Correction to: Characterization of n -rectifiability in terms of Jones' square function: part I

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ABSTRACT. In the author's paper "Characterization of n -rectifiability in terms of Jones' square function: Part I", Calc. Var. PDE. (2015), no. 4, 3643–3665, there is a gap in the proof of Main Lemma 2.1. In this note, this gap is corrected.

1 Introduction

In [2], it is shown that if μ is an n -rectifiable measure in $\mathbb{R}^d$, then

$$\int_0^\infty \beta_{2,\mu}^n(x, r)^2 \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

(for the notation, see the aforementioned paper). An essential step for the proof is the following result (Main Lemma 2.1 in [2]):

Main Lemma *Let μ be a finite Borel measure on $\mathbb{R}^d$ and let $\Gamma \subset \mathbb{R}^d$ be an n -dimensional Lipschitz graph in $\mathbb{R}^d$. Then*

$$\int_0^\infty \alpha_\mu^n(x, r)^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma.$$

The proof of this result in [2] has a gap because of an incorrect statement just above (2.22) in that paper. In this note, this gap is corrected. The new arguments are along the same lines as in [2], but they require many changes of technical type.

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2 The proof

2.1 Preliminaries

Given two finite Borel measures σ, μ on $\mathbb{R}^d$ and a closed ball $B \subset \mathbb{R}^d$, we set

$$\text{dist}_B(\sigma, \mu) := \sup \left\{ \left| \int f d\sigma - \int f d\mu \right| : \text{Lip}(f) \leq 1, \text{supp}(f) \subset B \right\},$$

where $\text{Lip}(f)$ stands for the Lipschitz constant of f . We also set

$$\alpha_\mu^n(B) = \frac{1}{r(B)^{n+1}} \inf_{a \geq 0, L} \text{dist}_B(\mu, a\mathcal{H}_L^n),$$

where the infimum is taken over all the constants $a \geq 0$ and all the n -planes L which intersect B . If the n -plane L is fixed, we write

$$\alpha_{\mu, L}^n(B) = \frac{1}{r(B)^{n+1}} \inf_{a \geq 0} \text{dist}_B(\mu, a\mathcal{H}_L^n).$$

We will drop the exponent n , and we will write $\alpha_\mu(x, r)$ and $\alpha_{\mu, L}(x, r)$ instead of $\alpha_\mu^n(\bar{B}(x, r))$ and $\alpha_{\mu, L}^n(\bar{B}(x, r))$ to shorten the notation.

2.2 The case when μ is supported on Γ

We will prove first the following partial result, which can be deduced by stopping time arguments analogous to some of the ones in [2]. However, for completeness, we show here the detailed proof.

Lemma 2.1 *Let $\Gamma \subset \mathbb{R}^d$ be an n -dimensional Lipschitz graph in $\mathbb{R}^d$, and let μ be a finite Borel measure supported on Γ . Then*

$$\int_0^\infty \alpha_\mu(x, r)^2 \frac{dr}{r} < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma.$$

To prove the preceding result, we need to introduce the following “ Γ -cubes” associated with Γ . We assume that Γ equals the graph of a Lipschitz function $A : \mathbb{R}^n \rightarrow \mathbb{R}^{d-n}$. We say that $Q \subset \Gamma$ is a Γ -cube if it is a subset of the form $Q = \Gamma \cap (Q_0 \times \mathbb{R}^{d-n})$, where $Q_0 \subset \mathbb{R}^n$ is an n -dimensional cube. We denote $\ell(Q) := \ell(Q_0)$. We say that Q is a dyadic Γ -cube if Q_0 is a dyadic cube. The center of Q is the point $x_Q = (x_{Q_0}, A(x_{Q_0}))$, where x_{Q_0} is the center of Q_0 . The collection of dyadic Γ -cubes Q with $\ell(Q) = 2^{-j}$ is denoted by $\mathcal{D}_{\Gamma, j}$. Also, we set $\mathcal{D}_\Gamma = \bigcup_{j \in \mathbb{Z}} \mathcal{D}_{\Gamma, j}$ and $\mathcal{D}_\Gamma^k = \bigcup_{j \geq k} \mathcal{D}_{\Gamma, j}$.

Given a Γ -cube Q , we denote by B_Q a closed ball concentric with Q with $r(B_Q) = 3 \text{diam}(Q)$. Note that B_Q contains Q and is centered on Γ . We set

$$\alpha_\mu(Q) := \alpha_\mu(B_Q).$$

By standard methods, it follows that Lemma 2.1 is an immediate consequence of the following more precise result:

Lemma 2.2 *Let Γ be an n -dimensional Lipschitz graph in $\mathbb{R}^d$, let μ be a finite Borel measure supported on Γ , and let $\mathcal{D}_\Gamma$ be the lattice of Γ -cubes introduced above. Denoting by L_Q the n -plane that minimizes $\alpha_{\mathcal{H}^n|_\Gamma}(Q)$, we have*

$$\sum_{Q \in \mathcal{D}_\Gamma : x \in Q} \alpha_{\mu, L_Q}(Q)^2 < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma.$$

Proof First note that if $\mu = g \mathcal{H}^n|_\Gamma$ with $g \in L^\infty(\mathcal{H}^n|_\Gamma)$, then the conclusion of the lemma is true. Indeed, it turns out that the measure $\sigma := (1 + g) \mathcal{H}^n|_\Gamma$ is n -AD-regular and thus

$$\sum_{Q \in \mathcal{D}_\Gamma: x \in Q, \ell(Q) \leq 1} \alpha_{\sigma, L_Q}(Q)^2 < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma,$$

by the results from [1]. Then, for any $Q \in \mathcal{D}_\Gamma$, we have

$$\alpha_{\mu, L_Q}(Q) \leq \alpha_{\mathcal{H}^n|_\Gamma, L_Q}(Q) + \alpha_{\sigma, L_Q}(Q),$$

and thus

$$\begin{aligned} \sum_{Q \in \mathcal{D}_\Gamma: x \in Q, \ell(Q) \leq 1} \alpha_{\mu, L_Q}(Q)^2 &\lesssim \sum_{Q \in \mathcal{D}_\Gamma: x \in Q, \ell(Q) \leq 1} \alpha_{\mathcal{H}^n|_\Gamma, L_Q}(Q)^2 \\ &+ \sum_{Q \in \mathcal{D}_\Gamma: x \in Q, \ell(Q) \leq 1} \alpha_{\sigma, L_Q}(Q)^2 < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma, \end{aligned}$$

On the other hand, using just that μ is a finite measure, it follows easily that

$$\sum_{Q \in \mathcal{D}_\Gamma: x \in Q, \ell(Q) \geq 1} \alpha_{\mu, L_Q}(Q)^2 < \infty \quad \text{for all } x \in \Gamma,$$

and so we have

$$\sum_{Q \in \mathcal{D}_\Gamma: x \in Q} \alpha_{\mu, L_Q}(Q)^2 < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma.$$

In the case when μ is an arbitrary finite Borel measure supported on Γ , we consider a suitable Calderón-Zygmund decomposition of μ with respect to $\mathcal{H}^n|_\Gamma$. So for some big $\lambda > 0$, we let $\mathcal{B}$ be the maximal family of Γ -cubes P such that $\mu(P) > \lambda \mathcal{H}^n(P)$, and then we have

$$\mu = g \mathcal{H}^n|_\Gamma + \nu,$$

where

$$g \mathcal{H}^n|_\Gamma = \mu|_G + \sum_{P \in \mathcal{B}} \frac{\mu(P)}{\mathcal{H}^n(P)} \mathcal{H}^n|_P,$$

with $G = \mathbb{R}^d \setminus \bigcup_{P \in \mathcal{B}} P$, and also

$$\nu = \sum_{P \in \mathcal{B}} \nu_P, \quad \nu_P = \mu|_P - \frac{\mu(P)}{\mathcal{H}^n(P)} \mathcal{H}^n|_P.$$

By standard arguments, it is clear that $\|g\|_{L^\infty(\mu)} \lesssim \lambda$, and thus if we denote $\sigma = g d\mathcal{H}^n|_\Gamma$, then

$$\sum_{Q \in \mathcal{D}_\Gamma: x \in Q} \alpha_{\sigma, L_Q}(Q)^2 < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma. \quad (2.1)$$

Let $\tilde{G} = \Gamma \setminus \bigcup_{P \in \mathcal{B}} 3B_P$, and observe that

$$\mathcal{H}^n(\Gamma \setminus \tilde{G}) = \sum_{P \in \mathcal{B}} \mathcal{H}^n(3B_P) \leq c \sum_{P \in \mathcal{B}} \mathcal{H}^n(P) \leq \frac{c}{\lambda} \sum_{P \in \mathcal{B}} \mu(P) \leq \frac{c}{\lambda} \|\mu\|.$$

Since λ can be taken arbitrarily big, to prove the lemma, it suffices to show that

$$\sum_{Q \in \mathcal{D}_\Gamma: x \in Q} \alpha_{\mu, L_Q}(Q)^2 < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \tilde{G}. \quad (2.2)$$

Denote by $\mathcal{G}$ the family of those Γ -cubes which are not contained in $\bigcup_{P \in \mathcal{B}} 3B_P$. Notice that, for $x \in \tilde{G}$, all the Γ -cubes in the sum in (2.2) are from $\mathcal{G}$. For a given $Q \in \mathcal{G}$, we have

$$\alpha_{\mu, L_Q}(Q) \leq \alpha_{\sigma, L_Q}(Q) + \frac{1}{\ell(Q)^{n+1}} \sup_f \left| \int f \, dv \right|,$$

where the supremum is taken over all functions f which are 1-Lipschitz and supported on B_Q . To estimate the last integral on the right hand side, we write

$$\left| \int f \, dv \right| \leq \sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \left| \int f \, dv_P \right|.$$

Since $\int f \, dv_P = 0$ for all $P \in \mathcal{B}$, we have

$$\left| \int f \, dv_P \right| \leq \int |f - f(x_P)| \, d|v_P| \lesssim \ell(P) \|v_P\| \lesssim \ell(P) \mu(P).$$

Thus,

$$\alpha_{\mu, L_Q}(Q) \lesssim \alpha_{\sigma, L_Q}(Q) + \sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \frac{\ell(P) \mu(P)}{\ell(Q)^{n+1}}.$$

So, because of (2.1), to prove the lemma, it suffices to show that

$$\sum_{Q \in \mathcal{G}: x \in Q} \left(\sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \frac{\ell(P) \mu(P)}{\ell(Q)^{n+1}} \right)^2 < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \tilde{G}. \quad (2.3)$$

To prove (2.3), first we take into account that if $P \in \mathcal{B}$ is such that $P \cap B_Q \neq \emptyset$, then

$$r(B_P) \leq r(B_Q), \quad (2.4)$$

because otherwise $Q \subset B_Q \subset 3B_P$, which contradicts the fact that $Q \in \mathcal{G}$. Now, from (2.4), we deduce that $P \subset B_P \subset 3B_Q$, and taking also into account that $\mu(P) \lesssim \lambda \ell(P)^n$, we get

$$\sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \frac{\ell(P) \mu(P)}{\ell(Q)^{n+1}} \lesssim \lambda \sum_{P \in \mathcal{B}: P \subset 3B_Q} \frac{\ell(P)^{n+1}}{\ell(Q)^{n+1}} \lesssim \lambda.$$

Therefore,

$$\sum_{Q \in \mathcal{G}: x \in Q} \left(\sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \frac{\ell(P) \mu(P)}{\ell(Q)^{n+1}} \right)^2 \lesssim \lambda \sum_{Q \in \mathcal{G}: x \in Q} \sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \frac{\ell(P) \mu(P)}{\ell(Q)^{n+1}}.$$

Now we can show that the last sum is finite for $\mathcal{H}^n$ -a.e. $x \in \tilde{G}$ by integrating this with respect to $\mathcal{H}^n|_\Gamma$. Then we obtain

$$\begin{aligned} \int_\Gamma \sum_{Q \in \mathcal{G}: x \in Q} \sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \frac{\ell(P) \mu(P)}{\ell(Q)^{n+1}} d\mathcal{H}^n(x) &\approx \sum_{Q \in \mathcal{G}} \sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \frac{\ell(P) \mu(P)}{\ell(Q)} \\ &= \sum_{P \in \mathcal{B}} \mu(P) \sum_{Q \in \mathcal{G}: P \cap B_Q \neq \emptyset} \frac{\ell(P)}{\ell(Q)}. \end{aligned}$$

Recalling that $r(B_Q) \geq r(B_P)$ for every Q in the last sum, we get $\sum_{Q \in \mathcal{G}: P \cap B_Q \neq \emptyset} \frac{\ell(P)}{\ell(Q)} \lesssim 1$, and so we deduce that

$$\int_\Gamma \sum_{Q \in \mathcal{G}: x \in Q} \sum_{P \in \mathcal{B}: P \cap B_Q \neq \emptyset} \frac{\ell(P) \mu(P)}{\ell(Q)^{n+1}} d\mathcal{H}^n(x) \lesssim \sum_{P \in \mathcal{B}} \mu(P) \leq \|\mu\|,$$

which implies (2.3) and completes the proof of the lemma. $\square$

2.3 The approximating measure σ for the general case

To prove the main lemma in full generality, we consider a Whitney decomposition of $\mathbb{R}^d \setminus \Gamma$ as in Section 2.3 of [2]. That is, we have a family $\mathcal{W}$ of dyadic cubes in $\mathbb{R}^d$ with disjoint interiors such that

$$\bigcup_{Q \in \mathcal{W}} Q = \mathbb{R}^d \setminus \Gamma,$$

and moreover, there are some constants $R > 20$ and $D_0 \geq 1$ such the following holds for every $Q \in \mathcal{W}$:

- (i) $10Q \subset \mathbb{R}^d \setminus \Gamma$;
- (ii) $RQ \cap \Gamma \neq \emptyset$;
- (iii) there are at most D_0 cubes $Q' \in \mathcal{W}$ such that $10Q \cap 10Q' \neq \emptyset$. Further, for such cubes Q' , we have $\ell(Q') \approx \ell(Q)$.

From properties (i) and (ii), it is clear that $\text{dist}(Q, \Gamma) \approx \ell(Q)$. We assume that the Whitney cubes are small enough so that

$$\text{diam}(Q) < \text{dist}(Q, \Gamma). \quad (2.5)$$

This can be achieved by replacing each cube $Q \in \mathcal{W}$ by its descendants $P \in \mathcal{D}_k(Q)$, for some fixed $k \geq 1$, if necessary. From (2.5), we infer that if $Q \in \mathcal{W}$ intersects some ball $B(y, r)$ with $y \in \Gamma$, then

$$\text{diam}(Q) \leq r, \quad (2.6)$$

and thus

$$Q \subset B(y, 3r). \quad (2.7)$$

For a given Borel measure μ , we denote

$$M_n \mu(x) = \sup_{r>0} \frac{\mu(B(x, r))}{r^n}.$$

From the growth condition $\mathcal{H}^n(\Gamma \cap B(x, r)) \leq c r^n$ for all x and $r > 0$, it follows easily that the maximal operator M_n is bounded from the space of finite signed Radon measures

$M(\mathbb{R}^d)$ into $L^{1,\infty}(\mathcal{H}^n|_\Gamma)$. As a consequence, for any arbitrary finite Borel measure μ in $\mathbb{R}^d$, $M_n\mu(x) < \infty$ for $\mathcal{H}^n$ -a.e $x \in \Gamma$.

Denote by Π_Γ the projection on Γ given by $\Pi_\Gamma(x) = (x_1, \dots, x_n, A(x_1, \dots, x_n))$. To each cube $Q \in \mathcal{W}$ we associate a ball $\tilde{B}_Q$ with radius $r(\tilde{B}_Q) = \frac{1}{4}\ell(Q)$, centered in $\tilde{x}_Q := \Pi_\Gamma(x_Q)$, where x_Q is the center of Q . In particular, we have $\text{dist}(Q, \tilde{B}_Q) \approx \ell(Q) \approx r(\tilde{B}_Q)$.

Let $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth non-negative radial function supported in $B(0, 1)$ which equals 1 in $B(0, 1/2)$. For each $Q \in \mathcal{W}$, we consider the function $\varphi_Q : \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$\varphi_Q(x) = \varphi\left(\frac{x - \tilde{x}_Q}{r(\tilde{B}_Q)}\right)$$

and then we set

$$g_Q(x) = \frac{\mu(Q)}{\|\varphi_Q\|_{L^1(\mathcal{H}^n|_\Gamma)}} \varphi_Q(x), \quad (2.8)$$

and also

$$g(x) = \sum_{Q \in \mathcal{W}} g_Q(x).$$

Notice that $\int g_Q d\mathcal{H}^n|_\Gamma = \mu(Q)$ and also that

$$\|g_Q\|_\infty \lesssim \frac{\mu(Q)}{\ell(Q)^n}, \quad \|\nabla g_Q\|_\infty \lesssim \frac{\mu(Q)}{\ell(Q)^{n+1}}.$$

Then we consider the following measure σ on Γ , which approximates $\mu|_{\Gamma^c}$, in a sense:

$$\sigma = g d\mathcal{H}^n|_\Gamma = \sum_{Q \in \mathcal{W}} g_Q d\mathcal{H}^n|_\Gamma. \quad (2.9)$$

Next, for some constant $A > 10$ to be chosen below, we also consider the auxiliary function

$$G_A(x) = \sum_{Q \in \mathcal{W}} \frac{\mu(Q)}{\ell(Q)^n} \chi_{A\tilde{B}_Q}.$$

Notice that, for some absolute constant C ,

$$g(x) \leq C G_A(x) \quad \text{for all } x \in \Gamma.$$

Also,

$$\|G_A\|_{L^1(\mathcal{H}^n|_\Gamma)} \leq C \sum_{Q \in \mathcal{W}} \frac{\mu(Q) A^n r(\tilde{B}_Q)^n}{\ell(Q)^n} \leq C A^n \|\mu\|,$$

and thus $G_A(x) < \infty$ for $\mathcal{H}^n$ -a.e. $x \in \Gamma$.

2.4 The α_μ coefficients of the Γ -cubes

Lemma 2.3 *Assume A big enough. Then, for each $Q \in \mathcal{D}_\Gamma$, we have*

$$\begin{aligned} \alpha_\mu(Q) &\lesssim \alpha_{\mu|_{\Gamma}, L_Q}(Q) + \alpha_{\sigma, L_Q}(Q) + \int_{6B_Q} \frac{\text{dist}(x, \Gamma)}{\ell(Q)^{n+1}} d\mu(x) \\ &\quad + \inf_{x \in Q} G_A(x) \alpha_{\mathcal{H}^n|_\Gamma}(Q) + \sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}}, \end{aligned}$$

where L_Q is the n -plane that minimizes $\alpha_{\mathcal{H}^n|_\Gamma}(Q)$.

Proof Note first that $\alpha_\mu(Q) \leq \alpha_{\mu|_{\Gamma}, L_Q}(Q) + \alpha_{\mu|_{\Gamma^c}, L_Q}(Q)$. So we just have to deal with $\alpha_{\mu|_{\Gamma^c}, L_Q}(Q)$. To estimate this coefficient, we consider a 1-Lipschitz function f supported on B_Q , and we write

$$\int f d\mu|_{\Gamma^c} = \sum_{P \in \mathcal{W}} \int f d(\mu|_P - g_P \mathcal{H}^n|_\Gamma) + \int f d\sigma,$$

where g_P is the function defined in (2.8) and σ the measure in (2.9).

So, for any constant $a_Q \in \mathbb{R}$, we get

$$\alpha_{\mu|_{\Gamma^c}, L_Q}(Q) \leq \alpha_{\sigma, L_Q}(Q) + \frac{1}{\ell(Q)^{n+1}} \sup_f \left| \sum_{P \in \mathcal{W}} \int f d(\mu|_P - g_P \mathcal{H}^n|_\Gamma) + a_Q \int f d\mathcal{H}^n|_{L_Q} \right|,$$

where the supremum is taken over all 1-Lipschitz functions supported on B_Q .

We denote by I_a the subfamily of the cubes from $\mathcal{W}$ which intersect $2B_Q$, and $I_b = \mathcal{W} \setminus I_a$, and we split the last sum above as follows:

$$\begin{aligned} &\left| \sum_{P \in \mathcal{W}} \int f d(\mu|_P - g_P \mathcal{H}^n|_\Gamma) + a_Q \int f d\mathcal{H}^n|_{L_Q} \right| \\ &\leq \sum_{P \in I_a} \left| \int f d(\mu|_P - g_P \mathcal{H}^n|_\Gamma) \right| + \left| \sum_{P \in I_b} \int f d(\mu|_P - g_P \mathcal{H}^n|_\Gamma) + a_Q \int f d\mathcal{H}^n|_{L_Q} \right| \\ &=: S_a + S_b. \end{aligned}$$

First we deal with the sum S_a . For each $P \in I_a$, since $\int g_P d\mathcal{H}^n|_\Gamma = \mu(P)$, we deduce that

$$\begin{aligned} \left| \int f d(\mu|_P - g_P \mathcal{H}^n|_\Gamma) \right| &\leq \left| \int_P (f(x) - f(x_P)) d\mu(x) \right| \\ &\quad + \left| \int (f(x_P) - f(x)) g_P(x) \mathcal{H}^n|_\Gamma(x) \right|. \end{aligned}$$

To deal with the first integral on the right-hand side, we take into account that for $x \in P$ we have

$$|f(x) - f(x_P)| \leq \|\nabla f\|_\infty |x - x_P| \leq c \ell(P). \quad (2.10)$$

Concerning the second integral, recall that $\text{supp } g_P \subset \Gamma \cap \bar{B}(x_P, c \ell(P))$, and thus we also have $|x - x_P| \leq c \ell(P)$ in the domain of integration, so that (2.10) holds in this case too.

Therefore,

$$\left| \int f d(\mu|_P - g_P \mathcal{H}^n|_\Gamma) \right| \leq c \ell(P) \mu(P) \approx \int_P \text{dist}(x, \Gamma) d\mu(x),$$

where we took into account that $\text{dist}(x, \Gamma) \approx \ell(P)$ for every $x \in P$. Recall that $\text{supp } f \subset B_Q$ and since $P \in I_a$, then $P \subset 6B_Q$ by the argument in (2.7). Thus,

$$S_a = \sum_{P \in I_a} \left| \int f d(\mu|_P - g_P \mathcal{H}^n|_\Gamma) \right| \leq c \int_{6B_Q} \text{dist}(x, \Gamma) d\mu(x). \quad (2.11)$$

Next we consider the sum S_b . For each $P \in I_b$, we have $P \cap 2B_Q = \emptyset$, and so $\int f d\mu|_P = 0$. Therefore,

$$\begin{aligned} S_b &= \left| \sum_{P \in I_b} \int f g_P d\mathcal{H}^n|_\Gamma - a_Q \int f d\mathcal{H}^n|_{L_Q} \right| \\ &\leq \left| \sum_{P \in I_b} \int f g_P d(\mathcal{H}^n|_\Gamma - c_Q \mathcal{H}^n|_{L_Q}) \right| + \int |f| \left| \sum_{P \in I_b} c_Q g_P - a_Q \right| d\mathcal{H}^n|_{L_Q} \\ &=: S_{b,1} + S_{b,2}, \end{aligned}$$

where c_Q is the constant minimizing $\alpha_{\mathcal{H}^n|_\Gamma, L_Q}(Q)$. To deal with $S_{b,1}$ we take into account that $\sum_{P \in I_b} f g_P$ is a Lipschitz function supported in B_Q , and $f g_P$ vanishes unless $\tilde{B}_P \cap B_Q \neq \emptyset$. To shorten notation, we write $P \sim Q$ if $P \in I_b$ is such that $\tilde{B}_P \cap B_Q \neq \emptyset$. It is easy to check that $P \sim Q$ implies that $\ell(P) \gtrsim \ell(Q)$. Then we have

$$\begin{aligned} \left\| \nabla \left(\sum_{P \in I_b} f g_P \right) \right\|_\infty &\leq \sum_{P \sim Q} (\|g_P\|_\infty + c \ell(Q) \|\nabla g_P\|_\infty) \\ &\lesssim \sum_{P \sim Q} \left(\frac{\mu(P)}{\ell(P)^n} + \frac{\ell(Q) \mu(P)}{\ell(P)^{n+1}} \right) \lesssim \sum_{P \sim Q} \frac{\mu(P)}{\ell(P)^n}. \end{aligned}$$

From the fact that $\ell(P) \gtrsim \ell(Q)$, it also follows that, for A big enough, we have

$$\sum_{P \sim Q} \frac{\mu(P)}{\ell(P)^n} \lesssim \inf_{x \in Q} G_A(x). \quad (2.12)$$

Therefore,

$$S_{b,1} \lesssim \alpha_{\mathcal{H}^n|_\Gamma}(Q) \ell(Q)^{n+1} \inf_{x \in Q} G_A(x).$$

Finally, we turn our attention to the term $S_{b,2}$. Choosing $a_Q = c_Q \sum_{P \sim Q} g_P(x_Q)$ and using that $|c_Q| \lesssim 1$, we obtain

$$S_{b,2} \lesssim \int \left| f \sum_{P \sim Q} [g_P - g_P(x_Q)] \right| d\mathcal{H}^n|_{L_Q}.$$

Note now that, for $x \in B_Q$,

$$\sum_{P \sim Q} |g_P(x) - g_P(x_Q)| \lesssim \sum_{P \sim Q} \|\nabla g_P\|_\infty \ell(Q) \lesssim \sum_{P \sim Q} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}},$$

and so, using also that $|f| \leq C \ell(Q) \chi_{B_Q}$,

$$S_{b,2} \lesssim \sum_{P \sim Q} \frac{\mu(P) \ell(Q)^{n+2}}{\ell(P)^{n+1}}.$$

Gathering all the estimates above, we obtain

$$\begin{aligned} \alpha_{\mu|_{\Gamma^c}, L_Q}(Q) &\lesssim \alpha_{\sigma, L_Q}(Q) + \int_{6B_Q} \frac{\text{dist}(x, \Gamma)}{\ell(Q)^{n+1}} d\mu(x) + \inf_{x \in Q} G_A(x) \alpha_{\mathcal{H}^n|_{\Gamma}}(Q) \\ &\quad + \sum_{P \sim Q} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}}, \end{aligned}$$

which completes the proof of the lemma. $\square$

2.5 Proof of the Main Lemma

It is immediate to check that the statement in the main lemma is equivalent to the fact that

$$\sum_{Q \in \mathcal{D}_{\Gamma}} \alpha_{\mu}(Q)^2 \chi_Q(x) < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma. \quad (2.13)$$

To prove this estimate, we intend to use Lemma 2.3. To this end, observe that, by Lemma 2.1, we know that

$$\sum_{Q \in \mathcal{D}_{\Gamma}} (\alpha_{\mu|_{\Gamma}, L_Q}(Q)^2 + \alpha_{\sigma, L_Q}(Q)^2) \chi_Q(x) < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma.$$

Also, for every $x \in \Gamma$,

$$\sum_{Q \in \mathcal{D}_{\Gamma}: Q \ni x} \inf_{y \in Q} G_A(y)^2 \alpha_{\mathcal{H}^n|_{\Gamma}}(Q)^2 \leq G_A(x)^2 \sum_{Q \in \mathcal{D}_{\Gamma}: Q \ni x} \alpha_{\mathcal{H}^n|_{\Gamma}}(Q)^2 < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma,$$

since $G_A(x) < \infty$ and $\sum_{Q \in \mathcal{D}_{\Gamma}: Q \ni x} \alpha_{\mathcal{H}^n|_{\Gamma}}(Q)^2 < \infty$ for $\mathcal{H}^n$ -a.e. $x \in \Gamma$.

Next we show that

$$\sum_{Q \in \mathcal{D}_{\Gamma}} \left(\sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}} \right)^2 \chi_Q(x) < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma. \quad (2.14)$$

To this end, note that, as in (2.12),

$$\sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}} \chi_Q(x) \lesssim \sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P)}{\ell(P)^n} \chi_Q(x) \lesssim G_A(x).$$

Hence,

$$\sum_{Q \in \mathcal{D}_{\Gamma}} \left(\sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}} \right)^2 \chi_Q(x) \lesssim G_A(x) \sum_{Q \in \mathcal{D}_{\Gamma}} \sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}} \chi_Q(x).$$

Thus to prove (2.14), it suffices to show that

$$\sum_{Q \in \mathcal{D}_\Gamma} \sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}} \chi_Q(x) < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma. \quad (2.15)$$

By Fubini, we have

$$\begin{aligned} & \int \sum_{Q \in \mathcal{D}_\Gamma} \sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}} \chi_Q(x) d\mathcal{H}^n|_\Gamma(x) \\ & \approx \sum_{Q \in \mathcal{D}_\Gamma} \sum_{\substack{P \in \mathcal{W}: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\mu(P) \ell(Q)}{\ell(P)^{n+1}} \ell(Q)^n \\ & \approx \sum_{P \in \mathcal{W}} \mu(P) \sum_{\substack{Q \in \mathcal{D}_\Gamma: P \cap 2B_Q = \emptyset, \\ \tilde{B}_P \cap B_Q \neq \emptyset}} \frac{\ell(Q)^{n+1}}{\ell(P)^{n+1}} \lesssim \sum_{P \in \mathcal{W}} \mu(P) = \|\mu\|, \end{aligned}$$

where in the last inequality we took into account that the Γ -cubes $Q \in \mathcal{D}_\Gamma$ such that $P \cap 2B_Q = \emptyset$ and $\tilde{B}_P \cap B_Q \neq \emptyset$ satisfy $Q \subset c\tilde{B}_P$ for some $c > 1$, because $\ell(Q) \lesssim \ell(P)$. So (2.15) is proved.

Finally, to complete the proof of this lemma, by Lemma 2.3, it remains to show that

$$\sum_{Q \in \mathcal{D}_\Gamma} \left(\int_{6B_Q} \frac{\text{dist}(y, \Gamma)}{\ell(Q)^{n+1}} d\mu(y) \right)^2 \chi_Q(x) < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma. \quad (2.16)$$

The arguments for this are almost the same as the ones in [2]. We repeat them here for the reader's convenience. By Cauchy–Schwarz, we get

$$\left(\int_{6B_Q} \frac{\text{dist}(y, \Gamma)}{\ell(Q)^{n+1}} d\mu(y) \right)^2 \leq \mu(6B_Q) \int_{6B_Q} \left(\frac{\text{dist}(y, \Gamma)}{\ell(Q)^{n+1}} \right)^2 d\mu(y).$$

Since $\frac{\mu(6B_Q)}{\ell(Q)^n} \lesssim M_n \mu(x)$ for all $x \in Q$, and $M_n \mu(x) < \infty$ for $\mathcal{H}^n$ -a.e. $x \in \Gamma$, it turns that, to prove (2.16), it suffices to show that

$$\sum_{Q \in \mathcal{D}_\Gamma} \int_{6B_Q} \frac{\text{dist}(y, \Gamma)^2}{\ell(Q)^{n+2}} d\mu(y) \chi_Q(x) < \infty \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in \Gamma. \quad (2.17)$$

The integral of the left-hand side of (2.17) with respect to $\mathcal{H}^n|_\Gamma$ does not exceed

$$c \sum_{Q \in \mathcal{D}_\Gamma} \int_{6B_Q} \frac{\text{dist}(y, \Gamma)^2}{\ell(Q)^{n+2}} d\mu(y) \ell(Q)^n = c \sum_{Q \in \mathcal{D}_\Gamma} \int_{6B_Q} \frac{\text{dist}(y, \Gamma)^2}{\ell(Q)^2} d\mu(y).$$

By Fubini, this equals

$$c \int \text{dist}(y, \Gamma)^2 \sum_{Q \in \mathcal{D}_\Gamma} \chi_{6B_Q}(y) \frac{1}{\ell(Q)^2} d\mu(y).$$

Notice now that

$$\sum_{Q \in \mathcal{D}_\Gamma} \chi_{6B_Q}(y) \frac{1}{\ell(Q)^2} = \sum_{Q \in \mathcal{D}_\Gamma: y \in 6B_Q} \frac{1}{\ell(Q)^2} \lesssim \frac{1}{\text{dist}(y, \Gamma)^2},$$

because the condition $y \in 6B_Q$ implies that $\text{dist}(y, \Gamma) \leq r(B_Q) \approx \ell(Q)$. Thus,

$$\sum_{Q \in \mathcal{D}_\Gamma} \int_{6B_Q} \frac{\text{dist}(y, \Gamma)^2}{\ell(Q)^2} d\mu(y) \lesssim \int \frac{\text{dist}(y, \Gamma)^2}{\text{dist}(y, \Gamma)^2} d\mu(y) = \|\mu\|.$$

Hence, (2.16) follows and we are done. $\square$

References

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