Calc. Var. (2019) 58:28
https://doi.org/10.1007/s00526-018-1472-1

Calculus of Variations

Positive solutions for nonlinear singular elliptic equations of *p*-Laplacian type with dependence on the gradient

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Received: 23 April 2018 / Accepted: 7 December 2018 / Published online: 3 January 2019

Abstract

In this paper, we study a nonlinear Dirichlet problem of p-Laplacian type with combined effects of nonlinear singular and convection terms. An existence theorem for positive solutions is established as well as the compactness of solution set. Our approach is based on Leray–Schauder alternative principle, method of sub-supersolution, nonlinear regularity, truncation techniques, and set-valued analysis.

Mathematics Subject Classification 35J60 · 35J91 · 35J92 · 35D30 · 35D35

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with C^2 boundary. In this paper, we investigate the following singular elliptic equation with Dirichlet boundary condition, p-Laplace differential operator, and a nonlinear convection term (i.e., the reaction function depends on the solution u and its gradient ∇u):

$$\begin{cases} -\Delta_p u(x) = f(x, u(x), \nabla u(x)) + g(x, u(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Communicated by P. Rabinowitz.

This project has received funding from the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 823731 – CONMECH. It is supported by the National Science Center of Poland under Maestro Project No. UMO-2012/06/A/ST1/00262, National Science Center of Poland under Preludium Project No. 2017/25/N/ST1/00611, NNSF of China Grant No. 11671101, NSF of Guangxi Grant No. 2018GXNSFDA138002, Special Funds of Guangxi Distinguished Experts Construction Engineering, International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under Grant No. 3792/GGPJ/H2020/2017/0. D. Motreanu received Visiting Professor fellowship from CNPQ/Brazil PV- 400633/2017-5.

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Here Δ_p stands for the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

with $1 < p < \infty$ and gradient operator ∇. For the convection term $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, a suitable growth condition $H(f)$ in Sect. 3 is required. The semilinear function $g\colon \Omega \times (0,\infty) \to \mathbb{R}$ is singular at $s = 0$, that is,

$$\lim_{s\to 0^+} g(x,s) = +\infty.$$

In order to emphasize the main ideas, we suppose that $p < N$. The case $N \leq p$ can be handled along the same lines. As usual, we denote $p^* := \frac{Np}{N-p}$, which is the Sobolev critical exponent. The solution of problem (1) is understood in the weak sense as described below.

Definition 1 We say that $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of problem (1) if

$$\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\,dx = \int_\Omega \big[f(x,u(x),\nabla u(x)) + g(x,u(x))\big]v(x)\,dx$$

for all $v \in W_0^{1,p}(\Omega)$.

If $p = 2$, problem (1) reduces to the semilinear Dirichlet elliptic equation with a singular term and gradient dependence considered by Faraci and Puglisi [14]:

$$\begin{cases} -\Delta u(x) = f(x,u(x),\nabla u(x)) + g(x,u(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

A typical case in (1) and (2) is when the singular term is in the form $g(x,u(x)) = h(x)u(x)^{-\mu}$ for $\mu > 0$ and a suitable function h, which gives rise to the nonlinear Dirichlet elliptic equation with combined effects of singular and convection terms

$$\begin{cases} -\Delta_p u(x) = f(x,u(x),\nabla u(x)) + h(x)u(x)^{-\mu} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Elliptic equations with singular terms represent a class of hot-point problems because they appear in applications to chemical catalysts processes, non-Newtonian fluids, and in models for the temperature of electrical conductors, see, e.g., [4,11]. An extensive literature is devoted to such problems, especially from the point of view of theoretical analysis. For instance, Ghergu and Rădulescu [21] established several existence and nonexistence results for boundary value problems with singular term and parameters; Gasínski and Papageorgiou [20] studied a nonlinear Dirichlet problem with a singular term, a $(p-1)$-sublinear term, and a Carathéodory perturbation; Hirano et al. [23] proved Brezis–Nirenberg type theorems for a singular elliptic problem. More details on the topics related to singular problems can be found in Crandall et al. [8], Cîrstea et al. [7], Dupaigne et al. [12], Kaufmann and Medri [25], D'Ambrosio and Mitidieri [9], Carl et al. [6], Giacomoni et al. [22], Gasiński and Papageorgiou [19], Bai et al. [2], Carl [5] and the references therein.

On the other hand, as another challenging topic, elliptic problems with convection terms have been considered in various frameworks. Amongst the results we mention: Faraci et al. [13] proved the existence of a positive solution and of a negative solution for a quasi-linear elliptic problem with dependence on the gradient; Faria et al. [15] proved the existence of a positive solution for a quasi-linear elliptic problem involving the (p,q)-Laplacian and a convection term; Zeng et al. [39] proved the existence of positive solutions for a generalized elliptic inclusion problem driven by a nonhomogeneous partial differential operator with Dirichlet boundary condition and a convection multivalued term; Papageorgiou et al. [35] proved that a nonlinear boundary value problem driven by a nonhomogeneous differential operator has at least five nontrivial smooth solutions, four of constant sign, and one nodal. For other results in this area the reader may consult: Motreanu et al. [32], Motreanu and Tanaka [33], Averna et al. [1], Faria et al. [16], Gasiński and Papageorgiou [18], and the references therein.

In this paper, under verifiable conditions, we provide the existence of positive solutions for problem (1). It is for the first time when such a result is obtained for problem (1), in particular (3), exhibiting singular and convection terms in the nonlinear case $p \neq 2$. The approach uses the method of subsolution-supersolution, truncation techniques, nonlinear regularity theory, Leray–Schauder alternative principle, and set-valued analysis. It is worth mentioning that in our analysis of problem (1) we strongly rely on multi-valued mappings arguments. Specifically, the multi-valued setting offers an efficient way to handle the smallest solution of the constructed auxiliary problem. This is another trait of novelty in our paper. The compactness of the solution set of problem (1) is proved, too.

We briefly describe the main ideas in our approach. Corresponding to a fixed smooth function w, we associate to the original statement (1) an intermediate problem replacing the gradient ∇u in $f(x, u, \nabla u)$ with ∇w and keeping unchanged the singular term. For the intermediate problem, a positive subsolution $\underline{u}$ is constructed independently of w and is shown the existence of a solution greater than $\underline{u}$. We are thus enabled to consider the set-valued mapping $\mathscr{S}$ assigning to w the set $\mathscr{S}(w)$ of all such solutions of the intermediate problem. On the basis of the properties of the set-valued mapping $\mathscr{S}$ we can prove that the mapping Γ defined by $\Gamma(w)$ equal to the minimal element of $\mathscr{S}(w)$ is compact. The positive solution of the original problem is obtained by applying Leray–Schauder alternative principle to the mapping Γ. At this point we need the following smallness condition

$$c_1 + c_2\lambda_1^{\frac{p-1}{p}} < \lambda_1,$$

where the constants $c_1 > 0$ and $c_2 > 0$ are the coefficients of $|u|$ and $|\nabla u|$, respectively, in the subcritical growth condition of $f(x, u, \nabla u)$, while λ_1 is the first eigenvalue of $-\Delta u$ on $W_0^{1,p}(\Omega)$. This condition requires a certain compatibility between the growth of $f(x, u, \nabla u)$ and the geometry of the bounded domain Ω imposing some restrictions on Ω as can be seen from known estimates from above and from below for λ_1. For instance, if Ω is the ball $B(0, R)$ in $\mathbb{R}^N$ of radius $R > 0$ and centered at the origin, we have the estimates

$$\frac{Np}{R^p} \leq \lambda_1 \leq \frac{(p+1)\ldots(p+N)}{N!R^p}.$$

We refer to Benedikt and Drábek [3] and Kajikiya [24] for estimates of λ_1 on different bounded domains $\Omega \subset \mathbb{R}^N$ related to geometric quantities.

The rest of the paper is organized as follows. In Sect. 2 we present the needed preliminary material. Section 3 is devoted to establishing our results.

2 Mathematical background

Let $1 < p < \infty$ and p' defined by $\frac{1}{p} + \frac{1}{p'} = 1$. The Lebesgue space $L^p(\Omega)$ is endowed with the standard norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, dx \right)^{\frac{1}{p}} \quad \text{for all } u \in L^p(\Omega).$$

The Sobolev space $W_0^{1,p}(\Omega)$ is equipped with the usual norm

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We denote by $C^k(\overline{\Omega})$ for $k \in \mathbb{N}$ the space of real-valued k-times continuously differentiable functions u in Ω such that the partial derivatives $D^\alpha u$ continuously extend to $\overline{\Omega}$ for all $|\alpha| \leq k$. The space $C^k(\overline{\Omega})$ is endowed with the norm

$$\|u\|_{C^k(\overline{\Omega})} := \max_{|\alpha| \leq k} \sup_{x \in \overline{\Omega}} |D^\alpha u(x)|.$$

We shall also use the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) \, : \, u = 0 \quad \text{on } \partial\Omega\}$$

and its cone of nonnegative functions

$$C_0^1(\overline{\Omega})_+ = \{u \in C_0^1(\overline{\Omega}) \, : \, u \geq 0 \quad \text{in } \Omega\},$$

which has a nonempty interior in $C_0^1(\overline{\Omega})$ given by

$$\operatorname{int}(C_0^1(\overline{\Omega})_+) = \left\{ u \in C_0^1(\overline{\Omega}) \, : \, u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial \nu} < 0 \ \text{ on } \partial\Omega \right\},$$

where the notation $\partial u / \partial \nu$ stands for the normal derivative of u with the unit outer normal ν to $\partial\Omega$.

For clarity regarding arguments that involve order we recall the following notions.

Definition 2 Let $(P, \leq)$ be a partially ordered set.

(i) A subset $E \subset P$ is called upward directed, if for each pair $u, v \in E$ there exists $w \in E$ with $w \geq u$ and $w \geq v$.
(ii) A subset $E \subset P$ is called downward directed, if for each pair $u, v \in E$ there exists $w \in E$ such that $w \leq u$ and $w \leq v$.

For any $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$. If $u \in W_0^{1,p}(\Omega)$, one has

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

The gradients of these functions are equal to

$$\nabla u^+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) \leq 0, \end{cases}$$

$$\nabla u^-(x) = \begin{cases} -\nabla u(x) & \text{if } u(x) < 0 \\ 0 & \text{if } u(x) \geq 0, \end{cases}$$

$$\nabla |u|(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) = 0 \\ -\nabla u(x) & \text{if } u(x) < 0. \end{cases}$$

Given the functions $u_1, u_2\colon \Omega \to \mathbb{R}$, we utilize the notation $\{u_1 > u_2\} = \{x \in \Omega : u_1(x) > u_2(x)\}$, and accordingly $\{u_1 \geq u_2\}$. For a subset $K \subset \Omega$, its characteristic function is denoted by χ_K, which means

$$\chi_K(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{elsewhere.} \end{cases}$$

We recall the eigenvalue problem for the p-Laplacian with Dirichlet boundary condition

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2} u(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The first eigenvalue denoted λ_1 is positive, isolated, simple, and has the following variational characterization

$$\lambda_1 = \inf \left\{ \frac{\|u\|^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0 \right\}.$$

Finally, we review some background material of set-valued analysis. More details can be found in [10,17,31,34,38].

Definition 3 Let X and Y be topological spaces. A set-valued mapping $F\colon X \to 2^Y$ is called

(i) upper semicontinuous (u.s.c., for short) at $x \in X$, if for every open set $O \subset Y$ with $F(x) \subset O$ there exists a neighborhood $N(x)$ of x such that

$$F(N(x)) := \cup_{y \in N(x)} F(y) \subset O;$$

when this holds for every $x \in X$, F is called upper semicontinuous;

(ii) lower semicontinuous (l.s.c., for short) at $x \in X$, if for every open set $O \subset Y$ with $F(x) \cap O \neq \emptyset$ there exists a neighborhood $N(x)$ of x such that

$$F(y) \cap O \neq \emptyset \ \text{ for all } y \in N(x);$$

when this holds for every $x \in X$, F is called lower semicontinuous;

(iii) continuous at $x \in X$, if F is both upper semicontinuous and lower semicontinuous at $x \in X$; when this holds for every $x \in X$, F is called continuous.

Proposition 4 *The following properties are equivalent:*

(i) *F is u.s.c.;*

(ii) *for every closed subset $C \subset Y$, the set*

$$F^-(C) := \{x \in X \mid F(x) \cap C \neq \emptyset\}$$

is closed in X;

(iii) *for every open subset $O \subset Y$, the set*

$$F^+(O) := \{x \in X \mid F(x) \subset O\}$$

is open in X.

Proposition 5 *The following properties are equivalent:*

(a) *F is l.s.c.;*

(b) *if $u \in X$, $\{u_\lambda\}_{\lambda \in J} \subset X$ is a net such that $u_\lambda \to u$, and $u^* \in F(u)$, then for each $\lambda \in J$ there is $u_\lambda^* \in F(u_\lambda)$ with $u_\lambda^* \to u^*$ in Y.*

Proposition 6 *Let X, Y be topological spaces and let $F\colon X \to 2^Y$ be u.s.c. with compact values. Let $\{u_\lambda\}_{\lambda\in J}$ be a net in X with $u_\lambda \to u$ and let $u^*_\lambda \in F(u_\lambda)$ for each $\lambda \in J$. Then there exist $u^* \in F(u)$ and a subnet $\{u^*_\gamma\}$ of $\{u^*_\lambda\}$ such that $u^*_\gamma \to u^*$.*

An essential tool in the sequel is the Leray–Schauder alternative principle (or Schaefer's fixed point theorem), see, e.g., Gasiński and Papageorgiou [17, p. 827].

Theorem 7 *Let X be a Banach space and let $C \subset X$ be nonempty and convex. Assume that $\Gamma\colon C \to C$ is a compact mapping, i.e., Γ is continuous and maps bounded sets into relatively compact sets. Then it holds exactly one of the following statements:*

(a) *Γ has a fixed point;*
(b) *the set $\Lambda(\Gamma) := \{u \in C \,:\, u = t\Gamma(u) \text{ for some } t \in (0,1)\}$ is unbounded.*

3 Existence of positive solutions

Our assumptions on the data in problem (1) are as follows.
$\underline{H(f)}$: The convection term $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ verifies:

(i) f is a Carathéodory function, i.e., $x \mapsto f(x, s, \xi)$ is measurable for each $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, and $(s, \xi) \mapsto f(x, s, \xi)$ is continuous for a.e. $x \in \Omega$;
(ii) for each constant $M > 0$, there exist constants $c_M > 0$ and $0 < d_M < \lambda_1$ with

$$|f(x, s, \xi)| \le c_M + d_M |s|^{p-1}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$ with $|\xi| \le M$.

$\underline{H(g)}$: The singular term $g\colon \Omega \times (0, +\infty) \to [0, +\infty)$ satisfies:

(i) g is a Carathéodory function;
(ii) $g(x, \cdot)$ is nonincreasing on the interval $(0, 1)$ for a.e. $x \in \Omega$, $g(x, s) \ge g(x, 1)$ for a.e. $x \in \Omega$, all $s < 1$, and $g(\cdot, 1)$ is not identically zero;
(iii) there exist a function $\vartheta \in \operatorname{int}(C^1_0(\overline{\Omega})_+)$, and constants $q > \max\{N, p'\}$ as well as $\varepsilon_0 > 0$ such that

$$x \mapsto g(x, \varepsilon\vartheta(x)) \in L^q(\Omega) \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Remark 8 Hypotheses $H(f)$ and $H(g)$ permit to construct a sub-supersolution for intermediate problem (4), see below. Condition $H(f)$ was employed in Faraci et al. [13], whereas condition $H(g)$ was dealt with in Faraci and Puglisi [14] and goes back to Perera and Silva [36] and Perera and Zhang [37].

Examples of singular functions fulfilling all the requirements in $H(g)$ can be constructed with any $\gamma > 0$ and $h \in L^q(\Omega)_+$. For instance, one can take Ω to be an open ball in $\mathbb{R}^N$ and choose any function as

$$g(x, s) = h(x)s^{-\gamma};$$
$$g(x, s) = h(x)e^{\frac{1}{s^\gamma}};$$
$$g(x, s) = \begin{cases} -h(x)\ln(s) & \text{if } s \le e^{-1} \\ h(x)\frac{e^{-\gamma}}{s^\gamma} & \text{if } s > e^{-1}, \end{cases}$$

with $s \in (0, 1)$ and appropriately extending for $s > 1$, and suitable corresponding functions h on Ω (see [36,37]).

For $w \in C_0^1(\overline{\Omega})$ fixed, we first focus on an intermediate singular Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = f(x, u(x), \nabla w(x)) + g(x, u(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

Definition 9 We say that

(i) $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of problem (4) if

$$\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega \big[f(x, u(x), \nabla w(x)) + g(x, u(x))\big]v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$;

(ii) $\overline{u} \in W^{1,p}(\Omega)$ with $\overline{u} \geq 0$ on $\partial\Omega$ (in the sense of trace) is a supersolution of problem (4) if

$$\int_\Omega |\nabla \overline{u}(x)|^{p-2}(\nabla \overline{u}(x), \nabla v(x))_{\mathbb{R}^N}\, dx \geq \int_\Omega \big[f(x, \overline{u}(x), \nabla w(x)) + g(x, \overline{u}(x))\big]v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$ with $v(x) \geq 0$ for a.e. $x \in \Omega$.

(iii) $\underline{u} \in W^{1,p}(\Omega)$ with $\underline{u} \leq 0$ on $\partial\Omega$ (in the sense of trace) is a subsolution of problem (4) if

$$\int_\Omega |\nabla \underline{u}(x)|^{p-2}(\nabla \underline{u}(x), \nabla v(x))_{\mathbb{R}^N}\, dx \leq \int_\Omega \big[f(x, \underline{u}(x), \nabla w(x)) + g(x, \underline{u}(x))\big]v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$ with $v(x) \geq 0$ for a.e. $x \in \Omega$.

The next lemma is essential for our development.

Lemma 10 *If $u_1, u_2 \in W^{1,p}(\Omega)$ are two supersolutions for problem (4), then the function $u := \min\{u_1, u_2\} \in W^{1,p}(\Omega)$ is also a supersolution for problem (4).*

Proof Let $u_1, u_2 \in W^{1,p}(\Omega)$ be supersolutions for problem (4). Corresponding to any $\varepsilon > 0$, consider the truncation $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ given by

$$\eta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{t}{\varepsilon} & \text{if } 0 < t < \varepsilon \\ 1 & \text{otherwise,} \end{cases}$$

which is Lipschitz continuous. From Marcus and Mizel [30], we know about the composition

$$\eta_\varepsilon(u_2 - u_1) \in W^{1,p}(\Omega)$$

that

$$\nabla(\eta_\varepsilon(u_2 - u_1)) = \eta_\varepsilon'(u_2 - u_1)\nabla(u_2 - u_1).$$

For any function $v \in C_0^\infty(\Omega)_+ := \{v \in C_0^\infty(\Omega) \,:\, v(x) \geq 0 \text{ for a.e. } x \in \Omega\}$, we have

$$\eta_\varepsilon(u_2 - u_1)v \in W_0^{1,p}(\Omega) \text{ with } \big(\eta_\varepsilon(u_2 - u_1)v\big)(x) \geq 0 \text{ for a.e. } x \in \Omega,$$

and

$$\nabla\big(\eta_\varepsilon(u_2 - u_1)v\big) = v\nabla(\eta_\varepsilon(u_2 - u_1)) + \eta_\varepsilon(u_2 - u_1)\nabla v.$$

The definition of supersolution for problem (4) yields

$$\int_\Omega |\nabla u_i(x)|^{p-2}(\nabla u_i(x), \nabla h(x))_{\mathbb{R}^N}\, dx \geq \int_\Omega f(x, u_i(x), \nabla w(x))h(x)\, dx + \int_\Omega g(x, u_i(x))h(x)\, dx$$

for all $h \in W_0^{1,p}(\Omega)$ with $h(x) \geq 0$ a.e. $x \in \Omega$, $i = 1, 2$. Inserting $h = \eta_\varepsilon(u_2 - u_1)v$ for $i = 1$ and $h = \big(1 - \eta_\varepsilon(u_2 - u_1)\big)v$ for $i = 2$, and then summing up the resulting inequalities, we get

$$\begin{aligned}
&\int_\Omega |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla\big(\eta_\varepsilon(u_2 - u_1)v\big)(x))_{\mathbb{R}^N}\, dx \\
&\quad + \int_\Omega |\nabla u_2(x)|^{p-2}\big(\nabla u_2(x), \nabla\big(\big(1 - \eta_\varepsilon(u_2 - u_1)\big)v\big)(x)\big)_{\mathbb{R}^N}\, dx \\
&\geq \int_\Omega g(x, u_1(x)) + f(x, u_1(x), \nabla w(x))\big(\eta_\varepsilon(u_2 - u_1)v\big)(x)\, dx \\
&\quad + \int_\Omega g(x, u_2(x)) + f(x, u_2(x), \nabla w(x))\big(\big(1 - \eta_\varepsilon(u_2 - u_1)\big)v\big)(x)\, dx.
\end{aligned}$$

We note that

$$\begin{aligned}
&\int_\Omega |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla\big(\eta_\varepsilon(u_2 - u_1)v\big)(x))_{\mathbb{R}^N}\, dx \\
&\quad = \frac{1}{\varepsilon}\int_{\{0<u_2-u_1<\varepsilon\}} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla(u_2 - u_1)(x))_{\mathbb{R}^N}\, v(x)\, dx \\
&\qquad + \int_\Omega |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^N}\big(\eta_\varepsilon(u_2 - u_1)\big)(x)\, dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla\big(\big(1 - \eta_\varepsilon(u_2 - u_1)\big)v\big)(x))_{\mathbb{R}^N}\, dx \\
&\quad = -\frac{1}{\varepsilon}\int_{\{0<u_2-u_1<\varepsilon\}} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla(u_2 - u_1)(x))_{\mathbb{R}^N}\, v(x)\, dx \\
&\qquad + \int_\Omega |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^N}\big(1 - \eta_\varepsilon(u_2 - u_1)\big)(x)\, dx.
\end{aligned}$$

Altogether, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^N}\big(\eta_\varepsilon(u_2 - u_1)\big)(x)\, dx \\
&\quad + \int_\Omega |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^N}\big(1 - \eta_\varepsilon(u_2 - u_1)\big)(x)\, dx \\
&\geq \int_\Omega [g(x, u_1(x)) + f(x, u_1(x), \nabla w(x))]\big(\eta_\varepsilon(u_2 - u_1)v\big)(x)\, dx \\
&\quad + \int_\Omega [g(x, u_2(x)) + f(x, u_2(x), \nabla w(x))]\big(\big(1 - \eta_\varepsilon(u_2 - u_1)\big)v\big)(x)\, dx.
\end{aligned}$$

Now we pass to the limit as $\varepsilon \to 0^+$. Using Lebesgue's Dominated Convergence Theorem (see, e.g., [31, Theorem 2.38]) and

$$\eta_\varepsilon\big((u_2 - u_1)(x)\big) \to \chi_{\{u_1<u_2\}}(x) \quad \text{for a.e. } x \in \Omega \quad \text{as } \varepsilon \to 0^+,$$

we find

$$
\begin{aligned}
&\int_{\{u_1<u_2\}} |\nabla u_1(x)|^{p-2}(\nabla u_1(x), \nabla v(x))_{\mathbb{R}^N}\, dx \\
&\quad + \int_{\{u_1\geq u_2\}} |\nabla u_2(x)|^{p-2}(\nabla u_2(x), \nabla v(x))_{\mathbb{R}^N}\, dx \\
&\geq \int_{\{u_1<u_2\}} [g(x, u_1(x)) + f(x, u_1(x), \nabla w(x))]v(x)\, dx \\
&\quad + \int_{\{u_1\geq u_2\}} [g(x, u_2(x)) + f(x, u_2(x), \nabla w(x))]v(x)\, dx \qquad (5)
\end{aligned}
$$

for all $v \in C_0^\infty(\Omega)_+$. Notice that $u = \min\{u_1, u_2\} \in W^{1,p}(\Omega)$ with $u \geq 0$ on $\partial\Omega$ and

$$
\nabla u(x) = \begin{cases} \nabla u_1(x) & \text{for a.e. } x \in \{u_1 < u_2\} \\ \nabla u_2(x) & \text{for a.e. } x \in \{u_1 \geq u_2\}. \end{cases}
$$

Combining with (5) leads to

$$
\begin{aligned}
&\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx \geq \int_\Omega g(x, u(x))v(x)\, dx \\
&\quad + \int_\Omega f(x, u(x), \nabla w(x))v(x)\, dx \qquad (6)
\end{aligned}
$$

for all $v \in C_0^\infty(\Omega)_+$. The density of $C_0^\infty(\Omega)_+$ into $W_0^{1,p}(\Omega)_+ := \{u \in W_0^{1,p}(\Omega) : u \geq 0 \text{ a.e. on } \Omega\}$ ensures that (6) holds true for all $v \in W_0^{1,p}(\Omega)_+$, so u is also a supersolution of problem (4). □

Similarly, we can prove the corresponding statement for subsolutions.

Lemma 11 *If $v_1, v_2 \in W^{1,p}(\Omega)$ are two subsolutions for problem (4), then the function $v = \max\{v_1, v_2\} \in W^{1,p}(\Omega)$ is also a subsolution for problem (4).*

Denote by $\overline{U}_w \subset W^{1,p}(\Omega)$ and $\underline{U}_w \subset W^{1,p}(\Omega)$ the supersolution set and subsolution set of problem (4), respectively. The following result is a direct consequence gathering Lemmata 10 and 11.

Corollary 12 *The sets $\overline{U}_w$ and $\underline{U}_w$ are upward directed and downward directed, respectively.*

Next we establish the existence of subsolutions of problem (4).

Lemma 13 *Under the assumptions $H(g)$ and $H(f)$, there exists a subsolution $\underline{u}$ of problem (4).*

Proof Let $\vartheta \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$ be given in hypothesis $H(g)$(iii). Hence, there is $\varepsilon_1 > 0$ such that

$$
\|\varepsilon\vartheta\|_{L^\infty(\Omega)} \leq 1 \text{ for all } \varepsilon \in (0, \varepsilon_1).
$$

Then the monotonicity of g required in $H(g)$(ii) implies that

$$
0 \leq g(x, 1) \leq g(x, \varepsilon\vartheta(x)) \quad \text{for a.e. } x \in \Omega \text{ and } \varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\}).
$$

Condition $H(g)$(iii) entitles that for each $\varepsilon \in (0, \varepsilon_0)$ the function $x \mapsto g(x, \varepsilon\vartheta(x))$ belongs to $L^q(\Omega)$ for some $q > N$, which results in

$$x \mapsto g(x, 1) \in L^q(\Omega).$$

According to $H(g)$(iii), the function $x \mapsto g(x, 1)$ is not identically zero. Then there exists a unique $u^* \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$ that resolves the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = g(x, 1) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Set $\underline{u} = \alpha u^* \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, where $\alpha = \min\{1, \frac{1}{\|u^*\|_{L^\infty(\Omega)}}\}$. Using the monotonicity of g on $(0, 1)$ again, it turns out

$$0 \le g(x, 1) \le g(x, \underline{u}(x)) \quad \text{for a.e. } x \in \Omega. \tag{7}$$

Because of $\vartheta, \underline{u} \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$, we can choose $\varepsilon > 0$ small enough to fulfill

$$\underline{u} - \varepsilon\vartheta \in \operatorname{int}(C_0^1(\overline{\Omega})_+).$$

Taking into account hypothesis $H(g)$, we derive

$$\begin{cases} 0 \le g(x, \underline{u}(x)) \le g(x, \varepsilon\vartheta(x)) & \text{for a.e. } x \in \Omega \\ x \mapsto g(x, \underline{u}(x)) \in L^q(\Omega). \end{cases} \tag{8}$$

Since $q > N > (p^*)'$, we have $q' < p^*$, where for $r > 1$, we denote $r' = r/(r-1)$. Therefore we can use the Sobolev embedding theorem (see, e.g., [17, Theorem 2.5.3]), to infer that the embedding of $W_0^{1,p}(\Omega)$ into $L^{q'}(\Omega)$ is continuous. On account of (8) we deduce

$$x \mapsto g(x, \underline{u}(x))v(x) \in L^1(\Omega) \quad \text{for all } v \in W_0^{1,p}(\Omega).$$

From

$$\begin{aligned} \int_\Omega |\nabla \underline{u}(x)|^{p-2}(\nabla \underline{u}(x), \nabla v(x))_{\mathbb{R}^N}\, dx &= \alpha^{p-1} \int_\Omega |\nabla u^*(x)|^{p-2}(\nabla u^*(x), \nabla v(x))_{\mathbb{R}^N}\, dx \\ &= \alpha^{p-1} \int_\Omega g(x, 1)v(x)\, dx \quad \text{for all } v \in W_0^{1,p}(\Omega), \end{aligned}$$

and due to $\alpha \le 1$ and (7), one has

$$\int_\Omega |\nabla \underline{u}(x)|^{p-2}(\nabla \underline{u}(x), \nabla v(x))_{\mathbb{R}^N}\, dx \le \int_\Omega g(x, 1)v(x)\, dx \le \int_\Omega g(x, \underline{u}(x))v(x)\, dx \tag{9}$$

for all $v \in W_0^{1,p}(\Omega)$ with $v(x) \ge 0$ for a.e. $x \in \Omega$. In view of $f(x, s, \xi) \ge 0$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, by (9) it holds

$$\int_\Omega |\nabla \underline{u}(x)|^{p-2}(\nabla \underline{u}(x), \nabla v(x))_{\mathbb{R}^N}\, dx \le \int_\Omega \big[f(x, \underline{u}(x), \nabla w(x)) + g(x, \underline{u}(x))\big]v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$ with $v(x) \ge 0$ and for a.e. $x \in \Omega$. Consequently, $\underline{u}$ is a subsolution of problem (4), which completes the proof. □

Remark 14 From the proof Lemma 13 it is clear that the obtained subsolution $\underline{u}$ is independent of function w and belongs to $\operatorname{int}(C_0^1(\overline{\Omega})_+)$.

We are able to show the existence of positive solutions to auxiliary problem (4).

Lemma 15 *Assume that conditions $H(g)$ and $H(f)$ hold. Then problem (4) admits a positive solution u with regularity $u \in int(C_0^1(\overline{\Omega})_+)$, which is greater than the subsolution $\underline{u}$.*

Proof Consider the nonlinear singular truncated Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = \widehat{f}(x, u(x)) + \widehat{g}(x, u(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $\widehat{f}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widehat{g}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are truncated functions corresponding to f and g defined by

$$\widehat{f}(x, s) = \begin{cases} f(x, \underline{u}(x), \nabla w(x)) & \text{if } s \le \underline{u}(x) \\ f(x, s, \nabla w(x)) & \text{if } s > \underline{u}(x) \end{cases}$$

and

$$\widehat{g}(x, s) = \begin{cases} g(x, \underline{u}(x)) & \text{if } s \le \underline{u}(x) \\ g(x, s) & \text{if } s > \underline{u}(x) \end{cases}$$

for a.e. $x \in \Omega$ and $s \in \mathbb{R}$. Consider also the primitives $G\colon \Omega \times \mathbb{R} \to \mathbb{R}$ and $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$G(x, s) = \int_0^s \widehat{g}(x, t)\, dt \quad \text{and} \quad F(x, s) = \int_0^s \widehat{f}(x, t)\, dt$$

for a.e. $x \in \Omega$ and $s \in \mathbb{R}$. The energy functional $\mathscr{E}_w\colon W_0^{1,p} \to \mathbb{R}$ associated to problem (10) has the expression

$$\mathscr{E}_w(u) = \frac{1}{p}\|u\|^p - \int_\Omega G(x, u(x))\, dx - \int_\Omega F(x, u(x))\, dx \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Claim 1 *The energy functional $\mathscr{E}_w$ is of class C^1.*

Let $u, v \in W_0^{1,p}(\Omega)$ and $t > 0$. By Mean Value Theorem we may write

$$\begin{aligned} \frac{1}{tp}(\|u + tv\|^p - \|u\|^p) &= \frac{1}{tp}\left(\int_\Omega |\nabla(u + tv)(x)|^p\, dx - \int_\Omega |\nabla u(x)|^p\, dx\right) \\ &= \int_\Omega \big(|\nabla u(x) + t\tau\nabla v(x)|^{p-2}(\nabla u(x) + t\tau\nabla v(x)), \nabla v(x)\big)_{\mathbb{R}^N}\, dx \end{aligned}$$

with some $\tau \in (0, 1)$. Using Lebesgue's Dominated Convergence Theorem entails

$$\lim_{t\to 0^+} \frac{1}{tp}(\|u + tv\|^p - \|u\|^p) = \int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx.$$

The expressions of G and F imply

$$\begin{aligned} &\int_\Omega \frac{G(u + tv) + F(u + tv) - G(u) - F(u)}{t}\, dx = \int_\Omega \frac{\int_{u(x)}^{(u+tv)(x)} \widehat{g}(x, s)\, ds}{t}\, dx \\ &+ \int_\Omega \frac{\int_{u(x)}^{(u+tv)(x)} \widehat{f}(x, s)\, ds}{t}\, dx = \int_\Omega \widehat{g}(x, u(x) + \tau_1 v(x)) v(x)\, dx \\ &+ \int_\Omega \widehat{f}(x, u(x) + \tau_2 v(x)) v(x)\, dx \end{aligned}$$

with some $\tau_1, \tau_2 \in (0, t)$. Invoking Lebesgue's Dominated Convergence Theorem again, we obtain

$$\begin{aligned}\lim_{t\to 0^+}\int_\Omega \frac{G(u+tv)+F(u+tv)-G(u)-F(u)}{t}\,dx \\ =\int_\Omega [\widehat{g}(x,u(x))+\widehat{f}(x,u(x))]v(x)\,dx.\end{aligned}$$

Thus it holds true

$$\begin{aligned}\lim_{t\to 0^+}\frac{\mathscr{E}_w(u+tv)-\mathscr{E}_w(u)}{t} &= \int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x),\nabla v(x))_{\mathbb{R}^N}\,dx \\ &+\int_\Omega [\widehat{g}(x,u(x))+\widehat{f}(x,u(x))]v(x)\,dx \quad \text{for all } v\in W_0^{1,p}(\Omega).\end{aligned}$$

We can conclude that $\mathscr{E}_w$ is of class C^1 because $\widehat{g}(x,\cdot)$ and $\widehat{f}(x,\cdot)$ are continuous.

Claim 2 *The energy functional $\mathscr{E}_w$ is coercive.*

Through the definition of $\widehat{g}$, hypothesis $H(g)$(ii) and $\underline{u} \le 1$, the following estimate is valid

$$\begin{aligned}\mathscr{E}_w(u) &= \frac{1}{p}\|u\|^p - \int_\Omega\int_0^{u(x)} \widehat{g}(x,t)\,dt\,dx - \int_\Omega\int_0^{u(x)}\widehat{f}(x,t)\,dt\,dx \\ &\ge \frac{1}{p}\|u\|^p - \int_\Omega\int_0^{u(x)} g(x,\underline{u}(x))\,dt\,dx - \int_\Omega\int_0^{u(x)}\widehat{f}(x,t)\,dt\,dx.\end{aligned}$$

The definition of $\widehat{f}$ and growth condition $H(f)$(ii) render

$$\begin{aligned}\int_\Omega\int_0^{u(x)}\widehat{f}(x,t)\,dt\,dx &= \int_\Omega\int_0^{\underline{u}(x)}\widehat{f}(x,t)\,dt\,dx + \int_\Omega\int_{\underline{u}(x)}^{u(x)}\widehat{f}(x,t)\,dt\,dx \\ &\le \int_\Omega \underline{u}(x)f(x,\underline{u}(x),\nabla w(x))\,dx + \int_\Omega c_M(|u(x)|+|\underline{u}(x)|)\,dx \\ &+\frac{d_M}{p}\int_\Omega (|u(x)|^p+|\underline{u}(x)|^p)\,dx,\end{aligned}$$

where $M=\|w\|_{C_0^1(\overline{\Omega})}$, whence

$$\int_\Omega\int_0^{u(x)}\widehat{f}(x,t)\,dt\,dx \le C_1(1+\|u\|)+\frac{d_M}{\lambda_1 p}\|u\|^p,$$

with a constant $C_1>0$. Therefore we get

$$\mathscr{E}_w(u)\ge \frac{1}{p}\left(1-\frac{d_M}{\lambda_1}\right)\|u\|^p - C_1(1+\|u\|)-C_2,$$

where $C_2=\int_\Omega \underline{u}(x)g(x,\underline{u}(x))\,dx<\infty$. The smallness condition $d_M<\lambda_1$ (see $H(f)$(ii)) determines that the energy functional $\mathscr{E}_w$ is coercive.

Claim 3 *The energy functional $\mathscr{E}_w$ is weakly sequentially lower semicontinuous.*

Let $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. It follows from the convexity of the norm that

$$\liminf_{n\to\infty}\frac{1}{p}\|u_n\|^p \ge \frac{1}{p}\|u\|^p.$$

By Rellich-Kondrachov Embedding Theorem (see, e.g., [17, Theorem 2.5.17]), we have $u_n \to u$ in $L^p(\Omega)$, so along a relabeled subsequence $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. But, Lebesgue's Dominated Convergence Theorem confirms

$$\lim_{n\to\infty}\left(\int_\Omega G(x,u_n(x))\,dx + \int_\Omega F(x,u_n(x))\,dx\right) = \int_\Omega G(x,u(x))\,dx + \int_\Omega F(x,u(x))\,dx.$$

Claim 3 ensues.

On the basis of Claims 1–3 we are able to apply Weierstrass-Tonelli Theorem finding $u \in W_0^{1,p}(\Omega)$ such that

$$\mathscr{E}_w(u) = \inf_{v\in W_0^{1,p}(\Omega)} \mathscr{E}_w(v).$$

Claim 4 *If u is a critical point of $\mathscr{E}_w$, then $u \geq \underline{u}$ and u is a solution of problem (4).*

Let $u \in W_0^{1,p}(\Omega)$ be a critical point of $\mathscr{E}_w$, that is,

$$\mathscr{E}_w'(u) = 0.$$

This reads as

$$\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\,dx - \int_\Omega [\widehat{g}(x,u(x)) + \widehat{f}(x,u(x))]v(x)\,dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$, i.e., $u \in W_0^{1,p}(\Omega)$ is a solution of problem (10).

Inserting $v = (\underline{u} - u)^+$ in the above equality and in (9) produces

$$\int_{\{u<\underline{u}\}} |\nabla u(x)|^{p-2}(\nabla u(x), \nabla(\underline{u}-u)(x))_{\mathbb{R}^N}\,dx \geq \int_{\{u<\underline{u}\}} g(x,\underline{u}(x))(\underline{u}-u)(x)\,dx$$

and

$$\int_{\{u<\underline{u}\}} |\nabla \underline{u}(x)|^{p-2}(\nabla \underline{u}(x), \nabla(\underline{u}-u)(x))_{\mathbb{R}^N}\,dx \leq \int_{\{u<\underline{u}\}} g(x,\underline{u}(x))(\underline{u}-u)(x)\,dx,$$

due to $f(x,s,\xi) \geq 0$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$. We are led to

$$\int_{\{u<\underline{u}\}} (|\nabla u(x)|^{p-2}\nabla u(x) - |\nabla \underline{u}(x)|^{p-2}\nabla \underline{u}(x), \nabla(u-\underline{u})(x))_{\mathbb{R}^N}\,dx \leq 0,$$

which forces $u \geq \underline{u}$.

On the basis of Claim 4, by virtue of the definitions of $\widehat{g}$ and $\widehat{f}$, the solution u of (10) becomes a solution of problem (4). This completes the proof. □

Remark 16 The Moser iteration technique (see, e.g., [27, Theorem 4.1]) shows that each solution u of problem (4) is an element of $L^\infty(\Omega)$. Moreover, the nonlinear regularity theory in [26,28,29] ensures that u belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

We introduce the set-valued mapping $\mathscr{S}\colon C_0^1(\overline{\Omega}) \to 2^{C_0^1(\overline{\Omega})}$ as follows

$$\mathscr{S}(w) := \left\{u \in C_0^1(\overline{\Omega}) \ : \ u \text{is a solution of problem (4) with } u \geq \underline{u}\right\}.$$

Via Lemma 15 and Remark 16 we see that $\mathscr{S}$ is well-defined meaning that its values are nonempty.

Lemma 17 *Assume that $H(g)$ and $H(f)$ hold. Then the set-valued mapping $\mathscr{S}$ is compact, that is, $\mathscr{S}$ maps the bounded sets in $C_0^1(\overline{\Omega})$ into relatively compact subsets of $C_0^1(\overline{\Omega})$.*

Proof. Let B be a bounded subset of $C_0^1(\overline{\Omega})$, so there is a constant $M > 0$ such that

$$\|B\| := \sup_{w \in B} \|w\|_{C_0^1(\overline{\Omega})} \leq M.$$

For $w \in B$ and $u \in \mathscr{S}(w)$, we have

$$\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega [g(x, u(x)) + f(x, u(x), \nabla w(x))]v(x)\, dx$$

whenever $v \in W_0^{1,p}(\Omega)$. Setting $v = u$, it follows from hypotheses $H(g)$, $H(f)$, the property $u \geq \underline{u}$, (8), and Hölder's inequality that

$$\begin{aligned}
\|u\|^p &= \int_\Omega [g(x, u(x)) + f(x, u(x), \nabla w(x))]u(x)\, dx \\
&\leq \int_\Omega [g(x, \underline{u}(x)) + f(x, u(x), \nabla w(x))]u(x)\, dx \\
&\leq \int_\Omega [g(x, \varepsilon\vartheta(x)) + c_M + d_M u(x)^{p-1}]u(x)\, dx \\
&\leq d_M \|u\|_p^p + \|u\|_p (\|g(\cdot, \varepsilon\vartheta(\cdot))\|_{p'} + c_M |\Omega|^{\frac{1}{p}}).
\end{aligned}$$

Thanks to $\|u\|_p^p \leq \|u\|^p/\lambda_1$, we obtain

$$\|u\|^p \leq \frac{d_M}{\lambda_1}\|u\|^p + \frac{\|g(\cdot, \varepsilon\vartheta(\cdot))\|_{p'} + c_M|\Omega|^{\frac{1}{p}}}{\lambda_1^{\frac{1}{p}}}\|u\|.$$

The smallness condition $d_M < \lambda_1$ allows us to derive that $\mathscr{S}(B)$ is bounded in $W_0^{1,p}(\overline{\Omega})$. Through the nonlinear regularity theory in [26,28,29], there exists $\alpha \in (0, 1)$ such that $\mathscr{S}(B) \subset C^{1,\alpha}(\overline{\Omega})$ is bounded as well. Since $C^{1,\alpha}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, we infer that $\mathscr{S}(B)$ is relatively compact in $C_0^1(\overline{\Omega})$. □

The next results establish the continuity of $\mathscr{S}$.

Lemma 18 *Assume that $H(g)$ and $H(f)$ hold. Then the set-valued mapping $\mathscr{S}$ is upper semicontinuous.*

Proof According to Proposition 4, we must prove that for any closed subset C of $C_0^1(\overline{\Omega})$, the set

$$\mathscr{S}^-(C) = \left\{w \in C_0^1(\overline{\Omega}) \ : \ \mathscr{S}(w) \cap C \neq \emptyset\right\}$$

is closed in $C_0^1(\overline{\Omega})$. To this end, let $\{w_n\} \subset \mathscr{S}^-(C)$ satisfy $w_n \to w$ in $C_0^1(\overline{\Omega})$. For each $n \in \mathbb{N}$ there exists $u_n \in \mathscr{S}(w_n) \cap C$, so

$$\int_\Omega |\nabla u_n(x)|^{p-2}(\nabla u_n(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega g(x, u_n(x))v(x)\, dx \\ + \int_\Omega f(x, u_n(x), \nabla w_n(x))v(x)\, dx \tag{11}$$

for all $v \in W_0^{1,p}(\Omega)$. It follows from Lemma 17 that the sequence $\{u_n\}$ is relatively compact in $C_0^1(\overline{\Omega})$. Passing to a relabeled subsequence, we may assume that $u_n \to u$ in $C_0^1(\overline{\Omega})$. Recall that $u_n \geq \underline{u}$ and C is closed in $C_0^1(\overline{\Omega})$. Hence we have

$$u \geq \underline{u} \quad \text{and} \quad u \in C. \tag{12}$$

The continuity of $f(x, \cdot, \cdot)$ and $g(x, \cdot)$ implies

$$g(x, u_n(x)) \to g(x, u(x)) \quad \text{and} \quad f(x, u_n(x), \nabla w_n(x)) \to f(x, u(x), \nabla w(x))$$

for a.e. $x \in \Omega$ because $u_n \to u$ and $w_n \to w$ in $C_0^1(\overline{\Omega})$. From (8), $u_n \geq \underline{u}$ and $H(f)$(ii), corresponding to $v \in W_0^{1,p}(\Omega)$ we can find a function $h \in L^1_+(\Omega)$ satisfying

$$|[g(x, u_n(x)) + f(x, u_n(x), \nabla w_n(x))]v(x)| \leq h(x)$$

for a.e. $x \in \Omega$. Letting $n \to \infty$ in (11), by means of Lebesgue's Dominated Convergence Theorem, we see that

$$\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega [g(x, u(x)) + f(x, u(x), \nabla w(x))]v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$, thus u is a solution of problem (4). The latter and (12) reveal that $u \in \mathscr{S}(w) \cap C$, or in other terms, $w \in \mathscr{S}^-(C)$, achieving the proof that $\mathscr{S}$ is upper semicontinuous. □

Corollary 19 *Assume that $H(g)$ and $H(f)$ hold. If $\{w_n\}$ and $\{u_n\}$ are sequences in $C_0^1(\overline{\Omega})$ satisfying*

$$w_n \to w \quad \textit{as } n \to \infty \quad \textit{and} \quad u_n \in \mathscr{S}(w_n) \quad \textit{for all } n \in \mathbb{N},$$

then there exist $u \in \mathscr{S}(w)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u$ in $C_0^1(\overline{\Omega})$ as $k \to \infty$.

Proof It is straightforward to check that $\mathscr{S}$ has closed values. Then Lemma 17 guarantees that $\mathscr{S}$ has compact values. The desired conclusion is readily obtained from Lemma 18 and Proposition 6. □

Lemma 20 *Assume that $H(g)$ and $H(f)$ hold. Then the set-valued mapping $\mathscr{S}$ is lower semicontinuous.*

Proof In order to invoke Proposition 5, let $\{w_n\} \subset C_0^1(\overline{\Omega})$ satisfy $w_n \to w$ in $C_0^1(\overline{\Omega})$ and let $v \in \mathscr{S}(w)$. For each $n \in \mathbb{N}$, we formulate the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = g(x, v(x)) + f(x, v(x), \nabla w_n(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{13}$$

In view of $v \geq \underline{u}$, (8) and

$$\begin{cases} g(x, v(x)) + f(x, v(x), \nabla w_n(x)) \geq 0 & \text{for a.e. } x \in \Omega \\ g(\cdot, v(\cdot)) + f(\cdot, v(\cdot), \nabla w_n(\cdot)) \not\equiv 0, \end{cases}$$

it is clear that problem (13) has a unique solution $u_n^0 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. As in the proof of Lemma 17, we can verify that, since $w_n \to w$ in $C_0^1(\overline{\Omega})$, then the sequence $\{u_n^0\}$ is relatively compact in $C_0^1(\overline{\Omega})$. So, there exists a subsequence $\{u_{n_k}^0\}$ of $\{u_n\}$ such that $u_{n_k}^0 \to u$ in $C_0^1(\overline{\Omega})$ as $k \to \infty$ and u is the unique solution of the problem

$$\begin{cases} -\Delta_p u(x) = g(x, v(x)) + f(x, v(x), \nabla w(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We point out that $v \in \mathscr{S}(w)$ provides

$$\begin{cases} -\Delta_p v(x) = g(x, v(x)) + f(x, v(x), \nabla w(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

A simple comparison gives $u = v$. Since every subsequence $\{u_{n_k}^0\}$ of $\{u_n\}$ converges to the same limit v, it is true that

$$\lim_{n\to\infty} u_n^0 = v.$$

Next, for each $n \in \mathbb{N}$, we consider the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = g(x, u_n^0(x)) + f(x, u_n^0(x), \nabla w_n(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proceeding as before, we show that this problem has a unique solution u_n^1, which belongs to $\operatorname{int}(C_0^1(\overline{\Omega})_+)$, and

$$\lim_{n\to\infty} u_n^1 = v.$$

Continuing the process, we generate a sequence $\{u_n^k\}_{k,n\geq 1}$ such that

$$\begin{cases} -\Delta_p u_n^k(x) = g(x, u_n^{k-1}(x)) + f(x, u_n^{k-1}(x), \nabla w_n(x)) & \text{in } \Omega \\ u_n^k > 0 & \text{in } \Omega \\ u_n^k = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\lim_{n\to\infty} u_n^k = v \quad \text{for all } k \in \mathbb{N}. \tag{14}$$

Fix $n \geq 1$. As in the proof of Lemma 17, we notice that the sequence $\{u_n^k\}_{k\geq 1}$ is relatively compact in $C_0^1(\overline{\Omega})$, so we may suppose

$$u_n^k \to u_n \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } k \to \infty.$$

Then it appears that

$$\begin{cases} -\Delta_p u_n(x) = g(x, u_n(x)) + f(x, u_n(x), \nabla w_n(x)) & \text{in } \Omega \\ u_n > 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

and $u_n \geq \underline{u}$ (see Lemma 15), which amounts to saying that $u_n \in \mathscr{S}(w_n)$.

We carry on the proof by the nonlinear regularity [26,28,29], the convergence in (14), and the double limit lemma (see, e.g., [17, p. Proposition A.2.35]) to obtain

$$u_n \to v \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty.$$

We conclude that for every sequence $\{w_n\}$ in $C_0^1(\overline{\Omega})$ such that $w_n \to w$ in $C_0^1(\overline{\Omega})$ and for every $v \in \mathscr{S}(w)$ we can find a sequence $\{u_n\} \subset C_0^1(\overline{\Omega})$ satisfying $u_n \in \mathscr{S}(w_n)$ for each $n \in \mathbb{N}$ and $u_n \to u$ in $C_0^1(\overline{\Omega})$. Consequently, by Proposition 5, $\mathscr{S}$ is lower semicontinuous. □

The following statement summarizes Lemmata 17, 18 and 20.

Corollary 21 *Assume that $H(g)$ and $H(f)$ hold. Then the set-valued mapping $\mathscr{S}\colon C_0^1(\overline{\Omega}) \to 2^{C_0^1(\overline{\Omega})}$ is continuous in the sense of Definition 3(iii) and has compact values.*

For each $w \in C_0^1(\overline{\Omega})$, the set $\mathscr{S}(w)$ has a rich order structure.

Lemma 22 *Assume that $H(g)$ and $H(f)$ hold. Then for each $w \in C_0^1(\overline{\Omega})$, the set $\mathscr{S}(w)$ is downward directed in the sense of Definition 2.*

Proof For any $w \in C_0^1(\overline{\Omega})$, let $u_1, u_2 \in \mathscr{S}(w)$ and $u := \min\{u_1, u_2\}$. Consider the Dirichlet problem

$$\begin{cases} -\Delta_p u(x) = \widetilde{f}(x, u(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $\widetilde{f}\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\widetilde{f}(x,s) = \begin{cases} g(x, \underline{u}(x)) + f(x, \underline{u}(x), \nabla w(x)) & \text{if } s \leq \underline{u}(x) \\ g(x,s) + f(x,s,\nabla w(x)) & \text{if } \underline{u} < s < u(x) \\ g(x, u(x)) + f(x, u(x), \nabla w(x)) & \text{if } u(x) \leq s. \end{cases}$$

Arguing as in the proof of Lemma 15, we see that problem (15) admits a positive solution $\widetilde{u}$ with $\widetilde{u} \geq \underline{u}$.

We now show that $\widetilde{u} \leq u$. Since

$$\int_\Omega |\widetilde{u}(x)|^{p-2}(\widetilde{u}(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega \widetilde{f}(x, \widetilde{u}(x)) v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$, we may insert $v = (\widetilde{u} - u)^+$, which results in

$$\begin{aligned} &\int_{\{\widetilde{u}>u\}} |\nabla \widetilde{u}(x)|^{p-2}(\nabla \widetilde{u}(x), \nabla(\widetilde{u}-u)(x))_{\mathbb{R}^N}\, dx \\ &= \int_{\{\widetilde{u}>u\}} g(x,u(x))(\widetilde{u}-u)(x)\, dx + \int_{\{\widetilde{u}>u\}} f(x, u(x), \nabla w(x))(\widetilde{u}-u)(x)\, dx \\ &\leq \int_{\{\widetilde{u}>u\}} |\nabla u(x)|^{p-2}(\nabla u(x), \nabla(\widetilde{u}-u)(x))_{\mathbb{R}^N}\, dx. \end{aligned}$$

The last inequality holds because, by Lemma 10, u is a supersolution of problem (4). Observe that the obtained inequality ensures that $\widetilde{u} \leq u$. Then from (15) and the definition of $\widetilde{f}$ we deduce that $\widetilde{u} \in \mathscr{S}(w)$, which completes the proof. □

Theorem 23 *Assume that $H(g)$ and $H(f)$ hold. Then, for each $w \in C_0^1(\overline{\Omega})$ problem (4) admits a smallest solution u_w greater than the subsolution $\underline{u}$.*

Proof Lemma 22 asserts that for each $w \in C_0^1(\overline{\Omega})$ the ordered set $\mathscr{S}(w)$ is downward directed. Let B be a chain in $\mathscr{S}(w)$. We can find a sequence $\{u_n\} \subset B$ such that

$$\lim_{n\to\infty} u_n = \inf B.$$

Since every u_n is a solution of (4) with $u_n \geq \underline{u}$, Lemma 17 claims that the sequence $\{u_n\}$ is relatively compact in $C_0^1(\overline{\Omega})$. So, passing to a subsequence if necessary, there exists $v \in C_0^1(\overline{\Omega})$ such that

$$u_n \to v \text{ in } C_0^1(\overline{\Omega}) \text{ and } v \geq \underline{u}.$$

Therefore $v = \inf B$, which allows us to apply Zorn's Lemma (see, e.g., [38]) to provide a minimal element u_w for $\mathscr{S}(w)$.

We check that u_w is the smallest solution of (4) greater than the subsolution $\underline{u}$. Let $u \in \mathscr{S}(w)$. Since, as known from Lemma 22, the ordered set $\mathscr{S}(w)$ is downward directed, we can find $\widetilde{u} \in \mathscr{S}(w)$ verifying $\widetilde{u} \leq \min\{u_w, u\}$. However, the minimality of $u_w \in \mathscr{S}(w)$ entails

$$\underline{u} \leq u_w \leq \widetilde{u} \leq u,$$

which yields that u_w is the smallest solution greater than the subsolution $\underline{u}$. □

Theorem 23 demonstrates that the map $\Gamma: C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ given by

$$\Gamma(w) = u_w$$

is well defined.

Lemma 24 *Assume that $H(g)$ and $H(f)$ hold. Then, the map $\Gamma: C_0^1(\Omega) \to C_0^1(\Omega)$ is compact.*

Proof The fact that Γ maps the bounded subsets of $C_0^1(\overline{\Omega})$ into relatively compact subsets in $C_0^1(\overline{\Omega})$ is the direct consequence of Lemma 17. Indeed, if B is a bounded subset of $C_0^1(\overline{\Omega})$, then $\mathscr{S}(B)$ is relatively compact in $C_0^1(\overline{\Omega})$, so does $\Gamma(B) \subset \mathscr{S}(B)$.

It remains to verify that Γ is continuous. Let $\{w_n\} \subset C_0^1(\overline{\Omega})$ satisfy $w_n \to w$ and denote $u_n = \Gamma(w_n)$, which reads as

$$\begin{cases} -\Delta_p u_n(x) = f(x, u_n(x), \nabla w_n(x)) + g(x, u_n(x)) & \text{in } \Omega \\ u_n > 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Invoking Lemma 17 again, the sequence $\{u_n\}$ is relatively compact in $C_0^1(\overline{\Omega})$. Up to a subsequence, we may assume that $u_n \to u$ in $C_0^1(\overline{\Omega})$. It is obvious that $u \geq \underline{u}$ owing to $u_n \geq \underline{u}$. On the other hand, in the limit (16) yields

$$\begin{cases} -\Delta_p u(x) = f(x, u(x), \nabla w(x)) + g(x, u(x)) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

thus $u \in \mathscr{S}(w)$. The lower semicontinuity of $\mathscr{S}$ proved in Lemma 20 and the characterization of semicontinuity in Proposition 5 ensure that there exists a sequence $\{v_n\} \subset C_0^1(\overline{\Omega})$ with the properties

$$v_n \in \mathscr{S}(w_n) \text{ for each } n \in \mathbb{N}, \quad \text{and } v_n \to \Gamma(w) \in \mathscr{S}(w).$$

Notice that $u_n = \Gamma(w_n) \le v_n$ and $u \in \mathscr{S}(w)$. Letting $n \to \infty$ implies

$$\Gamma(w) \le u = \lim_{n\to\infty} u_n \le \lim_{n\to\infty} v_n = \Gamma(w),$$

that is $u = \Gamma(w)$, so the map Γ is continuous. □

We are now in a position to prove our main result.

Theorem 25 *Assume that $H(g)$ and $H(f)$ hold. If there exist positive constants c_0, c_1, c_2 with $c_1 + c_2\lambda_1^{\frac{p-1}{p}} < \lambda_1$ such that*

$$|f(x,s,\xi)| \le c_0 + c_1|s|^{p-1} + c_2|\xi|^{p-1} \quad \text{for a.e.} x \in \Omega, \text{ all } s \in \mathbb{R}, \text{ and } \xi \in \mathbb{R}^N,$$

then problem (1) admits a (weak, positive) solution $u \in int(C_0^1(\overline{\Omega})_+)$. Moreover, the (weak, positive) solution set of problem (1) is compact in $C_0^1(\overline{\Omega})$.

Proof First, let us emphasize that every solution of problem (1) must be positive. We claim that each solution of problem (1) is greater than the subsolution $\underline{u}$ of problem (4) constructed in Lemma 13. Let u be a solution of (1). This is expressed by

$$\int_\Omega |\nabla u(x)|^{p-2}(\nabla u(x), \nabla v(x))_{\mathbb{R}^N}\, dx = \int_\Omega [g(x,u(x)) + f(x,u(x),\nabla u(x))]v(x)\, dx$$

for all $v \in W_0^{1,p}(\Omega)$. Acting with $v = (\underline{u} - u)^+$ and using the monotonicity hypothesis $H(g)$(ii), nonnegativity of f and (9), one has

$$\begin{aligned}
&\int_{\{\underline{u}>u\}} |\nabla u(x)|^{p-2}(\nabla u(x), \nabla(\underline{u}-u)(x))_{\mathbb{R}^N}\, dx = \int_{\{\underline{u}>u\}} g(x,u(x))(\underline{u}-u)(x)\, dx \\
&\quad + \int_{\{\underline{u}>u\}} f(x,u(x),\nabla u(x))(\underline{u}-u)(x)\, dx \ge \int_{\{\underline{u}>u\}} g(x,u(x))(\underline{u}-u)(x)\, dx \\
&\ge \int_{\{\underline{u}>u\}} g(x,\underline{u}(x))(\underline{u}-u)(x)\, dx \\
&\ge \int_{\{\underline{u}>u\}} |\nabla \underline{u}(x)|^{p-2}(\nabla \underline{u}(x), \nabla(\underline{u}-u)(x))_{\mathbb{R}^N}\, dx.
\end{aligned}$$

From the above inequality we deduce that $u \ge \underline{u}$.

In order to justify that problem (1) possesses a (positive) solution we make use of Theorem 7. From Lemma 24, we know that Γ is a compact map. It remains to prove that the set

$$\Lambda(\Gamma) := \left\{u \in C_0^1(\overline{\Omega}) \,:\, u = t\Gamma(u) \text{ for some } t \in (0,1)\right\}$$

is bounded in $C_0^1(\overline{\Omega})$. For any $u \in \Lambda(\Gamma)$, we have $u = t\Gamma(u)$ for some $t \in (0,1)$, or equivalently

$$\begin{aligned}
\int_\Omega \left|\nabla \frac{u(x)}{t}\right|^{p-2}\left(\nabla \frac{u(x)}{t}, \nabla v(x)\right)_{\mathbb{R}^N} dx &= \int_\Omega f\left(x, \frac{u(x)}{t}, \nabla u(x)\right)v(x)\, dx \\
&\quad + \int_\Omega g\left(x, \frac{u(x)}{t}\right)v(x)\, dx
\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. From this equation we get that $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$. Choosing $v = \frac{u}{t}$ and using $H(f)$ and Hölder's inequality provide

$$\|u\|^p \leq t^p \int_\Omega \Big[\frac{c_0 u(x)}{t} + c_1 \frac{u(x)^p}{t^p} + c_2 \frac{|\nabla u(x)|^{p-1} u(x)}{t}\Big] dx$$
$$+ t^p \int_\Omega g\Big(x, \frac{u(x)}{t}\Big)\frac{u(x)}{t}\, dx \leq \int_\Omega g\Big(x, \frac{u(x)}{t}\Big) u(x)\, dx$$
$$+ c_0 |\Omega|^{\frac{p-1}{p}} \|u\|_p + c_1 \|u\|_p^p + c_2 \|u\|^{p-1}\|u\|_p.$$

Addressing hypothesis $H(g)$ (with an $\varepsilon > 0$ small enough) and the inequalities $\|u\|_p^p \leq \frac{\|u\|^p}{\lambda_1}$ and $u \geq \underline{u}$, we get the estimate

$$\begin{aligned}
\|u\|^p &\leq \int_\Omega g(x, \varepsilon\vartheta(x)) u(x)\, dx + c_0 |\Omega|^{\frac{1}{p}} \|u\|_p + c_1 \|u\|_p^p + c_2 \|u\|^{p-1} \|u\|_p \\
&\leq \big(\|g(\cdot, \varepsilon\vartheta(\cdot))\|_{p'} + c_0 |\Omega|^{\frac{1}{p}}\big)\|u\|_p + c_1 \|u\|_p^p + c_2 \|u\|^{p-1}\|u\|_p \\
&\leq c_1 \frac{\|u\|^p}{\lambda_1} + c_2 \frac{\|u\|^p}{\lambda_1^{1/p}} + \big(\|g(\cdot, \varepsilon\vartheta(\cdot))\|_{p'} + c_0 |\Omega|^{1/p}\big) \frac{\|u\|}{\lambda_1^{1/p}}.
\end{aligned}$$

The imposed smallness condition $c_1 + c_2 \lambda_1^{(p-1)/p} < \lambda_1$ and $p > 1$ enable us to infer that $\Lambda(\Gamma)$ is bounded in $W_0^{1,p}(\Omega)$. Then, as before, we can apply the nonlinear regularity theory (see [26,28,29]) to confirm that $\Lambda(\Gamma)$ is bounded in $C_0^1(\overline{\Omega})$. Through Theorem 7 we conclude that problem (1) has at least one positive solution $u \in \mathrm{int}(C_0^1(\overline{\Omega})_+)$.

The final step in the proof is to show that the solution set for problem (1) is compact in $C_0^1(\overline{\Omega})$. It is straightforward to verify that the solution set of problem (1) is closed in $C_0^1(\overline{\Omega})$. From the proof of the first part we know that it is bounded in $W_0^{1,p}(\Omega)$. Then the nonlinear regularity theory (see [26,28,29]) indicates that it is bounded in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$, so relatively compact in $C_0^1(\overline{\Omega})$. The proof is thus complete. □

Acknowledgements The authors thank the Referee for relevant comments and suggestion.

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