Bifurcation from infinity for elliptic problems on $\mathbb{R}^N$

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Abstract

In the paper the asymptotic bifurcation of solutions to a parameterized stationary semilinear Schrödinger equation involving a potential of the Kato-Rellich type is studied. It is shown that the bifurcation from infinity occurs if the parameter is an eigenvalue of the hamiltonian lying below the asymptotic bottom of the bounded part of the potential. Thus the bifurcating solution are related to bound states of the corresponding Schrödinger equation. The argument relies on the use of the (generalized) Conley index due to Rybakowski and resonance assumptions of the Landesman-Lazer or sign-condition type.

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1 Introduction

We study a parameterized elliptic problem

$$\begin{cases} -\Delta u(x) + V(x)u(x) = \lambda u(x) + f(x, u(x)), & x \in \mathbb{R}^N, \lambda \in \mathbb{R}, \\ u \in H^1(\mathbb{R}^N), \end{cases} \quad (1.1)$$

related to a nonlinear Schrödinger equation (1.9) and its bound states of the form (1.10). Solutions to (1.1) may also be interpreted as stationary states of the corresponding reaction-diffusion equation (1.8).

We are interested in a characterization of asymptotic bifurcation for (1.1).

Definition 1.1 A parameter $\lambda_0 \in \mathbb{R}$ is a point of bifurcation from infinity or asymptotic bifurcation of solutions to (1.1) if there exists a sequence $(\lambda_n, u_n)_{n=1}^\infty$ such that $\lambda_n \rightarrow \lambda_0$, $u_n \in H^1(\mathbb{R}^N)$ is a weak solution of (1.1) with $\lambda = \lambda_n$ for each $n \geq 1$, and $\|u_n\|_{H^1} \rightarrow +\infty$.

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The study of asymptotic bifurcation, apparently started by M. Krasnoselskii [22], who introduced the notion of an asymptotically linear operator, and P. Rabinowitz [33], as well as the study of bifurcation from zero (i.e. from the zero solution), have been conducted by numerous authors from both the abstract and application viewpoints (e.g. by Toland, Dancer, Mawhin, Schmitt, Ward and many others; see e.g. [12,25,39,44,45]). These problems are related since it is often possible to adapt ideas and techniques coming from the study of bifurcation from zero to asymptotic bifurcation; this was effectively employed by Toland in [44] and in [33,43] via the so-called Toland inversion. Most of applications to PDEs were concerned with bifurcation and multiplicity of solutions to elliptic problems of the form $-\Delta u = \lambda u + f(x, u)$ on a bounded domain $\Omega \subset \mathbb{R}^N$ together with various boundary conditions (see e.g. [3,16,24]). A careful analysis of interactions (i.e. crossing) of λ with the (purely discrete) spectrum of $-\Delta$ subject to the boundary condition along with appropriate behavior of f such as, for instance, the so-called ‘sign condition’, leads to the existence and multiplicity of solution. In [25] (see also [8,9,39]) it was pointed out that a *condition of the Landesman-Lazer type* could substitute the sign condition. The topological tools used depend on the parity of the crossed eigenvalue of $-\Delta$: roughly speaking topological degree techniques are exploited if λ crosses an eigenvalue of odd multiplicity while variational methods are used in the case of even multiplicity.

The problem of bifurcation of solutions to elliptic problems on $\mathbb{R}^N$ is not that well-recognized. A detailed study of bifurcation from zero is given e.g. in [14,32,42], while questions of asymptotic bifurcation were dealt with in [17], [43] (see also the references therein) and [23]. An important issue of the spectral theory of elliptic equations on $\mathbb{R}^N$, as opposed to its counterpart on bounded domains, is that the spectrum of $-\Delta + V(x)$ is not discrete in general and, depending on the potential, may be quite complicated. Results from [17,23,43] show that the existence of asymptotic bifurcation at an eigenvalue λ_0 relies on the appropriate relationship between λ_0 , f and the essential spectrum of $-\Delta + V(x)$ inasmuch as bound states of the Schrödinger equation correspond to energies below the bottom of the essential spectrum.

Let us now present the standing assumptions. As concerns the potential generating the hamiltonian

$$\mathbf{A} := -\Delta + V(x)$$

we assume that

$$V \in L^\infty(\mathbb{R}^N) + L^p(\mathbb{R}^N), \text{ i.e. } V = V_\infty + V_0, \text{ where} \quad (1.2)$$

$$\begin{aligned} V_\infty \in L^\infty(\mathbb{R}^N) \text{ and } V_0 \in L^p(\mathbb{R}^N), \quad p \geq 2 \text{ if } N = 1, \quad p > 2 \text{ if } \\ N = 2 \text{ and } p \geq N \text{ for } N \geq 3, \end{aligned} \quad (1.3)$$

and, as concerns the nonlinear interaction term, we assume that $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

$$|f(x, u)| \leq m(x) \text{ for all } u \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N, \quad (1.4)$$

$$|f(x, u) - f(x, v)| \leq l(x)|u - v| \text{ for all } u, v \in \mathbb{R} \text{ and for a.e. } x \in \mathbb{R}^N, \quad (1.5)$$

where $m \in L^2(\mathbb{R}^N)$, $l = l_0 + l_\infty$ with l_0 satisfying (1.3) (with l_0 instead of V_0) and $l_\infty \in L^\infty(\mathbb{R}^N)$.

Remark 1.2 Observe that V belongs to the so-called *Kato class* of potentials K_N considered by Aizenman and Simon (see [37, A.2]) since $L^r(\mathbb{R}^N) \subset K_N$ whenever $r \geq 2$ with $r > N/2$, $N \geq 2$, or a slightly more general class considered in [19]. If, for instance, V is the *Coulomb*

type potential, i.e. $V(x) := c/|x - x_0|^\alpha$ for $x \neq x_0$, where $x_0 \in \mathbb{R}^N$, $c \in \mathbb{R}$ and $\alpha \in [0, 1/2)$ if $N = 1$, $\alpha \in [0, 1)$ for $N = 2$ and $\alpha \in [0, 1)$ for $N \geq 3$, then V satisfies conditions (1.2) and (1.3) since one may take $V_0 = \chi V$ and $V_\infty = (1 - \chi)V$, here χ is the characteristic function of the unit ball in $\mathbb{R}^N$ around x_0 . $\square$

Since $\lim_{|s| \rightarrow +\infty} f(x, s)/s = 0$ for $x \in \mathbb{R}^N$, one expects that, as in the classical situation (see e.g. [33]), if λ approaches an eigenvalue of $\mathbf{A}$, then solutions to (1.1) bifurcate from infinity as the result of a produced resonance phenomenon. Indeed: as we shall see in Theorem 4.1, the necessary condition for λ_0 lying beyond the essential spectrum of the hamiltonian for inducing asymptotic bifurcation is that $\lambda_0 \in \sigma_p(\mathbf{A})$ the point spectrum of the hamiltonian. Conversely, if λ_0 is an isolated eigenvalue of *odd* multiplicity, then the asymptotic bifurcation occurs. In order to provide sufficient conditions for asymptotic bifurcation from an isolated eigenvalue of *even* multiplicity, one needs to impose additional assumptions concerning the behavior of f at infinity: the so-called *Landesman-Lazer type* or *strong resonance* conditions.

The *Landesman-Lazer type conditions* state that either

$$\begin{cases} \check{f}_+(x) \geq 0 \text{ and } \hat{f}_-(x) \leq 0 \text{ for a.e. } x \in \mathbb{R}^N, \\ \text{there is a set of positive measure on which none of } \check{f}_+ \text{ and } \hat{f}_- \text{ vanishes,} \end{cases} \quad ((LL)_+)$$

or

$$\begin{cases} \hat{f}_+(x) \leq 0 \text{ and } \check{f}_-(x) \geq 0 \text{ for a.e. } x \in \mathbb{R}^N, \\ \text{there is a set of positive measure on which none of } \hat{f}_+ \text{ and } \check{f}_- \text{ vanishes,} \end{cases} \quad ((LL)_-)$$

where $\hat{f}_\pm(x) := \limsup_{s \rightarrow \pm\infty} f(x, s)$ and $\check{f}_\pm(x) := \liminf_{s \rightarrow \pm\infty} f(x, s)$ for a.e. $x \in \mathbb{R}^N$.

Remark 1.3 Conditions of this type has been considered by many authors; see e.g. [15] for a relatively up-to-date survey. Observe (see also the proof of Lemma 5.2) that $(LL)_+$ (resp. $(LL)_-$), together with the so-called *unique continuation property*, imply that

$$\int_{\mathbb{R}^N} (\check{f}_+ \varphi^+ - \hat{f}_- \varphi^-) dx > 0 \quad \left(\text{resp. } \int_{\mathbb{R}^N} (\hat{f}_+ \varphi^+ - \check{f}_- \varphi^-) dx < 0 \right) \quad (1.6)$$

for any eigenfunction φ of the hamiltonian $\mathbf{A}$ and $\varphi^\pm = \max\{0, \pm\varphi\}$. Clearly (1.6) is the classical Landesman-Lazer condition (see e.g. [15, eq. (LL)]); one can easily check by proof-inspection that each of the conditions stated in (1.6) is actually sufficient for our purposes. $\square$

The so-called *sign conditions* or *strong resonance conditions* are fulfilled if $k_\pm(x) := \lim_{s \rightarrow \pm\infty} sf(x, s)$ exists for a.a. $x \in \mathbb{R}^N$, $k_\pm \in L^\infty(\mathbb{R}^N)$ and either

$$\begin{cases} sf(x, s) \geq 0 \text{ for a.a. } x \in \mathbb{R}^N \text{ and all } s \in \mathbb{R}, \\ \text{and there is a set of positive measure on which } k_\pm \text{ is positive,} \end{cases} \quad ((SR)_+)$$

or

$$\begin{cases} sf(x, s) \leq 0 \text{ for a.a. } x \in \mathbb{R}^N \text{ and all } s \in \mathbb{R}, \\ \text{and there is a set of positive measure on which } k_\pm \text{ is negative.} \end{cases} \quad ((SR)_-)$$

As we shall see (comp. Lemma 5.2) both assumption $(LL)_\pm$ and $(SR)_\pm$ lead to the geometric condition (5.2) concerning inward (or outward) behavior of the nonlinearity with respect to eigenspaces of $\mathbf{A}$. Such conditions were already studied on an abstract level in [25, Eq. (2.3) or (2.4)], [6] and [21]. A discussion of some other resonance conditions and their role is provided in [5].

Our main result is as follows. Let

$$\alpha_\infty := \lim_{R \rightarrow \infty} \operatorname{ess\,inf}_{|x| \geq R} V_\infty(x), \quad (1.7)$$

be the asymptotic bottom of the potential V_∞ .

Theorem 1.4 *Suppose that $\lambda_0 \in \sigma(\mathbf{A})$. If either*

- (i) λ_0 is an isolated eigenvalue of odd multiplicity; or
 - (ii) $\lambda_0 < \alpha_\infty$ ¹ and one of conditions $(LL)_\pm$ or $(SR)_\pm$ holds,
- then λ_0 is a point of bifurcation from infinity for (1.1).*

Remark 1.5 (1) It is clear that if (λ_n, u_n) is a sequence bifurcating from infinity at λ_0 , then $u_n \in H^2(\mathbb{R}^N)$ and $\|u_n\|_{H^2} \rightarrow +\infty$. In Theorem 4.1 we show that under the assumptions of the above theorems also both sequences $(\|u_n\|_{L^2})$ and $(\|\nabla u_n\|_{L^2})$ tend to infinity; moreover these sequences have the same growth rate.

(2) Theorem 1.4 complements and generalizes results concerning the asymptotic bifurcation for equations of the form (1.1) from [43] and [23]. In [23] problem (1.1) was studied when $V \in L^\infty(\mathbb{R}^N)$ (i.e., $V_0 \equiv 0$) and under hypotheses which, together with the ansatz (f_4) (see [23, p. 415]), imply our standing assumptions with one important difference in comparison to (1.4): in the setting of [23], the bounding function $m \in L^\infty(\mathbb{R}^N)$. In [43] a similar problem is very thoroughly investigated with $f(x, u) = h(x) + \tilde{f}(u)$, where $h \in L^2(\mathbb{R}^N)$, $\tilde{f}$ is Lipschitz and $\tilde{f}(u)/u \rightarrow 0 = \tilde{f}'(0)$ as $|u| \rightarrow +\infty$ (see the assumption (G) in [43]). Hence, the results of this paper apply to a more general class of potentials but consider a different type of nonlinearities. It is also noteworthy that in both papers [43] and [23] the asymptotic bifurcation occurs at an eigenvalue λ_0 of $\mathbf{A}$ provided the distance $\operatorname{dist}(\lambda_0, \sigma_e(\mathbf{A}))$ of λ_0 to $\sigma_e(\mathbf{A})$, the essential spectrum of the hamiltonian, is larger than the Lipschitz constant of the nonlinearity g (in [43] a bit more restrictive bound is necessary). Such a condition was also implicitly contained in [12, Assumption D]. If the multiplicity of λ_0 is odd, then the proofs from [23, 43] use the degree theory (via the Toland inversion in [43]), while for an eigenvalue of even multiplicity the existence of asymptotic bifurcation in [23] relies on a variational approach based on the Morse theory. In [17] the principal eigenvalue (being simple) of the linearization at infinity is shown to be a point of asymptotic bifurcation and the result is obtained by the Toland inversion.

In our approach the physically relevant unbounded part V_0 of the potential is not trivial, but, at least in case the multiplicity of λ_0 is even, we need that $\lambda_0 < \alpha_\infty$ which, as we shall see, implies that λ_0 lies below the bottom of $\sigma_e(\mathbf{A})$ since $\sigma_e(\mathbf{A}) \subset [\alpha_\infty, \infty)$. We do not require any relations of the distance $\operatorname{dist}(\lambda_0, \sigma_e(\mathbf{A}))$ with the Lipschitz constant, but instead we make use of the estimate (1.4). If $V_0 \neq 0$ (making V look like a potential well) is sufficiently deep and steep, then $\sigma(\mathbf{A}) \cap (-\infty, \alpha_\infty) \neq \emptyset$ (this holds for instance if V is the Coulomb type potential from Remark 1.2; see also eg. [34, Theorem XIII.6] and [40]).

(3) Our attitude to the first part of Theorem 1.4 is based on the Leray-Schauder degree theory; in this context condition (1.5) is not necessary since the continuity of the Nemytskii operator generated by f is sufficient. In the second part we shall rely on the Conley index theory applied to the semiflow generated by the parabolic equation

$$u_t = \Delta u - V(x)u + \lambda u + f(x, u), \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}, \quad t > 0, \quad (1.8)$$

related to (1.1). We shall show that our assumptions imply that this semiflow is well-defined and its Conley indices ‘at infinity’ change when the parameter λ crosses λ_0 . To meet the quite

¹ We shall see that this also implies that λ_0 is an isolated eigenvalue of finite multiplicity.

demanding requirements concerning compactness issues (i.e. the so-called *admissibility* of the semiflow with respect to bounded sets) we adopt some ideas of Prizzi [30,31]. The use of the (generalized) Conley type index of Rybakowski [36] in the context of bifurcation has been started by Ward [45,46] and applied for elliptic problems on bounded domains. Quite recently this approach has been thoroughly complemented and expanded in [24] (see also the rich bibliography therein) and applied to bifurcation problems on bounded domains. To the best of our knowledge the present paper is the first one to employ Conley index to the asymptotic bifurcation for elliptic problems in $\mathbb{R}^N$. $\square$

Let us now discuss the physical context of the studied problem. We consider the externally driven *nonlinear Schrödinger equation* of the form

$$i\psi_t = -\Delta\psi + V(x)\psi - W'(x, \psi), \quad (1.9)$$

and its bound states, i.e. *wave-functions* $\psi : [0, +\infty) \times \mathbb{R}^N \rightarrow \mathbb{C}$ that vanish at infinity; here V satisfies assumptions (1.2) and (1.3), $W : \mathbb{R}^N \times \mathbb{C} \rightarrow \mathbb{R}$ and $W'(x, z) := \frac{\partial}{\partial z_1} W(x, z) + i \frac{\partial}{\partial z_2} W(x, z)$, $x \in \mathbb{R}^N$, $z = z_1 + iz_2$. One usually assumes that W depends on $x \in \mathbb{R}^N$ and $|z|$ only, i.e. $W(x, z) = H(x, |z|)$ where $H : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R}$ has the form

$$H(x, s) = \int_0^s h(x, \xi) d\xi, \quad x \in \mathbb{R}^N, \quad s \geq 0,$$

and $h : \mathbb{R}^N \times [0, +\infty) \rightarrow \mathbb{R}$ is a Carathéodory function satisfying conditions analogous to (1.4) and (1.5). Therefore for all $x \in \mathbb{R}^N$

$$W'(x, z) = h(x, |z|) \frac{z}{|z|} \quad \text{for } z \in \mathbb{C} \setminus \{0\}, \quad W'(x, 0) = 0.$$

Problems concerning (1.9) play an important role in different physical contexts, especially in the description of macroscopic quantum systems like, for instance, plasma physics, nonlinear optics and others – see e.g. [28], [41]. For appropriate choice of h the equation (1.9) has *standing wave solutions*, i.e. satisfying the ansatz

$$\psi(t, x) = e^{-i\lambda t} u(x), \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad (1.10)$$

with the time-independent profile $u \in H^1(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$. Substituting (1.10) into (1.9) and putting for $x \in \mathbb{R}^N$ and $u \in \mathbb{R}$

$$f(x, u) := h(x, |u|) \frac{u}{|u|} \quad \text{if } u \neq 0, \quad f(x, 0) = 0, \quad (1.11)$$

we get (1.1) along with our standing assumptions; clearly any solution $(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ gives via (1.10) a bound state ψ for (1.9).

The *energy* (see [4]) of a wave-function ψ satisfying (1.9), given by

$$E(\psi) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla\psi|^2 + V(x)|\psi|^2) dx - \int_{\mathbb{R}^N} W(x, \psi) dx$$

is time invariant and, in case (1.10),

$$E(\psi) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbb{R}^N} H(x, |u|) dx.$$

Theorem 1.6 Suppose that $\lambda_0 < \alpha_\infty$, where α_∞ is given by (1.7), $\lambda_0 \in \sigma(-\Delta + V)$ and one of the following conditions is satisfied:

- (i)₊ for a.a. $x \in \mathbb{R}^N$, $\check{h}(x) := \liminf_{\xi \rightarrow +\infty} h(x, \xi) \geq 0$ and $\check{h}$ is positive on a set of positive measure;
- (i)₋ for a.a. $x \in \mathbb{R}^N$, $\hat{h}(x) := \limsup_{\xi \rightarrow +\infty} h(x, \xi) \leq 0$ and $\hat{h}$ is negative on a set of positive measure;
- (ii)₊ for a.a. $x \in \mathbb{R}^N$ and all $\xi \geq 0$, $h(x, \xi) \geq 0$ and $\lim_{\xi \rightarrow +\infty} \xi h(x, \xi)$ is positive on a set of positive measure;
- (ii)₋ for a.a. $x \in \mathbb{R}^N$ and all $\xi \geq 0$, $h(x, \xi) \leq 0$ and $\lim_{\xi \rightarrow +\infty} \xi h(x, \xi)$ is negative on a set of positive measure.

Then there is a sequence (ψ_n) of bound states of (1.9) of the form $\psi_n(t, x) = e^{-i\lambda_n t} u_n(x)$ for $t \geq 0$, $x \in \mathbb{R}^N$, where $\lambda_n \in \mathbb{R}$, $u_n \in H^1(\mathbb{R}^N)$ for all $n \geq 1$, $\lambda_n \rightarrow \lambda_0$ and $\|u_n\|_{H^1} \rightarrow +\infty$. If $\lambda_0 \neq 0$, then $|E(\psi_n)| \rightarrow +\infty$.

Proof: It is easy to see that if f is given by (1.11), then condition (i)_± (resp. (ii)_±) implies $(LL)_\pm$ (resp. $(SR)_\pm$); hence, in view of Theorem 1.4, there is a sequence (λ_n, u_n) of solutions to (1.1), yielding via (1.10) the existence of the required sequence of bound states ψ_n . Observe that

$$\begin{aligned} E(\psi_n) &= \frac{1}{2} \left(\lambda_n \|u_n\|_{L^2}^2 + \int_{\mathbb{R}^N} (h(x, |u_n|) |u_n| - 2H(x, |u_n|)) dx \right) \\ &\geq \frac{1}{2} \lambda_n \|u_n\|_{L^2}^2 - 2\|m\|_{L^2} \|u_n\|_{L^2} \rightarrow +\infty \end{aligned}$$

when $\lambda_0 > 0$ and $E(\psi_n) \rightarrow -\infty$ if $\lambda_0 < 0$. $\square$

The paper is organized as follows. Section 2 is devoted to basic notation and a brief exposition of the Conley index theory. In Section 3 we construct the semiflow related to the considered problem, study its basic properties such as continuity and admissibility; we also recall a linearization method to compute the Conley index of the set of bounded trajectories. Section 4 deals with necessary conditions as well as further properties of bifurcating sequences. Finally, Section 5 is devoted to the proof of the main results.

2 Preliminaries

By $L^p(\Omega)$, $1 \leq p \leq \infty$, and $H^k(\Omega)$, $k \in \mathbb{N}$, we denote the standard Lebesgue and Sobolev spaces on an open domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with their standard norms and inner products. For brevity, in the sequel we will write L^p or H^k instead of $L^p(\mathbb{R}^N)$ and $H^k(\mathbb{R}^N)$.

If (X, A) is a topological pair with a closed and nonempty $A \subset X$, then X/A denotes the quotient space, obtained by collapsing the subset A to a point $[A]$. Pointed spaces (X, x_0) and (Y, y_0) are *homotopy equivalent* or have the same homotopy type if there are pointed maps $f: (X, x_0) \rightarrow (Y, y_0)$ and $g: (Y, y_0) \rightarrow (X, x_0)$ such that $f \circ g$ (resp. $g \circ f$) is homotopic to the identity on (Y, y_0) (resp. on (X, x_0)). The homotopy class represented by a space (X, x_0) is denoted by $[(X, x_0)]$.

2.1 Conley index due to Rybakowski

We shall briefly recall a version of the Conley index due to Rybakowski (see [35] or [36]). Let $\Phi: [0, +\infty) \times X \rightarrow X$ be a semiflow on a complete metric space X . We write $\Phi_t(x) := \Phi(t, x)$ and $\Phi_{[0,t]}(x) := \{\Phi_s(x) \mid 0 \leq s \leq t\}$ for $t \geq 0$, $x \in X$. A continuous $u: J \rightarrow X$,

where $J \subset \mathbb{R}$ is an interval, is a solution of Φ if $u(t+s) = \Phi_t(u(s))$ for all $t \geq 0$ and $s \in J$ such that $t+s \in J$. If, in addition $0 \in J$ and $u(0) = x$, then u is a solution through x .

If $a \in \mathbb{R}$ and $u: [a, +\infty) \rightarrow X$ is a solution of Φ , then the ω -limit set of u is defined by

$$\omega(u) := \{x = \lim_{n \rightarrow \infty} u(t_n) \mid t_n \geq a, t_n \rightarrow +\infty\};$$

if $u: (-\infty, a] \rightarrow X$ is a solution of Φ , then the α -limit set of u is defined by

$$\alpha(u) := \{x = \lim_{n \rightarrow \infty} u(t_n) \mid t_n \leq a, t_n \rightarrow -\infty\}.$$

Note that both sets $\omega(u)$ and $\alpha(u)$ are closed.

Let $N \subset X$. We define the *invariant part* $\text{Inv}_\Phi(N)$ of N by

$$x \in \text{Inv}_\Phi(N) \iff \text{there is a solution } u: \mathbb{R} \rightarrow N \text{ through } x.$$

A set $K \subset X$ is a Φ -invariant or *invariant* (w.r.t. Φ) if $\text{Inv}_\Phi(K) = K$. A set K is an *isolated invariant* if there exists an *isolating neighborhood* of K , i.e. $N \subset X$ such that $K = \text{Inv}_\Phi(N) \subset \text{int } N$.

A set $N \subset X$ is Φ -admissible or *admissible* (w.r.t. Φ) if, for any sequences (t_n) in $[0, +\infty)$, (x_n) in X such that $t_n \rightarrow +\infty$ and $\Phi_{[0, t_n]}(x_n) \subset N$, the sequence of end-points $(\Phi_{t_n}(x_n))$ has a convergent subsequence. It is easy to see that if $N \subset X$ is Φ -admissible, then the invariant part $\text{Inv}_\Phi(N)$ is compact.

Suppose that $\{\Phi^\lambda\}_{\lambda \in \Lambda}$, where Λ is a metric space, is a family of semiflows on X . This family is *continuous* if the map $[0, +\infty) \times X \times \Lambda \ni (t, x, \lambda) \mapsto \Phi_t^\lambda(x)$ is continuous. A set $N \subset X$ is *admissible* w.r.t. $\{\Phi^\lambda\}$ if, for any sequences (t_n) in $[0, +\infty)$, (x_n) in X and (λ_n) such that $t_n \rightarrow +\infty$, $\lambda_n \rightarrow \lambda_0$ in Λ and $\Phi_{[0, t_n]}^{\lambda_n}(x_n) \subset N$, the sequence $(\Phi_{t_n}^{\lambda_n}(x_n))$ has a convergent subsequence.

Let $\mathcal{I}(X)$ be the family of all pairs (Φ, K) , where Φ is a semiflow on X and a set $K \subset X$ is isolated invariant w.r.t. Φ having a Φ -admissible isolating neighborhood. If $(\Phi, K) \in \mathcal{I}(X)$, then the Conley *homotopy index* $h(\Phi, K)$ of K relative to Φ is defined by

$$h(\Phi, K) := [(B/B^-, [B^-])],$$

where B is an *isolating block* of K (relative to Φ ; see [35] for the details) with the *exit set* $B^- \neq \emptyset$; if $B^- = \emptyset$ we put $h(\Phi, K) := [(B \cup \{a\}, a)]$ where a is an arbitrary point out of B . In particular, $h(\Phi, \emptyset) = \bar{0}$ where $\bar{0} := [(\{a\}, a)]$.

Let us enumerate several important properties of homotopy index:

- (H1) for any $(\Phi, K) \in \mathcal{I}(X)$, if $h(\Phi, K) \neq \bar{0}$, then $K \neq \emptyset$;
- (H2) if $(\Phi, K_1), (\Phi, K_2) \in \mathcal{I}(X)$ and $K_1 \cap K_2 = \emptyset$, then $(\Phi, K_1 \cup K_2) \in \mathcal{I}(X)$ and $h(\Phi, K_1 \cup K_2) = h(\Phi, K_1) \vee h(\Phi, K_2)$;
- (H3) for any $(\Phi_1, K_1) \in \mathcal{I}(X_1)$ and $(\Phi_2, K_2) \in \mathcal{I}(X_2)$, $(\Phi_1 \times \Phi_2, K_1 \times K_2) \in \mathcal{I}(X_1 \times X_2)$ and $h(\Phi_1 \times \Phi_2, K_1 \times K_2) = h(\Phi_1, K_1) \wedge h(\Phi_2, K_2)$;
- (H4) if the family of semiflows $\{\Phi^\lambda\}_{\lambda \in [0, 1]}$ is continuous and there exists an admissible (with respect to this family) N such that $K_\lambda = \text{Inv}_{\Phi^\lambda}(N) \subset \text{int } N$, $\lambda \in [0, 1]$, then

$$h(\Phi^0, K_0) = h(\Phi^1, K_1).$$

In a linear case the following formula for computation of the Conley index is used.

Theorem 2.1 (See [35, Ch. I, Th. 11.1]) Assume that a C_0 semigroup $\{T(t)\}_{t \geq 0}$ of bounded linear operators on a Banach space X is hyperbolic (see e.g. [13, Def. V.1.14]). If the

dimension $\dim X_u = k$ of the unstable subspace X_u ⁽²⁾ is finite, then $\Phi: [0, +\infty) \times X \rightarrow X$, given by $\Phi(t, x) := T(t)x$ for $x \in X$ and $t \geq 0$, is a semiflow on X , $\{0\}$ is the maximal bounded invariant set with respect to Φ , $(\Phi, \{0\}) \in \mathcal{I}(X)$ and $h(\Phi, \{0\}) = \Sigma^k$ where $\Sigma^k = [(S^k, \bar{s})]$ is the homotopy type of the pointed k -dimensional sphere. $\square$

3 Admissibility and compactness properties of semiflow

Let us consider problems (1.1) in its abstract form

$$(\mathbf{A} - \lambda \mathbf{I})u = \mathbf{F}(u), \quad u \in H^2, \quad \lambda \in \mathbb{R}, \quad (3.1)$$

where $\mathbf{I}$ is the identity on L^2 , with the linear operator $\mathbf{A}: D(\mathbf{A}) \subset L^2 \rightarrow L^2$ given by

$$D(\mathbf{A}) := H^2, \quad \mathbf{A} := \mathbf{A}_0 + \mathbf{V}_0 + \mathbf{V}_\infty, \quad \text{where:} \quad (3.2)$$

$$\mathbf{A}_0 u := -\Delta u, \quad \text{i.e.,} \quad \mathbf{A}_0 u := -\sum_{j=1}^N \frac{\partial^2 u}{\partial x_j^2} \quad \text{for } u \in D(\mathbf{A}_0) = D(\mathbf{A}), \quad (3.3)$$

$$\mathbf{V}_\infty u := V_\infty \cdot u \quad \text{for } u \in D(\mathbf{V}_\infty) := L^2 \quad \text{and} \quad (3.4)$$

$$\mathbf{V}_0 u := V_0 \cdot u \quad \text{for } u \in D(\mathbf{V}_0) := L^q, \quad \text{where } q \text{ is given by (3.8) below;} \quad (3.5)$$

and $\mathbf{F}: H^1 \rightarrow L^2$ is the superposition operator generated by f , i.e.:

$$\mathbf{F}(u) := f(\cdot, u(\cdot)), \quad \text{for } u \in L^2. \quad (3.6)$$

Let us discuss the above abstract setting.

Remark 3.1 (1) By [27, Th. 7.3.5], $\mathbf{A}_0$ is self-adjoint and sectorial. Clearly $\mathbf{V}_\infty$ is a bounded linear operator. By [13, Proposition III.1.12] $\mathbf{A}_0 + \mathbf{V}_\infty$, defined on $D(\mathbf{A}_0 + \mathbf{V}_\infty) = D(\mathbf{A})$, is sectorial, too. By the Kato-Rellich theorem (see [40, Theorem 8.5]) it is self-adjoint. It is also clear that

$$s_\infty := \inf \sigma(\mathbf{A}_0 + \mathbf{V}_\infty) = \inf_{u \in H^1, \|u\|_{L^2}=1} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty(x)u^2) dx,$$

i.e. $\sigma(\mathbf{A}_0 + \mathbf{V}_\infty) \subset [s_\infty, +\infty)$. In view of the Persson theorem [29, Theorem 2.1] we have that

$$\begin{aligned} s_\infty^* &:= \inf \sigma_e(\mathbf{A}_0 + \mathbf{V}_\infty) \\ &= \lim_{R \rightarrow \infty} \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty(x)u^2) dx \mid u \in C_0^\infty(\{|x| \geq R\}), \|u\|_{L^2} = 1 \right\}. \end{aligned}$$

It is immediate to see that $\alpha_\infty \leq s_\infty^*$ (recall (1.7)). Therefore

$$\sigma_e(\mathbf{A}_0 + \mathbf{V}_\infty) \subset [\alpha_\infty, +\infty). \quad (3.7)$$

At most instances $\alpha_\infty < s_\infty^*$ (see [29]); if, however, $\lim_{R \rightarrow \infty} \text{esssup}_{|x| \geq R} |V_\infty(x) - \alpha_\infty| = 0$, then $\sigma_e(\mathbf{A}_0 + \mathbf{V}_\infty) = [\alpha_\infty, +\infty)$.

(2) Let p be as in (1.3) and let

$$q := \frac{2p}{p-2} \quad \text{if } p > 2, \quad q := \infty \quad \text{for } p = 2. \quad (3.8)$$

² The unstable space X_u is equal to $\text{Ker } P$, where P is the spectral projection corresponding to $\{\lambda \in \sigma(T(t_0)) \mid |\lambda| < 1\}$ for some $t_0 > 0$, or the closed subspace in X corresponding $\{\lambda \in \sigma(A) \mid \text{Re } \lambda < 0\}$, where A is the generator of $\{T(t)\}$.

Observe that, in view of the Sobolev embeddings (see [1, Theorem 4.12]), our assumptions imply that for any $N \geq 1$, $H^1 \hookrightarrow L^q$ (continuous embeddings) and, in view of the Rellich-Kondrachov theorem (see [1, Theorem 6.3]), $H^2(\Omega)$ is compactly embedded in $L^q(\Omega)$ provided $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain.

(3) By the above, $H^1 \hookrightarrow D(\mathbf{V}_0) = L^q$. In view of the Hölder inequality $\mathbf{V}_0$ is well-defined and, as the operator $L^q \rightarrow L^2$, continuous. It is symmetric, hence, closable. In view of Lemma 3.2 below, $\mathbf{V}_0$ is relatively $(\mathbf{A}_0 + \mathbf{V}_\infty)$ -compact. Therefore, by [13, Corollary III.2.17 (ii)], $\mathbf{A}$ is sectorial and, in view of [40, Proposition 8.14 (ii), Theorem 8.5], $\mathbf{A}$ is self-adjoint; see also [34, Corollary XIII.4.2]. Hence $\sigma(\mathbf{A}) \subset \mathbb{R}$.

(4) The relative compactness of $\mathbf{V}_0$ w.r.t. $\mathbf{A}_0 + \mathbf{V}_\infty$ (see Lemma 3.2) implies, in view of the Weyl theorem (see e.g. [38, Theorem 1.4.6] or [40, Theorem 8.15]) and (1.7), that

$$\sigma_e(\mathbf{A}) = \sigma_e(\mathbf{A}_0 + \mathbf{V}_\infty) \subset [\alpha_\infty, +\infty). \quad (3.9)$$

Therefore $\sigma(\mathbf{A}) \cap (-\infty, \alpha_\infty)$ is contained in the discrete part of the spectrum $\sigma_d(\mathbf{A})$; hence it consists of at most countable number of isolated eigenvalues with finite multiplicity.

(5) Observe that in view of (1.4) $\mathbf{F}$ is well-defined and continuous as an operator $L^2 \rightarrow L^2$ since

$$\|\mathbf{F}(u)\|_{L^2} \leq \|m\|_{L^2}, \quad u \in L^2, \quad (3.10)$$

and, by (1.5),

$$\begin{aligned} \|\mathbf{F}(u) - \mathbf{F}(v)\|_{L^2} &\leq \|(l_0 + l_\infty)|u - v|\|_{L^2} \\ &\leq \|l_0\|_{L^p} \|u - v\|_{L^q} + \|l_\infty\|_{L^\infty} \|u - v\|_{L^2} \\ &\leq L \|u - v\|_{H^1}, \end{aligned} \quad (3.11)$$

for $u, v \in H^1$, with an appropriately chosen Lipschitz constant L . Clearly, if $u \in H^1$, then $u \in L^2 \cap L^q$ and $\max\{\|u\|_{L^2}, \|u\|_{L^q}\} \leq \text{const.} \|u\|_{H^1}$ ⁽³⁾. Hence $\mathbf{F}$ is Lipschitz continuous as a map $H^1 \rightarrow L^2$.

(6) By [20, Theorem 3.3.3] (comp. [7, Chapter 3]), the sectoriality of $\mathbf{A}$, conditions (3.10) and (3.11) imply that for each $\bar{u} \in H^1$ and $\lambda \in \mathbb{R}$ there is a unique global solution u of

$$\dot{u} = -\mathbf{A}u + \lambda u + \mathbf{F}(u), \quad t > 0, \quad \lambda \in \mathbb{R}, \quad u \in H^1, \quad (3.12)$$

i.e. a continuous function $u = u(\cdot; \bar{u}, \lambda) : [0, +\infty) \rightarrow H^1$ such that $u \in C((0, +\infty), H^2) \cap C^1((0, +\infty), L^2)$, $u(0) = \bar{u}$ and (3.12) holds for all $t > 0$. $\square$

Lemma 3.2 *The operator $\mathbf{V}_0$ is relatively $(\mathbf{A}_0 + \mathbf{V}_\infty)$ -compact, i.e. $D(\mathbf{A}_0 + \mathbf{V}_\infty) \subset D(\mathbf{V}_0)$ and $\mathbf{V}_0$ is compact as a map on $D(\mathbf{A}_0 + \mathbf{V}_\infty)$ endowed with the graph-norm.*

Proof In view of Remark 3.1 (2), $D(\mathbf{A}_0 + \mathbf{V}_\infty) = H^2(\mathbb{R}^N) \subset L^q = D(\mathbf{V}_0)$. Assume that a sequence $(u_n)_{n=1}^\infty$ is bounded in the H^2 sense, i.e. $\sup \|u_n\|_{H^2} \leq R$ for some $R > 0$. Clearly $\sup \|u_n\|_{L^q} \leq \text{const.} R$. Let $v_n := \mathbf{V}_0 u_n$, $n \geq 1$; we will show that the set $\{v_n\}_{n=1}^\infty$ is precompact in L^2 . Take an arbitrary $\varepsilon > 0$. For any $n, k \geq 1$,

$$\begin{aligned} \int_{\{|x| \geq k\}} v_n^2 dx &\leq \left(\int_{\{|x| \geq k\}} |V_0|^p dx \right)^{2/p} \left(\int_{\{|x| \geq k\}} |u_n|^q dx \right)^{2/q} \\ &\leq \text{const.} R^q \left(\int_{\{|x| \geq k\}} |V_0|^p dx \right)^{2/p} < \varepsilon^2 \end{aligned} \quad (3.13)$$

³ Here and below by const. we denote an appropriate constant for which the given inequality holds; therefore const. may vary from one inequality to another.

provided k is large enough. Take such k , let $B := \{x \in \mathbb{R}^N \mid |x| < k\}$ and $u'_n = u_n|_B$, $n \geq 1$. Then $u'_n \in H^2(B)$, (u'_n) is bounded in $H^2(B)$ and, in view of the compactness of the embedding $H^2(B) \subset L^q(B)$, without loss of generality we may assume that $u'_n \rightarrow u'_0$ in $L^q(B)$ as $n \rightarrow \infty$. For $n \geq 0$ let

$$w_n = \begin{cases} V_0 u_n & \text{on } B, \\ 0 & \text{on } \mathbb{R}^N \setminus B. \end{cases}$$

Then $w_n \rightarrow w_0$ in L^2 and, by (3.13), $\|v_n - w_n\|_{L^2} < \varepsilon$. It follows that $\{v_n\}_{n=1}^\infty$ is precompact. $\square$

Remark 3.3 (1) The above argument shows actually that $\mathbf{V}_0$ is relatively $(\mathbf{A}_0 + \mathbf{V}_\infty)$ -compact if $p \geq 2$ for $N \leq 3$ and $p > N/2$ for $N \geq 3$; comp. [40, Theorem 8.19]. The restrictions put on p in (1.3) are necessary to ensure that $H^1 \subset L^q$.

(2) An argument similar to the one used in the above proof shows that a bounded subset $M \subset H^1$ is relatively compact in L^2 provided for any $\varepsilon > 0$ there is $R > 0$ such that

$$\forall u \in M \quad \int_{\{|x| \geq R\}} |u(x)|^2 dx < \varepsilon.$$

$\square$

In view of Remark 3.1 (6), for any $\lambda \in \mathbb{R}$, we are in a position to define $\Phi^\lambda : [0, \infty) \times H^1 \rightarrow H^1$ by putting

$$\Phi_t^\lambda(\bar{u}) := u(t; \bar{u}, \lambda), \quad \bar{u} \in H^1, \quad t \geq 0. \quad (3.14)$$

It is immediate to see that Φ^λ is a semiflow on H^1 . By invoking [30, Prop. 2.3] (comp. [7, Theorem 3.2.1], [10, Prop. 4.3]) we get the following continuity result.

Proposition 3.4 *Given sequences $(\bar{u}_n)$ in H^1 and $\lambda_n \rightarrow \lambda$ in $\mathbb{R}$,*

(i) *if $\bar{u}_n \rightarrow \bar{u}$ in H^1 , then $\Phi_t^{\lambda_n}(\bar{u}_n) \rightarrow \Phi_t^\lambda(\bar{u})$ uniformly with respect to t in compact subsets of $\mathbb{R}$; as a consequence the family $\{\Phi^\lambda\}_{\lambda \in \mathbb{R}}$ is continuous;*

(ii) *if $T > 0$, $R > 0$, $\|\Phi_t^{\lambda_n}(\bar{u}_n)\|_{H^1} \leq R$ for all $t \in [0, T]$ and $\bar{u}_n \rightarrow \bar{u}$ in L^2 , then $\Phi_t^{\lambda_n}(\bar{u}_n) \rightarrow \Phi_t^\lambda(\bar{u})$ uniformly with respect to t in compact subsets of $(0, T]$.* $\square$

Recall the standing assumptions and, as in Theorem 1.4 (i), suppose that

$$\begin{aligned} \lambda_0 & \text{ is an isolated eigenvalue of } \mathbf{A} \text{ of finite multiplicity and let} \\ 0 & < \delta < \text{dist}(\lambda_0, \sigma(\mathbf{A}) \setminus \{\lambda_0\}). \end{aligned} \quad (3.15)$$

Let $X_0 := \text{Ker}(\mathbf{A} - \lambda_0 \mathbf{I})$, $X_\pm$ be the closed subspaces of L^2 corresponding to $\sigma(\mathbf{A}) \cap (-\infty, \lambda_0)$, $\sigma(\mathbf{A}) \cap (\lambda_0, +\infty)$, respectively; let $X := X_- \oplus X_+$ ($\oplus$ stands for the orthogonal sum). It is clear that $X_0, X_\pm$ are $\mathbf{A}$ -invariant, $L^2 = X_0 \oplus X$, $\dim X_0, \dim X_- < \infty$ and $X_0, X_- \subset H^2$ since these spaces are spanned by a finite number of eigenfunctions. Let $\mathbf{Q}_\pm : L^2 \rightarrow L^2$ be the orthogonal projections onto $X_\pm$, $\mathbf{Q} := \mathbf{Q}_- + \mathbf{Q}_+$ and $\mathbf{P} := \mathbf{I} - \mathbf{Q}$. Observe that $\mathbf{P}, \mathbf{Q}_- \in \mathcal{L}(L^2, H^2)$, $\mathbf{Q}_+(H^2) \subset H^2 \cap X_+$ and $\mathbf{Q}_+|_{H^1} \in \mathcal{L}(H^1, H^1)$, i.e.

$$\|\mathbf{Q}|_{H^1}\|_{\mathcal{L}(H^1, H^1)} < \infty. \quad (3.16)$$

If $|\lambda - \lambda_0| \leq \delta$, then $\lambda \notin \sigma(\mathbf{A}|_X)$. Hence $(\mathbf{A} - \lambda \mathbf{I})|_X$ is invertible and the map

$$[\lambda_0 - \delta, \lambda_0 + \delta] \times X \ni (\lambda, w) \mapsto [(\mathbf{A} - \lambda \mathbf{I})|_X]^{-1} w \in X \cap H^2 \quad (3.17)$$

is continuous and $\|[(\mathbf{A} - \lambda \mathbf{I})|_X]^{-1} w\|_{H^2} \leq \text{const.} \|w\|_{L^2}$.

Lemma 3.5 *The map*

$$[\lambda_0 - \delta, \lambda_0 + \delta] \times L^2 \ni (\lambda, u) \mapsto \mathbf{G}(\lambda, u) := \mathbf{F}(\mathbf{P}u + [(\mathbf{A} - \lambda \mathbf{I})|_X]^{-1} \mathbf{Q}u) \in L^2$$

is completely continuous.

Proof The continuity of $\mathbf{G}$ is evident. Let sequence (u_n) in L^2 and (λ_n) in $[\lambda_0 - \delta, \lambda_0 + \delta]$ be bounded. Let $v_n = \mathbf{P}u_n$, $w_n := \mathbf{Q}u_n$, $\tilde{w}_n := [(\mathbf{A} - \lambda_n \mathbf{I})|_X]^{-1} w_n$ and $z_n := \mathbf{G}(\lambda_n, u_n)$, $n \geq 1$. Without loss of generality we may assume that $v_n \rightarrow v_0 \in X_0$. Take an arbitrary $\varepsilon > 0$. In view of (1.4) there is $R > 0$ such that for all $n \geq 1$

$$\int_{\{|x| \geq R\}} z_n^2 dx \leq \int_{\{|x| \geq R\}} m^2 dx < \varepsilon^2. \quad (3.18)$$

Let $B = \{x \in \mathbb{R}^N \mid |x| < R\}$, $v'_n := v_n|_B$, $\tilde{w}'_n := \tilde{w}_n|_B$, $n \geq 1$. Then $v'_n \rightarrow v'_0 := v_0|_B$; the sequence $(\tilde{w}'_n)$ is bounded in $H^2(B)$ and, thus, we may assume that $\tilde{w}'_n \rightarrow \tilde{w}'_0 \in L^2(B)$ as $n \rightarrow \infty$. For $n \geq 0$ let

$$z'_n = \begin{cases} f(x, v'_n(x) + \tilde{w}'_n(x)) & \text{on } B, \\ 0 & \text{on } \mathbb{R}^N \setminus B. \end{cases}$$

Then $z'_n \rightarrow z'_0$ in L^2 and, in view of (3.18), $\|z_n - z'_n\|_{L^2} < \varepsilon$. This implies that $\{z_n\}$ is precompact. $\square$

Now, in the context of Theorem 1.4 (ii) we suppose that

$$\lambda_0 \in \sigma(\mathbf{A}) \quad \text{and} \quad \lambda_0 < \alpha_\infty. \quad (3.19)$$

In view of Remark 3.1 (4), λ_0 is an isolated eigenvalue of finite multiplicity. Take $\delta > 0$ such that

$$0 < \delta < \min\{\alpha_\infty - \lambda_0, \text{dist}(\lambda_0, \sigma(\mathbf{A}) \setminus \{\lambda_0\})\}. \quad (3.20)$$

Lemma 3.6 (comp. [31, Proposition 2.2], [10]) *Let $R > 0$ and let $\delta > 0$ be given as in (3.20). There is $\alpha > 0$ and a sequence (α_n) with $\alpha_n \searrow 0$ such that if $u : [t_0, t_1] \rightarrow H^1$ is a solution of the semiflow Φ^λ corresponding to (3.12) for some $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ such that $\|\mathbf{Q}u(t)\|_{H^1} \leq R$ for all $t \in [t_0, t_1]$, then there is $n_0 \geq 1$ such that*

$$\forall n \geq n_0 \quad \int_{\mathbb{R}^N \setminus B(0, n)} |\mathbf{Q}u(t_1)|^2 dx \leq e^{-2\alpha(t_1 - t_0)} \|\mathbf{Q}u(t_0)\|_{L^2}^2 + \alpha_n. \quad (3.21)$$

Proof Since u is a solution of Φ^λ , we have $u(t + t_0) = \Phi_t^\lambda(u(t_0))$ for $t \in [0, t_1 - t_0]$, i.e., in the case of (3.12),

$$\dot{u}(t) = -\mathbf{A}u(t) + \lambda u(t) + \mathbf{F}(u(t)) \quad \text{for } t \in (t_0, t_1].$$

For $w := \mathbf{Q}u$ and $t \in (t_0, t_1]$ we have

$$\dot{w}(t) = -\mathbf{A}w(t) + \lambda w(t) + \mathbf{QF}(u(t)).$$

Let $\phi : [0, +\infty) \rightarrow [0, 1]$ be a smooth function such that $\phi(s) = 0$ if $s \in [0, 1/2]$ and $\phi(s) = 1$ if $s \geq 1$. Putting

$$\phi_n(x) := \phi(|x|^2/n^2), \quad x \in \mathbb{R}^N,$$

we get, for $t \in (t_0, t_1]$ and $n \geq 1$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle w(t), \phi_n w(t) \rangle_{L^2} &= \langle \phi_n w(t), \dot{w}(t) \rangle_{L^2} \\ &= \langle \phi_n w(t), -(\mathbf{A}_0 + \mathbf{V}_0 + \mathbf{V}_\infty - \lambda \mathbf{I})w(t) + \mathbf{QF}(u(t)) \rangle_{L^2} \\ &= I_1(t) + I_2(t) + I_3(t), \end{aligned}$$

where

$$\begin{aligned} I_1(t) &= \langle \phi_n w(t), -\mathbf{A}_0 w(t) \rangle_{L^2} = -\langle \nabla(\phi_n w(t)), \nabla w(t) \rangle_{L^2} = -\int_{\mathbb{R}^N} \phi_n(x) |\nabla w(t)|^2 dx + \\ &\quad - \frac{2}{n^2} \int_{\{\frac{\sqrt{2}}{2}n \leq |x| \leq n\}} \phi'(|x|^2/n^2) \langle w(t)x, \nabla w(t) \rangle_{\mathbb{R}^N} dx \\ &\leq \frac{2L_\phi}{n} \|w(t)\|_{L^2} \|w(t)\|_{H^1} \leq \frac{2L_\phi R^2}{n}, \end{aligned}$$

with

$$L_\phi := \sup_{s \in [0, +\infty)} |\phi'(s)|; \quad (3.22)$$

note that $L_\phi < \infty$.

In order to estimate the second term $I_2(t)$, take $0 < \eta \leq \frac{1}{2}(\alpha_\infty - \lambda_0 - \delta)$. By definition of α_∞ (see (1.7)), there is a positive integer n_0 such that $V_\infty(x) > \alpha_\infty - \eta$ for a.a. $|x| \geq \sqrt{2}n_0/2$. For $n \geq n_0$ we have

$$\begin{aligned} I_2(t) &= \langle \phi_n w(t), -(\mathbf{V}_0 + \mathbf{V}_\infty - \lambda \mathbf{I})w(t) \rangle_{L^2} \\ &= -\langle \phi_n w(t), (\mathbf{V}_\infty - \lambda \mathbf{I})w(t) \rangle_{L^2} - \langle \phi_n w(t), V_0 w(t) \rangle_{L^2} \\ &= -\int_{\{\frac{\sqrt{2}}{2}n \leq |x| \leq n\}} \phi_n(x) (V_\infty(x) - \lambda) |w(t)|^2 dx - \int_{\mathbb{R}^N} \phi_n(x) V_0(x) |w(t)|^2 dx \\ &\leq -\alpha \langle \phi_n w(t), w(t) \rangle_{L^2} + \text{const.} \|w(t)\|_{H^1}^2 \left(\int_{\{|x| \geq \frac{\sqrt{2}}{2}n\}} |V_0(x)|^p dx \right)^{1/p}, \end{aligned}$$

where $\alpha := \alpha_\infty - \lambda_0 - \delta - \eta > 0$; the last estimate follows in view of the Hölder inequality since $\|w(t)\|_{L^{2p/p-1}} \leq \text{const.} \|u(t)\|_{H^1}$. Finally for all $n \geq 1$

$$\begin{aligned} I_3(t) &= \langle \phi_n w(t), \mathbf{QF}(u(t)) \rangle_{L^2} \|w(t)\|_{L^2} (\|\phi_n \mathbf{F}(u(t))\|_{L^2} + \|\phi_n \mathbf{PF}(u(t))\|_{L^2}) \\ &\leq R \left(\left(\int_{\{|x| > \frac{\sqrt{2}}{2}n\}} |m(x)|^2 dx \right)^{1/2} + \kappa_n \right), \end{aligned} \quad (3.23)$$

where $\kappa_n := \sup \left\{ \left(\int_{\{|x| > \frac{\sqrt{2}}{2}n\}} |z(x)|^2 dx \right)^{1/2} \mid z \in \mathbf{P}(B(0, \|m\|_{L^2})) \right\}$ for $n \geq 1$. Since $\mathbf{P}(B(0, \|m\|_{L^2}))$ is relatively compact (as a bounded subset of the finite dimensional space) with respect to the L^2 topology, in view of the Kolmogorov-Riesz compactness criterion (see e.g. [18, Theorem 5]), we see that $\kappa_n \rightarrow 0^+$ as $n \rightarrow \infty$.

Combining these estimates we get that for any $n \geq n_0$

$$\frac{d}{dt} \langle w(t), \phi_n w(t) \rangle_{L^2} \leq -2\alpha \langle w(t), \phi_n w(t) \rangle_{L^2} + 2\tilde{\alpha}_n,$$

where

$$\begin{aligned}\tilde{\alpha}_n &:= \frac{2R^2 L_\phi}{n} + \text{const.} R^2 \left(\int_{\{|x| \geq \frac{\sqrt{2}}{2}n\}} |V_0(x)|^p dx \right)^{1/p} \\ &\quad + R \left(\int_{\{|x| \geq \frac{\sqrt{2}}{2}n\}} |m(x)|^2 dx \right)^{1/2} + R\kappa_n.\end{aligned}$$

Multiplying by $e^{2\alpha(t-t_0)}$ and integrating over $[t_0, t_1]$ one obtains

$$e^{2\alpha(t_1-t_0)} \langle w(t_1), \phi_n w(t_1) \rangle_{L^2} - \langle w(t_0), \phi_n w(t_0) \rangle_{L^2} \leq \frac{e^{2\alpha(t_1-t_0)} - 1}{\alpha} \tilde{\alpha}_n.$$

This clearly implies

$$\int_{\mathbb{R}^N \setminus B(0,n)} |w(t_1)|^2 dx \leq \langle w(t_1), \phi_n w(t_1) \rangle_{L^2} \leq e^{-2\alpha(t_1-t_0)} \|w(t_0)\|_{L^2}^2 + \alpha^{-1} \tilde{\alpha}_n,$$

which finally yields the assertion with $\alpha_n := \frac{\tilde{\alpha}_n}{\alpha}$. $\square$

Proposition 3.7 *Let $R > 0$, δ be as in Lemma 3.6 and M_R be the set of $\bar{u} \in H^1(\mathbb{R}^N)$ such that there exists a solution $u : (-\infty, 0] \rightarrow H^1(\mathbb{R}^N)$ of Φ^λ for some $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ with $u(0) = \bar{u}$ and $\|Qu(t)\|_{H^1} \leq R$ for all $t \leq 0$. Then QM_R is relatively compact in $L^2(\mathbb{R}^N)$.*

Proof We will use Remark 3.3 (2). Take $\varepsilon > 0$ and $t_0 < 0 = t_1$. In view of Lemma 3.6 there is $\alpha > 0$ and a sequence $\alpha_n \searrow 0^+$ (recall that α_n is independent of the choice of t_0) such that, for all $\bar{u} \in M_R$ and $n \geq n_0$,

$$\int_{\mathbb{R}^N \setminus B(0,n)} |Q\bar{u}|^2 dx \leq e^{2\alpha t_0} \|Qu(t_0)\|_{L^2}^2 + \alpha_n \leq e^{2\alpha t_0} R^2 + \alpha_n < \varepsilon,$$

where $u : (-\infty, 0] \rightarrow H^1(\mathbb{R}^N)$ is the solution of Φ^λ such that $u(0) = \bar{u}$, provided that $e^{2\alpha t_0} R^2 < \varepsilon/2$ and $\alpha_n < \varepsilon/2$ for $n \geq n_0$. $\square$

Remark 3.8 Conclusions of Lemma 3.6 and Proposition 3.7 stay true if the projection Q is replaced by the identity on $L^2(\mathbb{R}^N)$. $\square$

Corollary 3.9 (Comp. [30]) *Any bounded set $M \subset H^1$ is admissible with respect to $\{\Phi^\lambda\}_{\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]}$.*

Proof Take sequences $t_m \rightarrow \infty$, $(u_m) \in H^1$ and $\lambda_m \rightarrow \lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ such that $\Phi_{[0, t_m]}^{\lambda_m}(u_m) \subset M$ and $R > 0$ such that $M \subset D_{H^1}(0, R) := \{u \in H^1 \mid \|u\|_{H^1} \leq R\}$. With no loss of generality we may assume that $t_m > t_0$ for all m . Then, for all m ,

$$\Phi_{t_m}^{\lambda_m}(u_m) = \Phi_{t_0}^{\lambda_m}(z_m)$$

where $z_m := \Phi_{t_m - t_0}^{\lambda_m}(u_m)$. It follows from Lemma 3.6 that, for all $m, n \in \mathbb{N}$,

$$\int_{\mathbb{R}^N \setminus B(0,n)} |z_m(x)|^2 dx \leq e^{-2\alpha(t_m - t_0)} \|u_m\|_{L^2}^2 + \alpha_n \leq R^2 e^{-2\alpha(t_m - t_0)} + \alpha_n$$

where $\alpha_n \rightarrow 0^+$ as $n \rightarrow \infty$. This, in view of Remark 3.3 (2), means that the sequence (z_m) is relatively compact in L^2 . Now, by the weak relative compactness of bounded sets in H^1 , there exists $z \in H^1$ such that (up to a subsequence), $z_m \rightharpoonup z$ (weakly) in H^1 and $z_m \rightarrow z$ in L^2 . Thus, by Proposition 3.4, $\Phi_{t_m}^{\lambda_m}(u_m) = \Phi_{t_0}^{\lambda_m}(z_m) \rightarrow \Phi_{t_0}^\lambda(z)$. $\square$

Remark 3.10 (1) Observe that if $u : \mathbb{R} \rightarrow H^1$ is a full bounded solution of Φ^λ for some $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, then the set $u(\mathbb{R})$ is relatively compact (in H^1). Indeed: for any $(t_n) \in \mathbb{R}$ one has $u(t_n) = \Phi_n^\lambda(z_n)$ with $z_n = u(t_n - n)$, $n \in \mathbb{N}$, that are contained in a bounded set; hence, by Corollary 3.9, $(u(t_n))$ contains a convergent subsequence.

(2) Let the functional $J_\lambda : H^1 \rightarrow \mathbb{R}$, $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, be given by

$$J_\lambda(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2 - \lambda|u|^2) dx - \int_{\mathbb{R}^N} F(x, u) dx$$

where $F(x, s) := \int_0^s f(x, \tau) d\tau$. Then, for any solution $u : (t_0, t_1) \rightarrow H^1$ of Φ^λ , one has

$$\frac{d}{dt} [J_\lambda(u(t))] = -\|\dot{u}(t)\|_{L^2}^2 \quad \text{for each } t \in (t_0, t_1).$$

This means that J_λ is a Liapunov-function for Φ^λ , i.e. it decreases along solutions of Φ^λ . It is also clear that if a solution u is nonconstant, then so is $t \rightarrow J(u(t))$. Therefore, if $u : \mathbb{R} \rightarrow H^1$ is a full bounded solution of Φ^λ , then the limit sets $\alpha(u)$ and $\omega(u)$ consists only of equilibria of Φ^λ (see [35, Prop. 5.3]). $\square$

The following Conley index formula, obtained by linearization and Theorem 2.1, will be used in the sequel.

Proposition 3.11 (comp. [30, Theorem 3.3]) Under assumptions (1.3), (1.4) and (1.5), suppose that $\lambda \notin \sigma(\mathbf{A})$ and $\lambda < \alpha_\infty$. Denote by $K(\Phi^\lambda)$ the set of all $\bar{u} \in H^1$ such that there exists a bounded solution $u : \mathbb{R} \rightarrow H^1$ of Φ^λ such that $u(0) = \bar{u}$. Then $K(\Phi^\lambda)$ is bounded, isolated invariant with respect to Φ^λ , $(\Phi^\lambda, K(\Phi^\lambda)) \in \mathcal{I}(H^1)$ and the Conley index

$$h(\Phi^\lambda, K(\Phi^\lambda)) = \Sigma^{k(\lambda)}$$

where $k(\lambda)$ is the total multiplicity of the negative eigenvalues of $\mathbf{A} - \lambda \mathbf{I}$, i.e. eigenvalues of $-\Delta + V$ less than λ . $\square$

4 Necessary conditions

Below we provide necessary conditions for bifurcation from infinity and study additional properties of bifurcation sequences.

Theorem 4.1 If a bifurcation from infinity for (1.1) occurs at $\lambda_0 \notin \sigma_e(\mathbf{A})$, i.e., there is a sequence (u_n, λ_n) solving (1.1) with $\lambda = \lambda_n$, $\|u_n\|_{H^1} \rightarrow \infty$, $\lambda_n \rightarrow \lambda_0$, then λ_0 lies in $\sigma_p(\mathbf{A})$ the point spectrum of $\mathbf{A}$ and $\|\mathbf{P}u_n\|_{L^2}$, $\|\nabla \mathbf{P}u_n\|_{L^2} \rightarrow \infty$ as $n \rightarrow \infty$. This implies that $\|u_n\|_{L^2}$, $\|\nabla u_n\|_{L^2} \rightarrow \infty$, too. Moreover the sequences $(\|\mathbf{Q}u_n\|_{L^2})$ and $(\|\nabla \mathbf{Q}u_n\|_{L^2})$ are bounded.

If, additionally $\lambda_0 < \alpha_\infty$, then the sequences $(\|u_n\|_{L^2})$ and $(\|\nabla u_n\|_{L^2})$ have the same growth rates, i.e., there are constants $C_1, C_2 > 0$ such that, for all large n ,

$$C_1 \|u_n\|_{L^2} \leq \|\nabla u_n\|_{L^2} \leq C_2 \|u_n\|_{L^2}; \quad (4.1)$$

a similar estimate holds for $\|\mathbf{P}u_n\|_{L^2}$ and $\|\nabla \mathbf{P}u_n\|_{L^2}$ with large n .

Proof. Let $\rho_n := \|u_n\|_{H^1}$; we may assume that $\rho_n > 0$ for all n . Let $z_n := \rho_n^{-1} u_n$; then $\|z_n\|_{H^1} = 1$ and $\|z_n\|_{L^2} \leq \text{const}$. Suppose to the contrary that $\lambda_0 \notin \sigma_p(\mathbf{A})$. Since $\lambda_0 \notin \sigma_e(\mathbf{A})$, this implies that $\lambda_0 \in \rho(\mathbf{A})$, the resolvent set of $\mathbf{A}$. We have

$$(\mathbf{A} - \lambda_0 \mathbf{I})z_n = (\lambda_n - \lambda_0)z_n + \rho_n^{-1} \mathbf{F}(\rho_n z_n).$$

Clearly $v_n := (\lambda_n - \lambda_0)z_n + \rho_n^{-1} \mathbf{F}(\rho_n z_n) \rightarrow 0$ as $n \rightarrow \infty$ (in L^2). Hence $z_n = (\mathbf{A} - \lambda_0 \mathbf{I})^{-1} v_n \rightarrow 0$ in H^1 : a contradiction.

Since λ_0 is isolated in $\sigma(\mathbf{A})$, there is $c > 0$ such that for large n we have $\langle (\mathbf{A} - \lambda_n \mathbf{I})v, v \rangle_{L^2} \geq c\|v\|_{L^2}^2$ for $v \in X_+$ and $\langle (\mathbf{A} - \lambda_n \mathbf{I})w, w \rangle_{L^2} \leq -c\|w\|_{L^2}^2$ for $w \in X_-$. This implies that for large n

$$\begin{aligned} c\|\mathbf{Q}_{\pm}u_n\|_{L^2}^2 &\leq \pm \langle (\mathbf{A} - \lambda_n \mathbf{I})\mathbf{Q}_{\pm}u_n, \mathbf{Q}_{\pm}u_n \rangle_{L^2} = \pm \langle (\mathbf{A} - \lambda_n \mathbf{I})u_n, \mathbf{Q}_{\pm}u_n \rangle_{L^2} = \\ &= \pm \langle \mathbf{F}(u_n), \mathbf{Q}_{\pm}u_n \rangle_{L^2} \leq \|m\|_{L^2} \|\mathbf{Q}_{\pm}u_n\|_{L^2}. \end{aligned}$$

Therefore for large n

$$\|\mathbf{Q}u_n\|_{L^2} \leq 2c^{-1}\|m\|_{L^2}. \quad (4.2)$$

On the other hand

$$\|\nabla \mathbf{Q}u_n\|_{L^2}^2 + \langle (\mathbf{V} - \lambda_n \mathbf{I})\mathbf{Q}u_n, \mathbf{Q}u_n \rangle_{L^2} = \langle (\mathbf{A} - \lambda_n \mathbf{I})u_n, \mathbf{Q}u_n \rangle_{L^2} = \langle \mathbf{F}(u_n), \mathbf{Q}u_n \rangle_{L^2}.$$

Hence

$$\|\nabla \mathbf{Q}u_n\|_{L^2}^2 \leq \|V_{\infty} - \lambda_n\|_{L^{\infty}} \|\mathbf{Q}u_n\|_{L^2}^2 + \|V_0\|_{L^p} \|\mathbf{Q}u_n\|_{L^s}^2 + \|m\|_{L^2} \|\mathbf{Q}u_n\|_{L^2},$$

where $s := 2p/(p-1)$. Clearly, $s > 2$ and, if $N \geq 3$, one has also $s < 2^*_{N} = 2N/(N-2)$. In view of the Gagliardo-Nirenberg inequality (see Remark 4.2)

$$\begin{aligned} \|\nabla \mathbf{Q}u_n\|_{L^2}^2 &\leq \|V_{\infty} - \lambda_n\|_{L^{\infty}} \|\mathbf{Q}u_n\|_{L^2}^2 + C^2 \|V_0\|_{L^p} \|\nabla \mathbf{Q}u_n\|_{L^2}^{2\theta} \|\mathbf{Q}u_n\|_{L^2}^{2(1-\theta)} \\ &\quad + \|m\|_{L^2} \|\mathbf{Q}u_n\|_{L^2} \end{aligned} \quad (4.3)$$

for some $C > 0$ and $\theta \in (0, 1)$. This, together with (4.2), implies that the sequence $(\|\nabla \mathbf{Q}u_n\|_{L^2})$ is bounded.

The same argument (replacing $\mathbf{Q}$ in (4.3) by the identity $\mathbf{I}$) shows that $(\|\nabla u_n\|_{L^2})$ would be bounded if $(\|u_n\|_{L^2})$ were bounded. Since $\|u_n\|_{H^1} \rightarrow \infty$, we deduce therefore that $\|u_n\|_{L^2} \rightarrow \infty$. Now $\|\mathbf{P}u_n\|_{L^2}^2 = \|u_n\|_{L^2}^2 - \|\mathbf{Q}u_n\|_{L^2}^2$, so $\|\mathbf{P}u_n\|_{L^2} \rightarrow \infty$ in view of (4.2). This implies that also $\|\nabla \mathbf{P}u_n\|_{L^2} \rightarrow \infty$ because $\dim X_0 < \infty$. Since

$$\|\nabla u_n\|_{L^2} \geq \|\nabla \mathbf{P}u_n\|_{L^2} - \|\nabla \mathbf{Q}u_n\|_{L^2},$$

we finally infer that $\|\nabla u_n\|_{L^2} \rightarrow \infty$.

Now assume that $\lambda_0 < \alpha_{\infty}$. Take $\eta > 0$ such that $\lambda_0 + 3\eta < \alpha_{\infty}$ and $R > 0$ such that $V_{\infty}(x) \geq \alpha_{\infty} - \eta$ for a. a. $x \in \mathbb{R}^N$ with $|x| > R$. Then for large $n \geq 1$, $V_{\infty}(x) - \lambda_n > \eta$ a.e. on $\{x \in \mathbb{R}^N \mid |x| > R\}$.

For large n we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \eta \int_{\mathbb{R}^N} u_n^2 dx &\leq \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \\ &\quad + \int_{\{|x|>R\}} (V_{\infty}(x) - \lambda_n) u_n^2 dx + \eta \int_{\{|x|\leq R\}} u_n^2 dx \\ &= \int_{\mathbb{R}^N} (|\nabla u|^2 \\ &\quad + (V_{\infty}(x) - \lambda_n) u_n^2) dx + \int_{\{|x|\leq R\}} (\eta - V_{\infty}(x) + \lambda_n) u_n^2 dx. \end{aligned}$$

Hence

$$\|\nabla u_n\|_{L^2}^2 + \eta \|u_n\|_{L^2}^2 \leq - \int_{\mathbb{R}^N} V_0(x) u_n^2 dx + \int_{\{|x| \leq R\}} (\eta - V_\infty(x) + \lambda_n) u_n^2 dx + \int_{\mathbb{R}^N} f(x, u_n) u_n dx. \quad (4.4)$$

Take $\xi > 0$ such that $\xi \geq |\eta - V_\infty(x) - \lambda_n|$ for all large n and let $V_1(x) = \xi$ if $|x| \leq R$ and $V_1(x) = 0$ otherwise. Then $V_1 \in L^p$ and, by (4.4) we have

$$\|\nabla u_n\|_{L^2}^2 + \eta \|u_n\|_{L^2}^2 \leq \|V_0 + V_1\|_{L^p} \|u_n\|_{L^s}^2 + \|m\|_{L^2} \|u_n\|_{L^2}$$

and, again in virtue of the Gagliardo-Nirenberg inequality, we get that

$$\|\nabla u_n\|_{L^2}^2 + \eta \|u_n\|_{L^2}^2 \leq C^2 \|V_0 + V_1\|_{L^p} \|\nabla u_n\|_{L^2}^{2\theta} \|u_n\|_{L^2}^{2(1-\theta)} + \|m\|_{L^2} \|u_n\|_{L^2} \quad (4.5)$$

with constants $C > 0$ and $\theta \in (0, 1)$. For large n ,

$$\left(\frac{\|\nabla u_n\|_{L^2}}{\|u_n\|_{L^2}} \right)^2 + \eta \leq C^2 \|V_0 + V_1\|_{L^p} \left(\frac{\|\nabla u_n\|_{L^2}}{\|u_n\|_{L^2}} \right)^{2\theta} + 1$$

and

$$1 + \eta \left(\frac{\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}} \right)^2 \leq C^2 \|V_0 + V_1\|_{L^p} \left(\frac{\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}} \right)^{2(1-\theta)} + \frac{\|u_n\|_{L^2}}{\|\nabla u_n\|_{L^2}},$$

which gives the existence of $C_1, C_2 > 0$ satisfying (4.1). A similar argument shows that growth rates of $(\|\mathbf{P}u_n\|_{L^2})$ and $\|\nabla \mathbf{P}u_n\|_{L^2}$ are the same. $\square$

Remark 4.2 The Gagliardo-Nirenberg inequality (see [26] and [2]) states that given $1 < r < s$ (with $s < 2_N^* = \frac{2N}{N-2}$ if $N \geq 3$) there are $C > 0$ and $\theta \in (0, 1)$ such that for any $u \in H^1$

$$\|u\|_{L^s} \leq C \|\nabla u\|_{L^2}^\theta \|u\|_{L^r}^{1-\theta} \quad \text{for all } u \in H^1.$$

$\square$

Theorem 4.1 corresponds to [43, Theorem 5.2(i)] where a different situation is considered: $V \in L^\infty(\mathbb{R}^N)$, f is of the form given in Remark 1.5 (2) and $\text{dist}(\lambda_0, \sigma_e(\mathbf{A})) > 2 \cdot \sup_{u \neq 0} |\tilde{f}(u)/u| + l$ where l is the Lipschitz constant of $\tilde{f}$. Theorem 4.1 also shows that bifurcating sequences (u_n) are localized around the eigenspace $\text{Ker}(\mathbf{A} - \lambda_0 \mathbf{I})$ having mass $\|u_n\|_{L^2}$ and energy of the same growth rate as in [43, Theorem 5.2 (iii)] where the case of simple eigenvalue (with the mentioned assumptions on V and f) was considered.

5 Sufficient conditions - proof of Theorem 1.4

Recall the notation introduced in front of Lemma 3.5. We start with the proof of Theorem 1.4 (i): assume (3.15), let $\dim X_0$ be odd and suppose that there is no asymptotic bifurcation at λ_0 . Taking smaller $\delta > 0$ if necessary there is $r > 0$ such that for all $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ if $w \in H^2$ and $(\mathbf{A} - \lambda \mathbf{I})w = \mathbf{F}(w)$, then $\|w\|_{H^1} \leq r$.

Observe that $w \in H^2$, solves (3.1) with some $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, i.e. $(\mathbf{A} - \lambda \mathbf{I})w = \mathbf{F}(w)$, if and only if $u := \mathbf{P}w + (\mathbf{A} - \lambda \mathbf{I})\mathbf{Q}w \in L^2$ solves

$$\begin{aligned} u &= \mathbf{K}(\lambda, u) := (1 + \lambda - \lambda_0)\mathbf{P}u + \mathbf{F}(\mathbf{P}u + [(\mathbf{A} - \lambda \mathbf{I})|_X]^{-1}\mathbf{Q}u) \\ &= (1 + \lambda - \lambda_0)\mathbf{P}u + \mathbf{G}(\lambda, u); \end{aligned} \quad (5.1)$$

see Lemma 3.5. Here the nonlinearity $\mathbf{K} : [\lambda_0 - \delta, \lambda_0 + \delta] \times L^2 \rightarrow L^2$ is continuous and, in view of Lemma 3.5, completely continuous. Moreover (5.1) has no solutions if $|\lambda - \lambda_0| \leq \delta$ and $\|u\|_{L^2}$ is sufficiently large. Indeed if $u \in L^2$ solves (5.1), where $|\lambda - \lambda_0| \leq \delta$, then $w := \mathbf{P}u + [(\mathbf{A} - \lambda \mathbf{I})|_X]^{-1} \mathbf{Q}u$ solves (3.1), i.e., $\|\mathbf{P}u\|_{L^2} = \|\mathbf{P}w\|_{L^2} \leq \|w\|_{H^1} \leq r$. Hence $\|u\|_{L^2} \leq (1 + \delta)r + \|m\|_{L^2} := R_0$. Therefore the Leray-Schauder fixed-point index $\text{ind}_{LS}(\mathbf{K}(\lambda, \cdot), B)$, where B is the ball around 0 of radius $R > \max\{R_0, \delta^{-1}\|m\|_{L^2}\}$ in L^2 , is well-defined and independent of $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$. It is immediate to see that if $\lambda = \lambda_0 \pm \delta$, then $u \neq (1 + \lambda - \lambda_0)\mathbf{P}u + t\mathbf{G}(\lambda, u)$ for $u \notin B$ and $t \in [0, 1]$. Hence, in view of the homotopy invariance and the restriction property of the index, for $\lambda = \lambda_0 \pm \delta$

$$\text{ind}_{LS}(\mathbf{K}(\lambda, \cdot), B) = \text{ind}_{LS}((1 \pm \delta)\mathbf{P}, B) = \text{ind}_{LS}((1 \pm \delta)\mathbf{I}, B \cap X_0).$$

However

$$\text{ind}_{LS}((1 - \delta)\mathbf{I}, B \cap X_0) = 1, \quad \text{ind}_{LS}((1 + \delta)\mathbf{I}, B \cap X_0) = (-1)^{\dim X_0} = -1.$$

This is a contradiction. $\square$

Remark 5.1 The standard use of the Kuratowski-Whyburn lemma makes it easy to get a slightly better result in the context of Theorem 1.4 (i). Namely it appears that there exists a closed connected set $\Gamma \subset H^2 \times \mathbb{R}$ of solutions to (1.1) which contains a sequence (u_n, λ_n) such that $\|u_n\|_{H^2} \rightarrow \infty$, $\lambda_n \rightarrow \lambda_0$.

Now we shall pass to the proof of Theorem 1.4 (ii). We start with the geometric interpretation of the resonance assumptions in spirit of [6] and [21].

Lemma 5.2 Assume that $M \subset X$. If either

- (i) condition $(LL)_\pm$ holds and M is bounded in L^2 , or
- (ii) condition $(SR)_\pm$ holds and M relatively compact in L^2 , then there exist $R_0 > 0$ and $\alpha > 0$ such that for all $\bar{v} \in X_0$ with $\|\bar{v}\|_{L^2} \geq R_0$ and $\bar{w} \in M$

$$\pm \langle \bar{v}, \mathbf{F}(\bar{v} + \bar{w}) \rangle_{L^2} > \alpha. \quad (5.2)$$

Proof We carry out the proof for $(LL)_+$ and $(SR)_+$; other cases may be treated analogously. Suppose to the contrary that for any $n \in \mathbb{N}$ there are $\bar{v}_n \in X_0$ and $\bar{w}_n \in M$ such that $\|\bar{v}_n\|_{L^2} \geq n$ and

$$\langle \bar{v}_n, \mathbf{F}(\bar{v}_n + \bar{w}_n) \rangle_{L^2} \leq n^{-1}. \quad (5.3)$$

Let $\rho_n := \|\bar{v}_n\|_{L^2}$ and $\bar{z}_n := \rho_n^{-1}\bar{v}_n$, $n \in \mathbb{N}$. Since $\dim X_0 < \infty$, we may assume that $\|\bar{z}_n - \bar{z}_0\|_{L^2} \rightarrow 0$ as $n \rightarrow \infty$, where $\bar{z}_0 \in X_0$ and $\|\bar{z}_0\|_{L^2(\mathbb{R}^N)} = 1$. Therefore we may assume that $\bar{z}_n(x) \rightarrow \bar{z}_0(x)$ for a.a. $x \in \mathbb{R}^N$ and there is $\kappa \in L^2$ such that $|\bar{z}_n| \leq \kappa$ a.e. In view of the so-called *unique continuation property* (see e.g. [11, Proposition 3, Remark 2]), $\bar{z}_0 \neq 0$ a.e. Hence the set $\mathbb{R}^N \setminus (A_+ \cup A_-)$, where $A_\pm := \{x \in \mathbb{R}^N \mid \pm \bar{z}_0 > 0\}$, is of measure zero.

Dividing (5.3) by ρ_n we get

$$n^{-2} \geq \rho_n^{-1}/n \geq \langle \bar{z}_n, \mathbf{F}(\rho_n \bar{z}_n + \bar{w}_n) \rangle_{L^2} = \int_{\mathbb{R}^N} \bar{z}_n(x) f(x, \rho_n \bar{z}_n(x) + \bar{w}_n(x)) dx.$$

Assume (i); then $\rho_n^{-1}\bar{w}_n \rightarrow 0$ in L^2 since M is bounded. We may assume without loss of generality that $\rho_n^{-1}\bar{w}_n(x) \rightarrow 0$ for a.a. $x \in \mathbb{R}^N$. Hence $\bar{z}_n + \rho_n^{-1}\bar{w}_n \rightarrow \bar{z}_0$ a.e. This implies

that $\rho_n \bar{z}_n + \bar{w}_n \rightarrow \pm\infty$ for a.a. $x \in A_{\pm}$. Using (1.4) we are in a position to use the Fatou lemma to get

$$\begin{aligned} 0 &\geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \bar{z}_n f(x, \rho_n \bar{z}_n + \bar{w}_n) dx \geq \int_{\mathbb{R}^N} \liminf_{n \rightarrow \infty} \bar{z}_n f(x, \rho_n \bar{z}_n + \bar{w}_n) dx \geq \\ &\geq \int_{A_+} \check{f}_+ \bar{z}_0 dx + \int_{A_-} \check{f}_- \bar{z}_0 dx > 0, \end{aligned}$$

in view of Remark 1.3; this is a contradiction.

Assume (ii). Since now M is L^2 -precompact, we may assume that $\bar{w}_n \rightarrow \bar{w}_0 \in L^2(\mathbb{R}^N)$, $\bar{w}_n(x) \rightarrow \bar{w}_0(x)$ for a.e. $x \in \mathbb{R}^N$ and there is $\gamma \in L^2(\mathbb{R}^N)$ such that $|\bar{w}_n| \leq \gamma$ a.e. on $\mathbb{R}^N$ for all $n \in \mathbb{N}$.

Clearly $\langle \bar{v}_n, \mathbf{F}(\bar{v}_n + \bar{w}_n) \rangle_{L^2} = \langle \bar{v}_n + \bar{w}_n, \mathbf{F}(\bar{v}_n + \bar{w}_n) \rangle_{L^2} - \langle \bar{w}_n, \mathbf{F}(\bar{v}_n + \bar{w}_n) \rangle_{L^2}$. In view of $(SR)_+$, $\lim_{s \rightarrow \pm\infty} f(x, s) = 0$ for a.a. $x \in \mathbb{R}^N$. Hence, again by (1.4) and the Lebesgue dominated convergence theorem we have

$$\langle \bar{w}_n, \mathbf{F}(\bar{v}_n + \bar{w}_n) \rangle_{L^2} = \int_{\mathbb{R}^N} \bar{w}_n(x) f(x, \rho_n \bar{z}_n(x) + \bar{w}_n(x)) dx \rightarrow 0, \text{ as } n \rightarrow +\infty,$$

and, in view of (5.3), arguing as before

$$\begin{aligned} 0 &\geq \liminf_{n \rightarrow \infty} \langle \bar{v}_n + \bar{w}_n, \mathbf{F}(\bar{v}_n + \bar{w}_n) \rangle_{L^2} \\ &= \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} (\rho_n \bar{z}_n(x) + \bar{w}_n(x)) f(x, \rho_n \bar{z}_n(x) + \bar{w}_n(x)) dx > 0 \\ &\geq \int_{A_+} \check{k}_+(x) dx + \int_{A_-} \check{k}_-(x) dx > 0, \end{aligned}$$

we reach a contradiction. $\square$

The set of stationary points of the semiflow Φ^λ related to (3.12), where $|\lambda - \lambda_0| \leq \delta$ and δ is given by (3.20) will be denoted by $\mathcal{E}_\lambda$ and let

$$\mathcal{E} := \bigcup_{\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]} \mathcal{E}_\lambda.$$

Lemma 5.3 Suppose that there is $r > 0$ such that $\mathcal{E} \subset B_{H^1}(0, r)$ ⁽⁴⁾. Then there exists $R_\infty = R_\infty(r) > 0$ such that, for any bounded solution $u : \mathbb{R} \rightarrow H^1$ of Φ^λ with $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, one has

$$\sup_{t \in \mathbb{R}} \|\mathbf{Q}u(t)\|_{H^1} < R_\infty.$$

Proof Since $\delta < \text{dist}(\lambda_0, \sigma(\mathbf{A}) \setminus \{\lambda_0\})$, there is $c > 0$ such that $\sigma((\mathbf{A} - \lambda \mathbf{I})|_{X_-}) \subset (-\infty, -c)$ and $\sigma(\mathbf{A} - \lambda \mathbf{I})|_{X_+} \subset (c, +\infty)$ whenever $|\lambda - \lambda_0| \leq \delta$.

Fix $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ and let $\mathbf{B}_\pm := (\mathbf{A} - \lambda \mathbf{I})|_{X_\pm}$. Clearly $\mathbf{B}_+$ is sectorial and positive. By [7] (see Corollary 1.3.5 and comp. Corollary 1.3.4) the domain $D(\mathbf{B}_+^{1/2}) = D(\mathbf{A}_0^{1/2}) = H^1 \cap X_+$; thus, in view of [7, Proposition 1.3.6], there is $K > 0$ independent of $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ such that for all $\tau > 0$

$$\forall v \in X_+ \quad \|e^{-\tau \mathbf{B}_+} v\|_{H^1} = \|\mathbf{B}_+^{1/2} e^{-\tau \mathbf{B}_+} v\|_{L^2} \leq K \frac{e^{-c\tau}}{\tau^{1/2}} \|v\|_{L^2}, \quad \tau > 0, \quad (5.4)$$

⁴ $B_X(x, r)$ (resp. $D_X(x, r)$) stand for the open (resp. closed) ball at x of radius $r > 0$ in the Banach space X .

and

$$\forall v \in H^1 \cap X_+ \quad \|e^{-\tau \mathbf{B}_+} v\|_{H^1} \leq K e^{-c\tau} \|v\|_{H^1}, \quad (5.5)$$

where $\{e^{-\tau \mathbf{B}_+}\}_{\tau \geq 0}$ denotes the semigroup generated by $-\mathbf{B}_+$.

The semigroup $\{e^{-\tau \mathbf{B}_-}\}_{\tau \geq 0}$ generated by $\mathbf{B}_-$ is uniformly continuous, i.e. it extends to a strongly continuous group and there is $K' > 0$ independent of $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$ such that

$$\forall v \in X_- \quad \|e^{-\tau \mathbf{B}_-} v\|_{L^2} \geq \frac{1}{K'} e^{c\tau} \|v\|_{L^2}, \quad \tau \geq 0 \quad (5.6)$$

since $\sigma(\mathbf{B}_-) < -c$.

Now take a solution $u : \mathbb{R} \rightarrow H^1$ of the semiflow Φ^λ corresponding to (3.12). It is well-known that u is a mild solution (see [20]), i.e. the so-called Duhamel formula holds

$$u(t) = e^{-(t-s)(\mathbf{A}-\lambda \mathbf{I})} u(s) + \int_s^t e^{-(t-\tau)(\mathbf{A}-\lambda \mathbf{I})} \mathbf{F}(u(\tau)) d\tau \quad \text{for all } s, t \in \mathbb{R}, t > s, \quad (5.7)$$

where $\{e^{-\tau(\mathbf{A}-\lambda \mathbf{I})}\}_{\tau \geq 0}$ denotes the analytic semigroup generated by $-(\mathbf{A} - \lambda \mathbf{I})$.

Since, due to Remark 3.10 (2), $\alpha(u) \subset \mathcal{E}_\lambda$, there exists $t_u < 0$ such that $\|u(\tau)\|_{H^1} < 2r$ for all $\tau \leq t_u$. Thus, by (5.5), (3.16) and (5.4), for $t \geq t_u$

$$\begin{aligned} \|\mathbf{Q}_+ u(t)\|_{H^1} &\leq \|e^{-(t-t_u)\mathbf{B}_+} \mathbf{Q}_+ u(t_u)\|_{H^1} + \int_{t_u}^t \|e^{-(t-\tau)\mathbf{B}_+} \mathbf{Q}_+ \mathbf{F}(u(\tau))\|_{H^1} d\tau \leq \\ &K \left(\|\mathbf{Q}_+ \|_{\mathcal{L}(H^1, H^1)} e^{-c(t-t_u)} 2r + \int_{t_u}^t (t-\tau)^{-1/2} e^{-c(t-\tau)} \|\mathbf{Q}_+ \mathbf{F}(u(\tau))\|_{L^2} d\tau \right). \end{aligned}$$

In view of (1.4)

$$\|\mathbf{Q}_+ \mathbf{F}(u(\tau))\|_{L^2} \leq \|m\|_{L^2}, \quad t_u \leq \tau \leq t; \quad (5.8)$$

thus

$$\|\mathbf{Q}_+ u(t)\|_{H^1} \leq K \left(\|\mathbf{Q}_+ \|_{\mathcal{L}(H^1, H^1)} 2r + \|m\|_{L^2} \int_0^{+\infty} s^{-1/2} e^{-cs} ds \right) =: R'_{1,\infty}.$$

This means that $\|\mathbf{Q}_+ u(t)\|_{H^1} \leq R_{1,\infty} = \max\{2r, R'_{1,\infty}\}$ for all $t \in \mathbb{R}$.

Since, due to Remark 3.10 (2), $\omega(u) \subset \mathcal{E}_\lambda$ we can take $s_u \in \mathbb{R}$ such that $\|u(\tau)\|_{H^1} \leq 2r$, for all $\tau \geq s_u$, and observe that, in view of (5.7), we have for each $t < s_u$

$$\mathbf{Q}_- u(s_u) = e^{-(s_u-t)\mathbf{B}_-} \mathbf{Q}_- u(t) + \int_t^{s_u} e^{-(s_u-\tau)\mathbf{B}_-} \mathbf{Q}_- \mathbf{F}(u(\tau)) d\tau.$$

Hence, using (5.6), we get

$$\|\mathbf{Q}_- u(t)\|_{L^2} \leq K' \left(e^{c(t-s_u)} \|\mathbf{Q}_- u(s_u)\|_{L^2} + \int_t^{s_u} e^{c(t-\tau)} \|\mathbf{Q}_- \mathbf{F}(u(\tau))\|_{L^2} d\tau \right).$$

Again in view of (1.4)

$$\|\mathbf{Q}_- \mathbf{F}(u(\tau))\|_{L^2} \leq \|m\|_{L^2}, \quad t \leq \tau \leq s_u. \quad (5.9)$$

Therefore

$$\|\mathbf{Q}_- u(t)\|_{L^2} \leq K'(2r + \|m\|_{L^2} c^{-1}) =: R'_{2,\infty}$$

and thus $\|\mathbf{Q}_- u(t)\|_{L^2} \leq \tilde{R}_{2,\infty} := \max\{2r, R'_{2,\infty}\}$ for all $t \in \mathbb{R}$. Since X_- is finite dimensional, there is a constant $R_{2,\infty} > 0$ such that $\|\mathbf{Q}_- u(t)\|_{H^1} \leq R_{2,\infty}$ for all $t \in \mathbb{R}$. $\square$

Lemma 5.4 *If $u : [t_0, t_1] \rightarrow H^1(\mathbb{R}^N)$ is a solution of Φ^λ for some $\lambda \in \mathbb{R}$, then*

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 = (\lambda - \lambda_0) \|\mathbf{P}u(t)\|_{L^2}^2 + \langle \mathbf{P}u(t), \mathbf{F}(u(t)) \rangle_{L^2}, t \in (t_0, t_1).$$

when u solves (3.12).

Proof The symmetry of $\mathbf{A}$ implies that X_0 is orthogonal to the range $R(\mathbf{A} - \lambda_0 \mathbf{I})$ in L^2 . Hence

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 &= \langle \mathbf{P}u(t), \dot{u}(t) \rangle_{L^2} = \langle \mathbf{P}u(t), -(\mathbf{A} - \lambda_0 \mathbf{I})u(t) + (\lambda - \lambda_0)u(t) + \mathbf{F}(u(t)) \rangle_{L^2} \\ &= (\lambda - \lambda_0) \|\mathbf{P}u(t)\|_{L^2}^2 + \langle \mathbf{P}u(t), \mathbf{F}(u(t)) \rangle_{L^2} \end{aligned}$$

for all $t \in (t_0, t_1)$. $\square$

Proof of Theorem 1.4 (ii) Assume (3.19) and suppose to the contrary that λ_0 is not a point of bifurcation from infinity. Thus there are $r > 0$ and $\delta > 0$ satisfying condition (3.20) such that

$$\mathcal{E} \subset B_{H^1}(0, r). \quad (5.10)$$

By Proposition 3.11, there is $R > 0$ such that the $K(\Phi^{\lambda_0 \pm \delta}) \subset B_{H^1}(0, R)$. By Lemma 5.3 one has $R_\infty = R_\infty(r) \geq R$ such that, for any bounded solution of $u : \mathbb{R} \rightarrow H^1$ of Φ^λ , $|\lambda - \lambda_0| \leq \delta$, one has

$$\sup_{t \in \mathbb{R}} \|\mathbf{Q}u(t)\|_{H^1} < R_\infty. \quad (5.11)$$

Let M_{R_∞} be the set of all $\bar{u} \in H^1$ such that there exists a solution $u : (-\infty, 0] \rightarrow H^1$ of Φ^λ , $|\lambda - \lambda_0| \leq \delta$, with $u(0) = \bar{u}$ and $\|\mathbf{Q}u(t)\|_{H^1} \leq R_\infty$ for all $t \leq 0$. In view of Proposition 3.7, the set $M := \mathbf{Q}M_{R_\infty} \subset X$ is relatively compact in L^2 . By Lemma 5.2 there are $R_0 \geq R_\infty$ and $\alpha > 0$ such that for all $\bar{v} \in X_0 \setminus B_{L^2}(0, R_0)$ and $\bar{w} \in M$

$$\langle \bar{v}, \mathbf{F}(\bar{v} + \bar{w}) \rangle_{L^2} > \alpha \quad (5.12)$$

if $(LL)_+$ or $(SR)_+$ is satisfied, or

$$\langle \bar{v}, \mathbf{F}(\bar{v} + \bar{w}) \rangle_{L^2} < -\alpha \quad (5.13)$$

if $(LL)_-$ or $(SR)_-$ is satisfied. Put

$$B := \{\bar{u} \in H^1 \mid \|\mathbf{P}\bar{u}\|_{L^2} \leq R_0, \quad \|\mathbf{Q}\bar{u}\|_{H^1} \leq R_\infty\}.$$

Taking δ smaller if necessary we may assume that

$$\delta R_0^2 < \alpha. \quad (5.14)$$

Then, for any $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, B is an isolating neighborhood for the semiflow Φ^λ . To see this, suppose to the contrary that there is $\bar{u} \in \text{Inv}_{\Phi^\lambda}(B) \cap \partial B$. Hence there is a solution $u : \mathbb{R} \rightarrow B$ of Φ^λ through $\bar{u}$, i.e. $\bar{u} = u(0)$. Since u is bounded, we have $\|\mathbf{Q}\bar{u}\|_{H^1} < R_\infty$ in view of (5.11). Therefore $\|\mathbf{P}\bar{u}\|_{L^2} = R_0$. Let $\bar{u} = \bar{v} + \bar{w}$, where $\bar{v} := \mathbf{P}\bar{u}$ and $\bar{w} := \mathbf{Q}\bar{u}$. Then $\bar{v} \in X_0 \setminus B_{L^2}(0, R_0)$ and $\bar{w} \in M$. By Lemma 5.4,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 \Big|_{t=0} &= (\lambda - \lambda_0) \|\mathbf{P}u(0)\|_{L^2}^2 + \langle \mathbf{P}u(0), \mathbf{F}(u(0)) \rangle_{L^2} \\ &= (\lambda - \lambda_0) R_0^2 + \langle \bar{v}, \mathbf{F}(\bar{v} + \bar{w}) \rangle_{L^2}. \end{aligned}$$

Due to (5.14) and (5.12) (or (5.13))

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 \Big|_{t=0} > 0 \quad \left(\text{or } \frac{1}{2} \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 \Big|_{t=0} < 0 \right).$$

This contradicts the assumption $u(\mathbb{R}) \subset B$ and proves that B is an isolating neighborhood for the semiflows Φ^λ , $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$. Using the continuation property (H4) of the homotopy index, we obtain

$$h(\Phi^{\lambda_0 - \delta}, K_{\lambda_0 - \delta}) = h(\Phi^{\lambda_0 + \delta}, K_{\lambda_0 + \delta}) \quad (5.15)$$

where $K_\lambda := \text{Inv}_{\Phi^\lambda}(B)$ for $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$.

We also claim that, for $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, one has

$$K_\lambda = K(\Phi^\lambda). \quad (5.16)$$

Indeed, the inclusion $K_\lambda \subset K(\Phi^\lambda)$ is self-evident. Conversely, any bounded full solution $u : \mathbb{R} \rightarrow H^1(\mathbb{R}^N)$ of Φ^λ satisfies (5.11). Therefore if u leaves B , then for some $t \in \mathbb{R}$ we have $\|\mathbf{P}u(t)\|_{L^2} > R_0$. Put $t_- := \inf\{t \in \mathbb{R} \mid \|\mathbf{P}u(t)\|_{L^2} > R_0\}$ and $t_+ := \sup\{t \in \mathbb{R} \mid \|\mathbf{P}u(t)\|_{L^2} > R_0\}$. In view of (5.10) and the fact that $R_0 \geq R_\infty > r$ we see that $-\infty < t_- < t_+ < +\infty$. It is clear that $\|\mathbf{P}u(t_\pm)\|_{L^2} = R_0$ and

$$\|\mathbf{P}u(t)\|_{L^2} < R_0 \text{ for all } t \in (-\infty, t_-) \cup (t_+, +\infty),$$

which means that

$$\frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 \Big|_{t=t_-} \geq 0 \quad \text{and} \quad \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 \Big|_{t=t_+} \leq 0. \quad (5.17)$$

But on the other hand, as before,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 \Big|_{t=t_\pm} &= (\lambda - \lambda_0) \|\mathbf{P}u(t_\pm)\|_{L^2}^2 + \langle \mathbf{P}u(t_\pm), \mathbf{F}(u(t_\pm)) \rangle_{L^2} \\ &= (\lambda - \lambda_0) R_0^2 + \langle \bar{v}, \mathbf{F}(\bar{v} + \bar{w}) \rangle_{L^2} \end{aligned}$$

where $\bar{v} := \mathbf{P}u(t_\pm)$ and $\bar{w} := \mathbf{Q}u(t_\pm)$,

which together with (5.12) (or (5.13)) yields

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 \Big|_{t=t_\pm} > 0 \quad \left(\text{or } \frac{1}{2} \frac{d}{dt} \|\mathbf{P}u(t)\|_{L^2}^2 \Big|_{t=t_\pm} < 0 \right).$$

This contradicts one of the inequalities in (5.17) and shows (5.16). Therefore, by Proposition 3.11, one has

$$h(\Phi^{\lambda_0 \pm \delta}, K_{\lambda_0 \pm \delta}) = h(\Phi^{\lambda_0 \pm \delta}, K(\Phi^{\lambda_0 \pm \delta})) = \Sigma^{k(\lambda_0 \pm \delta)}.$$

and this together with (5.15) leads to a contradiction, since $k(\lambda_0 + \delta) - k(\lambda_0 - \delta) = \dim X_0 > 0$. $\square$

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