Calculus of Variations

Metric currents and the Poincaré inequality

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Abstract

We show that a complete doubling metric space (X, d, μ) supports a weak 1-Poincaré inequality if and only if it admits a pencil of curves (PC) joining any pair of points $s, t \in X$. This notion was introduced by S. Semmes in the 90's, and has been previously known to be a sufficient condition for the weak 1-Poincaré inequality. Our argument passes through the intermediate notion of a generalised pencil of curves (GPC). A GPC joining s and t is a normal 1-current T, in the sense of Ambrosio and Kirchheim, with boundary $\partial T = \delta_t - \delta_s$, support contained in a ball of radius $\sim d(s, t)$ around $\{s, t\}$, and satisfying $\|T\| \ll \mu$, with

$$\frac{d\|T\|}{d\mu}(y) \lesssim \frac{d(s, y)}{\mu(B(s, d(s, y)))} + \frac{d(t, y)}{\mu(B(t, d(t, y)))}.$$

We show that the 1-Poincaré inequality implies the existence of GPCs joining any pair of points in X. Then, we deduce the existence of PCs from a recent decomposition result for normal 1-currents due to Paolini and Stepanov.

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1 Introduction

Let (X, d, μ) be a complete metric space, where μ is a doubling locally finite Borel measure. It is known, see for example [7,9], that plenty of analysis can be conducted on (X, d, μ) whenever the *weak p-Poincaré inequality*

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \leq C \operatorname{diam}(B) \left(\frac{1}{\mu(\lambda B)} \int_{\lambda B} \rho^p \, d\mu \right)^{1/p} \qquad (1.1)$$

is satisfied for some $C, p, \lambda \geq 1$, for all balls $B \subset X$, for all locally integrable Borel functions $u \colon X \to \mathbb{R}$, and for all upper gradients ρ of u. So, it is worthwhile to find necessary and sufficient conditions for the validity of (1.1). One well-known sufficient condition is the existence of *pencils of curves*, introduced by Semmes [15] in the 90's. To motivate the results in the present paper, we first discuss Semmes' condition in some detail; our definition is the one given in Section 14.2 in [9], where the setting is somewhat more general than in Semmes' original work [15].

Definition 1.1 (*Pencils of curves*) The space (X, d, μ) admits *pencils of curves* (PC) if there exists a constant $C_0 \geq 1$ with the following property. For all distinct $s, t \in X$ there is a family $\Gamma_{s,t}$ of rectifiable curves $\gamma \subset B(s, C_0 d(s, t))$, each joining s to t and satisfying $\mathcal{H}^1(\gamma) \leq C_0 d(s, t)$, and a probability measure $\alpha_{s,t}$ on $\Gamma_{s,t}$ such that

$$\int_{\Gamma_{s,t}} \int_\gamma g \, d\mathcal{H}^1 \, d\alpha_{s,t}(\gamma) \leq C_0 \int_{B(s, C_0 d(s,t))} \left(\frac{g(y)}{\Theta(s, d(s, y))} + \frac{g(y)}{\Theta(t, d(t, y))} \right) d\mu(y).$$

for Borel functions $g \colon X \to [0, \infty]$. Here, and in the sequel, Θ stands for the 1-dimensional density

$$\Theta(x, r) = \frac{\mu(B(x, r))}{r}, \qquad x \in X, \ r > 0.$$

A doubling space (X, d, μ) admitting PCs satisfies the weak 1-Poincaré inequality. This was proven by Semmes for Q-regular spaces, see [15, Theorem B.15], and the general case can be found for instance in Heinonen's book [8, Chapter 4].

Of course, Semmes in [15] also gives sufficient conditions for finding PCs: his Standard Assumptions (see [15, Theorem 1.11] and above) include the space (X, d, μ) to be an orientable topological n-manifold, with $\mu = \mathcal{H}^n$. Moreover, X has to be locally contractible (for more precise statements, see [15, Definition 1.7] or [15, Definition 1.15], but also the discussion in [15, Remark A.35]). These assumptions are certainly not necessary for a space (X, d, μ) to admit PCs or support a Poincaré inequality; notably, the Laakso spaces [11] have PCs, hence satisfy (1.1) with $p = 1$, but are generally not integer-dimensional.

The main result of our paper shows that, in complete doubling metric measure spaces, the weak 1-Poincaré inequality implies the existence of PCs. We achieve this by passing through an intermediate notion, the *generalised pencils of curves*.

Definition 1.2 (*Generalised pencils of curves*) The space (X, d, μ) admits *generalised pencils of curves* (GPC in short) if there exists a constant $C_0 \geq 1$ with the following property. For all distinct $s, t \in X$, there exists a normal 1-current T on X (in the sense of Ambrosio and Kirchheim) satisfying the following three properties:

(P1) $\partial T = \delta_t - \delta_s$,
(P2) $\operatorname{spt} T \subset B(s, C_0 d(s, t))$, and

(P3) $\|T\| = w\,d\mu$ with

$$|w(y)| \leq C_0 \left[\frac{1}{\Theta(s, d(s, y))} + \frac{1}{\Theta(t, d(t, y))} \right] \qquad \text{for } \mu\text{-a.e. } y \in X.$$

The main novelty of the present paper is the following result:

Theorem 1.3 *Let (X, d, μ) be complete and doubling. Then X satisfies (1.1) with $p = 1$ if and only if X admits generalised pencils of curves.*

It turns out that the existence of GPCs implies the existence of PCs. This is a consequence of a recent decomposition result for normal 1-currents, due to Paolini and Stepanov [13]. Combined with Theorem 1.3, we obtain the following characterisation.

Theorem 1.4 *Let (X, d, μ) be complete and doubling. Then X satisfies (1.1) with $p = 1$ if and only if X admits pencils of curves.*

Remark 1.5 The first version of this paper only contained Theorem 1.3, as we were not aware of the decomposition result of Paolini and Stepanov. Shortly afterwards, Theorem 1.4 was obtained by Durand-Cartagena et al. [4, Theorem 3.7]. Their proof uses the modulus of curve families instead of metric currents. So, Theorem 1.4 first appeared in [4].

The structure of the paper is the following. In Sect. 2, we briefly recall the definition of, and some basic concepts related to, the metric currents of Ambrosio and Kirchheim. Then, in Sect. 3, we prove the easy "if" implication

$$\text{GPCs} \Longrightarrow \text{weak 1-PI}$$

of Theorem 1.3, mostly using classical methods in metric analysis. We chose to retain a direct proof of this implication, as it indicates how GPCs can be applied in practice. Another proof would be

$$\text{GPCs} \Longrightarrow \text{PCs} \Longrightarrow \text{weak 1-PI,}$$

where the first implication follows from Paolini and Stepanov's work. As explained above, the second implication follows from [15, Theorem B.15] in the Q-regular case, and in full generality from [8, Chapter 4].

Section 4 is the core of the paper, containing the proof of the "only if" implication of Theorem 1.3. In short, the idea is to translate the problem of finding currents in (X, d, μ) to finding "network flows" in certain graphs derived from δ-nets in X. The existence of such flows is guaranteed by the famous *max flow–min cut* theorem of Ford and Fulkerson [5]. Then, the main task will be to verify that there are no "small cuts" in the graph, and this can be done by using the weak 1-Poincaré inequality. Finally, in Sect. 5, we explain how to deduce Theorem 1.4 from Theorem 1.3 using the results of Paolini and Stepanov.

1.1 Basic notation

Open balls in a metric space (X, d) will be denoted by $B(x, r)$, with $x \in X$ and $r > 0$. A *measure* on (X, d) will always refer to a Borel measure μ with $\mu(B(x, r)) < \infty$ for all balls $B(x, r) \subset X$. The notation $A \lesssim B$ means that there exists a constant $C \geq 1$ such that $A \leq CB$: the constant C will typically depend on the "data" of the ambient space, for example the doubling constant of μ, or the constant in the Poincaré inequality (1.1) (whenever (1.1) is assumed to hold). The two-sided inequality $A \lesssim B \lesssim A$ is abbreviated to $A \sim B$.

2 Background on currents

The main result in the paper mentions currents in metric spaces, so we include here a brief introduction. We claim no originality for anything in this section. We use the definition of metric currents given by Ambrosio and Kirchheim, see Definition 3.1 in [1].

Let X be a complete metric space, and let $\mathrm{Lip}(X)$ and $\mathrm{Lip}_b(X)$ be the families of Lipschitz, and bounded Lipschitz functions on X. For $k \geq 1$, Let $\mathcal{D}^k(X) := \mathrm{Lip}_b(X) \times [\mathrm{Lip}(X)]^k$. We typically denote the $(k+1)$-tuples in $\mathcal{D}^k(X)$ by $(f, \pi_1, \ldots, \pi_k)$. Following Definition 2.2 in [1], we consider subadditive, positively 1-homogenous functionals $T \colon \mathcal{D}^k(X) \to \mathbb{R}$. These are denoted by $MF_k(X)$. We say $T \in MF_k(X)$ has *finite mass*, if there exists a finite Borel measure ν on X such that

$$|T(f, \pi_1, \ldots, \pi_k)| \leq \prod_{j=1}^{k} \mathrm{Lip}(\pi_j) \int_X |f| \, d\nu, \qquad (f, \pi_1, \ldots, \pi_k) \in \mathcal{D}^k(X). \quad (2.1)$$

Here $\mathrm{Lip}(\pi_j)$ denotes the *Lipschitz constant* of π_j, that is, the smallest constant $L \in [0, \infty)$ for which $|\pi_j(x) - \pi_j(y)| \leq L d(x, y)$ holds for all $x, y \in X$. For $k = 0$, the correct interpretation of (2.1) is $|T(f)| \leq \int_X |f| \, d\nu$. As in Definition 2.6 in [1], the minimal measure ν satisfying (2.1) is denoted by $\|T\|$ (this is well-defined, as discussed below (2.2) in [1]).

A *k-dimensional current*, or just a *k-current*, is then a $(k+1)$-multilinear functional $T \in MF_k(X)$ with finite mass, satisfying a few additional requirements which we will not need explicitly, see Definition 3.1 in [1]. If T is a k-current, so that $\|T\|$ is a finite Borel measure, then bounded Lipschitz functions are dense in $L^1(X, \|T\|)$, and in particular in the space of bounded Borel functions $B(X)$ equipped with the $L^1(\|T\|)$-norm. This fact, and (2.1), together imply that T has a canonical extension to $B(X) \times [\mathrm{Lip}(X)]^k$, which we also denote by T.

We review a few basic concepts related to currents.

Definition 2.1 (*Support*) The support $\mathrm{spt}(T)$ of a k-current T is the usual measure-theoretic support of $\|T\|$, namely

$$\mathrm{spt} \, \|T\| = \{x \in X : \|T\|(B(x, r)) > 0 \text{ for all } r > 0\}.$$

Definition 2.2 (*Boundary*) Let $T \in MF_k(X), k \geq 1$. Then $\partial T \in MF_{k-1}(X)$ is the functional defined by

$$\partial T(f, \pi_1, \ldots, \pi_{k-1}) = T(1, f, \pi_1, \ldots, \pi_{k-1}).$$

A *k*-current T is called *normal*, if ∂T is a $(k-1)$-current, in particular, ∂T has finite mass.

Definition 2.3 (*Push-forward*) Let X, Y be complete metric spaces, and let $\varphi \colon X \to Y$ be Lipschitz. For $T \in MF_k(X)$, we define the functional $\varphi_\sharp T \in MF_k(Y)$ by

$$\varphi_\sharp T(g, \pi_1, \ldots, \pi_k) = T(g \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_k \circ \varphi).$$

The following is a special instance of Definition 2.5 in [1].

Definition 2.4 (*Restriction*) Let X be a complete metric space, $T \in MF_1(X)$, and $g \in \mathrm{Lip}_b(X)$. Then we define an element $T \lfloor_g \in MF_1(X)$ by setting

$$T \lfloor_g (f, \pi_1) := T(fg, \pi_1).$$

If T is a 1-current, then Definition 2.4 can be extended to $g \in B(X)$ using the canonical extension of T to $B(X) \times \text{Lip}(X)$, see [1, p.11]. Moreover, in that case, $T \lfloor_g$ is again a current, see [1, p.16 and p.19]. If E is a Borel subset of X and $g = \chi_E$, we write $T \lfloor E$ for the restriction $T \lfloor_g$. We have

$$|T \lfloor E (f, \pi_1)| = |T(f \chi_E, \pi_1)| \leq \text{Lip}(\pi_1) \int_E |f| d\|T\| \tag{2.2}$$

for all $(f, \pi_1) \in \mathcal{D}^1(X)$. Since χ_E is merely Borel but not Lipschitz, (2.2) does not follow directly from the definition of $\|T\|$ given by (2.1), but it can be deduced by the density argument alluded to earlier, see [1, (2.3)]. Finally, (2.2) and the minimality of $\|T \lfloor E\|$ show that

$$\text{spt}\,(T \lfloor E) \subseteq \text{spt}\,(\|T\| \lfloor E) \subseteq \overline{E}.$$

We record the following lemma, which follows from general measure theory:

Lemma 2.5 *Assume that T is a k-current on a σ-compact metric space X. Then, for any Borel set $B \subset X$ and any $\epsilon > 0$, there exists a compact set $K \subset B$ with $\|T\|(B \setminus K) < \epsilon$.*

Proof By assumption $\|T\|$ is a finite Borel measure. The claim now follows from [12, Theorem 1.10], and the Note directly below it.   □

We next describe a simple example, which will be useful later on.

Example 2.6 Given a non-degenerate interval $[a, b] \subset \mathbb{R}$ we may define the 1-current $[\![a, b]\!]$ as follows:

$$[\![a, b]\!](f, \pi) = \int_a^b f(t) \pi'(t) \, dt,$$

where $(f, \pi) \in \text{Lip}_b(\mathbb{R}) \times \text{Lip}(\mathbb{R})$. This is a particular case of Example 3.2 in [1], and it is noted there that $\|[\![a, b]\!]\| = \mathcal{H}^1|_{[a,b]}$. The boundary of $[\![a, b]\!]$ is the measure (or 0-current) $\delta_b - \delta_a$, as shown by the following computation:

$$\partial [\![a, b]\!](f) = [\![a, b]\!](1, f) = \int_a^b f'(t) \, dt = f(b) - f(a) = \int f \, d[\delta_b - \delta_a].$$

Next, consider an isometric embedding $\gamma: [a, b] \to X$, where X is any complete metric space. Then $\gamma_\sharp [\![a, b]\!]$ defines a current in X given by (spelling out Definition 2.3)

$$\gamma_\sharp [\![a, b]\!](f, \pi) = [\![a, b]\!](f \circ \gamma, \pi \circ \gamma) = \int_a^b f(\gamma(t))(\pi \circ \gamma)'(t) \, dt.$$

It is noted in [1, (2.6)], and in the discussion directly below, that

$$\|\gamma_\sharp [\![a, b]\!]\| = \gamma_\sharp [\![a, b]\!] = \gamma_\sharp (\mathcal{H}^1 \lfloor_{[a,b]}) = \mathcal{H}^1 \lfloor_{\gamma([a,b])}. \tag{2.3}$$

The last equation follows from the isometry assumption. Finally, because boundary and push-forward commute by [1, (2.1)], we have

$$\partial (\gamma_\sharp [\![a, b]\!]) = \gamma_\sharp (\partial [\![a, b]\!]) = \delta_{\gamma(b)} - \delta_{\gamma(a)}. \tag{2.4}$$

We end the section by recalling (a slightly simplified) version of the compactness theorem for normal currents. The original reference, and the full version of the theorem, is [1, Theorem 5.2].

Theorem 2.7 (Compactness) *Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of normal k-currents with*

$$\sup_n \left(\|T_n\|(X) + \|\partial T_n\|(X) \right) < \infty,$$

and such that spt $T_n \subset K$ *for some fixed compact set* $K \subset X$, *for all* $n \in \mathbb{N}$. *Then there exists a subsequence* $(T_{n_m})_{m \in \mathbb{N}}$, *and a normal k-current T supported on K, such that*

$$\lim_{m \to \infty} T_{n_m}(f, \pi_1, \dots, \pi_k) = T(f, \pi_1, \dots, \pi_k), \qquad (f, \pi_1, \dots, \pi_k) \in \mathrm{Lip}_b(X) \times [\mathrm{Lip}(X)]^k.$$

3 Generalised pencils of curves imply the 1-Poincaré inequality

In this section we prove that if X as in Theorem 1.3 admits generalised pencils of curves, then it supports a weak 1-Poincaré inequality. It is well known, see for instance [9, Theorem 8.1.7], that doubling metric measure spaces which support a Poincaré inequality can be characterised in terms of the validity of pointwise inequalities between functions and their upper gradients. We will use the existence of GPCs to derive such an inequality between an arbitrary Lipschitz function $u : X \to \mathbb{R}$ and its (upper) *pointwise Lipschitz constant*

$$x \mapsto \mathrm{Lip}(u, x) := \limsup_{r \to 0} \sup_{d(y,x) \leq r} \frac{|u(x) - u(y)|}{r}.$$

The desired inequality will be based on the following lemma.

Lemma 3.1 *Let X be a complete σ-compact metric space and let $u \colon X \to \mathbb{R}$ be a Lipschitz function. Then, for any 1-current T, we have*

$$|\partial T(u)| \leq \int \mathrm{Lip}(u, x) \, d\|T\|(x).$$

Proof The idea of the proof is the following: By definition of ∂T and since $\|T\|$ is a finite Borel measure on X satisfying (2.1), we know that

$$|\partial T(u)| = |T(1, u)| \leq \int_X \mathrm{Lip}(u) d\|T\|.$$

The desired inequality in Lemma 3.1 is similar, but $\mathrm{Lip}(u)$ is replaced by the **pointwise** Lipschitz constant $\mathrm{Lip}(u, \cdot)$. To achieve this, we will essentially decompose X into pieces where $\mathrm{Lip}(u, \cdot)$ is almost constant.

We now turn to the details. Write

$$E := \{x \in X : \mathrm{Lip}(u, x) > 0\} \quad \text{and} \quad Z := \{x \in X : \mathrm{Lip}(u, x) = 0\}.$$

We perform countable decompositions of the sets E and Z. Consider

$$u_R(x) := \sup_{0 < r \leq R} \sup_{d(y,x) \leq r} \frac{|u(x) - u(y)|}{r},$$

fix $\epsilon > 0$, and define

$$E_{\delta, j} := \{x \in X : (1 + \epsilon)^j < u_R \leq (1 + \epsilon)^{j+1} \text{ for all } R \leq \delta\}, \quad j \in \mathbb{Z}, \ \delta > 0$$

and

$$Z_{\delta, j} := \{x \in X : u_R \leq 2^{-j} \text{ for all } R \leq \delta\}, \quad j \in \mathbb{N}, \ \delta > 0.$$

Note that $\mathrm{Lip}(u, x) = \lim_{R \to 0} u_R(x)$ and that for every $j \in \mathbb{Z}$ fixed, the sequences $(E_{1/i,j})_{i \in \mathbb{N}}$ and $(Z_{1/i,j})_{i \in \mathbb{N}}$ are nested. Then, we can write

$$X = E \cup Z \subset \bigcup_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{Z}} E_{1/i,j} \cup \bigcap_{j \in \mathbb{N}} \bigcup_{i \in \mathbb{N}} Z_{1/i,j}. \tag{3.1}$$

Fix $x \in X, i \in \mathbb{N}$. Notice that the restriction of u to the set

$$B(x, 1/(2i)) \cap E_{1/i,j} =: E^x_{1/(2i),j}$$

is $(1 + \epsilon)^{j+1}$-Lipschitz since, if $y, z \in E^x_{1/(2i),j}$, then

$$y, z \in E_{1/i,j} \quad \text{and} \quad d(y, z) \leq \frac{1}{i},$$

which implies that

$$\frac{|u(y) - u(z)|}{d(y, z)} \leq u_{1/i}(y) \leq (1 + \epsilon)^{j+1}. \tag{3.2}$$

Moreover,

$$\mathrm{Lip}(u, y) \geq (1 + \epsilon)^j, \qquad y \in E^x_{1/(2i),j}. \tag{3.3}$$

A similar argument as for (3.2) applies if $x \in X$, and

$$y, z \in B(x, 1/(2i)) \cap Z_{1/i,j} =: Z^x_{1/(2i),j}.$$

Then the conclusion is simply that the restriction of u to the set $Z^x_{1/(2i),j}$ is 2^{-j}-Lipschitz.

Since (X, d) is σ-compact, it is separable and hence we can pick a countable dense subset $\{x_n\}_{n \in \mathbb{N}} \subseteq X$. Then for arbitrarily small $\delta \in [0, 1/2)$, the family $\mathcal{E} \cup \mathcal{Z}$ with

$$\mathcal{E} := \left\{ E^{x_n}_{1/(2i),j} : i \in \mathbb{N}, j \in \mathbb{Z}, n \in \mathbb{N} \right\}$$

and

$$\mathcal{Z} := \left\{ Z^{x_n}_{1/(2i),j} : i \in \mathbb{N}, j \in \mathbb{N} \text{ with } j \geq -\log_2 \delta, n \in \mathbb{N} \right\}$$

constitutes a countable cover of X by Borel sets. Using the cover $\mathcal{E} \cup \mathcal{Z}$, we can further produce a countable **disjoint** Borel cover of X of the form $\{E_i\}_{i \in \mathbb{N}} \cup \{Z_i\}_{i \in \mathbb{N}}$ with the following properties. First, $E_i \subset E$ and $Z_i \subset Z$ for $i \in \mathbb{N}$. Second, the function u can be decomposed as

$$u = u\chi_E + u\chi_Z = \sum_{i \in \mathbb{N}} u\chi_{E_i} + \sum_{i \in \mathbb{N}} u\chi_{Z_i}, \tag{3.4}$$

where

$$u|_{E_i} \text{ is } L_i\text{-Lipschitz} \quad \text{and} \quad \mathrm{Lip}(u, x) \geq (1 - \epsilon)L_i \quad \text{for } x \in E_i \tag{3.5}$$

for some finite constants $L_i > 0$, and

$$u|_{Z_i} \text{ is } \delta\text{-Lipschitz}, \tag{3.6}$$

where $\delta > 0$ can be taken arbitrarily small. To achieve such a disjoint cover $\{E_i\}_{i \in \mathbb{N}} \cup \{Z_i\}_{i \in \mathbb{N}}$, consider first the family

$$\mathcal{E}' := \{A \cap E : A \in \mathcal{E}\}.$$

The sets in $\mathcal{E}'$ form a countable cover of E and we enumerate them as $(E'_i)_{i \in \mathbb{N}}$. Then we arrive at a disjoint cover of E by setting $E_1 := E'_1$ and defining iteratively $E_i := E'_i \setminus (E'_1 \cup \ldots \cup E'_{i-1})$ for $i \geq 2$. To obtain the sets $\{Z_i\}_{i \in \mathbb{N}}$, apply the same procedure with $\mathcal{Z}' = \{A \cap Z : A \in \mathcal{Z}\}$ in place of $\mathcal{E}'$. The decomposition (3.4) then holds by construction. To establish (3.5) and (3.6), we note that the properties (3.2) and (3.3) (and the 2^{-j}-Lipschitz continuity of u on $Z^x_{1/(2i),j}$) are preserved under taking subsets of $E^x_{1/(2i),j}$ (and $Z^x_{1/(2i),j}$).

Next, Lemma 2.5 allows us to remove for every $i \in \mathbb{N}$ a Borel set N_i from E_i (or similarly Z_i) such that

$$E_i \setminus N_i \text{ is compact} \quad \text{and} \quad \|T\|(N_i) < \frac{\epsilon}{2^{i+1}}.$$

Thus, we may assume that the sets $\{E_i\}_{i \in \mathbb{N}} \cup \{Z_i\}_{i \in \mathbb{N}}$ are compact, if we replace (3.4) by a decomposition

$$u = \sum_{i \in \mathbb{N}} u \chi_{E_i} + \sum_{i \in \mathbb{N}} u \chi_{Z_i} + u \chi_N,$$

where $N \subset X$ is a Borel set with $\|T\|(N) < \epsilon$.

Next, we use the McShane extension theorem to find Lipschitz functions $u_i^E, u_i^Z : X \to \mathbb{R}$ such that

$$u_i^E|_{E_i} = u|_{E_i} \quad \text{and} \quad u_i^Z|_{Z_i} = u|_{Z_i}$$

and

$$u_i^E \text{ is } L_i\text{-Lipschitz} \quad \text{and} \quad u_i^Z \text{ is } \delta\text{-Lipschitz}.$$

Then

$$u = \sum_{i \in \mathbb{N}} u_i^E \chi_{E_i} + \sum_{i \in \mathbb{N}} u_i^Z \chi_{Z_i} + u \chi_N,$$

and we can write

$$|\partial T(u)| = |T(1, u)| \leq \sum_{i \in \mathbb{N}} |T(\chi_{E_i}, u)| + \sum_{i \in \mathbb{N}} |T(\chi_{Z_i}, u)| + |T(\chi_N, u)|. \tag{3.7}$$

Since $\mathrm{Lip}(u) < \infty$, we have $|T(\chi_N, u)| \leq \mathrm{Lip}(u)\|T\|(N) < \mathrm{Lip}(u)\epsilon$. We now estimate the terms involving χ_{E_i}. By definition of the restriction operation, see Definition 2.4 and the comment below it, we can write

$$\left|T(\chi_{E_i}, u)\right| = \left|T\llcorner_{E_i}(1, u)\right|. \tag{3.8}$$

Recall that $u|_{E_i} = u_i^E|_{E_i}$. This is useful information since, according to [1, (3.6)], the values of a 1-current agree on (f, π_1) and (f', π_1') whenever $f = f'$ and $\pi_1 = \pi_1'$ on the support of T. Using that $\mathrm{spt}\left(T\llcorner_{E_i}\right) \subseteq E_i$, we apply this fact to the current $T\llcorner_{E_i}$ and the pairs $(f, \pi_1) = (1, u)$ and $(f', \pi_1') = (1, u_i^E)$, $i \in \mathbb{N}$. This shows that

$$\left|T\llcorner_{E_i}(1, u)\right| = \left|T\llcorner_{E_i}(1, u_i^E)\right| \tag{3.9}$$

Finally, by (2.2) and the property (3.5) of u_i^E, it holds for every $i \in \mathbb{N}$ that

$$\left|T\llcorner_{E_i}(1, u_i^E)\right| \leq \int_{E_i} L_i d\|T\| \leq \frac{1}{1-\epsilon} \int_{E_i} \mathrm{Lip}(u, x) d\|T\|(x). \tag{3.10}$$

Combining (3.8), (3.9), and (3.10), and using the pairwise disjointedness of the sets E_i, $i \in \mathbb{N}$, we conclude that

$$\sum_{i \in \mathbb{N}} |T(\chi_{E_i}, u)| \leq \frac{1}{1 - \epsilon} \int_X \mathrm{Lip}(u, x) d\|T\|(x)$$

Similar considerations give

$$\sum_{i \in \mathbb{N}} |T(\chi_{Z_i}, u)| \leq \delta \|T\|(Z).$$

Letting $\epsilon \to 0$ and $\delta \to 0$ in (3.7) completes the proof of Lemma 3.1. $\qquad\square$

We next apply Lemma 3.1 to deduce the validity of a weak 1-Poincaré inequality from the existence of GPCs.

Proof of "if" implication in Theorem 1.3 By a result of Keith [10], see also Theorem 8.4.2 in [9], it suffices to verify the Poincaré inequality for *a priori* Lipschitz continuous functions u and for the pointwise Lipschitz constant $\rho = \mathrm{Lip}(u, \cdot)$ instead of arbitrary upper gradients. So, let $u \colon X \to \mathbb{R}$ with $\mathrm{Lip}(u) < \infty$.

Recalling that

$$\Theta(x, r) = \frac{\mu(B(x, r))}{r}, \quad x \in X, \ r > 0,$$

we will first check that

$$|u(t) - u(s)| \lesssim \int_{B(s, C_0 d(s, t))} \left(\frac{\mathrm{Lip}(u, y)}{\Theta(s, d(s, y))} + \frac{\mathrm{Lip}(u, y)}{\Theta(t, d(t, y))} \right) d\mu(y) \qquad (3.11)$$

for distinct points $s, t \in X$. Start by fixing such points s, t, let T be a GPC joining s to t, and recall that $\mathrm{spt}\, T \subset B(s, C_0 d(s, t))$. Then,

$$|u(t) - u(s)| = |\partial T(u)| \leq \int \mathrm{Lip}(u, y) \, d\|T\|(y)$$
$$\leq C_0 \int_{B(s, C_0 d(s, t))} \left(\frac{\mathrm{Lip}(u, y)}{\Theta(s, d(s, y))} + \frac{\mathrm{Lip}(u, y)}{\Theta(t, d(t, y))} \right) d\mu(y),$$

using Lemma 3.1 and property (P3) in Definition 1.2. (Since a complete doubling metric space is proper, see [9, Lemma 4.1.14], and a proper metric space is σ-compact, the space (X, d) satisfies the assumptions of Lemma 3.1). This proves (3.11), which is (almost) a well-known sufficient condition for the weak 1-Poincaré inequality. The only technicality here is that we only know (3.11) for the particular upper gradient $\mathrm{Lip}(u, \cdot)$. To complete the proof, we will now briefly argue that that this suffices to imply the weak 1-Poincaré inequality in full generality.

Indeed, [8, Theorem 9.5] lists several conditions that imply weak Poincaré inequalities in doubling spaces (see also the references in [8]). Our estimate (3.11) shows that condition (2) in [8, Theorem 9.5] holds for $p = 1$, μ, u Lipschitz, the particular upper gradient $\rho = \mathrm{Lip}(u, \cdot)$, $C_2 = C_3 = C_0$. It then follows from the proof in [8] that also condition (3) in the theorem holds for the same pair (u, ρ) (by this, we mean that to obtain condition (3) for u and ρ, one only needs to have condition (2) for u and ρ, and no other upper gradients). We rephrase condition (3) in a slightly peculiar manner for future application: there exists a

constant $C \geq 1$ (again depending on C_0) such that if $2B := B(z, 2r)$ is any ball in X, and $x, y \in 2B$, then

$$|u(x) - u(y)| \lesssim d(x, y) \left(M_{Cd(x,y)}\rho(x) + M_{Cd(x,y)}\rho(y) \right). \tag{3.12}$$

Here M_R is the restricted maximal function

$$M_R\rho(x) = \sup_{r < R} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \rho(y) \, d\mu(y), \qquad R > 0.$$

Next we need to verify that inequality (3.12) implies that the very same pair (u, ρ) satisfies the weak 1-Poincaré inequality. To this end, we apply Theorem 8.1.18 in [9] (originally due to Hajłasz [6]) with $h = M_{4Cr}\rho$ and $Q = \log_2 C_\mu$ for the doubling constant C_μ of μ, to deduce that

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu^Q \lesssim r \left(\frac{1}{\mu(2B)} \int_{2B} [M_{4Cr}\rho]^{\frac{Q}{Q+1}} \, d\mu \right)^{\frac{Q+1}{Q}}.$$

Finally, following verbatim the argument on p. 224 of [9], we conclude that

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \lesssim r \left(\frac{1}{\mu(2B)} \int_{2B} [M_{4Cr}\rho]^{\frac{Q}{Q+1}} \, d\mu \right)^{\frac{Q+1}{Q}}$$

$$\lesssim r \left(\frac{1}{\mu(2CB)} \int_{2CB} [M_{4Cr}\rho]^{\frac{Q}{Q+1}} \, d\mu \right)^{\frac{Q+1}{Q}}$$

$$\lesssim \frac{1}{\mu(6CB)} \int_{6CB} \rho \, d\mu.$$

Hence the inequality (1.1) holds with $p = 1$ and $\lambda = 6C$ (depending on C_0) for all open balls $B \subset X$ and all pairs (u, ρ), where $u : X \to \mathbb{R}$ is Lipschitz and $\rho = \text{Lip}(u, \cdot)$. According to [9, Theorem 8.4.2], this shows that (X, d, μ) supports the weak 1-Poincaré inequality. □

4 From 1-Poincaré to generalised pencils of curves

4.1 An initial reduction

The main effort in the rest of the paper consists of proving the following statement:

Theorem 4.1 *Let (X, d, μ) be a complete doubling **and geodesic** metric measure space. If X supports a 1-Poincaré inequality, then it supports generalised pencils of curves.*

The assumptions in Theorem 4.1 are superficially stronger than in the remaining implication of Theorem 1.3, so we start by briefly discussing how Theorem 1.3 reduces to the special case in Theorem 4.1.

Proof of the "only if" part of Theorem 1.3, assuming Theorem 4.1 By [9, Corollary 8.3.16], if the complete doubling space (X, d, μ) supports a weak 1-Poincaré inequality, then d is biLipschitz equivalent to a geodesic metric d_g. Clearly, the measure μ remains doubling with respect to this new metric d_g. Further, by [9, Lemma 8.3.18], the space (X, d_g, μ) still supports a weak 1-Poincaré inequality. In fact, since (X, d_g) is geodesic, it follows from

[9, Remark 9.1.19] that (X, d_g, μ) even satisfies the weak 1-Poincaré inequality (1.1) with constant $\lambda = 1$; this is often called the 1-Poincaré inequality (without the attribute "weak").

We can thus apply Theorem 4.1 to (X, d_g, μ) in order to find a GPC between any pair of distinct points $s, t \in X$. Then T is also a GPC joining s to t in (X, d, μ), since the conditions (P1)–(P3) in Definition 1.2 are obviously invariant under bi-Lipschitz changes of metric. The least obvious is (P3), where one needs to recall that μ is a doubling measure, whence $\Theta_d(s, d(s, y)) \sim \Theta_{d_g}(s, d_g(s, y))$ and $\Theta_d(t, d(t, y)) \sim \Theta_{d_g}(t, d_g(t, y))$ for all $y \in X$.   □

As in the assumptions of Theorem 4.1, we now suppose that (X, d, μ) is a complete geodesic doubling metric measure space supporting the Poincaré inequality (1.1) with $p = 1$ and $\lambda = 1$.

4.2 Proof of Theorem 4.1

Fix two points $s, t \in X$, and write $B_0 := B(s, C_0 d(s, t))$ for the ball inside (the closure of) which we should find the current T as in Definition 1.2; the constant $C_0 \geq 1$ will be specified later, and its size only depends on the data of (X, d, μ), such as the doubling constant of μ, and the constant in the Poincaré inequality. We find the current T by initially constructing a sequence of approximating currents, each of them a sum of finitely many currents of the form discussed in Example 2.6. We start by defining a sequence of covers of B_0 by balls. For $n \in \mathbb{N}$, write $r_n := 2^{-n}$, and let $X_n \subset B_0$ be an r_n-net, that is, some maximal family of points $X_n \subset B_0$ satisfying $d(x, x') \geq r_n$ for all distinct $x, x' \in X_n$. We assume that r_n is far smaller than $d(s, t)$, and we require that

$$\{s, t\} \subset X_n. \tag{4.1}$$

We note that the collection of open balls $\mathcal{B}_n := \{B(x, 2r_n) : x \in X_n\}$ is now a cover of B_0. In fact, already the balls $B(x, r_n)$ would be a cover of B_0: by the maximality of X_n, for every $y \in B_0$ there exists $x \in X_n$ such that

$$y \in B(x, r_n). \tag{4.2}$$

Moreover, every ball $B \in \mathcal{B}_n$ only has boundedly many "neighbours":

$$\mathrm{card}\{B' \in \mathcal{B}_n : B \cap B' \neq \emptyset\} \lesssim 1. \tag{4.3}$$

This follows by using the doubling property of μ and the consequential relative lower volume decay (see [9, Lemma 8.1.13]), and noting that the balls $B(x, r_n/2)$, $x \in X_n$, are disjoint.

We will now construct a current T_n supported in $\Omega_n = B(s, C_0 d(s, t) + C r_n)$ for suitable constants $C_0, C \geq 1$ (depending on the constants in the 1-Poincaré inequality). The current T_n will be constructed using the *max flow–min cut* theorem from graph theory, and a subsequence of the currents T_n will eventually be shown, using the compactness theorem for normal currents, to converge to the desired current T supported on $\bar{B}_0$.

4.3 Graphs and flows

To apply the max flow–min cut theorem, we need to define a graph $G_n = (\mathcal{V}_n, E_n)$ associated to our problem, where $\mathcal{V}_n$ is a collection of vertices, and $E_n \subset \mathcal{V}_n \times \mathcal{V}_n$ is a collection of edges. We set $\mathcal{V}_n := X_n$, and

$$E_n := \{(x, x') \in \mathcal{V}_n \times \mathcal{V}_n : x \neq x' \quad \text{and} \quad B(x, 2r_n) \cap B(x', 2r_n) \neq \emptyset\}.$$

Note that $(x, x') \in E_n$ if and only if $(x', x) \in E_n$, and that the maximum degree of any vertex is uniformly bounded by (4.3). We also define a *capacity function* $c_n \colon E_n \to \mathbb{Q}_+$ satisfying

$$c_n(x, x') \sim \frac{\Theta(x, r_n)}{\Theta(s, d(s, x))} + \frac{\Theta(x', r_n)}{\Theta(t, d(t, x'))}, \quad (x, x') \in E_n, \tag{4.4}$$

if $\{x, x'\} \cap \{s, t\} = \varnothing$, and $c_n(x, x') \sim 1$ otherwise. Here $\Theta(x, r)$ is the (1-dimensional) density

$$\Theta(x, r) = \frac{\mu(B(x, r))}{r}, \quad x \in X, \ r > 0.$$

We do not specify the values of $c_n(x, x')$ more precisely: we will only use that $c_n(x, y)$ is a rational number within a constant multiple of the right hand side of (4.4). Note that $c_n(x, x') \sim c_n(x', x)$ for all $(x, x') \in E_n$; in fact, we may as well define $c_n(x, x') = c_n(x', x)$.

A *flow in* G_n is a function $f \colon E_n \to \mathbb{R}$ satisfying the following three conditions:

(F1) $f(x, x') = -f(x', x)$ for all $(x, x') \in E_n$,
(F2) $f(\{x\}, \mathcal{V}_n) = 0$ for all $x \in \mathcal{V}_n \setminus \{s, t\}$,
(F3) $f(x, x') \leq c_n(x, x')$ for all $(x, x') \in E_n$.

Informally, (F2) means that the total flow entering any vertex $x \in \mathcal{V}_n$ equals the total flow leaving x (except for the "source" and "sink" vertices s and t), and (F3) says that the flow is bounded by the capacity function. Here, and in the sequel, we write

$$f(\mathcal{U}, \mathcal{W}) = \sum_{e \in E_n(\mathcal{U}, \mathcal{W})} f(e),$$

where $E_n(\mathcal{U}, \mathcal{W}) = \{(x, x') \in E_n : x \in \mathcal{U} \text{ and } x' \in \mathcal{W}\}$, and analogously

$$c_n(\mathcal{U}, \mathcal{W}) = \sum_{e \in E_n(\mathcal{U}, \mathcal{W})} c_n(e).$$

The *norm* of a flow $f \colon E_n \to \mathbb{R}$ is defined to be the quantity

$$\|f\| := f(\{s\}, \mathcal{V}_n).$$

A *cut* is any pair $(\mathcal{S}, \mathcal{S}^c)$, where $\mathcal{S} \subset \mathcal{V}_n$ is a set with $s \in \mathcal{S}$ and $t \in \mathcal{S}^c$. As usual, $\mathcal{S}^c$ denotes the complement of $\mathcal{S}$ (in $\mathcal{V}_n$). The "total flow" of f over any cut $(\mathcal{S}, \mathcal{S}^c)$ equals $\|f\|$:

$$\|f\| = f(\mathcal{S}, \mathcal{S}^c). \tag{4.5}$$

In particular, $\|f\| = f(\mathcal{V}_n \setminus \{t\}, \{t\})$. For a proof of (4.5), see [5, Proposition 6.2.1]. Consequently, by (F3),

$$\|f\| \leq \min_{(\mathcal{S}, \mathcal{S}^c)} c_n(\mathcal{S}, \mathcal{S}^c), \tag{4.6}$$

where the min runs over all cuts $(\mathcal{S}, \mathcal{S}^c)$. In other words, the norm of any flow is bounded from above by the capacity of any cut in the graph.

A well-known theorem in graph theory due to Ford and Fulkerson [5] states that if c_n is integer-valued, then (4.6) is sharp: there exists a flow f with $\|f\| = \min c_n(\mathcal{S}, \mathcal{S}^c)$. We learned the theorem from Diestel's graph theory book, see [3, Theorem 6.2.2]. Our capacity c_n is not integer valued, but since $c_n(x, x') \in \mathbb{Q}_+$, and the cardinality of E_n is finite, we may assume that $c_n(x, x') \in \mathbb{N}$ by initially multiplying all quantities by a suitable integer.

The reader should view flows in G_n as discrete models for the current T_n: we will make the connection rigorous in Sect. 4.4. For now, we wish to find a uniform lower bound for the numbers $c_n(\mathcal{S}, \mathcal{S}^c)$, where $(\mathcal{S}, \mathcal{S}^c)$ is an arbitrary cut. We claim that

$$c_n(\mathcal{S}, \mathcal{S}^c) \gtrsim 1. \tag{4.7}$$

To this end, fix a cut $(\mathcal{S}, \mathcal{S}^c)$, and recall that $s \in \mathcal{S}$ and $t \in \mathcal{S}^c$ by definition. Also by definition,

$$\begin{aligned}
c_n(\mathcal{S}, \mathcal{S}^c) &= \sum_{(x,x') \in E_n(\mathcal{S}, \mathcal{S}^c)} c_n(x, x') \\
&\sim \sum_{(x,x') \in E_n(\mathcal{S}, \mathcal{S}^c)} \left(\frac{\Theta(x, r_n)}{\Theta(s, d(s, x))} + \frac{\Theta(x', r_n)}{\Theta(t, d(t, x'))} \right).
\end{aligned} \tag{4.8}$$

To be precise, (4.8) only holds if none of the edges in $E_n(\mathcal{S}, \mathcal{S}^c)$ start or end in $\{s, t\}$. We may assume this, since, for example, if $(s, x') \in E_n(\mathcal{S}, \mathcal{S}^c)$, then $c_n(\mathcal{S}, \mathcal{S}^c) \geq c_n(s, x') \gtrsim 1$, and (4.7) follows. In fact, the same argument holds a little more generally: if $(x, x') \in E_n(\mathcal{S}, \mathcal{S}^c)$ satisfies $\mathrm{dist}(x, \{s, t\}) \lesssim r_n$ or $\mathrm{dist}(x', \{s, t\}) \lesssim r_n$, then again $c_n(\mathcal{S}, \mathcal{S}^c) \gtrsim 1$, using the assumption that μ is doubling. So, without loss of generality, we assume that

$$\min\{\mathrm{dist}(x, \{s, t\}), \mathrm{dist}(x', \{s, t\})\} \geq Cr_n, \qquad (x, x') \in E_n(\mathcal{S}, \mathcal{S}^c). \tag{4.9}$$

where $C \geq 1$ is a suitable large constant to be specified later. If $C \geq 20$, say, then (4.9) has the following consequence:

$$d(s, y) \sim d(s, x) \sim d(s, x') \quad \text{and} \quad d(t, y) \sim d(t, x') \sim d(t, x) \tag{4.10}$$

for all $y \in \overline{B(x, 5r_n)} \cup \overline{B(x', 5r_n)}$, and all pairs $(x, x') \in E_n(\mathcal{S}, \mathcal{S}^c)$, since $d(x, x') \leq 4r_n$ for such pairs. Note that (4.10), combined with the doubling property of μ, also allows us to replace the denominators in (4.9) by some comparable quantities, as indicated by (4.10), for example

$$\Theta(t, d(t, x')) \sim \Theta(t, d(t, x)), \qquad (x, x') \in E_n(\mathcal{S}, \mathcal{S}^c). \tag{4.11}$$

Evidently, the proof of (4.7) should somehow use our only assumption: the Poincaré inequality (1.1) with $p = 1$. To this end, we define a Lipschitz function $u = u_n \colon B_0 \to \mathbb{R}$ associated to the cut $(\mathcal{S}, \mathcal{S}^c)$, using a Lipschitz partition of unity on B_0, subordinate to the cover $\mathcal{B}_n$. For $x \in \mathcal{V}_n$, let

$$\phi_x(y) = \chi_{B(x, 2r_n)}(y) \cdot \frac{2r_n - d(y, x)}{2r_n} \quad \text{and} \quad \psi_x := \frac{\phi_x}{\sum_{x' \in \mathcal{V}_n} \phi_{x'}}.$$

Then

$$\{y \in B_0 : \psi_x(y) > 0\} = B_0 \cap B(x, 2r_n), \tag{4.12}$$

and ψ_x and is (C/r_n)-Lipschitz on B_0: this is easy to check, noting that

$$\sum_{x' \in \mathcal{V}_n} \phi_{x'}(y) \sim 1, \qquad y \in B_0,$$

by (4.2) and the bounded overlap of the balls in $\mathcal{B}_n$. Evidently,

$$\sum_{x \in \mathcal{V}_n} \psi_x(y) = 1, \qquad y \in B_0,$$

so the family $\{\psi_x\}_{x \in \mathcal{V}_n}$ is the partition of unity on B_0 we were after. We set

$$u := \sum_{x \in \mathcal{S}} \psi_x.$$

Evidently, u takes values in $[0, 1]$, and is L_n-Lipschitz on B_0 for some $L_n \sim 1/r_n$. For $y \in X$ and any subset $\mathcal{U} \subset \mathcal{V}_n$, write

$$\mathcal{U}(y) := \{x \in \mathcal{U} : y \in B(x, 2r_n)\} = \{x \in \mathcal{U} : \phi_x(y) > 0\}$$

Clearly $\mathcal{S}(y), \mathcal{S}^c(y) \subset \mathcal{V}_n(y)$ and $\mathcal{V}_n(y) = \mathcal{S}(y) \cup \mathcal{S}^c(y)$ for all $y \in X$.

Lemma 4.2 *For $y \in B_0$, we have*

$$u(y) = 1 \Longleftrightarrow \mathcal{V}_n(y) = \mathcal{S}(y),$$

and

$$u(y) = 0 \Longleftrightarrow \mathcal{V}_n(y) = \mathcal{S}^c(y).$$

Proof If $u(y) = 1$, then

$$1 = \sum_{x \in \mathcal{S}} \psi_x(y) = \sum_{x \in \mathcal{S}} \frac{\phi_x(y)}{\sum_{x' \in \mathcal{V}_n} \phi_{x'}(y)} = \frac{\sum_{x \in \mathcal{S}(y)} \phi_x(y)}{\sum_{x' \in \mathcal{V}_n(y)} \phi_{x'}(y)}.$$

Hence

$$\sum_{x \in \mathcal{S}(y)} \phi_x(y) = \sum_{x' \in \mathcal{V}_n(y)} \phi_{x'}(y),$$

which forces $\mathcal{S}(y) = \mathcal{V}_n(y)$. The converse implication is clear.

If $u(y) = 0$, then $\psi_x(y) = 0$ for all $x \in \mathcal{S}$, so $\mathcal{S}(y) = \emptyset$. Consequently, $\mathcal{V}_n(y) \subset \mathcal{S}^c$, as claimed. The converse implication is again clear. $\square$

Corollary 4.3 *We have $u(s) = 1$ and $u(t) = 0$.*

Proof This is, in fact, a corollary of Lemma 4.2 and (4.9). Start with s: if $u(s) < 1$, then $\phi_x(s) > 0$ for some $x \in \mathcal{S}^c$, hence $s \in B(x, 2r_n)$ by (4.12). On the other hand, $s \in \mathcal{S}$ (by the very definition of a cut), so the fact that $s \in \mathrm{spt}\,\psi_x \subseteq B(x, 2r_n)$ implies $(s, x) \in E_n(\mathcal{S}, \mathcal{S}^c)$. This contradicts (4.9) as soon as $C \geq 2$, and hence we deduce that $u(s) = 1$.

The treatment of t is essentially symmetric: if $u(t) > 0$, then $\phi_x(t) > 0$ for some $x \in \mathcal{S}$. But since $t \in \mathcal{S}^c$, this implies that $(x, t) \in E_n(\mathcal{S}, \mathcal{S}^c)$, again violating (4.9). $\square$

Next, still using Lemma 4.2, we investigate where $\mathrm{Lip}(u, y) = 0$.

Lemma 4.4 *We have*

$$\{y \in B_0 : \mathrm{Lip}(u, y) \neq 0\} \subset \bigcup_{x \in \mathbf{Bd}(\mathcal{S})} B(x, 5r_n),$$

where

$$\mathbf{Bd}(\mathcal{S}) := \{x \in \mathcal{S} : \exists\, x' \in \mathcal{S}^c \text{ such that } (x, x') \in E_n\},$$

Proof Pick $y \in B_0$ with $\mathrm{Lip}(u, y) \neq 0$. We claim that this has the following consequence: *for all $x \in \mathcal{V}_n(y)$, the set $\mathcal{N}(x) := \{x\} \cup \{x' \in \mathcal{V}_n : (x, x') \in E_n\}$ intersects both $\mathcal{S}$ and $\mathcal{S}^c$.*

Assume to the contrary that there is some $x \in \mathcal{V}_n(y)$ with $\mathcal{N}(x) \subset \mathcal{S}$ or $\mathcal{N}(x) \subset \mathcal{S}^c$: we start with the case $\mathcal{N}(x) \subset \mathcal{S}$. Pick $z \in B_0 \cap B(x, 2r_n)$ arbitrarily, and consider any $x' \in \mathcal{V}_n(z)$. Then $z \in B(x', 2r_n)$ by definition, so

$$z \in B(x, 2r_n) \cap B(x', 2r_n) \implies (x, x') \in E_n \implies x' \in \mathcal{S}.$$

This shows that $\mathcal{V}_n(z) \subset \mathcal{S}$, hence $u(z) = 1$ by Lemma 4.2. But $z \in B_0 \cap B(x, 2r_n)$ was arbitrary, so we have inferred that $u \equiv 1$ on the neighbourhood $B_0 \cap B(x, 2r_n)$ of y. In particular $\mathrm{Lip}(u, y) = 0$, a contradiction.

Next, consider the case $\mathcal{N}(x) \subset \mathcal{S}^c$. As before, pick $z \in B_0 \cap B(x, 2r_n)$ arbitrarily, and deduce as above that $x' \in \mathcal{S}^c$ for all $x' \in \mathcal{V}_n(z)$. This implies by Lemma 4.2 that $u(z) = 0$, and hence $u \equiv 0$ on $B_0 \cap B(x, 2r_n)$. This contradicts $\mathrm{Lip}(u, y) \neq 0$.

Now that we have proven the claim in italics, we finish the proof of the lemma. Fix $y \in B_0$ with $\mathrm{Lip}(u, y) \neq 0$, pick any $x \in \mathcal{V}_n(y)$, and assume first that $x \in \mathcal{S}$. Then there exists $x' \in \mathcal{S}^c$ with $(x, x') \in E_n$. Hence $x \in \mathbf{Bd}(\mathcal{S})$ by definition, and $y \in B(x, 2r_n) \subset B(x, 5r_n)$, as claimed. Next, if $x \in \mathcal{S}^c$, then we have shown that there exists $x' \in \mathcal{S}$ with $(x', x) \in E_n$. This means that $x' \in \mathbf{Bd}(\mathcal{S})$. Since $B(x, 2r_n) \cap B(x', 2r_n) \neq \emptyset$, we infer that $y \in B(x', 5r_n)$, and the proof is complete.   □

We extend u to an L_n-Lipschitz map $X \to \mathbb{R}$ without change in the notation. Then, we note that $(u, \mathrm{Lip}(u, \cdot))$ is a function–upper gradient pair on X, and we apply Theorem 9.5 in [8], more precisely the implication "(4) $\implies$ (2)", which requires the space (X, d) to be geodesic. This implication gives the following estimate:

$$1 = |u(s) - u(t)| \lesssim \int_{B_0} \left(\frac{\mathrm{Lip}(u, y)}{\Theta(s, d(s, y))} + \frac{\mathrm{Lip}(u, y)}{\Theta(t, d(t, y))} \right) d\mu(y),$$

assuming that s, t are "deep enough inside" the ball B_0. This can be arranged by choosing $C_0 \geq 1$ in the definition of B_0 large enough. Then, from Lemma 4.4, we obtain further

$$1 \lesssim \frac{1}{r_n} \int_{B_0 \cap \bigcup_{x \in \mathbf{Bd}(\mathcal{S})} B(x, 5r_n)} \left(\frac{1}{\Theta(s, d(s, y))} + \frac{1}{\Theta(t, d(t, y))} \right) d\mu(y). \qquad (4.13)$$

Using the definition of $\mathbf{Bd}(\mathcal{S})$, and recalling (4.10)–(4.11), we may continue the estimate (4.13) as follows:

$$1 \lesssim \frac{1}{r_n} \sum_{x \in \mathbf{Bd}(\mathcal{S})} \left(\frac{\mu(B(x, r_n))}{\Theta(s, d(s, x))} + \frac{\mu(B(x, r_n))}{\Theta(t, d(t, x))} \right)$$

$$\leq \sum_{(x, x') \in E_n(\mathcal{S}, \mathcal{S}^c)} \left(\frac{\Theta(x, r_n)}{\Theta(s, d(s, x))} + \frac{\Theta(x', r_n)}{\Theta(t, d(t, x'))} \right) \lesssim c_n(\mathcal{S}, \mathcal{S}^c),$$

recalling (4.8) in the last inequality. This proves (4.7).

By multiplying the c_n by a constant (rational) factor, we may now arrange $c_n(\mathcal{S}, \mathcal{S}^c) \geq 1$ for all cuts $(\mathcal{S}, \mathcal{S}^c)$ with $s \in \mathcal{S}$ and $t \in \mathcal{S}^c$. Then, we are in a position to apply the max flow–min cut theorem: there exists a flow $f_n : E_n \to \mathbb{R}$ such that $\|f_n\| \geq 1$. Moreover, recalling (4.6), the norm of the flow f_n is bounded from above by the capacity of the cut $(\{s\}, \mathcal{V}_n \setminus \{s\})$. Since there are only boundedly many edges in E_n of the form (s, x), $x \in \mathcal{V}_n$, and the capacity of each one of them is $c_n(s, x) \sim 1$, we get

$$\|f_n\| \sim 1. \qquad (4.14)$$

4.4 Currents

In this section, we use the flow f_n constructed above to find a metric current T_n supported in a neighborhood of B_0. We define the current T_n as follows. For all edges $e = (x, x') \in E_n$, write $I_e := [0, d(x, x')]$, and let $\gamma_e \colon I_e \to X$ be an isometric embedding with $\gamma_e(0) = x$ and $\gamma_e(d(x, x')) = x'$. Then $\mathcal{H}^1(\gamma_e(I_e)) = d(x, x') \sim r_n$. We define

$$T_n := \sum_{e \in E_n} f_n(e) \gamma_{e\sharp} [\![I_e]\!],$$

where $[\![I_e]\!]$ is the current discussed in Example 2.6.

First, we compute the boundary of T_n, based on the facts that the boundary operation is linear, and we already know (recall (2.4)) the boundary of each term $\gamma_{e\sharp}[\![I_e]\!]$:

$$\partial T_n = \sum_{e \in E_n} f_n(e) \partial (\gamma_{e\sharp}[\![I_e]\!]) = \sum_{(x,y) \in E_n} f_n(x, y)[\delta_y - \delta_x]. \tag{4.15}$$

To simplify the expression further, we use the flow property (F2) of f_n, which says that

$$\sum_{(x,y) \in E_n} f_n(x, y) = 0, \qquad x \in \mathcal{V}_n \setminus \{s, t\}.$$

It follows, using also the flow property (F1), namely $f(x, y) = -f(y, x)$, that if $x \in \mathcal{V}_n \setminus \{s, t\}$, then the terms in (4.15) containing δ_x cancel out:

$$- \sum_{(x,y) \in E_n} f_n(x, y) \delta_x + \sum_{(y,x) \in E_n} f_n(y, x) \delta_x = -2 \sum_{(x,y) \in E_n} f_n(x, y) \delta_x = 0.$$

Consequently, all that remains in (4.15) are the terms containing s and t:

$$\partial T_n = -2 \sum_{(s,y) \in E_n} f_n(s, y) \delta_s - 2 \sum_{(t,y) \in E_n} f_n(t, y) \delta_t = 2\|f_n\| \delta_t - 2\|f_n\| \delta_s. \tag{4.16}$$

In the last equation, we again used $f_n(t, y) = -f_n(y, t)$, and the little proposition stated in (4.5) that $\|f_n\| = f(\mathcal{S}, \mathcal{S}^c)$ for any cut $(\mathcal{S}, \mathcal{S}^c)$, in particular for $(\mathcal{S}, \mathcal{S}^c) = (\mathcal{V}_n \setminus \{t\}, \{t\})$. Recalling (4.14), this yields that

$$\|\partial T_n\|(X) \le 4\|f_n\| \lesssim 1, \qquad n \in \mathbb{N}, \tag{4.17}$$

which in particular verifies that T_n is a normal current.

Next, we estimate the measures $\|T_n\|$ and find a uniform upper bound for $\|T_n\|(X)$. We recall from (2.3) that $\|\gamma_\sharp[\![I_e]\!]\| = \mathcal{H}^1 \llcorner_{\gamma_e(I_e)}$. It follows that

$$\|T_n\| \le \sum_{e \in E_n} |f_n(e)| \mathcal{H}^1 \llcorner_{\gamma_e(I_e)} \le \sum_{e \in E_n} c_n(e) \mathcal{H}^1 \llcorner_{\gamma_e(I_e)}. \tag{4.18}$$

Recalling that $|f_n(e)| \le c_n(e)$ (using the flow property (F3) and the fact that $c_n(x, x') = c_n(x', x)$ for all $(x, x') \in E_n$), we may now easily estimate $\|T_n\|(B)$ from above for all balls $B = B(x, r) \subset X$ with $r \ge r_n$. We claim that

$$\|T_n\|(B) \lesssim \int_{10B} \left(\frac{1}{\Theta(s, d(s, y))} + \frac{1}{\Theta(t, d(t, y))} \right) d\mu(y), \quad B = B(x, r) \subset X, \ r \ge r_n. \tag{4.19}$$

We start by disposing of a little technicality. Note that there are only boundedly many edges in E_n of the form (x, y) where $\min\{d(x, s), d(y, s)\} \le 5r_n$. We denote these edges by $E_n(s)$,

and we use the trivial estimate $c_n(e) \lesssim 1$ for all edges $e \in E_n(s)$. If B happens to intersect $\gamma_e(I_e)$ for one of the edges $e \in E_n(s)$, we estimate as follows:

$$\|T_n\|(B) \lesssim \sum_{e \in E_n \setminus E_n(s)} c_n(e)\mathcal{H}^1(\gamma_e(I_e) \cap B) + \sum_{e \in E_n(s)} r_n.$$

But if B intersects $\gamma_e(I_e)$ for an edge $e \in E_n(s)$, then $10B$ contains $B(s, r_n)$, and the second term above is bounded from above by the right hand side of (4.19):

$$\sum_{e \in E_n(s)} r_n \lesssim \int_{B(s,r_n)} \frac{d\mu(y)}{\Theta(s, d(s, y))} \leq \int_{10B} \left(\frac{1}{\Theta(s, d(s, y))} + \frac{1}{\Theta(y, d(t, y))} \right) d\mu(y),$$

using in the first inequality that $d(s, y) \sim r_n$ for $y \in B(s, r_n) \setminus B(s, r_n/2) =: A(s, r_n)$, and $\mu(A(s, r_n)) \sim \mu(B(s, r_n))$ by the doubling hypothesis (this also requires $A(s, r_n) \neq \emptyset$, which easily follows from the path connectedness of (X, d); see also [9, (8.1.17)]). We may dispose similarly of the situation where B meets $\gamma_e(I_e)$ for some $e \in E_n(t)$ (defined in the same way as $E_n(s)$). In other words, it remains to estimate the sum

$$\sum_{e \in E_n \setminus (E_n(s) \cup E_n(t))} c_n(e)\mathcal{H}^1(\gamma_e(I_e) \cap B) \lesssim \sum_{(x,y) \in E_n(B)} \left(\frac{\mu(B(x, r_n))}{\Theta(s, d(s, x))} + \frac{\mu(B(x, r_n))}{\Theta(t, d(t, x))} \right),$$

$$(4.20)$$

where $E_n(B) := \{e \in E_n \setminus [E_n(s) \cup E_n(t)] : \gamma_e(I_e) \cap B \neq \emptyset\}$. We used in (4.20) that

$$d(t, y) \sim d(t, x) \quad \text{and} \quad \mu(B(y, r_n)) \sim \mu(B(x, r_n)), \qquad (x, y) \in E_n(B).$$

To proceed, we recall again that every $x \in \mathcal{V}_n$ only has boundedly many neighbours in G_n, and all the vertices $x \in \mathcal{V}_n$ with at least one edge $(x, y) \in E_n(B)$ must lie at distance $\leq r_n$ from B, and at distance $\geq 5r_n$ from $\{s, t\}$. We denote the collection of such vertices by $\mathcal{V}_n(B)$. The observations above, and the bounded overlap of the balls $B(x, r_n)$, $x \in \mathcal{V}_n$, allow us to continue (4.20) as follows:

$$(4.20) \lesssim \sum_{x \in \mathcal{V}_n(B)} \left(\frac{\mu(B(x, r_n))}{\Theta(s, d(s, x))} + \frac{\mu(B(x, r_n))}{\Theta(t, d(t, x))} \right)$$

$$\lesssim \int_{10B} \left(\frac{1}{\Theta(s, d(s, y))} + \frac{1}{\Theta(t, d(t, y))} \right) d\mu(y).$$

Here we used, once again, that $d(s, y) \sim d(s, x)$ and $d(t, y) \sim d(t, x)$ for all $y \in B(x, r_n)$, whenever $\text{dist}(x, \{s, t\}) \geq 5r_n$. This concludes the proof of (4.19).

Finally, since the sets $\gamma_e(I_e)$, $e \in E_n$, are geodesics connecting vertices in $\mathcal{V}_n$, we infer that

$$\text{spt } T_n = \text{spt } \|T_n\| \subset B(s, C_0 d(s, t) + 5r_n) \subset 2B_0.$$

In particular, the supports of the currents T_n are contained in the fixed compact set $\overline{2B_0}$. Moreover, noting that $20B_0 = B(s, 20C_0 d(s, t)) \subset B(t, 21C_0 d(s, t))$, we can write either

$$20B_0 \subset \bigcup_{2^{-j} \leq 50C_0 d(s,t)} A(s, 2^{-j}) \quad \text{or} \quad 20B_0 \subset \bigcup_{2^{-j} \leq 50C_0 d(s,t)} A(t, 2^{-j}) \qquad (4.21)$$

For $j \in \mathbb{N}$ fixed, we can use the doubling property of the measure μ to estimate

$$\int_{A(s,2^{-j})} \frac{d\mu(y)}{\Theta(s, d(s, y))} = \int_{A(s,2^{-j})} \frac{d(s, y) \, d\mu(y)}{\mu(B(s, d(s, y)))} \lesssim 2^{-j} \frac{\mu(A(s, 2^{-j}))}{\mu(B(s, 2^{-j}))},$$

and a similar bound holds with s replaced by t. Hence, using first (4.19), and then splitting the integration domain as in (4.21), we obtain the following uniform upper bound for the masses of the currents T_n:

$$\|T_n\|(X) = \|T_n\|(2B_0) \lesssim \int_{20B_0} \left(\frac{1}{\Theta(s, d(s, y))} + \frac{1}{\Theta(t, d(t, y))} \right) d\mu(y)$$

$$\lesssim \sum_{2^{-j} \leq 50C_0 d(s,t)} 2^{-j} \left[\frac{\mu(A(s, 2^{-j}))}{\mu(B(s, 2^{-j}))} + \frac{\mu(A(t, 2^{-j}))}{\mu(B(t, 2^{-j}))} \right] \lesssim \operatorname{diam}(B_0).$$

(4.22)

Recalling also (4.17), we are now in a position to use the compactness theorem for normal currents, Theorem 2.7: there exists a normal current T, supported on $\bar{B}_0$, such that

$$\lim_{m \to \infty} T_{n_m}(g, \pi) = T(g, \pi), \qquad (g, \pi) \in \operatorname{Lip}_b(X) \times \operatorname{Lip}(X), \tag{4.23}$$

for some subsequence $(n_m)_{m \in \mathbb{N}}$. From the proof of the compactness theorem [1, Theorem 5.2] in the paper of Ambrosio and Kirchheim, in particular "Step 2", one can read that the measure $\|T\|$ is bounded from above by a certain measure which is obtained as a weak limit of the measures $\|T_{n_m}\|$. As a consequence, we infer that (4.19) holds for the measure $\|T\|$, and for all balls $B(x, r) \subset X$. This fact, combined with the Lebesgue differentiation theorem, shows that $\|T\|$ is absolutely continuous with respect to the measure μ, with Radon-Nikodym derivative bounded by

$$\frac{d\|T\|}{d\mu}(x) \lesssim \frac{1}{\Theta(s, d(s, x))} + \frac{1}{\Theta(t, d(t, x))}.$$

So, to conclude the proof of Theorem 4.1, it remains to show that $\partial T = C\delta_t - C\delta_s$ for some constant $C \sim 1$. From (4.23) we infer that

$$\partial T(g) = \lim_{m \to \infty} \partial T_{n_m}(g) = \lim_{m \to \infty} (\|f_{n_m}\| g(t) - \|f_{n_m}\| g(s)), \qquad g \in \operatorname{Lip}_b(X).$$

Apply this to a bump function g satisfying $g(t) = 1$ and $g(s) = 0$ to find that the numbers $\|f_{n_m}\| \sim 1$ converge to a limit $C = \partial T(g)$, which evidently also satisfies $C \sim 1$. Thus $\partial T(g) = Cg(t) - Cg(s)$ for all $g \in \operatorname{Lip}_b(X)$, which by definition means that $\partial T = C\delta_t - C\delta_s$. Finally, to be precise, the current mentioned in the definition of generalised pencil of curves can be obtained by dividing T by C. The proof of Theorem 4.1 is complete.

5 GPC and pencils of curves

In this section, we prove Theorem 1.4. As explained in the introduction, the implication

$$X \text{ admits PCs} \Longrightarrow X \text{ satisfies the weak } 1 - \text{Poincaré inequality}$$

is due to Semmes for Q-regular spaces (X, d, μ), see [15, Theorem B.15], and a proof of the general case can be found in [8, Chapter 4]. So, we concentrate on the reverse implication. By the main result of this paper, Theorem 1.3, it suffices to verify the following:

$$X \text{ admits GPCs} \Longrightarrow X \text{ admits PCs}.$$

The proof is a straightforward application of a decomposition result [13, Theorem 5.1] of Paolini and Stepanov, which we now recall for the reader's convenience. Let C, T be 1-currents on X. We say that C is a *subcurrent* of T, denoted $C \leq T$, if

$$\|T - C\|(X) + \|C\|(X) \leq \|T\|(X).$$

We say that C is a *cycle* of T, if $C \leq T$ and $\partial C = 0$. A 1-current T is *acyclic*, if its only cycle is the trivial 1-current $C = 0$.

By a *curve* in X, we mean the image of a Lipschitz map $\theta \colon [0, 1] \to X$. The space of curves in X is denoted by Γ. We slightly abuse notation by using the letter θ to denote both Lipschitz mappings $[0, 1] \to X$, and elements in Γ. There is a natural metric d_Γ on Γ, defined in [13, (2.1)], and Borel sets in Γ are defined using the topology induced by d_Γ. Following the terminology above [13, (4.1)], a finite positive Borel measure on Γ is called a *transport*. If $\theta \in \Gamma$ has an injective parametrisation, then θ is called an *arc*.

By [13, Theorem 5.1], a normal acyclic 1-current T on X is decomposable in arcs. Combining [13, Definition 4.4] and [13, Lemma 4.17], this means that there exists a transport η on Γ such that η almost every curve in Γ is an arc, and moreover the following equalities hold:

$$\|T\| = \int_\Gamma \mathcal{H}^1 \lfloor_\theta d\eta(\theta), \tag{5.1}$$

and

$$\eta(1) = (\partial T)^+ \quad \text{and} \quad \eta(0) = (\partial T)^-. \tag{5.2}$$

The measures $(\partial T)^+$ and $(\partial T)^-$ appearing in (5.2) are the positive and negative parts of the measure ∂T, and $\eta(i)$ is the measure defined by

$$\eta(i)(A) = \eta\{\theta \in \Gamma : \theta(i) \in A\}, \qquad i \in \{0, 1\}.$$

If the reader checks carefully [13, Lemma 4.17], then she will find the measure "$\mu_{[[\theta]]}$" (in our notation $\|\theta_\sharp[[0, 1]]\|$) in place of "$\mathcal{H}^1 \lfloor_\theta$" in (5.1), but these measures coincide if θ is an arc. Indeed, in that case [14, Lemma 2.2] shows that $\|\theta_\sharp[[0, 1]]\| = \theta_\sharp(|\dot\theta|\mathcal{L}^1)$, where $|\dot\theta|$ denotes the metric derivative. Then it can be seen for instance by [2, Theorem 4.1.6] that $\theta_\sharp(|\dot\theta|\mathcal{L}^1)$ agrees with $\mathcal{H}^1 \lfloor_\theta$.

Now, we apply the decomposition result to our concrete situation. Assume that $s, t \in X$, and T is a GPC joining s to t, as in Definition 1.2. There is no guarantee that T is acyclic, but [13, Proposition 3.8] comes to our rescue: it gives a cycle $C \leq T$ such that $T' = T - C$ is acyclic. Clearly

$$\partial T' = \partial T - \partial C = \partial T = \delta_t - \delta_s.$$

Moreover, $\|T'\| + \|C\| = \|T\|$ by [13, (3.1)], so in particular $\|T'\| \leq \|T\|$. It follows that T' is an acyclic GPC joining s to t. With this argument in mind, we may assume that T was acyclic to begin with.

Hence, a decomposition as in (5.1)–(5.2) exists. In particular, (5.2) implies that

$$\eta\{\theta \in \Gamma : \theta(1) = t\} = \eta(1)(\{t\}) = (\partial T)^+(\{t\}) = \delta_t(\{t\}) = 1,$$

and similarly $\eta\{\theta \in \Gamma : \theta(0) = s\} = 1$. So, there exists a set $\Gamma_{s,t} \subset \Gamma$ of arcs of full η measure such that $\theta(0) = s$ and $\theta(1) = t$ for all $\theta \in \Gamma_{s,t}$. Moreover, if $g \colon X \to [0, \infty]$ is a Borel function, then (5.1) shows that

$$\int_{\Gamma_{s,t}} \int_\theta g \, d\mathcal{H}^1 \, d\eta(\theta) = \int g \, d\|T\| \lesssim \int_{B(s, C_0 d(s,t))} \left(\frac{g(y)}{\Theta(s, d(s, y))} + \frac{g(y)}{\Theta(t, d(t, y))} \right) d\mu(y),$$

taking into account that spt $\|T\| \subset B(s, C_0 d(s, t))$ by (P2). The only point we are still missing from Definition 1.1 is the claim that the arcs in $\Gamma_{s,t}$ are quasiconvex. This need not be true to begin with, but we note that

$$\|T\|(X) \lesssim d(s, t).$$

This can be easily seen from (P2)–(P3). Alternatively, our specific construction for T gives this estimate, see (4.22). So, it follows from (5.1) that half of the arcs $\theta \in \Gamma$, relative to the measure η, satisfy the quasiconvexity requirement $\mathcal{H}^1(\theta) \lesssim d(s, t)$. The proof is now completed by restricting $\Gamma_{s,t}$ to this "good half" of the arcs.

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