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Calculus of Variations

Estimates for capacity and discrepancy of convex surfaces in sieve-like domains with an application to homogenization

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Abstract We consider the intersection of a convex surface Γ with a periodic perforation of $\mathbb{R}^d$, which looks like a sieve, given by $T_\varepsilon = \bigcup_{k\in\mathbb{Z}^d}\{\varepsilon k + a_\varepsilon T\}$ where T is a given compact set and $a_\varepsilon \ll \varepsilon$ is the size of the perforation in the ε-cell $(0, \varepsilon)^d \subset \mathbb{R}^d$. When ε tends to zero we establish uniform estimates for p-capacity, $1 < p < d$, of the set $\Gamma \cap T_\varepsilon$. Additionally, we prove that the intersections $\Gamma \cap \{\varepsilon k + a_\varepsilon T\}_k$ are uniformly distributed over Γ and give estimates for the discrepancy of the distribution. As an application we show that the thin obstacle problem with the obstacle defined on the intersection of Γ and the perforations, in a given bounded domain, is homogenizable when $p < 1 + \frac{d}{4}$. This result is new even for the classical Laplace operator.

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1 Introduction

In this paper we study the properties of the intersection of a convex surface Γ with a periodic perforation of $\mathbb{R}^d$ given by $T_\varepsilon = \bigcup_{k\in\mathbb{Z}^d}\{\varepsilon k + a_\varepsilon T\}$, where T is a given compact set and a_ε is the size of the perforation in the ε-cell. Our primary interest is to obtain good control of p-capacity $1 < p < d$ and discrepancy of distributions of the components of the intersection $\Gamma \cap T_\varepsilon$ in terms of ε when the size of perforations tends to zero. As an application of our analysis we get that the thin obstacle problem in periodically perforated domain $\Omega \subset \mathbb{R}^d$

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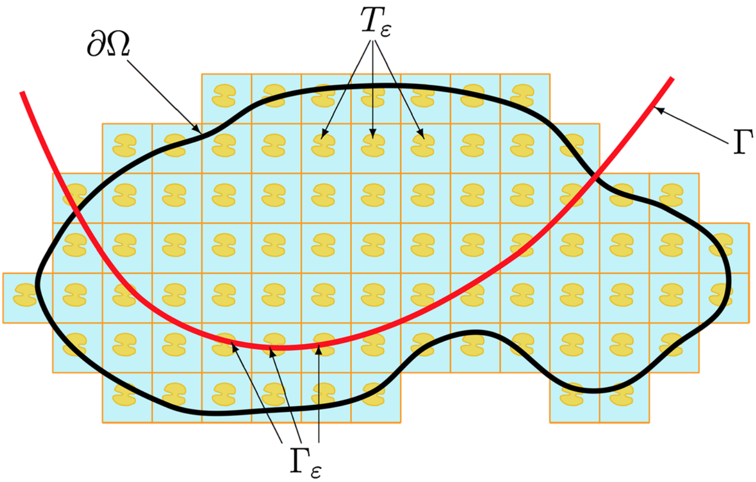

Fig. 1 The sieve-like configuration with convex Γ

with given strictly convex and C^2 smooth surface as the obstacle and p-Laplacian as the governing partial differential equation is homgenizable provided that $p < 1 + \frac{d}{4}$. Moreover, the limit problem admits a variational formulation with one extra term involving the mean capacity, see Theorem 3. The configuration of Γ, Γ_ε, T_ε and Ω is illustrated in Fig. 1.

This result is new even for the classical case $p = 2$ corresponding to the Laplace operator. Another novelty is contained in the proof of Theorem 2 where we use a version of the method of quasi-uniform continuity developed in [4].

1.1 Statement of the problem

Let

$$T_\varepsilon = \bigcup_{k \in \mathbb{Z}^d} \{\varepsilon k + a_\varepsilon T\},$$

and let

$$\Gamma_\varepsilon = \Gamma \cap T_\varepsilon.$$

We assume that Γ is a strictly convex surface in $\mathbb{R}^d$ that locally admits the representation

$$\{(x', g(x')) : x' \in Q'\}, \tag{1}$$

where $Q' \subset \mathbb{R}^{d-1}$ is a cube. For example, Γ may be a compact convex surface, or may be defined globally as a graph of a convex function.

Without loss of generality we assume that $x_d = g(x')$ because the interchanging of coordinates preserves the structure of the periodic lattice in the definition of T_ε. We will also study homogenization of the thin obstacle problem for the p-Laplacian with an obstacle defined on Γ_ε. Our goal is to determine the asymptotic behaviour, as $\varepsilon \to 0$, of the problem

$$\min \left\{ \int_\Omega |\nabla v|^p dx + \int_\Omega hv dx : v \in W_0^{1,p}(\Omega) \text{ and } v \geq \phi \text{ on } \Gamma_\varepsilon \right\}, \tag{2}$$

for given $h \in L^q(\Omega)$, $1/p + 1/q = 1$ and $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We make the following assumptions on Ω, T, Γ, d and p:

(A_1) $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain.
(A_2) The compact set T from which the holes are constructed must be sufficiently regular in order for the mapping

$$t \mapsto \mathrm{cap}(\{\Gamma + te\} \cap T)$$

to be continuous, where e is any unit vector. This is satisfied if, for example, T has Lipschitz boundary.
(A_3) The size of the holes is

$$a_\varepsilon = \varepsilon^{d/(d-p+1)}.$$

This is the critical size that gives rise to an interesting effective equation for (2).
(A_4) The exponent p in (2) is in the range

$$1 < p < \frac{d+4}{4}.$$

This is to ensure that the holes are large enough that we are able to effectively estimate the intersections between the surface Γ and the holes T_ε, of size a_ε. See the discussion following the estimate (15). In particular, if $p = 2$ then $d > 4$.

These are the assumptions required for using the framework from [4], though the (A_4) is stricter here.

1.2 Main results

The following theorems contain the main results of the present paper.

Theorem 1 *Suppose Γ is a C^2 convex surface. Let $I_\varepsilon \subset [0, 1)$ be an interval, let $Q' \subset \mathbb{R}^{d-1}$ be a cube and let*

$$A_\varepsilon = \#\left\{k' \in \mathbb{Z}^{n-1} \cap \varepsilon^{-1}Q' : \frac{g(\varepsilon k')}{\varepsilon} \in I_\varepsilon \quad (mod 1)\right\}.$$

Then

$$\left|\frac{A_\varepsilon}{N_\varepsilon} - |I_\varepsilon|\right| = O(\varepsilon^{\frac{1}{3}}),$$

where $N_\varepsilon = \#\{k' \in \mathbb{Z}^{d-1} \cap \varepsilon^{-1}Q'\}$.

Next we establish an important approximation result. We use the notation $T_\varepsilon^k = \varepsilon k + a_\varepsilon T$ and $\Gamma_\varepsilon^k = \Gamma \cap T_\varepsilon^k$.

Theorem 2 *Suppose Γ is a C^2 convex surface and P_x a support plane of Γ at the point $x \in \Gamma$. Then*

1° *the p-capacity of $P_x^k = P_x \cap T_\varepsilon^k$ approximates $\mathrm{cap}_p(\Gamma_\varepsilon^k)$ as follows*

$$\mathrm{cap}_p(\Gamma_\varepsilon^k) = \mathrm{cap}_p(P_x^k \cap \{a_\varepsilon T + \varepsilon k\}) + o(a_\varepsilon^{d-p}), \qquad (3)$$

where $x \in \Gamma_\varepsilon^k$.

2° *Furthermore, if* P_1 *and* P_2 *are two planes that intersect* $\{a_\varepsilon T + \varepsilon k\}$ *at a point* x*, with normals* ν_1, ν_2 *satisfying* $|\nu_1 - \nu_2| \le \delta$ *for some small* $\delta > 0$*, then*

$$|\operatorname{cap}_p(P_1 \cap \{a_\varepsilon T + \varepsilon k\}) - \operatorname{cap}_p(P_2 \cap \{a_\varepsilon T + \varepsilon k\})| \le c_\delta a_\varepsilon^{d-p}, \tag{4}$$

where $\lim_{\delta\to 0} c_\delta = 0$.

As an application of Theorems 1, 2 we have

Theorem 3 *Let* u_ε *be the solution of* (2)*. Then* $u_\varepsilon \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *as* $\varepsilon \to 0$*, where* u *is the solution to*

$$\min\left\{\int_\Omega |\nabla v|^p dx + \int_{\Gamma\cap\Omega} |(\phi - v)_+|^p \operatorname{cap}_{p,\nu(x)}(T) dH^{d-1} + \int_\Omega f v dx : v \in W_0^{1,p}(\Omega)\right\}. \tag{5}$$

In (5), $\nu(x)$ is the normal of Γ at $x \in \Gamma$ and $\operatorname{cap}_{p,\nu(x)}(T)$ is the mean p-capacity of T with respect to the hyperplane $P_{\nu(x)} = \{y \in \mathbb{R}^d : \nu(x) \cdot y = 0\}$, given by

$$\operatorname{cap}_{p,\nu(x)}(T) = \int_{-\infty}^{\infty} \operatorname{cap}_p(T \cap \{P_{\nu(x)} + t\nu(x)\}) dt, \tag{6}$$

where $\operatorname{cap}_p(E)$ denotes p-capacity of E with respect to $\mathbb{R}^d$.

Theorem 3 was proved by the authors in [4] under the assumption that Γ is a hyper plane, which was in turn a generalization of the paper [5]. In a larger context, Theorem 3 contributes to the theory of homogenization in non-periodic perforated domains, in that the support of the obstacle, Γ_ε, is not periodic. Another class of well-studied non-periodic perforated domains, not including that of the present paper, is the random stationary ergodic domains introduced in [1]. In the case of stationary ergodic domains the perforations are situated on lattice points, which is not the case for the set Γ_ε. The perforations, i.e. the components of Γ_ε, have desultory (though deterministic by definition) distribution. For the periodic setting [2] is a standard reference.

The proof of Theorem 3 has two fundamental ingredients. First the structure of the set Γ_ε is analysed using tools from the theory of uniform distribution, Theorem 1. We prove essentially that the components of Γ_ε are uniformly distributed over Γ with a good bound on the discrepancy. This is achieved by studying the distribution of the sequence

$$\{\varepsilon^{-1} g(\varepsilon k')\}_{k'}, \tag{7}$$

for g defined by (1) and $\varepsilon k' \in Q'$. Second, we construct a family of well-behaved correctors based on the result of Theorem 2.

The major difficulty that arises when Γ is a more general surface than a hyperplane is to estimate the discrepancy of the distribution of (the components of) Γ_ε over Γ, which is achieved through studying the discrepancy of $\{\varepsilon^{-1} g(\varepsilon k')\}_{k'}$. For a definition of discrepancy, see Sect. 2. In the framework of uniform convexity we can apply a theorem of Erdös and Koksma which gives good control of the discrepancy.

2 Discrepancy and the Erdös–Koksma theorem

In this section we formulate a general result for the uniform distribution of a sequence and derive a decay estimate for the corresponding discrepancy.

Definition 1 The discrepancy of the first N elements of a sequence $\{s_j\}_{j=1}^{\infty}$ is given by

$$D_N = \sup_{I \subset (0,1]} \left| \frac{A_N}{N} - |I| \right|,$$

where I is an interval, $|I|$ is the length of I and A_N is the number of $1 \le j \le N$ for which $s_j \in I \pmod 1$.

We first recall the Erdös–Turán inequality, see Theorem 2.5 in [7], for the discrepancy of the sequence $\{s_j\}_{j=1}^{\infty}$

$$D_N \le \frac{1}{n} + \frac{1}{N} \sum_{k=1}^{n} \frac{1}{k} \left| \sum_{j=1}^{N} e^{2\pi i f(j) k} \right| \tag{8}$$

where n is a parameter to be chosen so that the right hand side has optimal decay as $N \to \infty$. Observe that s_j is the j-th element of the sequence which in our case is $s_j = f(j)$ for a given function f and $N = \left[\frac{1}{\varepsilon}\right]$.

We employ the following estimate of Erdös and Koksma ([7], Theorem 2.7) in order to estimate the second sum in (8): let $a, b \in \mathbb{N}$ such that $0 < a < b$ then one has the estimate

$$\left| \sum_{j=1}^{N} e^{2\pi i f(j) k} \right| \le (|F_k'(b) - F_k'(a)| + 2) \left(3 + \frac{1}{\sqrt{\rho}} \right) \tag{9}$$

where $F_k(t) = kf(t)$ and $F_k''(t) \ge \rho > 0$ for some positive number ρ. In order to apply this result to our problem we first need to reduce the dimension of (7) to one. To do so let us assume that the obstacle Γ is given as the graph of a function $x_d = g(x')$ where g is strictly convex C^2 function such that

$$c_0 \delta_{\alpha,\beta} \le D_{x_\alpha x_\beta} g(x') \le C_0 \delta_{\alpha,\beta}, \quad 1 \le \alpha, \beta \le d-1 \tag{10}$$

for some positive constants $c_0 < C_0$.

Next we rescale the ε-cells and consider the normalised problem in the unit cube $[0, 1]^d$. The resulting function is $f(j) = \frac{g(\varepsilon j)}{\varepsilon}$, $j \in \mathbb{Z}^{d-1}$.

If $d = 2$ then we can directly apply (9) to the scaled function f above. Otherwise for $d > 2$ we need an estimate for the multidimensional discrepancy in terms of D_N introduced in Definition 1, a similar idea was used in [4] for the linear obstacle. Suppose for a moment that this is indeed the case. Then we can take $F_k(t) = kf(t)$ in (9) and noting

$$D_{x_\alpha} f(x') = kD_\alpha g(\varepsilon x'), \quad D^2_{x_\alpha} f(x') = k\varepsilon D^2_\alpha g(\varepsilon x') \ge k\varepsilon c_0, \quad 1 \le \alpha \le d-1 \tag{11}$$

one can proceed as follows

$$\begin{aligned}
\left| \sum_{j=1}^{N} e^{2\pi i f(j) k} \right| &\le (|kD_{x_\alpha} g(\varepsilon N) - kD_\alpha g(\varepsilon)| + 2) \left(3 + \frac{1}{\sqrt{k\varepsilon c_0}} \right) \\
&\le (k\varepsilon C_0(N-1) + 2) \left(3 + \frac{1}{\sqrt{k\varepsilon c_0}} \right) \\
&\le k \left(\varepsilon C_0(N-1) + \frac{2}{k} \right) \left(3 + \frac{1}{\sqrt{k\varepsilon c_0}} \right)
\end{aligned}$$

$$\leq k\left(\varepsilon C_0(N-1)+\frac{2}{k}\right)\left(3+\sqrt{\frac{N}{kc_0}}\right)$$
$$\leq \lambda k\left(1+\sqrt{\frac{N}{k}}\right)$$

for some tame constant $\lambda > 0$ independent of ε, k. Plugging this into (8) yields

$$\begin{aligned} D_N &\leq \frac{1}{n}+\frac{\lambda}{N}\sum_{k=1}^{n}\left(1+\sqrt{\frac{N}{k}}\right) \\ &= \frac{1}{n}+\frac{\lambda n}{N}+\frac{\lambda}{\sqrt{N}}\sum_{k=1}^{n}\frac{1}{\sqrt{k}} \\ &\leq \frac{1}{n}+\overline{\lambda}\sqrt{\frac{n}{N}}\left(1+\sqrt{\frac{n}{N}}\right) \end{aligned}$$

for another tame constant $\overline{\lambda} > 0$. Now to get the optimal decay rate we choose $\frac{1}{n} = \sqrt{\frac{n}{N}}$ which yields $N = n^3$ and hence

$$n = N^{\frac{1}{3}} \approx \frac{1}{\varepsilon^{\frac{1}{3}}}$$

and we arrive at the estimate

$$D_N = O(\varepsilon^{\frac{1}{3}}). \tag{12}$$

2.1 Proof of Theorem 1

Proof Suppose Q' is a cube of size r. Then there is a cube $Q'' \subset \mathbb{R}^{d-2}$ such that $Q' = [\alpha, \beta] \times Q'$, $\beta - \alpha = r$. We may rewrite A_ε as

$$A_\varepsilon = \sum_{k'' \in \varepsilon^{-1}Q'' \cap \mathbb{Z}^{d-2}} \#\left\{k_1 \in \mathbb{Z}\colon a \leq k_1 \leq b \text{ and } \varepsilon^{-1}g(\varepsilon k_1 + \varepsilon k'') \in I_\varepsilon \pmod 1\right\},$$

where $(k_1, k'') = k'$, a, b are the integer parts of $\varepsilon^{-1}\alpha$ and $\varepsilon^{-1}\beta$ respectively and $|(b-a) - \varepsilon^{-1}r| \leq 1$. We also note that $N_\varepsilon = (\varepsilon^{-1}r)^{d-1} + O(\varepsilon^{-1}r)^{d-2}$. Consider

$$A_\varepsilon^1(k'') = \#\left\{k_1 \in \mathbb{Z}\colon a \leq k_1 \leq b \text{ and } \varepsilon^{-1}g(\varepsilon k_1 + \varepsilon k'') \in I_\varepsilon \pmod 1\right\}.$$

Then we have

$$\frac{A_\varepsilon}{N_\varepsilon} - |I_\varepsilon| = \frac{1}{(\varepsilon^{-1}r)^{d-2}} \sum_{k'' \in \varepsilon^{-1}Q'' \cap \mathbb{Z}^{d-2}} \frac{A_\varepsilon^1(k'')}{(\varepsilon^{-1}r)} - |I_\varepsilon|. \tag{13}$$

For each k'' the function $h\colon s \to \varepsilon^{-1}g(\varepsilon s + \varepsilon k'')$ satisfies $|h'(s)| \leq C_1$ and $h''(s) \geq \rho\varepsilon$ for $a \leq s \leq b$. Thus we may apply the Erdös-Koksma Theorem as described above and conclude that

$$\left|\frac{A_\varepsilon^1(k'')}{(\varepsilon^{-1}r)} - |I_\varepsilon|\right| \leq C\varepsilon^{\frac{1}{3}}.$$

It follows that the modulus of the left hand side of (13) is bounded by $C\varepsilon^{\frac{1}{3}}$, proving the theorem. □

3 Correctors

The purpose of this section is to construct a sequence of correctors that satisfy the hypotheses given below. Once we have established the existence of these correctors, the proof of the Theorem 3 is identical to the planar case treated in [4].

H1 $0 \leq w_\varepsilon \leq 1$ in $\mathbb{R}^d$, $w_\varepsilon = 1$ on Γ_ε and $w_\varepsilon \rightharpoonup 0$ in $W^{1,p}_{\text{loc}}(\mathbb{R}^d)$,
H2 $\int_\Omega |\nabla w_\varepsilon|^p f dx \to \int\limits_\Gamma f(x) \operatorname{cap}_{p,\nu_x} d\mathcal{H}^{\lceil-\infty}$, for any $f \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$,
H3 (weak continuity) for any $\phi_\varepsilon \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{cases} \sup\limits_{\varepsilon>0} \|\phi_\varepsilon\|_{L^\infty(\Omega)} < \infty, \\ \phi_\varepsilon = 0 \text{ on } \Gamma_\varepsilon \text{ and } \phi_\varepsilon \rightharpoonup \phi \in W^{1,p}_0(\Omega), \end{cases}$$

we have

$$\langle -\Delta_p w_\varepsilon, \phi_\varepsilon \rangle \to \langle \mu, \phi \rangle$$

with

$$d\mu(x) = \operatorname{cap}_{p,\nu(x)} d\mathcal{H}^{\lceil-\infty} \llcorner \Gamma, \tag{14}$$

where $\operatorname{cap}_{p,\nu(x)}$ is given by (6) and $\mathcal{H}^{\int} \llcorner \Gamma$ is the restriction of $s-$dimensional Hausdorff measure on Γ.

Setting $\Gamma^k_\varepsilon := \Gamma \cap \{a_\varepsilon T + \varepsilon k\} \neq \emptyset$, we define w^k_ε by

$$\begin{aligned} \Delta_p w^k_\varepsilon &= 0 \quad \text{in } B_{\varepsilon/2}(\varepsilon k) \backslash \Gamma^k_\varepsilon, \\ w^k_\varepsilon &= 0 \quad \text{on } \partial B_{\varepsilon/2}(\varepsilon k), \\ w^k_\varepsilon &= 1 \quad \text{on } \Gamma^k_\varepsilon. \end{aligned}$$

Then it follows from the definition of cap_p [3] that

$$\int_{B_{\varepsilon/2}(\varepsilon k)} |\nabla w^k_\varepsilon|^p dx = \operatorname{cap}_p(\Gamma^k_\varepsilon) + o(a^{d-p}_\varepsilon).$$

Indeed, we have

$$\begin{aligned} \operatorname{cap}_p(\Gamma^k_\varepsilon, B_{\varepsilon/2}(\varepsilon k)) &= \inf \left\{ \int_{B_{\varepsilon/2}} |\nabla w|^p : w \in W^{1,p}_0(B_{\varepsilon/2}(\varepsilon k)) \text{ and } w = 1 \text{ on } \Gamma^k_\varepsilon \right\} \\ &= a^{d-p}_\varepsilon \inf \left\{ \int_{B_{\varepsilon/2a_\varepsilon}} |\nabla w|^p : w \in W^{1,p}_0(B_{\varepsilon/2a_\varepsilon} \text{ and } w = 1 \text{ on } \frac{1}{a_\varepsilon}\Gamma^k_\varepsilon \right\} \\ &= a^{d-p}_\varepsilon \left(\operatorname{cap}_p\left(\frac{1}{a_\varepsilon}\Gamma^k_\varepsilon\right) + o(1) \right) \\ &= \operatorname{cap}_p(\Gamma^k_\varepsilon) + o(a^{d-p}_\varepsilon). \end{aligned}$$

Note that $\operatorname{cap}_p(\Gamma^k_\varepsilon) = O(a^{d-p}_\varepsilon)$ since $\Gamma^k_\varepsilon = \Gamma \cap \{\varepsilon k + a_\varepsilon T\}$ and $\operatorname{cap}_p(tE) = t^{d-p} \operatorname{cap}_p(E)$ if $t \in \mathbb{R}_+$ and $E \subset \mathbb{R}^d$. If Q' is a cube in $\mathbb{R}^{d-1}$, the components of $\Gamma_\varepsilon \cap Q' \times \mathbb{R}$ are of the form $\Gamma^k_\varepsilon = \Gamma \cap \{(\varepsilon k', \varepsilon k_d) + a_\varepsilon T\}$ for $\varepsilon k' \in Q'$. In particular, $\Gamma^k_\varepsilon \neq \emptyset$ if and only if

$\varepsilon^{-1}g(\varepsilon k') \in I_\varepsilon \pmod 1$ where $|I_\varepsilon| = O(a_\varepsilon/\varepsilon)$. Thus Theorem 1 tells us that the number of components of $\Gamma_\varepsilon \cap Q' \times \mathbb{R}$ equals $A_\varepsilon = |I_\varepsilon|N_\varepsilon + N_\varepsilon O(\varepsilon^{\frac{1}{3}})$, or explicitly

$$\left| \frac{\frac{A_\varepsilon}{N_\varepsilon}}{\frac{a_\varepsilon}{\varepsilon}} - 1 \right| = \frac{O(\varepsilon^{\frac{1}{3}})}{\frac{a_\varepsilon}{\varepsilon}}. \tag{15}$$

Here we need to have $\varepsilon^{1/3} = o(|I_\varepsilon|)$, which is equivalent to (A_4). Since

$$\int_{B_{\varepsilon/2}(\varepsilon k)} |\nabla w_\varepsilon^k|^p dx = \mathrm{cap}_p(\Gamma_\varepsilon^k) + o(a_\varepsilon^{d-p}),$$

we get

$$\int_{\mathbb{R}\times Q'} |\nabla w_\varepsilon|^p dx \le C(|I_\varepsilon| N_\varepsilon \, \mathrm{cap}_p(\Gamma_\varepsilon^k)) \le C\frac{a_\varepsilon}{\varepsilon}\varepsilon^{1-d}|Q'|a_\varepsilon^{n-p} = C|Q'|.$$

Thus $\int_K |\nabla w_\varepsilon|^p$ is uniformly bounded on compact sets K. Since $w_\varepsilon(x) \to 0$ pointwise for $x \notin \Gamma$, **H1** follows.

When verifying $\mathbf{H_2}$ and $\mathbf{H_3}$ we will only prove that

$$\lim_{\varepsilon\to 0} \int_Q |\nabla w_\varepsilon|^p dx = \int_{\Gamma\cap Q} c_{\nu(x)} d\mathcal{H}^{d-1}(x), \quad \text{for all cubes } Q \subset \mathbb{R}^d. \tag{16}$$

Once this has been established the rest of the proof is identical to that given in [4].

4 Proof of Theorem 2

Proof **1°** Set $R_\varepsilon = \frac{\varepsilon}{2a_\varepsilon} \to \infty$, then after scaling we have to prove that

$$\int_{B_{R_\varepsilon}} |\nabla v_1|^p - \int_{B_{R_\varepsilon}} |\nabla v_2|^p = o(1) \tag{17}$$

uniformly in ε where

$$\begin{aligned} \Delta_p v_i &= 0 \quad \text{in } B_{R_\varepsilon} \setminus S_i, \\ v_i &= 0 \quad \text{on } \partial B_{R_\varepsilon}, \\ v_i &= 1 \quad \text{on } S_i. \end{aligned}$$

and $S_1 = \frac{1}{a_\varepsilon}\Gamma_\varepsilon^k$, $S_2 = \frac{1}{a_\varepsilon}P_x$.

We approximate v_i in the domain $B_{R_\varepsilon}\setminus D_i^t$ with D_i^t being a bounded domain with smooth boundary and $D_i^t \to S_i$ as $t \to 0$ in Hausdorff distance. Consider

$$\begin{aligned} \Delta_p v_i^t &= 0 \quad \text{in } B_{R_\varepsilon} \setminus D_i^t, \\ v_i^t &= 0 \quad \text{on } \partial B_{R_\varepsilon}, \\ v_i^t &= 1 \quad \text{on } \partial D_i^t. \end{aligned}$$

Observe that $\int_{B_{R_\varepsilon}\setminus D_i^t} |\nabla v_i^t|^p$, $i = 1, 2$ remain bounded as $t \to 0$ thanks to Caccioppoli's inequality. Indeed, $w = (1 - v_i^t)\eta \in W_0^{1,p}(B_5\setminus D_i^t)$ where $\eta \in C_0^\infty(B_5)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ in B_3. Using w as a test function we conclude that

$$\int_{B_5\setminus D_i^t} |\nabla v_i^t|^p \eta = \int_{B_5\setminus D_i^t} |\nabla v_i^t|^{p-2}\nabla v_i^t \nabla\eta(1 - v_i^t).$$

Since $\eta \equiv 1$ in B_3 then applying Hölder inequality we infer that $\int_{B_3\setminus D_i^t} |\nabla v_i^t|^p \le C\int_{B_5}(1-v_i^t)^p$. In $B_{R_\varepsilon}\setminus B_2$ the L^p we compare $W(x)=|x/2|^{\frac{p-d}{p-1}}$ with v_i. Note that our assumption A_4 implies that $p<d$. Moreover, since W is p-harmonic in $B_{R_\varepsilon}\setminus B_2$ then the comparison principle yields $v_i\le W$ in $B_{R_\varepsilon}\setminus B_2$. From the proof of Caccioppoli's inequality above choosing non-negative $\eta\in C^\infty(\mathbb{R}^d)$ such that $\eta\equiv 0$ in B_2, $\frac{1}{2}\le\eta\le 1$ in $B_{R_\varepsilon}\setminus B_3$, and $\eta=1$ in $\mathbb{R}^d\setminus B_{R_\varepsilon}$ and using $\eta v_i\in W_0^{1,p}(B_{R_\varepsilon}\setminus B_2)$ as a test function we infer

$$\int_{B_{R_\varepsilon}\setminus B_3}|\nabla v_i|^p\le\frac{C}{R_\varepsilon^p}\int_{B_{R_\varepsilon}\setminus B_2}v_i^p\le\frac{C}{R_\varepsilon^{\frac{1}{p-1}}}\to 0\quad\text{as } \varepsilon\to 0$$

where the last bound follows from the estimate $v_i\le W$. Combining these estimates we infer

$$\|v_i^t\|_{W^{1,p}(B_{R_\varepsilon})}\le K,\quad i=1,2 \tag{18}$$

for some tame constant K independent of t and ε. Thus, by construction $v_i^t\rightharpoonup v_i$ weakly in $W_0^{1,p}(B_{R_\varepsilon})$.

Let $\psi\in C^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\psi\supset D_1^t\cup D_2^t$ and $\psi\equiv 1$ in $\mathbb{R}^d\setminus B_2$. Then the function $\psi(v_1^t-v_2^t)\in W_0^{1,p}(B_{R_\varepsilon})$ and it vanishes on $\operatorname{supp}\psi\supset D_1^t\cup D_2^t$. Thus we have

$$\begin{aligned}&\int_{B_{R_\varepsilon}}(\nabla v_1^t|\nabla v_1^t|^{p-2}-\nabla v_2^t|\nabla v_2^t|^{p-2})(\nabla v_1^t-\nabla v_2^t)\psi\\&=-\int_{B_{R_\varepsilon}}(\nabla v_1^t|\nabla v_1^t|^{p-2}-\nabla v_2^t||\nabla v_2^t|^{p-2})(v_1^t-v_2^t)\nabla\psi\end{aligned}$$

Note that $v_1^t-v_2^t=0$ on $D_1^t\cap D_2^t$. Choosing a sequence ψ_n such that $1-\psi_m$ converges to the characteristic function $\chi_{D_1^t\cup D_2^t}$ of the set $D_1^t\cup D_2^t$ we conclude

$$\int_{B_{R_\varepsilon}}(\nabla v_1^t|\nabla v_1^t|^{p-2}-\nabla v_2^t|\nabla v_2^t|^{p-2})(\nabla v_1^t-\nabla v_2^t)=J_1+J_2 \tag{19}$$

where

$$\begin{aligned}J_1&=\int_{\partial D_1^t}(1-v_2^t)[\partial_\nu v_1^t|\nabla v_1^t|^{p-2}-\partial_\nu v_2^t|\nabla v_2^t|^{p-2}],\\J_2&=\int_{\partial D_2^t}(v_1^t-1)[\partial_\nu v_1^t|\nabla v_1^t|^{p-2}-\partial_\nu v_2^t|\nabla v_2^t|^{p-2}].\end{aligned}$$

Notice that on ∂D_i^t we have that $\nu=-\frac{\nabla\psi_m}{|\nabla\psi_m|}$ is the unit normal pointing inside D_i^t. We denote $n=-\nu$ and then we have that

$$\begin{aligned}-\int_{\partial D_1^t}(1-v_2^t)\partial_\nu v_2^t|\nabla v_2^t|^{p-2}&=\int_{\partial D_1^t}(1-v_2^t)\partial_n v_2^t|\nabla v_2^t|^{p-2}\\&=\int_{\partial(D_1^t\cap D_2^t)}(1-v_2^t)\partial_n v_2^t|\nabla v_2^t|^{p-2}\\&=\int_{D_1^t\setminus D_2^t}\operatorname{div}((1-v_2^t)\nabla v_2^t|\nabla v_2^t|^{p-2})\\&=-\int_{D_1^t\setminus D_2^t}|\nabla v_2^t|^p,\end{aligned}$$

and similarly

$$\int_{\partial D_2^t} (v_1^t - 1)\partial_\nu v_1^t |\nabla v_1^t|^{p-2} = -\int_{D_2^t \setminus D_1^t} |\nabla v_1^t|^p.$$

Setting

$$I = \int_{B_{R_\varepsilon}} (\nabla v_1^t |\nabla v_1^t|^{p-2} - \nabla v_2^t |\nabla v_2^t|^{p-2})(\nabla v_1^t - \nabla v_2^t) \tag{20}$$

and returning to (19) we infer

$$\begin{aligned} I &= -\int_{D_1^t \setminus D_2^t} |\nabla v_2^t|^p - \int_{D_2^t \setminus D_1^t} |\nabla v_1^t|^p + \int_{\partial D_1^t} (1 - v_2^t)\partial_\nu v_1^t |\nabla v_1^t|^{p-2} \\ &\quad - \int_{\partial D_2^t} (v_1^t - 1)\partial_\nu v_2^t |\nabla v_2^t|^{p-2} \\ &\le \int_{\partial D_1^t} (1 - v_2^t)\partial_\nu v_1^t |\nabla v_1^t|^{p-2} - \int_{\partial D_2^t} (v_1^t - 1)\partial_\nu v_2^t |\nabla v_2^t|^{p-2} \\ &\le \sup_{D_1^t}(1 - v_2^t) \int_{\partial D_1^t} |\partial_\nu v_1^t| |\nabla v_1^t|^{p-2} + \sup_{D_2^t}(1 - v_1^t) \int_{\partial D_2^t} |\partial_\nu v_2^t| |\nabla v_2^t|^{p-2}. \end{aligned}$$

But on ∂D_i^t we have $\partial_\nu v_i^t \ge 0$ (ν points inside D_i^t) because v_i^t attains its maximum on ∂D_i^t. Thus we can omit the absolute values of the normal derivatives and obtain

$$\begin{aligned} I &\le \sup_{D_1^t}(1 - v_2^t) \int_{\partial D_1^t} \partial_\nu v_1^t |\nabla v_1^t|^{p-2} + \sup_{D_2^t}(1 - v_1^t) \int_{\partial D_2^t} \partial_\nu v_2^t |\nabla v_2^t|^{p-2} \\ &= \sup_{D_1^t}(1 - v_2^t) \int_{B_{R_\varepsilon} \setminus D_1^t} \operatorname{div}(v_1 \nabla v_1^t |\nabla v_1^t|^{p-2}) + \sup_{D_2^t}(1 - v_1^t) \int_{B_{R_\varepsilon} \setminus D_2^t} \operatorname{div}(v_2 \nabla v_2^t |\nabla v_2^t|^{p-2}) \\ &= \sup_{D_1^t}(1 - v_2^t) \int_{B_{R_\varepsilon} \setminus D_1^t} |\nabla v_1^t|^p + \sup_{D_2^t}(1 - v_1^t) \int_{B_{R_\varepsilon} \setminus D_2^t} |\nabla v_2^t|^p. \end{aligned}$$

Recall that by Lemma 5.7 [6] there is a generic constant $M > 0$ such that

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\xi)(\xi - \eta) \ge M \begin{cases} |\xi - \eta|^p & \text{if } p > 2, \\ |\xi - \eta|^2(|\xi| + |\eta|)^{p-2} & \text{if } 1 < p \le 2 \end{cases} \tag{21}$$

for all $\xi, \eta \in \mathbb{R}^d$.

First suppose that $p > 2$ then applying inequality (21) to (20) yields

$$I \ge M \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p.$$

As for the case $1 < p \le 2$ then from (21) we have

$$I \ge M \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^2 (|\nabla v_1^t| + |\nabla v_2^t|)^{p-2}.$$

But, from Hölder's inequality and (18) we get

$$
\begin{aligned}
&\int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p \\
&= \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p (|\nabla v_1^t| + |\nabla v_2^t|)^{\frac{p(p-2)}{2}} (|\nabla v_1^t| + |\nabla v_2^t|)^{-\frac{p(p-2)}{2}} \\
&\le \left(\int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^2 (|\nabla v_1^t| + |\nabla v_2^t|)^{p-2} \right)^{\frac{p}{2}} \left(\int_{B_{R_\varepsilon}} (|\nabla v_1^t| + |\nabla v_2^t|)^p \right)^{1-\frac{p}{2}} \\
&\le \left(\frac{I}{M} \right)^{\frac{p}{2}} (2K)^{1-\frac{p}{2}}. \qquad (22)
\end{aligned}
$$

Therefore, there is a tame constant M_0 such that for any $p > 1$ we have

$$
\begin{aligned}
&\int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p \\
&\le M_0 \left[\sup_{D_1^t}(1 - v_2^t) \int_{B_{R_\varepsilon} \setminus D_1^t} |\nabla v_1^t|^p + \sup_{D_2^t}(1 - v_1^t) \int_{B_{R_\varepsilon} \setminus D_2^t} |\nabla v_2^t|^p \right]^{\min(1, \frac{p}{2})}.
\end{aligned}
$$

Letting $t \to 0$ we get

$$
\begin{aligned}
\int_{B_{R_\varepsilon}} |\nabla v_1 - \nabla v_2|^p &\le \liminf_{t \to 0} \int_{B_{R_\varepsilon}} |\nabla v_1^t - \nabla v_2^t|^p \\
&\le M_1 \liminf_{t \to 0} \left[\sup_{D_1^t}(1 - v_2^t) + \sup_{D_2^t}(1 - v_1^t) \right]^{\min(1, \frac{p}{2})}. \qquad (23)
\end{aligned}
$$

with some tame constant M_1.

Since $1 - v_i^t$ are nonnegative p-subsolutions in B_{R_ε}, from the weak maximum principle, Theorem 3.9 [6] we obtain

$$
\sup_{B_{\sigma r}(z)} (1 - v_i^t) \le \frac{C}{(1 - \sigma)^{n/p}} \left(⨍_{B_r(z)} (1 - v_i^t)^p \right)^{\frac{1}{p}}. \qquad (24)
$$

Take a finite covering of D_i^t with balls $B_r(z_k^i)$, $z_k^i \in S_i$, $r = 3a_\varepsilon$, $k = 1, \ldots, N$. Choose t small enough such that $D_j^t \subset \bigcup_{k=1}^N B_r(z_k^i)$ and applying (24) we obtain for $i, j \in \{1, 2\}$ with $i \neq j$

$$
\sup_{D_j^t}(1 - v_i^t) \le \max_k \sup_{B_r(z_k^i)} (1 - v_i^t) \le C \max_k \left(⨍_{B_{2r}(z_k^i)} (1 - v_i^t)^p \right)^{\frac{1}{p}}.
$$

Since $\|v_i^t\|_{W^{1,p}(B_3)} \le C$ uniformly for all $t > 0$ it follows that $v_i^t \to v_1$ strongly in $L^p(B_3)$ and v_i is quasi-continuous. In other words, for any positive number θ there is a set E_θ such that $\mathrm{cap}_p E_\theta < \theta$ and v_i is continuous in $B_2 \setminus E_\theta$. Notice that $E_\theta \subset S_1 \cup S_2$ and hence $\mathcal{H}^d(E_\theta) = 0$.

This yields

$$\lim_{t\to 0} \fint_{B_r(z_k^i)} (1-v_i^t)^p = \fint_{B_r(z_k^i)} (1-v_i)^p = \fint_{B_{2r}(z_k^i)\cap E_\theta} (1-v_i)^p + \fint_{B_{2r}(z_k^i)\setminus E_\theta} (1-v_i)^p = \fint_{B_{2r}(z_k^i)\setminus E_\theta} (1-v_i)^p \le C[\omega_i(6a_\varepsilon)]^p \tag{25}$$

where $\omega_i(\cdot)$ is the modulus of continuity of v_i on B_3 modulo the set E_θ. Thus

$$\int_{B_{R_\varepsilon}} |\nabla v_1 - \nabla v_2|^p \le C[\omega_1(6a_\varepsilon) + \omega_2(6a_\varepsilon)]^{p\min(1,\frac{p}{2})}.$$

Hence (17) is established. Rescaling back and noting that $a_\varepsilon^{d-p}\omega_i(a_\varepsilon) = o(a_\varepsilon^{d-p})$ the result follows. Observe that L^p norm of ∇v_i^t remains uniformly bounded in B_{R_ε} by (18) and hence the moduli of quasi-continuity in, say, B_3 do not depend on the particular choice of Γ_ε^k or the tangent plane P_x^k.

2° We recast the argument above but now for $S_1 = \frac{1}{a_\varepsilon}P_1$, $S_2 = \frac{1}{a_\varepsilon}P_2$. Squaring the inequality $|\nu_1 - \nu_2| \le \delta$ we get that $2\sin\frac{\beta}{2} \le \delta$ where β is the angle between P_1 and P_2. Since δ now measures the deviation of v_1^t from 1 on D_2^t, (resp. v_2^t on D_1^t) we conclude that the corresponding moduli of continuity of the limits v_1, v_2 (as $t \to 0$) modulo a set $E_\theta \subset S_1 \cup S_2$ with small $p-$capacity depend on δ, i.e.

$$\fint_{B_r(z_k^i)} (1-v_i)^p \le C[\omega_i(12\delta)]^p \tag{26}$$

where $B_r(z_k^i)$ provide a covering of D_i^t as above but now, say, $r = 6\delta$. Hence we can take $c_\delta = C(\omega_1(12\delta) + \omega_2(12\delta))$. □

5 Proof of Theorem 3

We now formulate our result on the local approximation of total capacity (say in Q') by tangent planes of Γ and prove (16).

Lemma 1 *Fix a cube $Q' \subset \mathbb{R}^{d-1}$ such that if $x = (x', x_d)$ and $y = (y', y_d)$ belong to Γ and $x', y' \in Q'$, then the normals ν_x, ν_y of Γ at x and y satisfy $|\nu_x - \nu_y| \le \delta$. Then for any $x = (x', x_d) \in \Gamma$ with $x' \in Q'$, there holds*

$$\lim_{\varepsilon\to 0} \sum_{k\in\mathbb{Z}^n : k'\in\varepsilon^{-1}Q'} \int_{B_\varepsilon^k} |\nabla w_\varepsilon^k|^p dx = [\mathrm{cap}_{p,\nu_x}(T) + O(C_\delta)]\mathcal{H}^{d-1}(\Gamma_{Q'}),$$

where $\lim_{\delta\to 0} C_\delta = 0$ *and* $\Gamma_{Q'} = \{x \in \Gamma : x' \in Q'\}$.

Proof Fix $x \in \Gamma_{Q'}$ and let P be the plane $\{y : y \cdot \nu_x = 0\}$, where ν_x is the normal of Γ at x. Suppose $k = (k', k_d) \in \mathbb{Z}^d$, $\varepsilon k' \in Q'$ and let P_{x^k} be the tangent plane to Γ at $x^k = (\varepsilon k', g(\varepsilon k'))$. Then Theorem 2 **1°** tells us that

$$\mathrm{cap}_p(\Gamma_\varepsilon^k) = \mathrm{cap}_p(P_{x^k} \cap T_\varepsilon^k) + o(a_\varepsilon^{d-p}).$$

If we set $P_\varepsilon^k = P + (-\varepsilon k', g(\varepsilon k'))$, then P_ε^k will intersect the point $(\varepsilon k', g(\varepsilon k'))$. By assumption, $|\nu_x - \nu_{x^k}| \le \delta$, so

$$\mathrm{cap}_p(P_\varepsilon^k \cap T_\varepsilon^k) = \mathrm{cap}_p(P_{x^k} \cap T_\varepsilon^k) + O(c_\delta a_\varepsilon^{d-p}),$$

by Theorem 2 **2**°. This gives $\mathrm{cap}_p(\Gamma_\varepsilon^k) = \mathrm{cap}_p(P_\varepsilon^k \cap T_\varepsilon^k) + O(c_\delta a_\varepsilon^{d-p})$. Since, by Theorem 1, the sequence $\{\varepsilon^{-1} g(\varepsilon k')\}_{k' \in \varepsilon^{-1} Q'}$ is uniformly distributed mod 1 with discrepancy of order $\varepsilon^{1/3}$, the rescaled planes $\varepsilon^{-1} P_\varepsilon^k$ have the same distribution mod 1, i.e. they are translates of P and the translates have the same distribution. Using the proof of Lemma 4 of [4], we conclude that

$$\lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}^n : k' \in \varepsilon^{-1} Q'} \mathrm{cap}_p(\{P_\varepsilon^k\} \cap T_\varepsilon^k) = \mathrm{cap}_{p,\nu_x}(T)\mathcal{H}^{d-1}(P_{Q'}),$$

where $P_{Q'} = \{x \in P : x' \in Q'\}$. Since we know that $\int_{B_\varepsilon^k} |\nabla w_\varepsilon^k|^p dx = \mathrm{cap}_p(\Gamma_\varepsilon^k) + o(a_\varepsilon^{d-p})$, the result follows from the fact that $\mathcal{H}^{d-1}(\Gamma_{Q'}) = (1 + O(c_\delta))\mathcal{H}^{d-1}(P_{Q'})$. □

Lemma 2

$$\lim_{\varepsilon \to 0} \int_Q |\nabla w_\varepsilon|^p dx = \int_{\Gamma \cap Q} \mathrm{cap}_{p,\nu_x}(T) d\mathcal{H}^{d-1}.$$

Proof The claim follows by decomposing the set $\{x' \in \mathbb{R}^{d-1} : (x', g(x')) \in \Gamma \cap Q\}$ into disjoint cubes $\{Q_j'\}$ that satisfy the hypothesis of Lemma 1. Since Γ is C^2, we can find a finite number of disjoint cubes $\{Q_j\}_{j=1}^{N(\delta)}$, such that $\mathcal{H}^{d-1}(\Gamma \cap Q \backslash \cup_j Q_j \cap \Gamma) = 0$ and Q_j' is as in Lemma 1. For all $x \in \Gamma \cap Q_j$ we have $x = (x', g(x))$ for $x' \in Q_j'$, after interchanging coordinate axes if necessary. Thus

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_Q |\nabla w_\varepsilon|^p dx &= \sum_j \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}^n : k' \in \varepsilon^{-1} Q_j'} \int_{B_\varepsilon^k} |\nabla w_\varepsilon^k|^p dx \\ &= \sum_{x^j \in Q_j'} [\mathrm{cap}_{p,\nu_{x^j}}(T) + O(C_\delta)]\mathcal{H}^{d-1}(\Gamma_{Q_j'}) \\ &= \int_{\Gamma \cap Q} \mathrm{cap}_{p,\nu(x)}(T) d\mathcal{H}^{d-1} + O(C_\delta),\end{aligned}$$

where in the last step we used that $\mathrm{cap}_{p,\nu(x)}(T) = \mathrm{cap}_{p,\nu_{x^j}}(T) + O(C_\delta)$ for all $x \in \Gamma_{Q_j'}$, by Lemma 1. Sending $\delta \to 0$ proves the lemma. □

Having established Lemma 2, the rest of the proof of $\mathbf{H_2}$ and $\mathbf{H_3}$ is carried out precisely as in [4], with Lemma 2 above replacing Lemma 4 in [4]. The proof of Theorem 3 from $\mathbf{H_1}$–$\mathbf{H_3}$ is given in section 4 of [4] when Γ is a hyper plane, and remains the same for the present case when Γ is a convex surface.

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