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Calculus of Variations

Asymmetric $(p, 2)$-equations with double resonance

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Abstract We consider a nonlinear Dirichlet elliptic problem driven by the sum of a p-Laplacian and a Laplacian [a $(p, 2)$-equation] and with a reaction term, which is superlinear in the positive direction (without satisfying the Ambrosetti–Rabinowitz condition) and sublinear resonant in the negative direction. Resonance can also occur asymptotically at zero. So, we have a double resonance situation. Using variational methods based on the critical point theory and Morse theory (critical groups), we establish the existence of at least three nontrivial smooth solutions.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.1}$$

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In this problem $2 < p$ and Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \forall u \in W_0^{1,p}(\Omega).$$

The reaction term $f(z,\zeta)$ is a measurable function on $\Omega \times \mathbb{R}$ and for almost all $z \in \Omega$, $f(z,\cdot) \in C^1(\mathbb{R})$. The interesting feature of our work here is that $f(z,\cdot)$ exhibits asymmetric behaviour as $\zeta \to \pm\infty$. More precisely, $f(z,\cdot)$ is $(p-1)$-superlinear as $\zeta \to +\infty$ but need not satisfy the usual for superlinear problems Ambrosetti–Rabinowitz condition. Instead, we use a weaker condition, which incorporates in the framework of our work also problems in which the forcing term is $(p-1)$-superlinear but with "slower" growth near $+\infty$. Such a function fails to satisfy the Ambrosetti–Rabinowitz condition. Near $-\infty$ the reaction term $f(z,\cdot)$ is $(p-1)$-sublinear and resonance can occur with respect to the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Resonance can occur also at zero. Thus, our problem exhibits double resonance.

Problems with asymmetric reaction term of the form described above, were studied by Arcoya and Villegas [4], Cuesta et al. [11], D'Aguì et al. [12], de Figueiredo and Ruf [14], Motreanu et al. [24,25], de Paiva and Presoto [29], Papageorgiou and Rădulescu [31], Recôva and Rumbos [35].

In problem (1.1), the differential operator $u \longmapsto -\Delta_p u - \Delta u$ is nonhomogeneous and this is a source of difficulties in the analysis on (1.1). We mention that $(p,2)$-Laplace equations (that is, elliptic problems driven by the sum of a p-Laplacian and a Laplacian), arise naturally in problems of mathematical physics. We mention the works of Benci et al. [6] (quantum physics) and Cherfils and Il′yasov [8] (plasma physics). Recently there have been some existence and multiplicity results for such equations. In this direction, we mention the works of Aizicovici et al. [2], Barile and Figueiredo [5], Cingolani and Degiovanni [9], Gasiński and Papageorgiou [17,20], Gasiński et al. [21], Mugnai and Papageorgiou [28], Papageorgiou and Rădulescu [31,32], Papageorgiou and Smyrlis [33], Sun [37] and Sun et al. [38]. Of the aforementioned works, only Papageorgiou and Rădulescu [31] and Gasiński and Papageorgiou [20] deal with problems having an asymmetric reaction term. However, the conditions are different, since in [31] it is assumed that $f(z,\cdot)$ is $(p-1)$-sublinear in both directions (crossing nonlinearity) and no resonance is allowed asymptotically at $-\infty$ or near zero (nonresonant problem; see Theorem 12 in [31]), while in [20], $f(z,\cdot)$ is $(p-1)$-sublinear as $\zeta \to +\infty$ and $(p-1)$-superlinear as $\zeta \to -\infty$. A more general problem with a (p,q)-Laplacian operator was studied in Gasiński and Papageorgiou [18,19].

Our approach combines variational methods based on the critical point theory, together with Morse theory (critical groups theory) and the use of suitable truncation and comparison techniques. In the next section, for the convenience of the reader, we review the main mathematical tools which we will use in the sequel.

2 Mathematical background

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Given $\varphi \in C^1(X)$, we say that φ satisfies the Cerami condition, if the following is true:

Every sequence $\{u_n\}_{n\geqslant 1} \subset X$, such that $\left\{\varphi(u_n)\right\}_{n\geqslant 1} \subset \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \longrightarrow 0 \quad \text{in} X^*$, admits a strongly convergent subsequence.

Evidently this is a compactness type condition on the functional φ which compensates for the fact that the ambient space which in applications is infinite dimensional, is not locally compact. Using this condition, one can prove a deformation theorem from which the minimax theory of the critical values of φ follows. One of the most important results in this theory is the so called mountain pass theorem due to Ambrosetti and Rabinowitz [3]. Here we state it in a slightly stronger form (see Gasiński and Papageorgiou [15]).

Theorem 2.1 *If X is a Banach space, $\varphi \in C^1(X;\mathbb{R})$ satisfies the Cerami condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \varrho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\left\{\varphi(u) : \|u - u_0\| = \varrho\right\} = m_\varrho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t))$, where

$$\Gamma = \left\{\gamma \in C([0,1]; X) : \gamma(0) = u_0,\ \gamma(1) = u_1\right\},$$

then $c \geqslant m_\varrho$ and c is a critical value of φ, that is there exists $\widehat{u} \in X$ such that

$$\varphi'(\widehat{u}) = 0 \quad and \quad \varphi(\widehat{u}) = c.$$

In the analysis of problem (1.1), in addition to the Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$, we will also use the Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\right\}.$$

This is an ordered Banach space with order cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all} z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior, given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\}.$$

Here $\frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ (the normal derivative of u). The space $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$ and in $H_0^1(\Omega)$.

We will also use some elementary facts on the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. So, we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $1 < p < +\infty$. We say that $\widehat{\lambda} \in \mathbb{R}$ is an *eigenvalue* of $(-\Delta_p, W_0^{1,p}(\Omega))$, provided (2.1) admits a nontrivial solution $\widehat{u} \in W_0^{1,p}(\Omega)$, which is known as an *eigenfunction* corresponding to $\widehat{\lambda}$. There exists a smallest eigenvalue $\widehat{\lambda}_1(p) > 0$ which has the following properties

- we have
$$\widehat{\lambda}_1(p) = \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), u \neq 0\right\}; \tag{2.2}$$
- $\widehat{\lambda}_1(p)$ is isolated (that is, we can find $\varepsilon > 0$ such that $(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p) + \varepsilon)$ contains no eigenvalues of $(-\Delta_p, W_0^{1,p}(\Omega))$);
- $\widehat{\lambda}_1(p)$ is simple (that is, if $\widehat{u}, \widehat{v} \in W_0^{1,p}(\Omega)$ are two eigenfunctions corresponding to $\widehat{\lambda}_1(p)$, we have $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R}\setminus\{0\}$).

In (2.2) the infimum is realized on the corresponding one dimensional eigenspace. It is clear from (2.2) that the elements of this eigenspace do not change sign. In what follows by $\widehat{u}_1(p)$ we denote the L^p-normalized (that is, $\|\widehat{u}_1(p)\|_p^p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(p)$. The nonlinear regularity theory and the nonlinear maximum principle (see, for example Gasiński and Papageorgiou [15, pp. 737, 738]) imply that $\widehat{u}_1(p) \in \operatorname{int} C_+$.

The Ljusternik–Schnirelmann minimax scheme, gives in addition to $\widehat{\lambda}_1(p)$ a whole strictly increasing sequence $\{\widehat{\lambda}_k(p)\}_{k\geqslant 1}$ of eigenvalues such that $\widehat{\lambda}_k(p) \longrightarrow +\infty$ as $k \to +\infty$. It is not known if this sequence exhausts the spectrum of $(-\Delta_p, W_0^{1,p}(\Omega))$. This is the case if $p = 2$ (linear eigenvalue problem) or if $N = 1$ (ordinary differential equation). For the linear eigenvalue problem ($p = 2$), every eigenvalue $\widehat{\lambda}_k(2)$, $k \geqslant 1$, has an eigenspace, denoted by $E(\widehat{\lambda}_k(2))$, which is a finite dimensional linear subspace of $H_0^1(\Omega)$. We have that

$$H_0^1(\Omega) = \overline{\bigoplus_{i\geqslant 1} E(\widehat{\lambda}_i(2))}.$$

Also, for every $k \geqslant 1$, we set

$$\overline{H}_k = \bigoplus_{i=1}^{k} E\left(\widehat{\lambda}_i(2)\right) \quad \text{and} \quad \widehat{H}_k = \overline{H}_k^{\perp} = \overline{\bigoplus_{i\geqslant k+1} E\left(\widehat{\lambda}_i(2)\right)}.$$

Then

$$H_0^1(\Omega) = \overline{H}_k \oplus \widehat{H}_k.$$

All the eigenvalues $\widehat{\lambda}_k(2)$, $k \geqslant 1$, admit variational characterizations

$$\widehat{\lambda}_1(2) = \inf\left\{\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in H_0^1(\Omega),\, u \neq 0\right\}, \tag{2.3}$$

$$\begin{aligned} \widehat{\lambda}_k(2) &= \sup\left\{\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : u \in \overline{H}_k,\, u \neq 0\right\} \\ &= \inf\left\{\frac{\|\nabla u\|_2^2}{\|u\|_2^2} :\; u \in \widehat{H}_{k-1},\, u \neq 0\right\} \quad \forall k \geqslant 2. \end{aligned} \tag{2.4}$$

Both the infimum and supremum are realized on $E(\widehat{\lambda}_k(2))$. Each eigenspace exhibits the *unique continuation property*, which says that, if $u \in E(\widehat{\lambda}_i(2))$ vanishes on a set of positive measure, then $u \equiv 0$. Standard regularity theory implies that $E(\widehat{\lambda}_i(2)) \subset C_0^1(\overline{\Omega})$.

The next lemma can be found in Motreanu et al. [26, p. 305]. It is an easy consequence of the properties of the eigenvalue $\widehat{\lambda}_1(p) > 0$ mentioned above.

Lemma 2.2 *If $\vartheta \in L^\infty(\Omega)_+$, $\vartheta(z) \leqslant \widehat{\lambda}_1(p)$ for almost all $z \in \Omega$ and the inequality is strict on a set of positive measure, then there exists $c_0 > 0$ such that*

$$\|\nabla u\|_p^p - \int_\Omega \vartheta(z)|u|^p\, dz \geqslant\; c_0\|\nabla u\|_p^p \quad \forall u \in W_0^{1,p}(\Omega).$$

Let $A_p\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ be the nonlinear map defined by

$$\langle A_p(u), h\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,p}(\Omega).$$

This map has the following properties (see Gasiński and Papageorgiou [15, p. 746]).

Proposition 2.3 *The map $A_p\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)(1 < p < +\infty)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

if $u_n \longrightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and $\limsup\limits_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leqslant 0$,
then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$.

When $p = 2$, we write $A_2 = A$ and we have $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$. Let $f_0\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function with subcritical growth, that is,

$$|f_0(z, \zeta)| \leqslant a_0(z)(1 + |\zeta|^{r-1}) \quad \forall z \in \Omega,\ \text{all } \zeta \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leqslant p \end{cases}$$

(the critical Sobolev exponent). We set $F_0(z, \zeta) = \int_0^\zeta f_0(z, s)\, ds$ and consider the C^1-functional $\varphi_0\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F_0(z, u(z))\, dz \quad \forall u \in W_0^{1,p}(\Omega).$$

The next proposition is a special case of a more general result of Gasiński and Papageorgiou [16]. Its proof is an outgrowth of the nonlinear regularity theory (see Lieberman [23]).

Proposition 2.4 *If $u_0 \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\varrho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \forall h \in C_0^1(\overline{\Omega}), \quad \|h\|_{C_0^1(\overline{\Omega})} \leqslant \varrho_0,$$

then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and it is a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\varrho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \forall h \in W_0^{1,p}(\Omega), \quad \|h\| \leqslant \varrho_1.$$

Hereafter, by $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. Because of the Poincaré inequality, we can have

$$\|u\| = \|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

Also, by $|\cdot|_N$ we denote that Lebesgue measure on $\mathbb{R}^N$.

For $\zeta \in \mathbb{R}$, we set $\zeta^\pm = \max\{\pm\zeta, 0\}$. Then given $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

Given a measurable function $h\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ (for example, a Carathéodory function), we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \forall u \in W_0^{1,p}(\Omega),$$

the Nemytskii (or superposition) map corresponding to the function $h(z, \zeta)$.

Finally, we recall some basic facts about critical groups (Morse theory). For details we refer to the book of Motreanu et al. [26].

So, let X be a Banach space, $\varphi \in C^1(X;\mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets

$$\begin{aligned}
\varphi^c &= \big\{u \in X : \varphi(u) \leqslant c\big\},\\
K_\varphi &= \big\{u \in X : \varphi'(u) = 0\big\},\\
K_\varphi^c &= \big\{u \in K_\varphi : \varphi(u) = c\big\}.
\end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$ and $k \geqslant 0$. By $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Given $u \in K_\varphi$ isolated with $\varphi(u) = c$ (that is $u \in K_\varphi^c$), the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \forall k \geqslant 0,$$

where U is a neighbourhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that the above definition of critical groups is independent of the particular choice of the neighbourhood U.

Suppose that φ satisfies the Cerami condition and $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \geqslant 0.$$

The second deformation theorem (see Gasiński and Papageorgiou [15, p. 628]), implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Let $\varphi \in C^1(X;\mathbb{R})$ and assume that φ satisfies the Cerami condition and that K_φ is finite. We define

$$\begin{aligned}
M(t, u) &= \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, u) t^k \quad \forall t \in \mathbb{R},\ u \in K_\varphi,\\
P(t, \infty) &= \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, \infty) t^k \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Then the *Morse relation* says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \forall t \in \mathbb{R}, \tag{2.5}$$

where $Q(t) = \sum\limits_{k \geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

3 Multiplicity theorem

In this section we prove a multiplicity theorem for problem (1.1) producing three nontrivial smooth solutions.

To obtain the first two solutions, we will not need the continuous differentiability of $f(z, \cdot)$. So, our hypothesis on the reaction term $f(z, \zeta)$ are the following:
H(f): $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $|f(z, \zeta)| \leqslant a(z)(1 + |\zeta|^{r-1})$ for almost all $z \in \Omega$, all $\zeta \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$;
(ii) if $F(z, \zeta) = \int_0^\zeta f(z, s)\, ds$, then

$$\lim_{\zeta \to +\infty} \frac{F(z, \zeta)}{\zeta^p} = +\infty$$

uniformly for almost all $z \in \Omega$ and there exist $q \in ((r - p) \max\{\frac{N}{p}, 1\}, p^*)$ and $\xi_0 > 0$ such that

$$0 < \xi_0 \leqslant \liminf_{\zeta \to +\infty} \frac{f(z, \zeta)\zeta - pF(z, \zeta)}{\zeta^q}$$

uniformly for almost all $z \in \Omega$;

(iii) there exist $\xi_1 > 0$ and $c_1 > 0$ such that

$$-\xi_1 \leqslant \liminf_{\zeta \to -\infty} \frac{f(z, \zeta)}{|\zeta|^{p-2}\zeta} \leqslant \limsup_{\zeta \to -\infty} \frac{f(z, \zeta)}{|\zeta|^{p-2}\zeta} \leqslant \widehat{\lambda}_1(p)$$

uniformly for almost all $z \in \Omega$ and

$$-c_1 \leqslant f(z, \zeta)\zeta - pF(z, \zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0;$$

(iv) there exist integer $m \geqslant 2$ and $\delta > 0$ such that

$$\widehat{\lambda}_m(2)\zeta^2 \leqslant f(z, \zeta)\zeta \leqslant \widehat{\lambda}_{m+1}(2)\zeta^2 \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \delta.$$

Remark 3.1 Hypotheses $H(f)(ii)$ and (iii) imply that the reaction term $f(z, \cdot)$ exhibits an asymmetric behaviour as $\zeta \to \pm\infty$. So, $f(z, \cdot)$ is $(p - 1)$-superlinear as $\zeta \to +\infty$ [see hypothesis $H(f)(ii)$] and $f(z, \cdot)$ is $(p - 1)$-sublinear as $\zeta \to -\infty$ [see hypothesis $H(f)(iii)$]. Note that the $(p - 1)$-superlinearity in the positive direction, is not expressed using the common is such cases (unilateral) Ambrosetti–Rabinowitz condition. We recall that the Ambrosetti–Rabinowitz condition (unilateral version that is, valid only in the positive semiaxis), says that there exist $\tau > p$ and $M > 0$ such that

$$0 < \tau F(z, \zeta) \leqslant f(z, \zeta)\zeta \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant M \tag{3.1}$$

and

$$\operatorname*{ess\,sup}_{\Omega} F(\cdot, M) > 0 \tag{3.2}$$

(see Ambrosetti and Rabinowitz [3] and Mugnai [27]). Integrating (3.1) and using (3.2), we obtain

$$c_2\zeta^\tau \leqslant F(z, \zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant M, \tag{3.3}$$

for some $c_2 > 0$. From (3.3) and (3.1), we see that $f(z, \cdot)$ has at least $(\tau - 1)$-polynomial growth near $+\infty$ and so

$$\lim_{\zeta \to +\infty} \frac{F(z, \zeta)}{\zeta^p} = +\infty \quad \text{and} \quad \lim_{\zeta \to +\infty} \frac{f(z, \zeta)}{\zeta^{p-1}} = +\infty$$

uniformly for almost all $z \in \Omega$. Hypothesis $H(f)(ii)$ is weaker than the unilateral Ambrosetti–Rabinowitz condition [see (3.1) and (3.2)]. Indeed, we may take $\tau > (r-p)\max\{\frac{N}{p}, 1\}$ and then using (3.1) we have

$$\frac{f(z,\zeta)\zeta - pF(z,\zeta)}{\zeta^\tau} = \frac{f(z,\zeta)\zeta - \tau F(z,\zeta)}{\zeta^\tau} + (\tau - p)\frac{F(z,\zeta)}{\zeta^\tau} \geqslant (\tau-p)\frac{F(z,\zeta)}{\zeta^\tau} \geqslant (\tau-p)c_2 > 0$$

[see (3.1) and (3.3)]. So, assuming the unilateral Ambrosetti–Rabinowitz condition, we have just seen that hypothesis $H(f)(ii)$ holds. Our hypothesis allows the consideration of $(p-1)$-superlinear at $+\infty$ nonlinearities with slower growth, which fail to satisfy the Ambrosetti–Rabinowitz condition (see the examples below). Hypothesis $H(f)(iii)$ implies that in the negative direction $f(z,\cdot)$ is $(p-1)$-sublinear and asymptotically at $-\infty$ we can have resonance with respect to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. Hypothesis $H(f)(iv)$ says that at zero we can have resonance with respect to any nonprincipal eigenvalue of $(-\Delta, H_0^1(\Omega))$.

Example 3.2 The following functions satisfy hypotheses $H(f)$. For the sake of simplicity, we drop the z-dependence.

(a) For $2 < p < \tau < p^*$, $m \geqslant 2$, $\widetilde{c}_1 = \widehat{\lambda}_1(p) - \widehat{\lambda}_m(2)$, $\widetilde{c}_2 = \widehat{\lambda}_m(2) - 1$, we consider

$$f_1(\zeta) = \begin{cases} \widehat{\lambda}_1(p)|\zeta|^{p-2}\zeta + \widetilde{c}_1 & \text{if } \zeta < -1, \\ \widehat{\lambda}_m(2)\zeta & \text{if } -1 \leqslant \zeta \leqslant 1, \\ \zeta^{\tau-1} + \widetilde{c}_2 & \text{if } 1 < \zeta. \end{cases}$$

This function satisfies the unilateral Ambrosetti–Rabinowitz condition.

(b) For $2 < p$, $\widehat{c}_1 = \widehat{\lambda}_1(p) - \widehat{\lambda}_m(p)$, $\widehat{c}_2 = \widehat{\lambda}_m(2) - \frac{1}{p}$, we consider

$$f_2(\zeta) = \begin{cases} \widehat{\lambda}_1(p)|\zeta|^{p-2}\zeta + \widehat{c}_1 & \text{if } \zeta < -1, \\ \widehat{\lambda}_m(2)\zeta & \text{if } -1 \leqslant \zeta \leqslant 1, \\ \zeta^{p-1}\left(\ln\zeta + \frac{1}{p}\right) + \widehat{c}_2 & \text{if } 1 < \zeta. \end{cases}$$

This function fails to satisfy the unilateral Ambrosetti–Rabinowitz condition.

Let $\varphi\colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\varphi(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z,u)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Proposition 3.3 *If hypotheses $H(f)$ hold, then the functional φ satisfies the Cerami condition.*

Proof Let $\{u_n\}_{n\geqslant 1} \subset W_0^{1,p}(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \quad \forall n \geqslant 1, \tag{3.4}$$

for some $M_1 > 0$ and

$$(1 + \|u_n\|)\varphi'(u_n) \longrightarrow 0 \quad \text{in } W^{-1,p'}(\Omega). \tag{3.5}$$

From (3.5), we have

$$\left| \langle A_p(u_n), h \rangle + \langle A(u_n), h \rangle - \int_\Omega f(z, u_n) h \, dz \right| \leqslant \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \forall h \in W_0^{1,p}(\Omega), \tag{3.6}$$

with $\varepsilon_n \searrow 0$. We will show that the sequence $\{u_n\}_{n \geqslant 1} \subset W_0^{1,p}(\Omega)$ is bounded. To this end note that

$$\begin{aligned} \frac{1}{p} \left\| \nabla u_n^+ \right\|_p^p + \frac{1}{2} \left\| \nabla u_n^+ \right\|_2^2 &= \frac{1}{p} \| \nabla u_n \|_p^p + \frac{1}{2} \| \nabla u_n \|_2^2 - \frac{1}{p} \left\| \nabla u_n^- \right\|_p^p - \frac{1}{2} \left\| \nabla u_n^- \right\|_2^2 \\ &\quad + \int_\Omega F(z, u_n) \, dz - \int_\Omega F(z, u_n) \, dz \\ &= \varphi(u_n) - \frac{1}{p} \left\| \nabla u_n^- \right\|_p^p - \frac{1}{2} \left\| \nabla u_n^- \right\|_2^2 + \int_\Omega F(z, u_n) \, dz \\ &\leqslant M_1 + \frac{1}{p} \left(\int_\Omega p F(z, u_n) \, dz - \left\| \nabla u_n^- \right\|_p^p - \frac{p}{2} \left\| \nabla u_n^- \right\|_2^2 \right) \\ &\leqslant M_1 + \frac{1}{p} \left(\int_\Omega p F(z, u_n) \, dz - \left\| \nabla u_n^- \right\|_p^p - \left\| \nabla u_n^- \right\|_2^2 \right) \end{aligned} \tag{3.7}$$

[see (3.4) and use the fact that $p > 2$]. In (3.6) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$- \left\| \nabla u_n^- \right\|_p^p - \left\| \nabla u_n^- \right\|_2^2 + \int_\Omega f(z, -u_n^-)(-u_n^-) \, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.8}$$

Using (3.8) in (3.7), we have

$$\frac{1}{p} \left\| \nabla u_n^+ \right\|_p^p + \frac{1}{2} \left\| \nabla u_n^+ \right\|_2^2 \leqslant M_2 + \frac{1}{p} \left(\int_\Omega \left(p F(z, u_n) - f(z, -u_n^-)(-u_n^-) \right) dz \right) \quad \forall n \geqslant 1,$$

for some $M_2 > 0$. Note that

$$F(z, u_n) = F(z, u_n^+) + F(z, -u_n^-) \quad \forall n \geqslant 1.$$

It follows that

$$\begin{aligned} \varphi(u_n^+) &\leqslant M_2 + \frac{1}{p} \left(\int_\Omega \left(p F(z, -u_n^-) - f(z, -u_n^-)(-u_n^-) \right) dz \right) \\ &\leqslant M_3 \quad \forall n \geqslant 1, \end{aligned} \tag{3.9}$$

for some $M_3 > 0$ [see hypothesis $H(f)(iii)$]. In (3.6) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$- \left\| \nabla u_n^+ \right\|_p^p - \left\| \nabla u_n^+ \right\|_2^2 + \int_\Omega f(z, u_n^+) u_n^+ \, dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.10}$$

On the other hand from (3.9), we have

$$\left\| \nabla u_n^+ \right\|_p^p + \frac{p}{2} \left\| \nabla u_n^+ \right\|_2^2 - \int_\Omega p F(z, u_n^+) \, dz \leqslant p M_3 \quad \forall n \geqslant 1. \tag{3.11}$$

We add (3.10) and (3.11) and recalling that $p > 2$, we infer that

$$\int_\Omega \left(f(z, u_n^+) u_n^+ - p F(z, u_n^+) \right) dz \leqslant M_4 \quad \forall n \geqslant 1, \tag{3.12}$$

for some $M_4 > 0$. Hypotheses $H(f)(i)$ and (ii) imply that we can find $\xi_2 \in (0, \xi_0)$ and $c_4 > 0$ such that

$$\xi_2 \zeta^q - c_4 \leqslant f(z, \zeta)\zeta - p F(z, \zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \geqslant 0. \tag{3.13}$$

Using (3.13) in (3.12), we infer that

$$\text{the sequence } \{u_n^+\}_{n\geqslant 1} \subset L^q(\Omega) \text{ is bounded.} \tag{3.14}$$

First we assume that $p \neq N$. From hypothesis $H(f)(ii)$, it is clear that without any loss of generality, we may assume that $q < r < p^*$. Let $t \in (0,1)$ be such that

$$\frac{1}{r} = \frac{1-t}{q} + \frac{t}{p^*}. \tag{3.15}$$

The interpolation inequality (see, for example Gasiński and Papageorgiou [15, p. 905]) implies that

$$\|u_n^+\|_r \ \leqslant\ \|u_n^+\|_q^{1-t}\|u_n^+\|_{p^*}^t,$$

so

$$\|u_n^+\|_r^r \ \leqslant\ M_5\|u_n^+\|^{tr} \quad \forall n \geqslant 1, \tag{3.16}$$

for some $M_5 > 0$ [see (3.14) and use the Sobolev embedding theorem]. In (3.6) we choose $h \in u_n^+ \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_n^+\|_p^p + \|\nabla u_n^+\|_2^2 \ \leqslant\ \varepsilon_n + \int_\Omega f(z,u_n^+)u_n^+\,dz \ \leqslant\ c_5\left(1+\|u_n^+\|^{tr}\right) \quad \forall n \geqslant 1, \tag{3.17}$$

for some $c_5 > 0$ [see hypothesis $H(f)(i)$ and (3.16)]. Using (3.15) and hypothesis $H(f)(ii)$, we see that $tr < p$, so

$$\text{the sequence } \{u_n^+\}_{n\geqslant 1} \subset W_0^{1,p}(\Omega) \text{ is bounded} \tag{3.18}$$

[see (3.17)].

Now assume that $N = p$. In this case $p^* = +\infty$, but the Sobolev embedding theorem says that $W_0^{1,p}(\Omega) \hookrightarrow L^\tau(\Omega)$ for all $\tau \in [1,+\infty)$. Let $\tau > r > q$ and choose $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{q} + \frac{t}{\tau},$$

so

$$tr = \frac{\tau(r-q)}{\tau-q}. \tag{3.19}$$

Note that

$$\frac{\tau(r-q)}{\tau-q} \ \longrightarrow\ r-q \quad \text{as } \tau \to +\infty = p^*. \tag{3.20}$$

Since, by hypothesis $H(f)(ii)$ we have $r-q < p$ (recall $N = p$), the previous argument remains valid if we replace p^* be $\tau > r$ big such that $tr < p$ [see (3.19) and (3.20)]. Then again we conclude that (3.18) holds.

Next we show that the sequence $\{u_n^-\}_{n\geqslant 1} \subset W_0^{1,p}(\Omega)$ is bounded. Arguing by contradiction, suppose that at least for a subsequence, we have $\|u_n^-\| \longrightarrow +\infty$. Let $y_n = \frac{u_n^-}{\|u_n^-\|}$ for all $n \geqslant 1$. Then

$$\|y_n\| = 1, \quad y_n \ \geqslant\ 0 \quad \forall n \geqslant 1.$$

So, we may assume that

$$y_n \ \longrightarrow\ y \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \ \longrightarrow\ y \quad \text{in } L^p(\Omega). \tag{3.21}$$

From (3.6) with $h = -u_n^- \in W_0^{1,p}(\Omega)$, we have

$$\left\|\nabla u_n^-\right\|_p^p + \left\|\nabla u_n^-\right\|_2^2 - \int_\Omega f(z, -u_n^-)(-u_n^-)\,dz \leqslant \varepsilon_n \quad \forall n \geqslant 1,$$

so

$$\|\nabla y_n\|_p^p + \frac{1}{\left\|u_n^-\right\|^{p-2}}\|\nabla y_n\|_2^2 - \int_\Omega \frac{N_f(-u_n^-)}{\left\|u_n^-\right\|^{p-1}}(-y_n)\,dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{3.22}$$

Hypotheses $H(f)(i)$ and (iii) imply that

$$|f(z,\zeta)| \leqslant c_6\left(1+|\zeta|^{p-1}\right) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0,$$

with $c_6 > 0$, so the sequence $\left\{\frac{N_f(-u_n^-)}{\|u_n^-\|^{p-1}}\right\}_{n\geqslant 1} \subset L^{p'}(\Omega)$ is bounded.

Therefore, by passing to a subsequence if necessary and using hypothesis $H(f)(iii)$, we obtain

$$\frac{N_f(-u_n^-)}{\left\|u_n^-\right\|^{p-1}} \longrightarrow -\eta y^{p-1} \quad \text{weakly in } L^{p'}(\Omega), \tag{3.23}$$

with $\eta \in L^\infty(\Omega)$, $\eta(z) \leqslant \widehat{\lambda}_1(p)$ for almost all $z \in \Omega$ (see Aizicovici et al. [1, proof of Proposition 16]). Therefore, if in (3.22) we pass to the limit as $n \to +\infty$ and use (3.21) and (3.23), then

$$\|\nabla y\|_p^p \leqslant \int_\Omega \eta(z) y^p\,dz \tag{3.24}$$

(recall that $p > 2$). If $\eta \not\equiv \widehat{\lambda}_1(p)$, then from (3.24) and Lemma 2.2, we have

$$c_0\|y\|^p \leqslant 0,$$

so $y = 0$. Then from (3.22), it follows that $\|\nabla y_n\|_p \longrightarrow 0$, so

$$y_n \longrightarrow 0 \quad \text{in } W_0^{1,p}(\Omega),$$

a contradiction to the fact that $\|y_n\| = 1$ for $n \geqslant 1$.

If $\eta(z) = \widehat{\lambda}_1(p)$ for almost all $z \in \Omega$, then from (3.24) and (2.2), we have

$$y = \vartheta \widehat{u}_1(p),$$

with $\vartheta \geqslant 0$. If $\vartheta = 0$, then $y = 0$ and so as above we reach a contradiction to the fact that $\|y_n\| = 1$ for $n \geqslant 1$. So, suppose that $\vartheta > 0$. Then $y \in \operatorname{int} C_+$ and so

$$u_n^-(z) \longrightarrow +\infty \quad \text{for almost all } z \in \Omega.$$

From (3.4) and (3.18), we have

$$\varphi\left(-u_n^-\right) \leqslant M_6 \quad \forall n \geqslant 1,$$

for some $M_6 > 0$, so

$$\frac{\widehat{\lambda}_1(p)}{p}\left\|u_n^-\right\|_p^p - \int_\Omega F(z, -u_n^-)\,dz + \frac{1}{2}\left\|\nabla u_n^-\right\|_2^2 \leqslant M_6 \quad \forall n \geqslant 1. \tag{3.25}$$

For almost all $z \in \Omega$ and all $\zeta \leqslant 0$, we have

$$\begin{aligned}\frac{d}{d\zeta}\left(\frac{F(z,\zeta)}{|\zeta|^p}\right) &= \frac{f(z,\zeta)|\zeta|^p - p|\zeta|^{p-2}\zeta F(z,\zeta)}{|\zeta|^{2p}}\\ &= \frac{f(z,\zeta)\zeta - pF(z,\zeta)}{|\zeta|^p\zeta} \leqslant -\frac{c_1}{|\zeta|^p\zeta}\end{aligned}$$

[see hypothesis $H(f)(iii)$], so for almost all $z \in \Omega$ and all $\zeta < y < 0$, we have

$$\frac{F(z,\zeta)}{|\zeta|^p} - \frac{F(z,y)}{|y|^p} \geqslant \frac{c_1}{p}\left(\frac{1}{|\zeta|^p} - \frac{1}{|y|^p}\right). \tag{3.26}$$

Hypothesis $H(f)(iii)$ implies that

$$\limsup_{\zeta \to -\infty} \frac{pF(z,\zeta)}{|\zeta|^p} \leqslant \widehat{\lambda}_1(p) \quad \text{uniformly for almost all } z \in \Omega. \tag{3.27}$$

So, if in (3.26) we pass to the limit as $\zeta \to -\infty$ and use (3.27), then

$$\frac{\widehat{\lambda}_1(p)}{p} - \frac{F(z,y)}{|y|^p} \geqslant -\frac{c_1}{p|y|^p} \quad \text{for almost all } z \in \Omega, \text{ all } y < 0,$$

so

$$\widehat{\lambda}_1(p)|y|^p - pF(z,y) \geqslant -c_1 \quad \text{for almost all } z \in \Omega, \text{ all } y \leqslant 0. \tag{3.28}$$

We return to (3.25) and use (3.28). Then

$$\frac{1}{2}\|\nabla u_n^-\|_2^2 \leqslant M_7 \quad \forall n \geqslant 1,$$

for some $M_7 > 0$, so

$$\frac{\widehat{\lambda}_1(2)}{2}\|u_n^-\|_2^2 \leqslant M_7 \quad \forall n \geqslant 1 \tag{3.29}$$

[see (2.3)]. But recall that $u_n^-(z) \longrightarrow +\infty$ for almost all $z \in \Omega$. Then using Fatou's lemma we contradict (3.29). This proves that the sequence $\{u_n^-\}_{n\geqslant 1} \subset W_0^{1,p}(\Omega)$ is bounded, and thus the sequence $\{u_n\}_{n\geqslant 1} \subset W_0^{1,p}(\Omega)$ is bounded [see (3.18)].

By passing to a suitable subsequence if necessary, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \longrightarrow u \quad \text{in } L^r(\Omega). \tag{3.30}$$

In (3.6) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to +\infty$. Using (3.30) we obtain

$$\lim_{n\to+\infty}\big(\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle\big) = 0,$$

so

$$\limsup_{n\to+\infty}\big(\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle\big) \leqslant 0,$$

(recall that A is monotone), thus

$$\limsup_{n\to+\infty}\langle A_p(u_n), u_n - u\rangle \leqslant 0,$$

[see (3.30)] and hence

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p}(\Omega)$$

(see Proposition 2.3).

This proves that functional φ satisfies the Cerami condition. □

We introduce the C^1-functional $\varphi_- : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_-(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F(z, -u^-)\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Proposition 3.4 *If hypotheses $H(f)$ hold, then the functional φ_- is coercive.*

Proof We argue indirectly. So, suppose that φ_- is not coercive. Then we can find a sequence $\{u_n\}_{n\geqslant 1} \subset W_0^{1,p}(\Omega)$ and $M_8 > 0$ such that

$$\varphi_-(u_n) \leqslant M_8 \quad \forall n \geqslant 1 \quad \text{and} \quad \|u_n\| \longrightarrow +\infty. \tag{3.31}$$

So, we have

$$\frac{1}{p}\|\nabla u_n\|_p^p + \frac{1}{2}\|\nabla u_n\|_2^2 - \int_\Omega F\left(z, -u_n^-\right) dz \leqslant M_8 \quad \forall n \geqslant 1. \tag{3.32}$$

Let $y_n = \frac{u_n}{\|u_n\|}$ for $n \geqslant 1$. Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so we may assume that

$$y_n \longrightarrow y \quad \text{weakly in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \longrightarrow y \quad \text{in } L^p(\Omega). \tag{3.33}$$

From (3.32), we have

$$\frac{1}{p}\|\nabla y_n\|_p^p + \frac{1}{2\|u_n\|^{p-2}}\|\nabla y_n\|_2^2 - \int_\Omega \frac{N_F\left(-u_n^-\right)}{\|u_n\|^p}\,dz \leqslant \frac{M_8}{\|u_n\|^p}. \tag{3.34}$$

Hypotheses $H(f)(i)$ and (iii) imply that

$$|F(z, \zeta)| \leqslant c_7\left(1 + |\zeta|^p\right) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \leqslant 0,$$

for some $c_7 > 0$, so the sequence $\left\{\frac{N_F(-u_n^-)}{\|u_n\|^p}\right\}_{n\geqslant 1} \subset L^1(\Omega)$ is uniformly integrable. Hence, by the Dunford–Pettis theorem, we may assume that

$$\frac{N_F\left(-u_n^-\right)}{\|u_n\|^p} \longrightarrow \mu \quad \text{weakly in } L^1(\Omega). \tag{3.35}$$

Using (3.27), we have

$$\mu = \frac{1}{p}\gamma(y^-)^p, \tag{3.36}$$

with $-\xi_1 \leqslant \gamma(z) \leqslant \widehat{\lambda}_1(p)$ for almost all $z \in \Omega$ (see Aizicovici et al. [1, proof of Proposition 16]). If in (3.34) we pass to the limit as $n \to +\infty$ and use (3.33), (3.35) and (3.36), then

$$\|\nabla y\|_p^p \leqslant \int_\Omega \gamma(z)(y^-)^p\,dz \tag{3.37}$$

(recall that $p > 2$), so

$$\|\nabla y^-\|_p^p \leqslant \int_\Omega \gamma(z)(y^-)^p\,dz. \tag{3.38}$$

If $\gamma \not\equiv \widehat{\lambda}_1(p)$, then from (3.38) and Lemma 2.2, we have $c_0\|y^-\|^p \leqslant 0$ so $y \geqslant 0$. Using this in (3.37), we obtain

$$\|\nabla y\|_p^p \leqslant 0,$$

so $y = 0$. Then from (3.34) it follows that

$$\|\nabla y_n\|_p \longrightarrow 0,$$

so $y_n \longrightarrow 0$ in $W_0^{1,p}(\Omega)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geqslant 1$.

If $\gamma(z) = \widehat{\lambda}_1(p)$ for almost all $z \in \Omega$, then from (3.38), we have

$$\|\nabla y^-\|_p^p = \widehat{\lambda}_1(p)\|y^-\|_p^p,$$

so $y^- = \widetilde{\xi}\widehat{u}_1(p)$, $\widetilde{\xi} \geqslant 0$.

If $\widetilde{\xi} = 0$, then $y^- = 0$ and from (3.37) we also have $y^+ = 0$, hence $y = 0$. From this as above, we reach a contradiction to the fact that $\|y_n\| = 1$ for all $n \geqslant 1$.

If $\widetilde{\xi} > 0$, then $y^- \in \operatorname{int} C_+$ and so

$$u_n^-(z) \longrightarrow +\infty \quad \text{for all } z \in \Omega. \tag{3.39}$$

From (3.32), (2.2) and (2.3), we have

$$\frac{\widehat{\lambda}_1(p)}{p} \|u_n^-\|_p^p + \frac{\widehat{\lambda}_1(2)}{2} \|u_n^-\|_2^2 - \int_\Omega F\left(z, -u_n^-\right) dz \leqslant M_8 \quad \forall n \geqslant 1,$$

so

$$\frac{\widehat{\lambda}_1(2)}{2} p \|u_n^-\|_2^2 \leqslant \int_\Omega \left(pF(z, -u_n^-) - \lambda_1(p)(u_n^-)^p\right) dz \leqslant c_1|\Omega|_N \quad \forall n \geqslant 1 \tag{3.40}$$

[see (3.28)]. From (3.39) and Fatou's lemma, we have

$$\frac{\widehat{\lambda}_1(2)}{2} p\|u_n^-\|_2^2 \longrightarrow +\infty. \tag{3.41}$$

From (3.40) and (3.41) we reach a contradiction. This proves the coercivity of φ_-. ☐

Using Proposition 3.4 and the direct method of the calculus of variations, we can produce a negative smooth solution.

Proposition 3.5 *If hypotheses $H(f)$ hold, then problem* (1.1) *admits a negative solution $u_0 \in -\operatorname{int} C_+$ which is a local minimizer of φ.*

Proof From Proposition 3.4 we know that the functional φ_- is coercive. Also, using the Sobolev embedding theorem, we see that φ_- is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_-(u_0) = \inf_{u \in W_0^{1,p}(\Omega)} \varphi_-(u). \tag{3.42}$$

Since $\widehat{u}_1(2) \in \operatorname{int} C_+$, we can find $t \in (0, 1)$ small such that

$$t\widehat{u}_1(2)(z) \in [0, \delta] \quad \forall z \in \overline{\Omega},$$

with $\delta > 0$ as in hypothesis $H(f)(iv)$. From that hypothesis, we have

$$\begin{aligned} \frac{\widehat{\lambda}_m(2)}{2} t^2\widehat{u}_1(2)(z)^2 &\leqslant F(z, -t\widehat{u}_1(2)(z)) \\ &\leqslant \frac{\widehat{\lambda}_{m+1}(2)}{2} t^2\widehat{u}_1(2)(z)^2 \quad \text{for a.a. } z \in \Omega. \end{aligned} \tag{3.43}$$

Therefore

$$\begin{aligned}\varphi_{-}\left(t\widehat{u}_1(2)\right) &\leqslant \frac{t^p}{p}\left\|\nabla\widehat{u}_1(2)\right\|_p^p + \frac{t^2}{2}\widehat{\lambda}_1(2) - \frac{t^2}{2}\widehat{\lambda}_m(2)\\ &= \frac{t^p}{p}\left\|\nabla\widehat{u}_1(2)\right\|_2^2 + \frac{t^2}{2}\left(\widehat{\lambda}_1(2) - \widehat{\lambda}_m(2)\right)\end{aligned}$$

[see (3.37) and recall that $\|\widehat{u}_1(2)\|_2 = 1$]. Since $m \geqslant 2$ and $p > 2$, choosing $t \in (0, 1)$ even smaller, we have

$$\varphi_{-}\left(t\widehat{u}_1(2)\right) < 0,$$

so

$$\varphi_{-}(u_0) \;<\; 0 = \varphi_{-}(0)$$

[see (3.42)], hence $u_0 \neq 0$. From (3.42), we have

$$\varphi_{-}'(u_0) = 0,$$

so

$$A_p(u_0) + A(u_0) = N_f(-u_0^{-}). \tag{3.44}$$

On (3.44) we act with $u_0^{+} \in W_0^{1,p}(\Omega)$. Then

$$\left\|\nabla u_0^{+}\right\|_p^p + \left\|\nabla u_0^{+}\right\|_2^2 = 0,$$

so $u_0 \leqslant 0$ and $u_0 \neq 0$.

So, equation (3.44) becomes

$$A_p(u_0) + A(u_0) = N_f(u_0),$$

thus

$$\begin{cases} -\Delta_p u_0(z) - \Delta u_0(z) \;=\; f\left(z, u_0(z)\right) \quad \text{in } \Omega,\\ u_0|_{\partial\Omega} = 0.\end{cases} \tag{3.45}$$

From Ladyzhenskaya and Uraltseva [22, p. 289], we have that $u_0 \in L^\infty$. Then Theorem 1 of Lieberman [23] implies that $u_0 \in (-C_+)\backslash\{0\}$.

Let $a(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}^N$. Then $a \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ and

$$\operatorname{div} a\left(\nabla u\right) = \Delta_p u + \Delta u \quad \forall u \in W_0^{1,p}(\Omega)$$

(recall that $p > 2$). We have

$$\nabla a(y) = |y|^{p-2}\left(I + (p-2)\frac{y \otimes y}{|y|^2}\right) + I \quad \forall y \in \mathbb{R}^N,$$

so

$$\left(\nabla a(y)\xi, \xi\right)_{\mathbb{R}^N} \;\geqslant\; |\xi|^2 \quad \forall y, \xi \in \mathbb{R}^N.$$

Then we can use the tangency principle of Pucci and Serrin [34, p. 35] on (3.38) and infer that

$$u_0(z) \;<\; 0 \quad \forall z \in \Omega.$$

Using the boundary point theorem of Pucci and Serrin [34, p. 120], we have

$$u_0 \;\in\; -\operatorname{int} C_+.$$

Note that

$$\varphi|_{-C_+} = \varphi_-|_{-C_+}.$$

So, $u_0 \in -\operatorname{int} C_+$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ. Then Proposition 2.4 implies that u_0 is a local $W_0^{1,p}(\Omega)$-minimizer of φ. □

Corollary 3.6 *If hypotheses $H(f)$ hold and $u_0 \in -\operatorname{int} C_+$ is the negative solution from Proposition 3.5, then*

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0.$$

Using $u_0 \in -\operatorname{int} C_+$ from Proposition 3.5 and the mountain pass theorem (see Theorem 2.1), we can produce a second nontrivial solution for problem (1.1). Of course we assume that K_φ is finite or otherwise we already have infinitely many of smooth solutions.

First we compute the critical groups of φ at zero.

Proposition 3.7 *If hypotheses $H(f)$ hold, then*

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \geqslant 0,$$

with $d_m = \dim \bigoplus\limits_{i=1}^{m} E(\widehat{\lambda}_i(2))$.

Proof Recall that

$$\overline{H}_m = \bigoplus_{i=1}^{m} E\left(\widehat{\lambda}_i(2)\right), \quad \widehat{H}_m = \overline{H}_m^{\perp} = \overline{\bigoplus_{i \geqslant m+1} E\left(\widehat{\lambda}_i(2)\right)}$$

and

$$H_0^1(\Omega) = \overline{H}_m \oplus \widehat{H}_m.$$

Then every $u \in H_0^1(\Omega)$ can be written in a unique way as

$$u = \overline{u} + \widehat{u} \quad \text{with } \overline{u} \in \overline{H}_m,\ \widehat{u} \in \widehat{H}_m.$$

Let $\widetilde{\psi}\colon H_0^1(\Omega) \longrightarrow \mathbb{R}$ be the C^2-functional defined by

$$\widetilde{\psi}(u) = \frac{1}{2}\|\nabla\widehat{u}\|_2^2 - \frac{1}{2}\|\nabla\overline{u}\|_2^2 \quad \forall u \in H_0^1(\Omega).$$

Evidently, we have

$$\widetilde{\psi}|_{\widehat{H}_m} \geqslant 0 \quad \text{and} \quad \widetilde{\psi}|_{\overline{H}_m} \leqslant 0.$$

Therefore from Proposition 2.3 of Su [36], we have

$$C_k\left(\widetilde{\psi}, 0\right) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \geqslant 0, \tag{3.46}$$

with $d_m = \dim \bigoplus\limits_{i=1}^{m} E(\widehat{\lambda}_i(2))$.

Next, let $\lambda \in (\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2))$ and let $\widetilde{\varphi}\colon H_0^1(\Omega) \longrightarrow \mathbb{R}$ be the C^2-functional defined by

$$\widetilde{\varphi}(u) = \frac{1}{2}\,\|\nabla u\|_2^2 - \frac{\lambda}{2}\|u\|_2^2 \quad \forall u \in H_0^1(\Omega).$$

Evidently $K_{\widetilde{\varphi}} = \{0\}$ and $\widetilde{\varphi}$ satisfies the Cerami condition. Let $\widehat{\varphi} = \widetilde{\varphi}|_{W_0^{1,p}(\Omega)}$. The density of the embedding of $W_0^{1,p}(\Omega)$ into $H_0^1(\Omega)$ implies that

$$C_k(\widetilde{\varphi}, 0) = C_k(\widehat{\varphi}, 0) \quad \forall k \geqslant 0 \tag{3.47}$$

(see Chang [7, p. 14] and Palais [30]). Note that hypotheses $H(f)(i)$ and (iv) imply that

$$|f(z,\zeta)| \leqslant \widehat{\lambda}_{m+1}(2)\zeta + c_8|\zeta|^{r-1} \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}, \tag{3.48}$$

with $c_8 > 0$. Then we have

$$\begin{aligned} |\varphi(u) - \widehat{\varphi}(u)| &\leqslant \frac{1}{p}\|\nabla u\|_p^p + \frac{\widehat{\lambda}_{m+1}(2) - \lambda}{2}\|u\|_2^2 + c_9\|u\|_r^r \\ &\leqslant c_{10}\left(\|u\|^p + \|u\|^2 + \|u\|^r\right) \end{aligned}$$

for some $c_9, c_{10} > 0$ [see (3.48)]. It follows that

$$|\varphi(u) - \widehat{\varphi}(u)| \leqslant c_{11}\|u\|^2 \quad \forall u,\ \|u\| < 1, \tag{3.49}$$

for some $c_{11} > 0$. Also, for all $h \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} \left|\langle \varphi'(u) - \widehat{\varphi}'(u), h\rangle\right| &\leqslant \|\nabla u\|_p^{p-1}\|\nabla h\|_p + \left(\widehat{\lambda}_{m+1}(2) - \lambda\right)\|u\|_2\|h\|_2 + c_{12}\|u\|_r^{r-1}\|h\|_r \\ &\leqslant c_{13}\left(\|u\|^{p-1} + \|u\| + \|u\|^{r-1}\right)\|h\| \end{aligned}$$

for some $c_{12}, c_{13} > 0$ (use the Sobolev embedding theorem). Again we have

$$\left|\langle \varphi'(u) - \widehat{\varphi}'(u), h\rangle\right| \leqslant c_{14}\|u\|\|h\| \quad \forall u,\ \|u\| < 1,$$

for some $c_{14} > 0$, so

$$\left\|\varphi'(u) - \widehat{\varphi}'(u)\right\|_* \leqslant c_{14}\|u\|. \tag{3.50}$$

From (3.49), (3.50) and the continuity of the critical groups with respect to the C^1-topology (see Corvellec and Hantoute [10, Theorem 5.1]), we have

$$C_k(\varphi, 0) = C_k(\widehat{\varphi}, 0) \quad \forall k \geqslant 0,$$

so

$$C_k(\varphi, 0) = C_k(\widetilde{\varphi}, 0) \quad \forall k \geqslant 0 \tag{3.51}$$

[see (3.47)]. Consider the homotopy

$$h(t,u) = (1-t)\widetilde{\varphi}(u) + t\widetilde{\psi}(u) \quad \forall (t,u) \in [0,1] \times H_0^1(\Omega).$$

By $\langle \cdot, \cdot \rangle_0$ we denote the duality brackets for the pair $(H^{-1}(\Omega), H_0^1(\Omega))$. Then for $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$ (here $\delta > 0$ is as in hypothesis $H(f)(iv)$) we have

$$\begin{aligned} \langle \widetilde{\varphi}'(u), \widehat{u} - \overline{u}\rangle_0 &= \|\nabla \widehat{u}\|_2^2 - \|\nabla \overline{u}\|_2^2 - \int_\Omega f(z,u)(\widehat{u} - \overline{u})\,dz \\ &\geqslant \|\nabla \widehat{u}\|_2^2 - \widehat{\lambda}_{m+1}(2)\|\widehat{u}\|_2^2 - \|\nabla \overline{u}\|_2^2 + \widehat{\lambda}_m(2)\|\overline{u}\|_2^2 \\ &\geqslant 0 \end{aligned} \tag{3.52}$$

[see hypothesis $H(f)(iv)$ and (2.4)]. Also for all $u \in H_0^1(\Omega)$, we have

$$\langle \widetilde{\psi}'(u), \widehat{u} - \overline{u}\rangle_0 = \|\nabla \widehat{u}\|_2^2 + \|\nabla \overline{u}\|_2^2 = \|\nabla u\|_2^2. \tag{3.53}$$

Using (3.52) and (3.53) we see that for $u \in C_0^1(\overline{\Omega})$ with $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$, we have

$$\langle h'_u(t,u), \widehat{u} - \overline{u}\rangle_0 \;=\; (1-t)\langle \widetilde{\varphi}'(u), \widehat{u} - \overline{u}\rangle_0 + t\langle \widetilde{\psi}'(u), \widehat{u} - \overline{u}\rangle_0 \;\geqslant\; t\|\nabla u\|_2^2.$$

So, if $t > 0$, then $h'_u(t,u) \neq 0$ for all $u \in C_0^1(\overline{\Omega})$, $u \neq 0$, $\|u\|_{C_0^1(\overline{\Omega})} \leqslant \delta$.

For $t = 0$, we have $h(0,u) = \widetilde{\varphi}(u)$ for all $u \in H_0^1(\Omega)$ and $K_{\widetilde{\varphi}} = \{0\}$. So, we can use the homotopy invariance property of critical groups (see Corvellec and Hantoute [10, Theorem 5.2]) and have that

$$C_k(\widetilde{\varphi}|_{C_0^1(\overline{\Omega})}, 0) \;=\; C_k(\widetilde{\psi}|_{C_0^1(\overline{\Omega})}, 0) \quad \forall k \geqslant 0,$$

so

$$C_k(\widetilde{\varphi}, 0) = C_k(\widetilde{\psi}, 0) \quad \forall k \geqslant 0$$

(see Chang [7] and Palais [30]), thus

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \geqslant 0$$

[see (3.51) and (3.46)]. □

Now we are ready to produce the second nontrivial smooth solution.

Proposition 3.8 *If hypotheses $H(f)$ hold, then problem (1.1) admits a second nontrivial solution $\widehat{u} \in C_0^1(\overline{\Omega})$.*

Proof From Proposition 3.5 we have a solution $u_0 \in -\mathrm{int}\, C_+$ which is a local minimizer of the functional φ. So, we can find $\varrho \in (0,1)$ small such that

$$\varphi(u_0) \;<\; \inf\{\varphi(u) : \|u - u_0\| = \varrho\} \;=\; m_\varrho \tag{3.54}$$

(see Aizicovici et al. [1, proof of Proposition 29]). Because of hypothesis $H(f)(ii)$ we have

$$\varphi(t\widehat{u}_1(p)) \;\longrightarrow\; -\infty \quad \text{as } t \to +\infty. \tag{3.55}$$

Moreover, from Proposition 3.3, we know that

$$\varphi \text{ satisfies the Cerami condition.} \tag{3.56}$$

Because of (3.54), (3.55) and (3.56), we can use the mountain pass theorem (see Theorem 2.1) and find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \;\in\; K_\varphi \quad \text{and} \quad m_\varrho \;\leqslant\; \varphi(\widehat{u}),$$

so $\widehat{u}$ is a solution of (1.1), hence $\widehat{u} \in C_0^1(\overline{\Omega})$ (nonlinear regularity; see Lieberman [23]) and $\widehat{u} \neq u_0$ [see (3.54)]. Since $\widehat{u} \in K_\varphi$ is of mountain pass type, we have

$$C_1(\varphi, \widehat{u}) \;\neq\; 0 \tag{3.57}$$

(see Motreanu et al. [26, p. 176]).

On the other hand from Proposition 3.7, we have

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \geqslant 0, \tag{3.58}$$

with $d_m = \dim \bigoplus_{i=1}^m E(\widehat{\lambda}_i(2)) \geqslant 2$. Comparing (3.57) and (3.58) we conclude that $\widehat{u} \neq 0$. □

We can produce a third nontrivial smooth solution provided we strengthen the regularity of f. To this end, first we compute the critical groups of φ at infinity. For this we do not need additional assumptions on f.

Proposition 3.9 *If hypotheses $H(f)$ hold, then*

$$C_k(\varphi, \infty) = 0 \quad \forall k \geqslant 0.$$

Proof Consider the set

$$\partial B_1^+ = \left\{u \in W_0^{1,p}(\Omega) : \|u\| = 1,\ u^+ \neq 0\right\}$$

and the deformation $h \colon [0, 1] \times \partial B_1^+ \longrightarrow \partial B_1^+$ defined by

$$h(t, u) = \frac{(1-t)u + t\widehat{u}_1(p)}{\|(1-t)u + t\widehat{u}_1(p)\|} \quad \forall (t, u) \in [0, 1] \times \partial B_1^+.$$

We have

$$h(0, \cdot) = \mathrm{id}|_{\partial B_1^+} \quad \text{and} \quad h(1, u) = \frac{\widehat{u}_1(p)}{\|\widehat{u}_1(p)\|} \in \partial B_1^+ \quad \forall u \in \partial B_1^+,$$

so ∂B_1^+ is contractible in itself.

Hypothesis $H(f)(ii)$ implies that given $\xi > 0$, we can find $M_9 = M_9(\xi) > 0$ such that

$$F(z, \zeta) \geqslant \frac{\xi}{p}\zeta^p \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta > M_9. \tag{3.59}$$

Hypothesis $H(f)(iii)$ implies that we can find $c_{15} > 0$ and $M_{10} > 0$ such that

$$-\frac{c_{15}}{p}|\zeta|^p \leqslant F(z, \zeta) \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta < -M_{10}. \tag{3.60}$$

Finally hypothesis $H(f)(i)$ implies that we can find $c_{16} > 0$ such that

$$|F(z, \zeta)| \leqslant c_{16} \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in [-M_{10}, M_{10}]. \tag{3.61}$$

Let $t \geqslant 1$ and $u \in \partial B_1^+$ and consider the sets

$$\begin{aligned}
\Omega_+ &= \{z \in \Omega : tu(z) > M_9\},\\
\Omega_- &= \{z \in \Omega : tu(z) < -M_{10}\},\\
\Omega_0 &= \{z \in \Omega : -M_{10} \leqslant u(z) \leqslant M_9\}.
\end{aligned}$$

We have

$$\begin{aligned}
\varphi(tu) &= \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^2}{2}\|\nabla u\|_2^2 - \int_\Omega F(z, tu)\,dz\\
&= \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^2}{2}\|\nabla u\|_2^2 - \int_{\Omega_+} F(z, tu)\,dz\\
&\quad - \int_{\Omega_-} F(z, tu)\,dz - \int_{\Omega_0} F(z, tu)\,dz\\
&\leqslant \frac{t^p}{p}\|\nabla u\|_p^p + \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{t^p\xi}{p}\int_{\Omega_+} u^p\,dz + \frac{t^p c_{15}}{p}\int_{\Omega_-} |u|^p\,dz + c_{16}|\Omega|_N\\
&\leqslant \frac{t^p}{p}\left(c_{17} - \xi\int_{\Omega_+} u^p\,dz\right) + \frac{t^2}{2}\|\nabla u\|_2^2 + c_{16}|\Omega|_N.
\end{aligned} \tag{3.62}$$

for some $c_{17} > 0$ [see (3.59), (3.60) and (3.61) and recall that $\|u\| = 1$]. Since $u^+ \neq 0$ (recall that $u \in \partial B_1^+$), we can find $t_0 > 0$ and $\widehat{\xi}_0 > 0$ such that

$$\int_{\Omega_+} u^p \, dz \geqslant \widehat{\xi}_0 \quad \forall t \geqslant t_0. \tag{3.63}$$

Using (3.63) in (3.62), we obtain

$$\varphi(tu) \leqslant \frac{t^p}{p}\left(c_{17} - \xi\widehat{\xi}_0\right) + \frac{t^2}{2}\|\nabla u\|_2^2 + c_{16}|\Omega|_N \quad \forall t \geqslant t_0. \tag{3.64}$$

Choose $\xi > \frac{c_{17}}{\widehat{\xi}_0}$. Then from (3.64) and since $p > 2$, we infer that

$$\varphi(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty. \tag{3.65}$$

From (3.13) and hypothesis $H(f)(iii)$, we see that there exists $c_{18} > 0$ such that

$$-c_{18} \leqslant f(z,\zeta)\zeta - pF(z,\zeta) \quad \text{for almost all } z \in \Omega, \text{ all } \zeta \in \mathbb{R}. \tag{3.66}$$

Using the chain rule, (3.66) and since $p > 2$, we have

$$\begin{aligned}
\frac{d}{dt}\varphi(tu) = \langle\varphi'(tu), u\rangle &= \frac{1}{t}\langle\varphi'(tu), tu\rangle \\
&= \frac{1}{t}\left(\|\nabla(tu)\|_p^p + \|\nabla(tu)\|_2^2 - \int_\Omega f(z,tu)(tu)\,dz\right) \\
&\leqslant \frac{1}{t}\left(\|\nabla(tu)\|_p^p + \|\nabla(tu)\|_2^2 - \int_\Omega pF(z,tu)\,dz + c_{18}|\Omega|_N\right) \\
&\leqslant \frac{1}{t}\left(p\varphi(tu) + c_{18}|\Omega|_N\right).
\end{aligned}$$

Because of (3.65), for $t \geqslant 1$ big, we will have

$$\frac{d}{dt}\varphi(tu) < 0.$$

Let $\eta < \min\{-\frac{c_{18}|\Omega|_N}{p}, \inf_{\overline{B}_1^+} \varphi\}$. The implicit function theorem implies that there is a unique $\sigma \in C(\partial B_1^+)$, $\sigma \geqslant 1$ such that

$$\varphi(tu) = \begin{cases} > \eta & \text{if } t \in [0, \sigma(u)), \\ = \eta & \text{if } t = \sigma(u), \\ < \eta & \text{if } t > \sigma(u). \end{cases} \tag{3.67}$$

From (3.67) and the choice of η, we have

$$\varphi^\eta \subseteq \{tu : u \in \partial B_1^+, \ t \geqslant \sigma(u)\}.$$

Let $E_+ = \{tu : u \in \partial B_1^+, \ t \geqslant 1\}$. We have $\varphi^\eta \subseteq E_+$. Let $\widehat{h} \colon [0,1] \times E_+ \longrightarrow E_+$ be the deformation defined by

$$\widehat{h}(\tau, tu) = \begin{cases} (1-\tau)tu + \tau\sigma(u)u & \text{if } t \in [1, \sigma(u)], \\ tu & \text{if } t > \sigma(u). \end{cases}$$

We have

$$\widehat{h}(1, E_+) \subseteq \varphi^\eta \quad \text{and} \quad \widehat{h}(\tau, \cdot)|_{\varphi^\eta} = \mathrm{id}|_{\varphi^\eta}$$

[see (3.67)], so φ^η is a strong deformation retract of E_+, and thus

$$H_k\left(W_0^{1,p}(\Omega), \varphi^\eta\right) = H_k\left(W_0^{1,p}(\Omega), E_k\right) \quad \forall k \geqslant 0. \tag{3.68}$$

Using the radial retraction and Theorem 6.5 of Dugundji [13, p. 325], we see that

$$E_+ \text{ and } \partial B_1^+ \text{ are homotopy equivalent,}$$

so

$$H_k\left(W_0^{1,p}(\Omega), E_+\right) = H_k\left(W_0^{1,p}(\Omega), \partial B_1^+\right) \quad \forall k \geqslant 0. \tag{3.69}$$

Recall that ∂B_1^+ is contractible in itself. Hence

$$H_k\left(W_0^{1,p}(\Omega), \partial B_1^+\right) = 0 \quad \forall k \geqslant 0$$

(see Motreanu et al. [26, p.147]), thus

$$H_k\left(W_0^{1,p}(\Omega), \varphi^\eta\right) = 0 \quad \forall k \geqslant 0$$

[see (3.68) and (3.69)], so

$$C_k(\varphi, \infty) = 0 \quad \forall k \geqslant 0$$

(choosing $\eta < 0$ even more negative if necessary). □

Now we introduce the stronger regularity conditions on f.
H(f)': $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function such that $f(z, 0) = 0$ for almost all $z \in \Omega$, $f(z, \cdot) \in C^1(\mathbb{R})$ and (i) $|f'_\zeta(z, \zeta)| \leqslant a(z)(1 + |\zeta|^{r-2})$ for almost all $z \in \Omega$, all $\zeta \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $p < r < p^*$; (ii)–(iv) are the same as the corresponding hypotheses $H(f)(ii)$–(iv).

Then we can have the full multiplicity theorem for problem (1.1).

Theorem 3.10 *If hypotheses $H(f)'$ hold, then problem* (1.1) *admits at least three nontrivial solutions*

$$u_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad \widehat{u}, \widehat{y} \in C_0^1(\overline{\Omega}).$$

Proof From Proposition 3.8 we already have two nontrivial smooth solutions

$$u_0 \in -\operatorname{int} C_+ \quad \text{and} \quad \widehat{u} \in C_0^1(\overline{\Omega}).$$

Hypotheses $H(f)'$ imply that $\varphi \in C^2(W_0^{1,p}(\Omega))$. Also, recall that

$$C_1(\varphi, \widehat{u}) \neq 0$$

[see (3.57)]. So, from Papageorgiou and Smyrlis [33] (see also Papageorgiou and Rădulescu [32]), we have

$$C_k(\varphi, \widehat{u}) = \delta_{k,1}\mathbb{Z} \quad \forall k \geqslant 0. \tag{3.70}$$

Also, from Corollary 3.6, we have

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0. \tag{3.71}$$

Finally, from Propositions 3.7 and 3.9, we have

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \geqslant 0, \tag{3.72}$$

$$C_k(\varphi, \infty) = 0 \quad \forall k \geqslant 0. \tag{3.73}$$

Suppose that $K_\varphi = \{0, u_0, \widehat{u}\}$. Then from (3.70), (3.71), (3.72), (3.73) and the Morse relation with $t = -1$ [see (2.5)], we have

$$(-1)^{d_m} + (-1)^0 + (-1)^1 = 0,$$

so $(-1)^{d_m} = 0$, a contradiction.

So, there exists $\widehat{y} \in K_\varphi$, $\widehat{y} \notin \{0, u_0, \widehat{u}\}$. This means that $\widehat{y}$ is the third nontrivial solution of problem (1.1) and using Theorem 1 of Lieberman [23], we conclude that $\widehat{y} \in C_0^1(\overline{\Omega})$. $\square$

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