x)t_1\varphi_1(x), \text{ for almost every } x \in \Omega. \tag{4.10}$$

Consequently

$$\int_{\Omega - B_{\frac{r}{2}}(x_0)} \hat{H}_\mu(x,t_1\varphi_1(x))dx \le t_1\|\varphi_1\|_\infty\|\eta\|_1. \tag{4.11}$$

Moreover we have that $t_1\varphi_1(x) \ge T_1$ and $\psi(x) = 1$, for every $x \in \bar{B}_{\frac{r}{2}}(x_0)$. Hence, using (4.10) one more time, for almost every $x \in B_{\frac{r}{2}}(x_0)$, we get

$$\hat{H}_\mu(x,t_1\varphi_1(x)) = H_\mu(x,t_1\varphi_1(x)) - A\int_0^{t_1\varphi_1(x)} \chi_1(s)ds \le t_1\varphi_1(x)\eta(x) - Ad,$$

where $d = [(T_1 - T_0) - 2\delta_1] > 0$. Therefore

$$\int_{B_{\frac{r}{2}}(x_0)} \hat{H}_\mu(x,t_1\varphi_1(x))dx \le t_1\|\varphi_1\|_\infty\|\eta\|_1 - Ad|B_{\frac{r}{2}}(x_0)|.$$

From the above inequality, (4.9)-(4.11) and $|\lambda - \lambda_1| < \mu$, we get

$$\hat{I}_{\lambda,\mu}(t_1\varphi_1) \ge \big[-\frac{t_1{}^p}{p\lambda_1} - 2t_1\|\varphi_1\|_\infty\|\eta\|_1 + Ad|B_{\frac{r}{2}}(x_0)|\big]\mu.$$

The proof of Lemma 4.2 is a direct consequence of the above inequality and the estimate (4.8). □

Proof of Theorem 1.5: We fix $A > 0$ given by Lemma 4.2. By (4.1) and properties (P_3) and (P_5) of Lemma 4.1, we have that

$$\int_\Omega \hat{h}_0(x,t\varphi_1)\varphi_1 dx \le -a < 0, \text{ when } |t - t_1| < \delta, \text{ or } t_3 - \delta < t \le t_3. \tag{4.12}$$

□

In view of property (P_4) of Lemma 4.1, $\hat{h}_\mu$ satisfies $(\hat{H}_1)$ and (H_μ^+) on (t_2, t_3). Moreover $\hat{f}$ satisfies (F_1) on the interval $[0, T_2]$. Hence by Theorem 2.1–Remark 2.5, there exist $[\hat{t}_2, \hat{t}_3] \subset (t_2, t_3)$, $\mu_1^*, \nu_1^* > 0$ such that, for every $\mu \in (0, \mu_1^*)$ and $|\lambda - \lambda_1| < \mu\nu_1^*$, $\hat{I}_{\lambda,\mu}$ has a critical point $\hat{u}_\mu = \hat{t}_\mu\varphi_1 + \hat{v}_\mu$, $\hat{t}_\mu \in (\hat{t}_2, \hat{t}_3)$, $\hat{v}_\mu \in X$. Moreover

$$\hat{I}_{\lambda,\mu}(\hat{u}_\mu) < \min\{\hat{I}_{\lambda,\mu}(t_2\varphi_1), \hat{I}_{\lambda,\mu}(t_3\varphi_1)\}. \tag{4.13}$$

On the other hand, using that $\hat{h}_\mu$ satisfies $(\hat{H}_1)$ and (H_μ^-) on (t_1, t_2), arguing as in the proof of Lemma 3.1, we may assume that there exists $t_0 \in (t_1, t_2)$ such that

$$\min\{\hat{I}_{\lambda,\mu}(t_0\varphi_1 + v); v \in X\} > \max\{\hat{I}_{\lambda,\mu}(t_1\varphi_1), \hat{I}_{\lambda,\mu}(t_2\varphi_1)\}. \tag{4.14}$$

Now we define

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma} \max_{0\le\tau\le1} \hat{I}_{\lambda,\mu}(\gamma(\tau)), \tag{4.15}$$

where

$$\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega));\ \gamma(0) = t_1\varphi_1,\ \gamma(1) = t_2\varphi_1\}. \tag{4.16}$$

From (4.14), we have

$$c_{\lambda,\mu} > \max\{\hat{I}_{\lambda,\mu}(t_1\varphi_1), \hat{I}_{\lambda,\mu}(t_2\varphi_1)\}. \tag{4.17}$$

Since $\hat{I}_{\lambda,\mu}$ satisfies the Palais-Smale condition, by (4.17) and the mountain pass theorem, we have that $c_{\lambda,\mu}$ is a critical value of $\hat{I}_{\lambda,\mu}$, i. e., there is $u_\mu = t_\mu\varphi_1 + v_\mu$, $t_\mu \in \mathbb{R}$, $v_\mu \in X$ such that $\hat{I}_{\lambda,\mu}(u_\mu) = c_{\lambda,\mu}$ and $\hat{I}'_{\lambda,\mu}(u_\mu) = 0$.

We claim that there exists $\mu^* \in (0, \mu_1^*)$ and $\nu^* \in (0, \nu_1^*)$ such that

$$\begin{cases} u_\mu = t_\mu\varphi_1 + v_\mu,\ \text{with } t_\mu \in (t_1 + \delta/2, t_3 - \delta/2),\ v_\mu \in X; \\ 0 < T_1 \le u_\mu(x),\ \text{a.e. in } B_r(x_0); \\ \|u_\mu\|_\infty \le T_2. \end{cases} \tag{4.18}$$

For proving the above claim it suffices to verify that given sequences $(\mu_n) \subset (0, \mu_1^*)$, $(\lambda_n) \subset \mathbb{R}$ and $(u_n) \subset W_0^{1,p}(\Omega)$ such that $\mu_n \to 0$, $|\lambda_n - \lambda_1|/\mu_n \to 0$, as $n \to \infty$, with u_n a critical point of $\hat{I}_{\lambda_n,\mu_n}$ at level c_{λ_n,μ_n}, for every $n \in \mathbb{N}$, we may find, up to a subsequence, $n_0 \in \mathbb{N}$ such that, for every $n \ge n_0$,

$$\begin{cases} u_n = t_n\varphi_1 + v_n,\ \text{with } t_n \in (t_1 + \delta/2, t_3 - \delta/2),\ v_n \in X; \\ 0 < T_1 \le u_n(x),\ \text{a.e. in } B_r(x_0); \\ \|u_n\|_\infty \le T_2. \end{cases} \tag{4.19}$$

Defining $g_n(x) = \lambda_n\hat{f}(u_n(x)) + \mu_n\hat{h}_{\mu_n}(x, u_n(x))$, for every $x \in \Omega$, and arguing as in the proof of Theorem 1.2, we may suppose that there exists $t \in \mathbb{R}$ such that

$$\begin{cases} u_n \to t\varphi_1 \text{ strongly in } W_0^{1,p}(\Omega),\ \text{as } n \to \infty; \\ u_n \to t\varphi_1 \text{ strongly in } C(\bar{\Omega}),\ \text{as } n \to \infty. \end{cases} \tag{4.20}$$

Moreover $-\Delta_p(t\varphi_1) = \lambda_1\hat{f}(t\varphi_1)$ in Ω. Using that u_n is a nonnegative function in Ω and arguing as in the proof of Theorem 1.5, we obtain that $t \in [0, t_3]$. Next we assert that actually $t \in [0, t_3 - \delta]$. Indeed, arguing by contradiction, we suppose that $t \in (t_3 - \delta, t_3]$. Using that $\hat{f}$ satisfies (4.5) and that u_n is a critical point of $\hat{I}_{\lambda_n,\mu_n}$, we get

$$0 = \langle \hat{I}'_{\lambda_n,\mu_n}(u_n), u_n\rangle \ge \Big[\frac{(\lambda_1 - \lambda_n)}{\lambda_1\mu_n}\|u_n\|^p - \int_\Omega \hat{h}_{\mu_n}(x, u_n)u_n dx\Big]\mu_n.$$

Consequently, from (4.20) and (4.12) we get $0 \geq at > 0$. This contradiction implies that effectively $t \in [0, t_3 - \delta]$. A similar argument implies that $t \notin (t_1 - \delta, t_1 + \delta)$.

We note that by $t \in [0, t_3 - \delta]$ and (4.20), we find $n_1 \in \mathbb{N}$ such that

$$\|u_n\|_\infty < t_3\|\varphi_1\|_\infty = T_2, \text{ for every } n \geq n_1. \tag{4.21}$$

Our next step is to verify that $t \notin [0, t_1 - \delta]$. Arguing by contradiction one more time, we suppose that $t \in [0, t_1 - \delta]$. On this case, by the first inequality on (4.2), the choice of T_0, (4.20) and (4.21), we obtain that $u_n \in D_{\lambda_n,\mu_n}$, for n sufficiently large. Hence, by Lemma 4.2, $c_{\lambda_n,\mu_n} = \hat{I}_{\lambda_n,\mu_n}(u_n) < \hat{I}_{\lambda_n,\mu_n}(t_1\varphi_1)$. However, this fact contradicts (4.17). We conclude that $t \notin [0, t_1 - \delta]$.

Considering the results above verified, we have that $t \in (t_1 + \delta, t_3 - \delta)$. The first limit in (4.20) implies that there exists $n_2 \in \mathbb{N}$ such that $t_n \in (t_1 + \delta/2, t_3 - \delta/2)$ for every $n \geq n_2$. Moreover, considering the second limit in (4.20), the second inequality in (4.2) and $t \geq t_1 + \delta$, we may assume that $u_n(x) \geq T_1 > 0$, almost everywhere in $\bar{B}_r(x_0)$, whenever $n \geq n_2$. Taking $n_0 = \max\{n_1, n_2\}$, we obtain (4.19). The proof of the claim (4.18) is complete.

As a direct consequence of the claim proved above and property (P_1) of Lemma 4.1, we obtain that for every $\mu \in (0, \mu_1^*)$ and $|\lambda - \lambda_1| < \nu^*\mu$, u_μ is a solution of problem (1.1). Moreover, $u_\mu = t_\mu\varphi_1 + v_\mu$, with $t_\mu \in (t_1, t_3)$, $v_\mu \in X$ and $\hat{I}_{\lambda,\mu}(u_\mu) = c_{\lambda,\mu}$, $c_{\lambda,\mu}$ defined by (4.15)-(4.16).

Applying a similar argument we have that problem (1.1) has a solution $\hat{u}_\mu = \hat{t}_\mu\varphi_1 + \hat{v}_\mu$, with $\hat{t}_\mu \in (t_2, t_3)$ and $\hat{v}_\mu \in X$, such that $\hat{u}_\mu$ is a critical point of $\hat{I}_{\lambda,\mu}$ and it satisfies (4.13). We conclude the proof of Theorem 1.5 by observing that the estimates (4.13) and (4.17) imply that $u_\mu \neq \hat{u}_\mu$.

5 Applications

In this section we present applications of our main results on the existence and multiplicity of solutions for quasilinear indefinite problems depending on a parameter.

5.1 Application 1

In our first application we consider the existence of solutions for the following problem

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u + \beta b_1(x)|u|^{q_1-2}uf_1(u) + b_2(x)|u|^{q_2-2}uf_2(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$; $\lambda, \beta > 0$; $1 < q_1 < q_2$, with $q_2 > p$, $f_1, f_2 \in C(\mathbb{R}, \mathbb{R})$ and $b_1, b_2 \in L^\sigma(\Omega)$, with $\sigma > \max\{N/p, 1\}$.

It's worthwhile mentioning that indefinite semilinear ($p = 2$) problems have been the object of an intense research since the works by Alama and Tarantello [2], Berestycki et al. [12] and Ouyang [26]—see [1,3,16–19,25] and references there in. For the corresponding results for indefinite problems with p-Laplacian operator we would like to mention the articles [20,21,32].

Setting

$$\lim_{s\to 0} f_1(s) = f_1(0), \quad \lim_{s\to 0} f_2(s) = f_2(0),$$
$$r_1 := f_1(0)\int_\Omega b_1\varphi_1^{q_1}dx \quad \text{and}$$
$$r_2 := f_2(0)\int_\Omega b_2\varphi_1^{q_2}dx,$$

we establish the following result:

Proposition 5.1 *If $1 < q_1 < q_2, q_2 > p$ and $r_1 > 0 > r_2$, then*

- (i) *there exist positive constants β^* and ν^* such that problem (5.1) has a nonnegative nontrivial solution for every $\beta \in (0, \beta^*)$ and $|\lambda - \lambda_1| < \nu^*\beta^{\frac{q_2-p}{q_2-q_1}}$;*
- (ii) *if $p < q_1$, there exist positive constants β^{**} and ν^{**} such that problem (5.1) has two nonnegative nontrivial solutions for every $\beta \in (0, \beta^{**})$ and $\lambda_1 - \beta^{\frac{q_2-p}{q_2-q_1}}\nu^{**} < \lambda < \lambda_1$.*

Proof Rescaling the solution u by $u = \beta^{\frac{1}{q_2-q_1}}\omega$, we obtain that u is a nonnegative nontrivial solution of problem (5.1) if and only if ω is a nonnegative nontrivial solution of the problem

$$\begin{cases} -\Delta_p\omega = \lambda|\omega|^{p-2}\omega + \mu h_\mu(x,\omega) \text{ in } \Omega, \\ \omega = 0, \text{ on } \partial\Omega, \end{cases} \tag{5.2}$$

where $\mu = \beta^{\frac{q_2-p}{q_2-q_1}}$ and $h_\mu(x, s) : \Omega \times \mathbb{R} \to \mathbb{R}$, for $\mu \geq 0$, is given by

$$h_\mu(x,s) = b_1(x)(s^+)^{q_1-1}f_1(\mu^{\frac{1}{q_2-p}}s^+) + b_2(x)(s^+)^{q_2-1}f_2(\mu^{\frac{1}{q_2-p}}s^+)$$

for every $(x, s) \in \Omega\times\mathbb{R}$. We have that h_μ is a family of Carathéodory functions. Furthermore, h_μ satisfies (H_1) and Definition 1.1-(ii). Considering the function $\Phi_0 : \mathbb{R} \to \mathbb{R}$ given by

$$\Phi_0(t) = \int_\Omega h_0(x, t\varphi_1)\varphi_1 dx = r_1(t^+)^{q_1-1} + r_2(t^+)^{q_2-1},$$

we use that $1 < q_1 < q_2$ and $r_2 < 0 < r_1$ to find $0 < t_1 < t_2$ such that h_0 satisfies the hypotheses (H_0^+).

- (i) A direct application of Theorem 1.2 implies that there exist $\mu^*, \nu^* > 0$ such that problem (5.2) has a nonnegative nontrivial solution for every $\mu \in (0, \mu^*)$ and $|\lambda - \lambda_1| < \nu^*\mu$. Taking $\beta^* = (\mu^*)^{\frac{q_2-q_1}{q_2-p}}$, the proof follows from this result.
- (ii) Using the definition of h_μ and the inequality $q_1 < q_2$, we find $c > 0$ such that $|h_\mu(x,s)| \leq c(|b_1(x)| + |b_2(x)|)|s|^{q_1-1}$, for every $0 \leq \mu \leq 1$ and $|s| \leq 1$. Since $b_1, b_2 \in L^\sigma(\Omega)$, $\sigma > \max\{N/p, 1\}$ and $q_1 > p$, we have that h_μ satisfies (H_2). The item (ii) is a consequence of Theorem 3.2. □

5.2 Application 2

We conclude this section by presenting an application of Corollary 1.6 when h_μ is given by

$$h_\mu(x,s) = \sum_{i=0}^{m} a_\mu^i(x,s)s^i, \text{ for every } s \in \mathbb{R}, \text{ a. e. in } \Omega, \tag{5.3}$$

with a_μ^i being a Carathéodory function satisfying the following hypothesis:

(A) $a^i_\mu(x,s)$ satisfies the hypotheses (H_1) and, for every $R>0$,

$$a^i_\mu(x,s)\to a^i_0(x),\ \text{ as } \mu\to 0 \text{ uniformly for } |s|\le R,\ \text{ a.e. in } \Omega.$$

Considering $h_0(x,s)=\sum_{i=0}^m a^i_0(x)s^i$, the associated function Φ, given by

$$\Phi(t)=\int_\Omega h_0(x,t\varphi_1)\varphi_1 dx=\sum_{i=0}^m t^i\int_\Omega a^i_0(x)\varphi_1^{i+1}dx=\sum_{i=0}^m d_i t^i,$$

where $d_i=\int_\Omega a^i_0(x)\varphi_1^{i+1}dx,\ i=1,\cdots,m$, is a polynomial function in the variable t. Note that under the condition (A), h_μ satisfies (H_1) and the item (ii) in Definition 1.1.
As a consequence of Corollary 1.6, we may state a result on the existence of multiple solutions for problem 1.1 depending on the number of roots of odd multiplicity of Φ:

Proposition 5.2 *Suppose (A), $a^0_\mu(x,0)\ge 0$ in Ω and $dm<0$. Then, if the function Φ has k roots of odd multiplicity, $\tau_1,\ldots,\tau_k\in(0,\infty)$, then there exist positive constants μ^* and ν^* such that, for every $0<\mu<\mu^*$ and $|\lambda-\lambda_1|<\mu\nu^*$, problem (1.1) has k weak solutions $u_1,\ldots,u_k$.*

Remark 5.3 A typical model for h_μ satisfying the hypothesis of Proposition 5.2 is obtained by considering $a^i_\mu(x,s)=a^i_0(x)\exp(\mu^{\beta_i}s^{r_i}),\ \beta_i>0, r_i\ge 0,\ 1\le i\le m.$

Proof Without loss of generality we may suppose that $\tau_1<\tau_2<\cdots<\tau_k$. From the hypothesis of Proposition 5.2 we may write

$$\Phi(t)=(t-\tau_1)^{2n_1-1}\cdots(t-\tau_k)^{2n_k-1}(t-c_1)^{2z_1}\cdots(t-c_l)^{2z_l}p(t),$$

with $n_1,\ldots,n_k,z_1,\ldots,z_l\in\mathbb{N}$ and $p(t)$ a product of irreducible quadratic polynomials. As a direct consequence of above expression, we may find $t_1,\ldots,t_{k+1}$ such that $t_1\in(-\infty,\tau_1)$, $t_{k+1}\in(\tau_k,\infty)$ and $t_i\in(\tau_{i-1},\tau_i)$, $i=2,\ldots,k$, $c_j\notin(t_i,t_{i+1})$, for every $i=1,\ldots,k$ and $j=1,\ldots,l$; $\Phi(t_i)\Phi(t_{i+1})<0$, $i=1,\ldots,k$; and $\Phi(t_{k+1})<0$. Hence h_μ satisfies $(H_\mu)_k$.

Noting that h_μ also satisfies (H_1), we may apply Corollary 1.6 to find positive constants μ^* and ν^* such that, for every $0<|\mu|<\mu^*$ and $|\lambda-\lambda_1|<\mu\nu^*$, Problem (1.1) has k solutions $u_i=\hat t_i\varphi_1+v_i$, with $\hat t_i\in(t_i,t_{i+1})$ and $v_i\in X$, $i=1,\ldots,k$, and the proof of Proposition 5.2 is complete. □

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Appendix

Proof of Theorem 2.6: By Stampacchia method [33], there is $M=M(\|g\|_\sigma)$ such that

$$\|u\|_\infty\le M. \tag{6.1}$$

Next, given $k\ge 0$ and a ball B_ρ of $\mathbb{R}^N$, $\rho>0$, we consider $\eta(x)=\xi^p|u-k|^+$, where $\xi:B_\rho\to[0,1]$ is a smooth function of compact support on B_ρ. Setting $A_{k,\rho}=\{x\in B_\rho\cap\Omega;\,u(x)>k\}$ and considering that $\eta\in W_0^{1,p}(\Omega)$, from (2.23), we get

$$\int_{A_{k,\rho}} \xi^p |\nabla u|^p dx + p \int_{A_{k,\rho}} |u-k| \xi^{p-1} |\nabla u|^{p-2} \nabla u \cdot \nabla \xi dx = \int_{A_{k,\rho}} g(x) \xi^p (u-k) dx.$$

Consequently, by Hölder inequality,

$$\int_{A_{k,\rho}} \xi^p |\nabla u|^p dx \le p \int_{A_{k,\rho}} \xi^{p-1} |\nabla u|^{p-1} |u-k| |\nabla \xi| dx + \|u\|_\infty \|g\|_\sigma |A_{k,\rho}|^{1-\frac{1}{\sigma}}.$$

Next, using the Young inequality, we get

$$p \xi^{p-1} |\nabla u|^{p-1} |u-k| |\nabla \xi| \le \frac{1}{2} |u-k|^p |\nabla \xi|^p + 2^{p-1} (p-1)^{p-1} |u-k|^p |\nabla \xi|^p.$$

Combining the above inequalities, we obtain

$$\int_{A_{k,\rho}} \xi^p |\nabla u|^p dx \le 2^p (p-1)^{p-1} \int_{A_{k,\rho}} |u-k|^p |\nabla \xi|^p dx + 2 \|u\|_\infty \|g\|_\sigma |A_{k,\rho}|^{1-\frac{1}{\sigma}}. \tag{6.2}$$

Now, given $\delta > 0$, we let $k \ge 0$ be such that $\sup_{B_\rho \cap \Omega} [u(x) - \delta] \le k$. From these values of k, we take ξ such that $\xi(x) = 1$ for every $x \in B_{\rho - \mu\rho}$, for $0 < \mu < 1$, in such a way that $|\nabla \xi| < \frac{c}{\mu\rho}$. Then, from (6.2) we may write

$$\begin{aligned} \int_{A_{k,(1-\mu)\rho}} |\nabla u|^p dx &\le \frac{2^p (p-1)^p c^p}{(\mu\rho)^p} \sup_{A_{k,\rho}} |u-k|^p |A_{k,\rho}| + 2 \|u\|_\infty \|g\|_\sigma |A_{k,\rho}|^{1-\frac{1}{\sigma}} \\ &= \left[\frac{2^p (p-1)^{p-1} c^p}{\mu^p \rho^p} |A_{k,\rho}|^{\frac{1}{\sigma}} \sup_{A_{k,\rho}} |u-k|^p + 2 \|u\|_\infty \|g\|_\sigma \right] |A_{k,\rho}|^{1-\frac{1}{\sigma}} \\ &\le \left[\frac{2^p (p-1)^{p-1} c}{\mu^p \rho^p} |c_N|^{\frac{1}{\sigma}} \rho^{\frac{N}{\sigma}} \sup_{A_{k,\rho}} |u-k|^p + 2 \|u\|_\infty \|g\|_\sigma \right] |A_{k,\rho}|^{1-\frac{1}{\sigma}} \end{aligned}$$

or, equivalently,

$$\int_{A_{k,(1-\mu)\rho}} |\nabla u|^p dx \le \gamma \left[\frac{\max |u-k|^p}{\mu^p \rho^{p(1-\frac{N}{p\sigma})}} + 1 \right] |A_{k,\rho}|^{1-\frac{p}{\sigma p}},$$

where $\gamma = \max\{2\|u\|_\infty \|g\|_\sigma, c_N^{\frac{1}{\sigma}} 2^p (p-1)^{p-1}\}$.

Observing that we obtain the same estimate for the function $-u(x)$ for every $k > \sup_{B_\rho \cap \Omega} [-u(x) - \delta]$, by taking $\eta(x) = \xi^p (-u-k)^+$, we may assert that $u \in \mathcal{B}_p(\overline{\Omega}, M, \gamma, \delta, \frac{1}{\sigma p})$, where $\mathcal{B}_p$ is as defined in [22, p. 90].

Since $\sigma p > N$, we may invoke Theorem 7.1 in [22] to find that there is $c > 0$ and $\alpha \in (0,1)$ such that $\|u\|_{C^{0,\alpha}(\overline{\Omega})} \le c$ with the constants c and α depending on $p, M, \gamma, \delta, \sigma$ and Ω. In view of (6.1), this concludes the proof of Theorem 2.6. □

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